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Exact solution of the D_3 non-Abelian anyon chain

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Commuting transfer matrices for linear chains of interacting non-Abelian anyons from the two-dimensional irreducible representation of the dihedral group D_3 [or, equivalently, the integer sector of the $su(2)_4$ spin-1 chain] are constructed using the spin-anyon correspondence to a D_3-symmetric formulation of the XXZ Heisenberg spin chain. The spectral problem is solved using discrete inversion identities satisfied by these transfer matrices and functional Bethe ansatz methods. The resulting spectrum can be related to that of the XXZ spin-1/2 Heisenberg chain with boundary conditions depending on the topological sector of the anyon chain. The properties of this model in the critical regime are studied by finite size analysis of the spectrum. In particular, points in the phase diagram where the anyon chain realizes some of the rational $\mathbb{Z}_2$ orbifold theories are identified.

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I. INTRODUCTION

There has been growing interest recently in the properties of the exotic quasiparticles arising in topologically ordered systems such as certain fractional quantum Hall states [1] or two-dimensional frustrated quantum magnets [2–4]. One-dimensional lattice models for these non-Abelian anyons have been constructed starting from a consistent set of fusion and braiding rules and the collective phases realized in many-body anyon models with pairwise interactions have been studied, mostly numerically, to identify a variety of critical phases and the conformal field theories describing their low energy properties, see, e.g., Refs. [5–8]. The gapless states in these phases are protected by the topological symmetry present in these chains and can be realized at interfaces between different topological phases where they provide insights into the nature of the adjacent quantum liquids [9–11].

Integrable models are known to be another important source (complementary to numerical approaches) of unbiased information on the properties of correlated many-body systems, in particular in low-dimensional systems where quantum fluctuations are strong. In fact, some of the anyonic lattice models mentioned above can be derived in the Hamiltonian limit of a class of integrable statistical lattice models with interactions around the face (also called IRF or face models), such as the restricted solid on solid (RSOS) models [12] and their generalizations [13,14]. Motivated by this connection several integrable chains of interacting non-Abelian anyons have been constructed and studied recently, see, e.g., Refs. [15,16].

Here we consider one of the deformations of $su(2)$ quantum spin chains leading to anyonic models studied by Gils *et al.* [7], namely the $su(2)_{k=4}$ spin-1 or, equivalently, D_3 anyon model, see also Refs. [17,18]. As pointed out in Ref. [19] this model can be related to one of the paradigms of integrable quantum chains, i.e., the XXZ Heisenberg spin chain, under a particular spin-anyon (or face-vertex) correspondence [13,20]. Based on this correspondence we have been able to embed the anyonic Hamiltonian into a family of commuting operators generated by an IRF transfer matrix. Interestingly, and unlike the usual $su(2)$ spin-1 chain with bilinear and biquadratic exchange or, e.g., the model of interacting $so(5)_2$ non-Abelian anyons [8], this embedding is possible for arbitrary coupling constants. Specifically, the spectrum of the anyon chain can be related to that of the XXZ spin chain subject to properly chosen boundary conditions depending on the topological sector of the former. The independence of thermodynamic properties on these boundary conditions allows us to relate the phase diagram of the anyon chain to that known from the exact solution of the Heisenberg model, see Fig. 1.

We find that, while the low energy properties of the anyon and the spin chain in the critical phases are both described by conformal field theories (CFTs) with central charge $c = 1$ their operator content is different as a consequence of the change in boundary conditions and the presence of additional selection rules on a $U(1)$ charge (i.e., the magnetization for the spin chain) for the anyon model: The continuum limit of the critical Heisenberg model is known to be a free boson with compactification radius r_G depending on the anisotropy. The critical properties of the anyon model, on the other side, are $\mathbb{Z}_2$ orbifolds of a boson compactified on a circle with radius $\tilde{r} = 3\,r_G$.[1] For rational values of $\tilde{r}^2$ these CFTs have an extended symmetry allowing for their formulation in terms of a finite number of primary fields. From the exact solution of the D_3 anyon chain the location of these special points in the phase diagram can be given in terms of the coupling constant of the microscopic model. These include the special rational CFTs corresponding to compactification radii $\tilde{r} = \sqrt{p/2}$ with integer $p = 1,2,\ldots,9$ that have been identified previously in the numerical investigation of Ref. [7], see Fig. 1. We note that the sequence of these field theories for *arbitrary* integer p is realized for the coupling constant corresponding to the dual radius, $\tilde{r} \leftrightarrow 1/(2\tilde{r})$.

The paper is organized as follows: First we recall some basic facts related to the integrability of the XXZ Heisenberg chain (or six-vertex model) and its formulation as a D_3 symmetric spin chain. The D_3 group algebra can be extended to form a quasitriangular Hopf algebra which allows us to construct related chains of anyons obeying D_3 fusion rules with an integrability structure closely related to that of the XXZ

[1] A similar relation exists for another model related to the XXZ chain, i.e. the critical Ashkin-Teller quantum chain [21]: in the continuum limit its partition function is identical to that of the orbifold model with $\tilde{r}_{\mathrm{AT}} = 2\,r_G$ [22,23].

