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SU(2) reduction in $\mathcal{N} = 4$ supersymmetric mechanics

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We perform an $su(2)$ Hamiltonian reduction of the general $su(2)$-invariant action for a self-coupled (4, 4, 0) supermultiplet. As a result, we elegantly recover the $\mathcal{N} = 4$ supersymmetric mechanics with spin degrees of freedom which was recently constructed in [S. Fedoruk, E. Ivanov, and O. Lechtenfeld, Phys. Rev. D **79**, 105015 (2009)]. This observation underscores the exceptional role played by the root supermultiplet in $\mathcal{N} = 4$ supersymmetric mechanics.

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I. INTRODUCTION

In a recent paper [1], $\mathcal{N} = 4$ superconformal mechanics with n bosonic and $4n$ fermionic degrees of freedom has been endowed with a potential term through a coupling to auxiliary supermultiplets with $4n$ bosonic and $4n$ fermionic components [2]. This combination gave rise to an $OSp(4|2)$ supersymmetric n-particle Calogero model. Subsequently, the one-particle case, i.e. $OSp(4|2)$ superconformal mechanics, was analyzed on the classical and quantum level [3]. Simultaneously, it was demonstrated that the potential-generating strategy works perfectly for the most general $D(2, 1; \alpha)$ superconformal one-particle mechanics [4]. It is quite satisfying how the spin degrees of freedom appear in the bosonic sector, with only first time derivatives in the action. Thus, the proposed coupling of two different $\mathcal{N} = 4$ supermultiplets provides a simple and elegant way to incorporate spin degrees of freedom in supersymmetric mechanics.

In both previous treatments [3,4], on mass shell all components of the basic (1, 4, 3) supermultiplet are expressed through those of the "auxiliary" (4, 4, 0) one. It seems that just this auxiliary supermultiplet plays a fundamental role in the construction. It is therefore natural to inquire whether the these models can be reformulated *purely* in terms of (4, 4, 0) supermultiplets. Of course, such a reformulation has to be supplied with a Hamiltonian reduction, which would reduce the four physical bosons to one boson plus spin variables. Alternatively, the passage from SU(2)-symmetric (4, 4, 0) models to general (1, 4, 3) models via gauging was described in [2] using harmonic superspace.

Incidentally, spin degrees of freedom have appeared in a bosonic system after Hamiltonian reduction (on the Lagrangian level) via the second Hopf map $S^7/S^3 \simeq S^4$ [5]. In the bosonic sector this reduced system resembles those in [3,4], besides the presence of four additional bosonic variables.

In the present paper we realize the above ideas and rederive the $\mathcal{N} = 4$ supersymmetric "spin mechanics" of [3,4] by an $su(2)$ Hamiltonian reduction applied to the general $su(2)$ invariant action for a self-coupled (4, 4, 0) supermultiplet. It is a further manifestation of the fundamental importance of the root supermultiplet [6] in $\mathcal{N} = 4$ supersymmetric mechanics [2,7,8].

II. SU(2) REDUCTION

Our point of departure is a quartet of real $\mathcal{N} = 4$ superfields Q^{ia} with $i, a = 1, 2$ defined in the $\mathcal{N} = 4$ superspace $\mathbb{R}^{(1|4)} = (t, \theta_i, \bar{\theta}^i)$ and subject to the constraints

$$D^{(i}Q^{j)a} = 0, \qquad \bar{D}^{(i}Q^{j)a} = 0 \quad \text{and} \quad (Q^{ia})^\dagger = Q_{ia}, \tag{2.1}$$

where the corresponding covariant derivatives have the form

$$D^i = \frac{\partial}{\partial\theta_i} + \mathrm{i}\bar{\theta}^i\partial_t, \qquad \bar{D}_i = \frac{\partial}{\partial\bar{\theta}^i} + \mathrm{i}\theta_i\partial_t \quad \text{so that } \{D^i, \bar{D}_j\} = 2\mathrm{i}\delta^i_j\partial_t. \tag{2.2}$$

This $\mathcal{N} = 4$ supermultiplet describes four bosonic and four fermionic but zero auxiliary variables off shell [9,10]. Let us now introduce the composite $\mathcal{N} = 4$ superfield[1]

$$X = 2(Q^{ia}Q_{ia})^{-1} \tag{2.3}$$

which, in virtue of (2.1), obeys the constraints [10]

$$D^iD_iX = \bar{D}_i\bar{D}^iX = [D^i, \bar{D}_i]X = 0. \tag{2.4}$$

The most general action for Q^{ia} is constructed by integrating an arbitrary superfunction $\tilde{\mathcal{F}}(Q^{ia})$ over the whole $\mathcal{N} = 4$ superspace. Here, we restrict ourselves to prepotentials of the form

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[1]We stress that $Q^{ia}Q_{ia} \sim \mathrm{e}^{-U}$ in the standard parametrization [10], where U is the superdilaton. Therefore, the new superfield X is well defined.