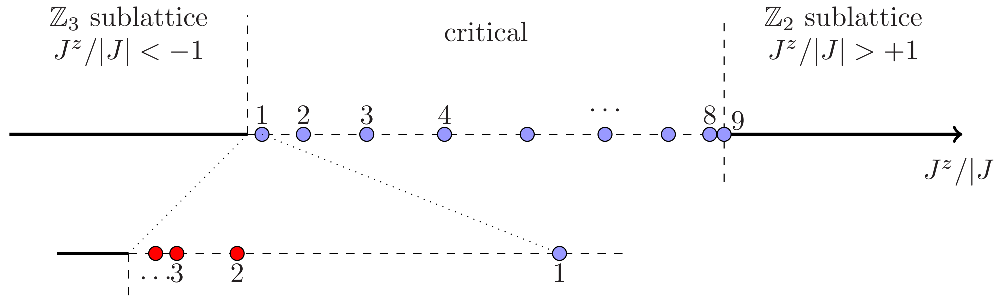


FIG. 1. Phase diagram of the D_3 anyon model: blue circles indicate the location of the points in the critical region corresponding to the rational $\mathbb{Z}_2$ orbifold CFTs with compactification radii $\tilde{r} = \sqrt{p/2}$ for $p = 1,2,3,\ldots,9$ as denoted by the labels. Similarly, red circles in the blow-up of the region around $J^z = -|J|$ indicate the sequence of rational orbifold theories at the dual radii $\tilde{r} = \sqrt{1/2p}$ for $p = 2,3,4,\ldots$. The CFT realized at the self-dual point, $p = 1$, is the Kosterlitz-Thouless theory.

model. Based on this structure we employ functional Bethe ansatz methods to solve the spectral problem of the D_3 anyon model. Using this solution we perform a finite-size analysis of the spectrum in the critical regime to identify the field theory describing the collective behavior of the anyons at low energies.

II. THE HEISENBERG SPIN-1/2 CHAIN

The XXZ Heisenberg Hamiltonian for spin-1/2 particles with nearest neighbor interaction on a one-dimensional lattice and for periodic boundary conditions is given as

$$H = \frac{1}{2}\sum_i \left(J\left(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y\right) + J^z\sigma_i^z\sigma_{i+1}^z\right). \qquad (2.1)$$

This model is closely connected to the classical integrable six-vertex model on the square lattice. It is defined in terms of local Boltzmann weights defined for each vertex with the state variables on the links connected to it. The integrability of this model relies on the Yang-Baxter equation (YBE) satisfied by the vertex weights,

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v). \qquad (2.2)$$

For the six-vertex-model the R matrix is

$$R(u) = \begin{pmatrix} a(u) & & & \\ & b(u) & c(u) & \\ & c(u) & b(u) & \\ & & & a(u) \end{pmatrix}, \qquad (2.3)$$

where $a(u) = \sinh(u + i\gamma)$, $b(u) = \sinh(u)$, and $c(u) = \sinh(i\gamma)$. Here u is the spectral parameter, and γ parameterizes the anisotropy. For $u = 0$ the R matrix becomes proportional to a permutation operator of two sites. As a consequence of the YBE the transfer matrix

$$\tau(u) = tr_0(R_{01}(u)R_{02}(u)\cdots R_{0L}(u)) \qquad (2.4)$$

commutes for different arguments and therefore generates a family of commuting operators. Among these $\tau(0) \propto e^{iP}$ is proportional to the translation operator allowing us to define the momentum operator in the lattice model; for periodic boundary conditions the eigenvalues of P are

$$P = \frac{2\pi}{L}n, \quad n \in \mathbb{N}. \qquad (2.5)$$

Higher integrals of motion are obtained by expansion of $\log\tau(u)$ around $u = 0$ with the XXZ Hamiltonian with $J^z/|J| = \cos\gamma$ given as

$$H \propto \frac{\partial}{\partial u}\log\tau(u)|_{u=0}. \qquad (2.6)$$

Note that for even L the XXZ Hamiltonian $H(J,J^z)$ is unitary equivalent to $H(-J,J^z)$. Therefore we will consider only the case $J = +1$ in most of the discussion below. Furthermore, the XXZ chain has a $U(1)$ symmetry resulting in the conservation of the total magnetization $\langle S^z\rangle = L/2 - M$, where S^z is the z component of the total spin. Hence the eigenstates of the transfer matrix can be assigned to different M sectors.

The XXZ model as a D_3 spin chain

Consider the dihedral group D_3, the group of symmetries of a triangle generated by rotation σ and reflection τ: $D_3 = \{\sigma,\tau|\sigma^3 = \tau^2 = 1\}$ [19]. This group has two one-dimensional irreducible representations (irreps) $(\pi_\pm, V_\pm)$ and one two-dimensional irrep (π_2, V_2). To formulate the XXZ model as a D_3 chain we identify the states spanning the representation spaces V_α with the states $|\sigma_1,\sigma_2\rangle \in \mathbb{C}^2\otimes\mathbb{C}^2$ of a system of two $SU(2)$ spins $\frac{1}{2}$ as

$$V_\pm \propto \{|\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle\}, \quad V_2 \propto \{|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle\}.$$

Now both the nearest neighbor XXZ Hamiltonian (2.1) and the R matrix (2.3) of the six-vertex model can be represented in terms of the projection operators $P^{(2,\pm)}$ onto the corresponding two-spin states:

$$\begin{aligned} h_{i,i+1} &= (J^z - J)P^{(2)}_{i,i+1} - 2J P^{(-)}_{i,i+1} \\ R(u) &= (\sinh(u) + \sinh(i\gamma))P^{(+)} \\ &\quad + (\sinh(u) - \sinh(i\gamma))P^{(-)} + \sinh(u + i\gamma)P^{(2)}. \end{aligned} \qquad (2.7)$$

Note that the periodic closure is compatible with this construction, therefore the XXZ Hamiltonian is D_3 symmetric.

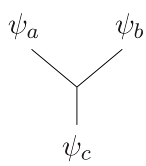
III. THE INTEGRABLE D_3 ANYON CHAIN

A. Anyon chain from braided fusion categories

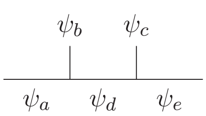
Mathematically, anyonic theories can be represented by braided tensor categories [4]. An anyon model consists of a collection of conserved topological objects ψ_a obeying a commutative fusion algebra

$$\psi_a \otimes \psi_b \simeq \oplus_c N_{ab}^c\, \psi_c \tag{3.1}$$

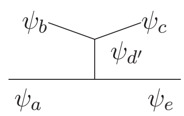
with non-negative integers N_{ab}^c (we assume that the tensor category is multiplicity free which corresponds to $N_{ab}^c \in \{0,1\}$). Fusion can be represented graphically



implying that ψ_c appears in the fusion of ψ_a and ψ_b, i.e., $N_{ab}^c = 1$. Associativity of the algebra allows for a reordering of the fusion of multiple anyons governed by so-called F moves (the analog of 6-j symbols)



$$= \sum_{d'} \left(F_e^{abc}\right)_{d'}^{d}$$



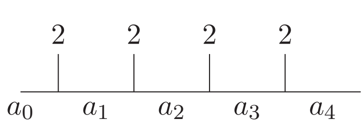
(3.2)

Here, we consider the fusion algebra obeyed by the finite-dimensional irreps (or anyons) of D_3 to construct an anyonic model related to the Heisenberg chain [19]. With (π_+, V_+) being the unique 'vacuum' the remaining nontrivial fusion rules are

$$\begin{aligned} V_2 \otimes V_2 &= V_+ \oplus V_2 \oplus V_-, \\ V_- \otimes V_- &= V_+, \quad V_- \otimes V_2 = V_2. \end{aligned} \tag{3.3}$$

Below we shall identify representation spaces with the anyonic charges labeled 2, $\pm$, respectively.

The D_3 anyon model is constructed from L copies of the two-dimensional anyon with charge 2 (which is, in fact, identical to the integer sector of the $su(2)_4$ spin-1 anyon chain considered in Refs. [7,17,18]). Based on the fusion rules we can construct the fusion path basis of the anyonic Hilbert space: The states $|a\rangle \equiv |a_0 a_1 a_2 \ldots a_L\rangle$ correspond to a sequence of anyonic charges subject to the constraint that a_{i+1} appears in $a_i \otimes 2$. It is convenient to represent these graphically as



The vertices in this diagram represent the fusion of two incoming anyons (top and left) resulting in the outgoing anyon (right). The fusion rules lead to a local condition on the possible neighboring labels

$$a_i a_{i+1} \in \{+2, -2, 2+, 2-, 22\}, \tag{3.4}$$

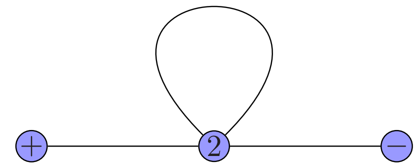


FIG. 2. Graphical representation of allowed neighboring anyon labels in fusion paths of D_3 anyons of type $2 = (\pi_2, V_2)$. Nodes of the graph correspond to the possible types of anyons which are connected via an edge if the two anyon labels can appear next to each other.

which is conveniently presented in the form of a graph with an adjacency matrix $A_{ab} = N_{a2}^b$, i.e.

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{3.5}$$

for the present case, such that allowed pairs of labels correspond to adjacent nodes, see Fig. 2. This constraint together with periodic boundary conditions $a_0 = a_L$ yields the total dimension of the Hilbert space of the L-site anyon chain $\dim(\mathcal{H}) = 2^L + (-1)^L$.

Nearest neighbor D_3-symmetric interactions can be expressed in terms of two-site projection operators on the different topological charges. They can be expressed in terms of the F moves (3.2) and their inverses $\bar{F}$ as

$$\tilde{P}^{(b)}_{i-1,i,i+1} = \left(\bar{F}^{a_{i-1}22}_{a_{i+1}}\right)^{b}_{a_i'} \left(F^{a_{i-1}22}_{a_{i+1}}\right)^{a_i}_{b} |a_{i-1}a_i'a_{i+1}\rangle\langle a_{i-1}a_ia_{i+1}|, \quad b \in \{2,\pm\}. \tag{3.6}$$

Note that an operator coupling anyons on neighboring sites i and $i+1$ acts on three neighboring labels, i.e., $O_{i,i+1} = \tilde{O}_{a_{i-1}a_ia_{i+1}}$ in the fusion path basis. Only the anyon label on site i may change under its action though.