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$$\tilde{\mathcal{F}}(Q^{ia}) = \mathcal{F}(X(Q^{ia})) \to S = -\frac{1}{8}\int dt d^4\theta \mathcal{F}(X). \quad (2.5)$$

The rationale for this selection is its manifest invariance under $su(2)$ transformations acting on the "a" index of Q^{ia}. This is the symmetry over which we are going to perform the Hamiltonian reduction.

In terms of components the action (2.5) reads

$$S = \int dt \Big\{ G\Big(\dot{x}^2 + \mathrm{i}(\dot{\eta}^i\bar{\eta}_i - \eta^i\dot{\bar{\eta}}_i) + \frac{1}{2}x^2\omega^{ij}\omega_{ij}\Big) - \mathrm{i}(2G + xG')\omega^{ij}\eta_i\bar{\eta}_j - \frac{1}{4}\Big(G'' + 6\frac{G'}{x} + 6\frac{G}{x^2}\Big)\eta^i\eta_i\bar{\eta}_j\bar{\eta}^j\Big\}, \quad (2.6)$$

where

$$x = X|, \qquad \eta^i = -\mathrm{i}D^iX|, \qquad \bar{\eta}_i = -\mathrm{i}\bar{D}_iX|, \qquad q^{ia} = \sqrt{X}Q^{ia}|, \qquad G = \mathcal{F}''(X)| \quad (2.7)$$

and

$$\omega_{ij} = \dot{q}_i^a q_{ja} + \dot{q}_j^a q_{ia}. \quad (2.8)$$

Here, as usual, $(\ldots)|$ denotes the $\theta_i = \bar{\theta}_i = 0$ limit.

To proceed we introduce the following substitution for the bosonic variables q^{ia} subject to $q^{ia}q_{ia} = 2$:

$$q^{11} = \frac{\mathrm{e}^{-(\mathrm{i}/2)\phi}}{\sqrt{1+\Lambda\bar{\Lambda}}}\Lambda, \qquad q^{21} = -\frac{\mathrm{e}^{-(\mathrm{i}/2)\phi}}{\sqrt{1+\Lambda\bar{\Lambda}}}, \qquad q^{22} = (q^{11})^\dagger, \qquad q^{12} = -(q^{12})^\dagger. \quad (2.9)$$

In terms of the new variables $(\phi, \Lambda, \bar{\Lambda})$, the $su(2)$ rotations $\delta q^{ia} = \gamma^{(ab)}q_b^i$ read [10]

$$\delta\Lambda = \gamma^{11}\mathrm{e}^{\mathrm{i}\phi}(1+\Lambda\bar{\Lambda}), \qquad \delta\bar{\Lambda} = \gamma^{22}\mathrm{e}^{-\mathrm{i}\phi}(1+\Lambda\bar{\Lambda}), \qquad \delta\phi = -2\mathrm{i}\gamma^{12} + \mathrm{i}\gamma^{22}\mathrm{e}^{-\mathrm{i}\phi}\Lambda - \mathrm{i}\gamma^{11}\mathrm{e}^{\mathrm{i}\phi}\bar{\Lambda}. \quad (2.10)$$

It is easy to check that

$$\omega^{11} = 2\frac{\dot{\Lambda} - \mathrm{i}\Lambda\dot{\phi}}{1+\Lambda\bar{\Lambda}}, \qquad \omega^{22} = (\omega^{11})^\dagger \quad \text{and} \quad \omega^{12} = \mathrm{i}\frac{1-\Lambda\bar{\Lambda}}{1+\Lambda\bar{\Lambda}}\dot{\phi} + \frac{\dot{\Lambda}\bar{\Lambda} - \Lambda\dot{\bar{\Lambda}}}{1+\Lambda\bar{\Lambda}} \quad (2.11)$$

are indeed invariant under (2.10), as is the whole action (2.6).

Next, we introduce the standard Poisson brackets

$$\{\pi, \Lambda\} = 1, \qquad \{\bar{\pi}, \bar{\Lambda}\} = 1, \qquad \{p_\phi, \phi\} = 1, \quad (2.12)$$

so that the generators of the transformations (2.10),

$$I_\phi = p_\phi, \qquad I = \mathrm{e}^{\mathrm{i}\phi}[(1+\Lambda\bar{\Lambda})\pi - \mathrm{i}\bar{\Lambda}p_\phi], \qquad \bar{I} = \mathrm{e}^{-\mathrm{i}\phi}[(1+\Lambda\bar{\Lambda})\bar{\pi} + \mathrm{i}\Lambda p_\phi], \quad (2.13)$$

will be the Noether constants of motion for the action (2.6). To perform the reduction over this SU(2) group we fix the Noether constants as (c.f. [5])

$$I_\phi = m \quad \text{and} \quad I = \bar{I} = 0, \quad (2.14)$$

which yields

$$p_\phi = m \quad \text{and} \quad \pi = \frac{\mathrm{i}m\bar{\Lambda}}{1+\Lambda\bar{\Lambda}}, \qquad \bar{\pi} = -\frac{\mathrm{i}m\Lambda}{1+\Lambda\bar{\Lambda}}. \quad (2.15)$$