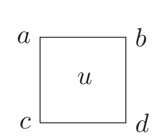
B. Spin-anyon correspondence

Based on these projection operators we can construct an anyonic quantum chain by exploiting the correspondence to the Heisenberg chain in terms of the D_3 description [19]. As in (2.7) we can define an R matrix

$$\begin{aligned} \tilde{R}_{i-1,i,i+1}(u) = &(\sinh(u) + \sinh(i\gamma))\tilde{P}^{(+)}_{i-1,i,i+1} \\ &+ (\sinh(u) - \sinh(i\gamma))\tilde{P}^{(-)}_{i-1,i,i+1} \\ &+ \sinh(u + i\gamma)\tilde{P}^{(2)}_{i-1,i,i+1}. \end{aligned} \tag{3.7}$$

The matrix elements of $\tilde{R}(u)$ are the Boltzmann weights W of a two-dimensional statistical model with interactions around the face (IRF)

$$\langle abd|\tilde{R}(u)|acd\rangle \equiv W\left(\begin{matrix} a & b \\ c & d \end{matrix}\middle|\, u\right) = \tag{3.8}$$



The nonzero ones are

$$
\begin{aligned}
&W\left(\begin{matrix}2 & 2\\ 2 & 2\end{matrix}\middle| u\right) = c(u)\\
&W\left(\begin{matrix}\pm & 2\\ 2 & 2\end{matrix}\middle| u\right) = W\left(\begin{matrix}2 & 2\\ 2 & \pm\end{matrix}\middle| u\right) = a(u)\\
&W\left(\begin{matrix}2 & \pm\\ 2 & 2\end{matrix}\middle| u\right) = W\left(\begin{matrix}2 & 2\\ \pm & 2\end{matrix}\middle| u\right) = \frac{1}{\sqrt{2}}b(u)\\
&W\left(\begin{matrix}\pm & 2\\ 2 & \pm\end{matrix}\middle| u\right) = (c(u)+b(u)),\\
&W\left(\begin{matrix}\pm & 2\\ 2 & \mp\end{matrix}\middle| u\right) = (c(u)-b(u))\\
&W\left(\begin{matrix}2 & \pm\\ \pm & 2\end{matrix}\middle| u\right) = \frac{1}{2}(c(u)+a(u)),\\
&W\left(\begin{matrix}2 & \pm\\ \mp & 2\end{matrix}\middle| u\right) = \frac{1}{2}(c(u)-a(u)).
\end{aligned}
\tag{3.9}
$$

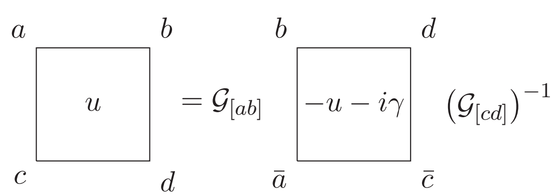
They satisfy the (face) Yang-Baxter equation

$$
\begin{aligned}
&\sum_g W\left(\begin{matrix}f & a\\ e & g\end{matrix}\middle| u-v\right) W\left(\begin{matrix}a & b\\ g & c\end{matrix}\middle| v\right) W\left(\begin{matrix}g & c\\ e & d\end{matrix}\middle| u\right)\\
&\quad = \sum_{g'} W\left(\begin{matrix}a & b\\ f & g'\end{matrix}\middle| u\right) W\left(\begin{matrix}f & g'\\ e & d\end{matrix}\middle| v\right) W\left(\begin{matrix}b & c\\ g' & d\end{matrix}\middle| u-v\right).
\end{aligned}
\tag{3.10}
$$

Additional properties of the Boltzmann weights are unitarity

$$
\sum_e W\left(\begin{matrix}a & e\\ d & c\end{matrix}\middle| u\right) W\left(\begin{matrix}a & b\\ e & c\end{matrix}\middle| -u\right) = \rho(u)\rho(-u)\,\delta_{bd}
\tag{3.11}
$$

with $\rho(u) = \sinh(u + i\gamma)$ and a generalized crossing symmetry



$$
W\left(\begin{matrix}a & b\\ c & d\end{matrix}\middle| u\right) = \mathcal{G}_{[ab]}\, W\left(\begin{matrix}b & d\\ \bar{a} & \bar{c}\end{matrix}\middle| -u-i\gamma\right)(\mathcal{G}_{[cd]})^{-1}
\tag{3.12}
$$

where we have defined a gauge transformation $\mathcal{G}$

$$
\mathcal{G}_{[\pm 2]} = -\sqrt{2}, \quad \mathcal{G}_{[2\pm]} = -\frac{1}{\sqrt{2}}, \quad \mathcal{G}_{[22]} = 1,
\tag{3.13}
$$

and the mapping $a \to \bar{a} \equiv f(a)$ with $f(\pm) = \mp$, $f(2) = 2$.

The initial conditions satisfied by the Boltzmann weights at the special points $u = 0, -i\gamma$ are

$$
\begin{aligned}
&W\left(\begin{matrix}a & b\\ c & d\end{matrix}\middle| 0\right) = \sinh(i\gamma)\,\delta_{cb},\\
&W\left(\begin{matrix}a & b\\ c & d\end{matrix}\middle| -i\gamma\right) = (\mathcal{G}_{[ca]})^{-1}\mathcal{G}_{[db]}\,\sinh(i\gamma)\,\delta_{a\bar{d}}.
\end{aligned}
\tag{3.14}
$$

As a consequence of the Yang-Baxter equation (3.10) the transfer matrix (for periodic boundary conditions $a_{L+1} = a_1$, $b_{L+1} = b_1$)

$$
\tilde{\tau}(u) = W\left(\begin{matrix}a_1 & a_2\\ b_1 & b_2\end{matrix}\middle| u\right) W\left(\begin{matrix}a_2 & a_3\\ b_2 & b_3\end{matrix}\middle| u\right)\cdots W\left(\begin{matrix}a_L & a_1\\ b_L & b_1\end{matrix}\middle| u\right)
\tag{3.15}
$$

commutes for different arguments and therefore generates a family of commuting operators on the Hilbert space of the D_3 anyon chain. The initial conditions (3.14) imply that among these the momentum operator is obtained from the generator of translations as

$$
p = -i\log(\tilde{\tau}(0)/\sinh^L(i\gamma)).
\tag{3.16}
$$

The Hamiltonian with local interactions is the logarithmic derivative of the transfer matrix at $u = 0$. In terms of the D_3 projection operators in the anyon basis it reads (up to a constant)

$$
\begin{aligned}
\tilde{H} &= \sinh(i\gamma)\,\partial_u \log\tilde{\tau}(u)|_{u=0}\\
&= \sum_i \left((\cos\gamma - 1)\tilde{P}^{(2)}_{i-1,i,i+1} - 2\tilde{P}^{(-)}_{i-1,i,i+1}\right),
\end{aligned}
\tag{3.17}
$$

where we have set $J = +1$ and $J^z = \cos\gamma$. The case $J = -1$ will be discussed below in the context of the low energy effective theory.

C. Symmetries

From the Boltzmann weights (3.9) it follows immediately that the face model has a Z_2 symmetry generated by the mapping $a \to \bar{a}$ on each site

$$
\hat{\sigma}\,|a_1 a_2 \cdots a_L\rangle = |\bar{a}_1\bar{a}_2\cdots\bar{a}_L\rangle
\tag{3.18}
$$

(for even L there are, in fact, two such symmetries related to the restriction of these parity operations on either of the two sublattices).

In addition, the spectrum of the anyon chain can be decomposed into sectors with different topological charge Y (related to the flux through a ring of anyons [7,9])

$$
\langle x'|Y|x\rangle = \prod_{i=1}^{L}\left(F^{2x_{i+1}2}_{x'_i}\right)^{x'_{i+1}}_{x_i}
\tag{3.19}
$$

where $x_i \in \{\pm,2\}$ and $|x\rangle$ ($\langle x|$) is a (dual) vector from the anyonic fusion path basis. The F moves can be related to the Boltzmann weights of the isotropic ($\gamma = 0$) anyon model through[2]

$$
\sinh(u)\left(F^{2d2}_a\right)^b_c = (-1)^{\delta_{c,-}}\,W_{\gamma=0}\left(\begin{matrix}a & b\\ c & d\end{matrix}\middle| u\right)(-1)^{\delta_{d,-}}.
\tag{3.20}
$$

For periodic boundary conditions the gauge factors cancel which implies that the topological charge can be obtained from

[2]Note that we do not rescale the spectral parameter $u \to \gamma u$ which is necessary to recover the rational Boltzmann weights describing the isotropic model.

the transfer matrix in the limit of infinite spectral parameter as

$$\lim_{u\to\infty} \tilde{\tau}_{\gamma=0}(u)\, \mathrm{e}^{-Lu} = Y. \tag{3.21}$$

Therefore Y is part of the hierarchy of commuting integrals, i.e., $[\tilde{\tau}_{\gamma=0}(u),Y]=0$. The eigenvalues of the topological charge [which, according to (3.21), determine the asymptotic behavior of the eigenvalues of $\tilde{\tau}_{\gamma=0}$] take values from the spectrum of the adjacency matrix (3.5) of the anyon model [24,25]. For the D_3 anyon model the eigenvalues of Y are 2, -1, and 0. The dimension of the corresponding eigenspaces is

$$\dim(\mathcal{H}_Y) = \begin{cases} \frac{1}{3}(2^{L-1}+(-1)^L) & \text{for} \quad Y=2 \\ \frac{2}{3}(2^{L-1}+(-1)^L) & \text{for} \quad Y=-1 \\ 2^{L-1} & \text{for} \quad Y=0 \end{cases}. \tag{3.22}$$