Conducting a Routh transformation over the variables $(\Lambda, \bar{\Lambda}, \phi)$, we reduce the action (2.6) to

$$\tilde{S} = S - \int dt\{\pi\dot{\Lambda} + \bar{\pi}\dot{\bar{\Lambda}} + p_\phi\dot{\phi}\} \quad (2.16)$$

and substitute the expressions (2.15) into $\tilde{S}$. A slightly lengthy but straightforward calculation gives

$$\tilde{S}_{\mathrm{red}} = \int dt \Big\{ G(\dot{x}^2 + \mathrm{i}(\dot{\eta}^i\bar{\eta}_i - \eta^i\dot{\bar{\eta}}_i)) - \frac{1}{4}\Big(G'' - \frac{3}{2}\frac{(G')^2}{G}\Big)\eta^2\bar{\eta}^2 - \frac{m^2}{4x^2G} - \frac{m(2G+xG')}{2x^2G(1+\Lambda\bar{\Lambda})}(2\Lambda\eta_1\bar{\eta}_1 - 2\bar{\Lambda}\eta_2\bar{\eta}_2 - (1-\Lambda\bar{\Lambda})(\eta_1\bar{\eta}_2 + \eta_2\bar{\eta}_1))\Big\}. \quad (2.17)$$

To ensure that the reduction constraints (2.15) are satisfied we add Lagrange multiplier terms,

$$S_{\mathrm{red}} = \tilde{S}_{\mathrm{red}} + \int dt \Big\{ m\dot{\phi} + \frac{\mathrm{i}m(\dot{\Lambda}\bar{\Lambda} - \Lambda\dot{\bar{\Lambda}})}{1+\Lambda\bar{\Lambda}}\Big\}. \quad (2.18)$$

Finally, by employing new variables $v^i = q^{i1}$ and $\bar{v}_i = (v^i)^\dagger$ we rewrite this action in the symmetric form

$$S_{\mathrm{red}} = \int dt \Big\{ G(\dot{x}^2 + \mathrm{i}(\dot{\eta}^i\bar{\eta}_i - \eta^i\dot{\bar{\eta}}_i)) - \frac{1}{4}\Big(G'' - \frac{3}{2}\frac{(G')^2}{G}\Big)\eta^2\bar{\eta}^2 - \frac{m^2}{4x^2G} + \mathrm{i}m(\dot{v}^i\bar{v}_i - v^i\dot{\bar{v}}_i) - \frac{m(2G+xG')}{2x^2G}v^i\bar{v}^j(\eta_i\bar{\eta}_j + \eta_j\bar{\eta}_i)\Big\} \quad \text{with} \quad v^i\bar{v}_i = 1. \quad (2.19)$$

Amazingly, this final action coincides with the one presented in [4] and specializes to the one derived in [3] for the choice of $G = 1$, which corresponds to OSp(4|2) symmetry.

We stress that the $su(2)$ reduction algebra, realized in (2.10), commutes with all (super)symmetries of the action (2.5). Therefore, all symmetry properties of the theory [including the $D(2, 1; \alpha)$ invariance for a properly chosen prepotential $\mathcal{F}$] are preserved in our reduction.

III. CONCLUSION

We have demonstrated that the novel $\mathcal{N} = 4$ supersymmetric "spin mechanics" of [1,3,4] is nicely interpreted as an $su(2)$ reduction of a self-interacting root supermultiplet with (4, 4, 0) component content. This procedure is remarkably simple and automatically successful.

An almost straightforward application of this insight is a similar $su(2)$ reduction applied to the $\mathcal{N} = 4$ "nonlinear" supermultiplet [10]. The resulting system will contain only spinor variables accompanied by four fermions. In this regard, one could also investigate the nonlinear root supermultiplet and its action [11].

Finally, we mention that our reduction will almost never work for the $\mathcal{N} = 8$ supersymmetric mechanics in the literature. The reason is simple: these systems do not possess any internal symmetry which commutes with all eight supersymmetries. This is also the situation discussed in [5]. The one positive exception is the "real" $\mathcal{N} = 8$, $d = 1$ hypermultiplet, which is obtained by dimensional reduction from $\mathcal{N} = 2$, $d = 4$ and requires $\mathcal{N} = 8$, $d = 1$ harmonic superspace [12,13]. We expect the corresponding $su(2)$ reduction to produce some spin extension of the recently constructed $\mathcal{N} = 8$ superconformal mechanics [14]. We intend to turn to this issue soon.

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