For $\gamma \neq 0$ the topological charge is not easily obtained from the corresponding transfer matrix of the anyon model. It is easily checked, however, that the Boltzmann weights (3.9) together with $W_{\gamma=0}(u)$ satisfy the following equations

$$\begin{aligned} &\sum_g W_{\gamma=0}\left(\begin{matrix} f & a \\ e & g \end{matrix}\middle| u-v\right) W_{\gamma=0}\left(\begin{matrix} a & b \\ g & c \end{matrix}\middle| v\right) W\left(\begin{matrix} g & c \\ e & d \end{matrix}\middle| u\right) \\ &\quad = \sum_{g'} W\left(\begin{matrix} a & b \\ f & g' \end{matrix}\middle| u\right) W_{\gamma=0}\left(\begin{matrix} f & g' \\ e & d \end{matrix}\middle| v\right) \\ &\qquad \times W_{\gamma=0}\left(\begin{matrix} b & c \\ g' & d \end{matrix}\middle| u-v\right). \end{aligned} \tag{3.23}$$

Together with (3.21) this equation can be used to show that Y commutes with the transfer matrix of the anyon model for arbitrary γ as well as the parity operator (3.18)

$$[\tilde{\tau}(u),Y]=0, \quad [\hat{\sigma},Y]=0. \tag{3.24}$$

Calculation of the transfer matrix eigenvalues for small L we find their large-u asymptotics in the different topological sectors to be

$$\begin{aligned} &\lim_{u\to\infty} \tilde{\tau}(u)\, \mathrm{e}^{-Lu} \\ &= \begin{cases} \mathrm{e}^{i\gamma(L-M)} + \mathrm{e}^{i\gamma M} + O(\mathrm{e}^{-2u}) & \text{for } Y=2 \\ \omega\, \mathrm{e}^{i\gamma(L-M)} + \omega^{-1}\, \mathrm{e}^{i\gamma M} + O(\mathrm{e}^{-2u}) & \text{for } Y=-1 \\ O(\mathrm{e}^{-u}) & \text{for } Y=0 \end{cases}. \end{aligned} \tag{3.25}$$

Here the integer M takes values such that $L-2M$ is an even (odd) multiple of 3 for anyon chains of even (odd) length, and ω is a primitive cube root of unity. Similarly, we observe that the parity of the transfer matrix eigenstates is connected to the corresponding topological charge: It is found to be $\sigma=(-1)^L$ in the sectors $Y=2,-1$ and $\sigma=-(-1)^L$ for $Y=0$.

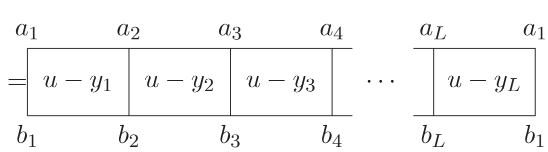
IV. SOLUTION OF THE SPECTRAL PROBLEM

As a consequence of the Yang-Baxter equation for the local Boltzmann weights the D_3 anyon chain is integrable. Unlike the situation for vertex models such as the six-vertex model and the related XXZ (or D_3) spin chain, the spectral problem for the anyon chain cannot be solved by means of the algebraic Bethe ansatz. Instead we will follow Refs. [26,27] and derive exact inversion identities satisfied by the transfer matrix of inhomogeneous generalizations of the D_3 anyon chain to derive Bethe equations for its spectrum.

A. Inhomogeneous chain and inversion identities

Since the Boltzmann weights satisfy the Yang-Baxter equation (3.10) with the difference property the transfer matrix of a model with generic inhomogeneities $\{y_k\}_{k=1}^L$

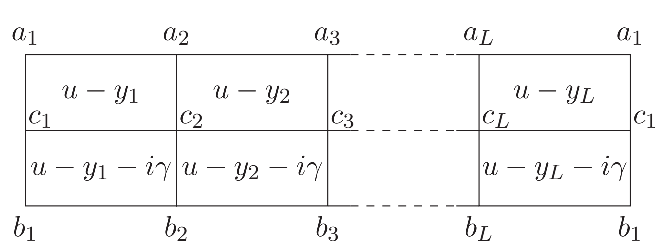
$\tilde{\tau}(u|\{y_k\}) =$



(4.1)

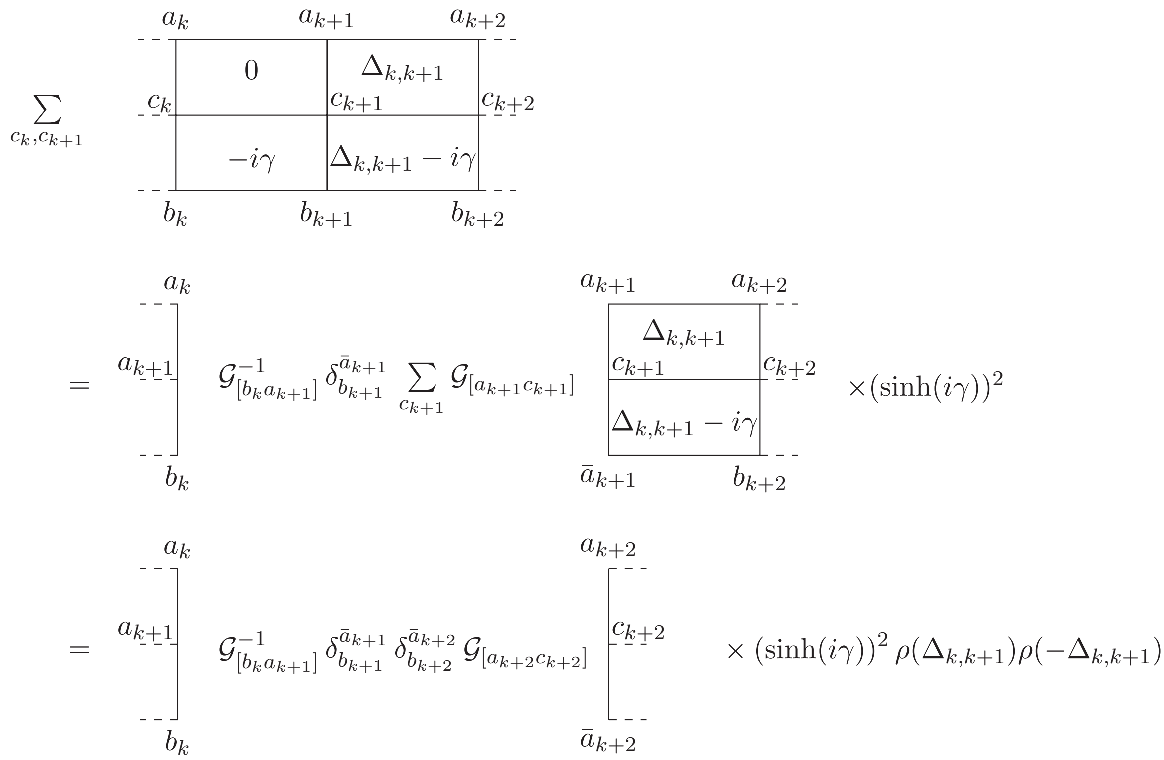
also commutes for different spectral parameters, $[\tilde{\tau}(u|\{y_k\}),\tilde{\tau}(v|\{y_k\})]=0$. Consider the following product of these transfer matrices:

$\tilde{\tau}(u-i\gamma)\,\tilde{\tau}(u) = \sum_c$



(4.2)

where the summation is over the possible anyon labels $\{c_j\}_{j=1}^L$ on the inner nodes. Choosing the spectral parameter u to take the value y_k from the set of inhomogeneities we can use the initial condition (3.14) of the Boltzmann weights together with crossing and unitarity (3.12), (3.11) to obtain



(4.3)

with $\Delta_{k\ell} = y_k - y_\ell$. Iterating this operation we obtain the following set of discrete inversion identities satisfied by the transfer matrix

$$\tilde{\tau}(y_k - i\gamma)\,\tilde{\tau}(y_k) = \hat{\sigma}\prod_{\ell=1}^{L}\rho(y_k - y_\ell)\rho(y_\ell - y_k), \qquad k = 1,\ldots,L. \tag{4.4}$$

Periodic boundary conditions imply

$$\prod_{\ell=1}^{L}\tilde{\tau}(y_k) = \prod_{k,\ell=1}^{L}\rho(y_k - y_\ell)\,\mathbf{1},$$
$$\prod_{\ell=1}^{L}\tilde{\tau}(y_k - i\gamma) = \hat{\sigma}^L\prod_{k,\ell=1}^{L}\rho(y_k - y_\ell). \tag{4.5}$$

Therefore only $L-1$ of the identities (4.4) are independent.

As a consequence of the commutativity of the transfer matrices and the parity operator $\hat{\sigma}$, the eigenvalues of $\tilde{\tau}(u|\{y_k\})$ satisfy similar identities, i.e.,

$$\Lambda(y_k - i\gamma)\,\Lambda(y_k) = \pm(-1)^L\prod_{\ell=1}^{L}\rho(y_k - y_\ell)\rho(y_\ell - y_k), \qquad k = 1,\ldots,L-1, \tag{4.6}$$

where the sign $+$ $(-)$ has to be chosen for the topological sectors $Y = 2, -1$ $(Y = 0)$. Equations (4.4) or (4.6) together with information on the analytical properties of the transfer matrix allow us to compute its eigenvalues: by construction they are Fourier polynomials in e^{2u} of degree $\leqslant L/2$. The coefficients of the leading terms $e^{\pm Lu}$ are determined by the topological charge according to (3.25). The remaining $L-1$ coefficients can be computed by solving the quadratic equations (4.6).

B. TQ equation for the D_3 anyon chain

For an efficient calculation of transfer matrix eigenvalues for large systems the inversion identities are not suitable. Note, however, that for a generic choice of inhomogeneities, i.e. $y_k - y_\ell \neq 0, \pm i\gamma$ for $k \neq \ell$ they are formally equivalent to a TQ-type equation

$$\Lambda(u)q(u) = \Delta_+(u)q(u - i\gamma) + \Delta_-(u)q(u + i\gamma) \tag{4.7}$$

restricted to the discrete set of points $u \in \{y_k, y_k - i\gamma\}$ provided that the functions $\Delta_\pm$ factorize the right hand side of (4.6) as

$$\Delta_+(y_k)\Delta_-(y_k - i\gamma) = \pm(-1)^L\prod_{\ell=1}^{L}\rho(y_k - y_\ell)\rho(y_\ell - y_k) \tag{4.8}$$

and satisfy $\Delta_+(y_k - i\gamma) = \Delta_-(y_k) = 0$. In the present case these conditions are met by the choice

$$\Delta_+(u) = \omega_+\prod_{\ell=1}^{L}\sinh(u - y_\ell + i\gamma),$$
$$\Delta_-(u) = \omega_-\prod_{\ell=1}^{L}\sinh(u - y_\ell), \tag{4.9}$$

provided that $\omega_+\omega_- = 1$ (-1) in the topological sectors $Y = 2,-1$ $(Y = 0)$.

TQ equations such as (4.7) which are valid for arbitrary $u \in \mathbb{C}$ are obtained in the Bethe ansatz formulations of the spectral problem of integrable systems such as the six-vertex model [28]. Provided that they allow for a sufficiently simple (e.g., polynomial) ansatz for the functions $q(u)$ they can be solved using the Bethe equations for the finitely many zeros of these functions.

1. Sectors $Y = 2, -1$

We take the Fourier polynomial

$$q(u) = \prod_{k=1}^{M} \sinh\left(u - u_k + \frac{i}{2}\gamma\right), \tag{4.10}$$

parameterized by M complex numbers u_k as ansatz for the q functions in these sectors. The large u asymptotics of the corresponding transfer matrix eigenvalues $\Lambda(u)$ as obtained from the TQ equation (4.7) together with the factorization (4.9) of $\Lambda(y_k - i\gamma)\Lambda(y_k)$,

$$\lim_{u\to\infty} \Lambda(u)\mathrm{e}^{-Lu+\sum_{\ell=1}^{L} y_\ell} = \omega_+ \mathrm{e}^{i\gamma(L-M)} + \omega_- \mathrm{e}^{i\gamma M}, \tag{4.11}$$

agrees with (3.25) provided that $L - 2M$ is an even (odd) multiple of 3 for even (odd) L and $\omega_+ = 1/\omega_-$ is 1 in the topological sector $Y = 2$, and ω, a primitive cube root of unity, for $Y = -1$. Analyticity of

$$\Lambda^{(Y)}(u) = \Delta_+^{(Y)}(u) \prod_{k=1}^{M} \frac{\sinh\left(u - u_k - \frac{i}{2}\gamma\right)}{\sinh\left(u - u_k + \frac{i}{2}\gamma\right)} + \Delta_-^{(Y)}(u) \prod_{k=1}^{M} \frac{\sinh\left(u - u_k + \frac{3i}{2}\gamma\right)}{\sinh\left(u - u_k + \frac{i}{2}\gamma\right)} \tag{4.12}$$

implies that the parameters u_k have to satisfy the Bethe equations

$$\frac{\omega_+}{\omega_-} \prod_{\ell=1}^{L} \frac{\sinh\left(u_k - y_\ell + \frac{i}{2}\gamma\right)}{\sinh\left(u_k - y_\ell - \frac{i}{2}\gamma\right)} = -\prod_{j=1}^{M} \frac{\sinh(u_k - u_j + i\gamma)}{\sinh(u_k - u_j - i\gamma)}, \quad k = 1,\ldots,M. \tag{4.13}$$

These equations coincide with those for the spin-1/2 XXZ (or D_3) spin chain for periodic (in the sector $Y = 2$) or twisted boundary conditions with (diagonal) twist $\omega_+/\omega_- = \omega^2$ for (for $Y = -1$). For the spin chain with these boundary conditions the number of Bethe roots M is related to the conserved $U(1)$ charge, i.e., the magnetization $\langle S^z\rangle = L/2 - M$. A particularly simple eigenstate of the XXZ model is the completely polarized state corresponding to $M = 0$. In the anyon chain such a state can be realized for chains of length L being multiples of 3. The corresponding eigenstates are in the subspace spanned by the fusion path states $|a_1 22 a_4 \ldots 22 a_{L-2} 22\rangle$ and its translations [7].[3] The a_i represent the state $|a_i\rangle = (|+\rangle - |-\rangle)/\sqrt{2}$ at position i. The translationally invariant zero momentum superposition belongs to the sector with topological charge $Y = 2$, the linear combinations with momentum $\pm 2\pi/3$ appear in the sector $Y = -1$.

Similarly, the fusion path states contributing to transfer matrix eigenstates with $M > 0$ (appearing for a chain of length L with $L - 2M$ being multiples of 3) can be built by concatenation of $L - 2M$ segments $|a22\rangle$ and, in addition, M two-site segments

$$|b2\rangle = \frac{1}{\sqrt{2}}(|+2\rangle + |-2\rangle) \quad \text{or} \quad |22\rangle. \tag{4.14}$$

2. Sector $Y = 0$

In the topological sector with $Y = 0$ the ansatz (4.10) for the q functions does not reproduce the asymptotic behavior (3.25) of the transfer matrix eigenvalues. Instead we choose

$$q(u) = \prod_{k=1}^{\widetilde{M}} \sinh \frac{1}{2}\left(u - u_k + \frac{i}{2}\gamma\right). \tag{4.15}$$

Again, the large-u behavior as obtained from the TQ equation (4.7)

$$\lim_{u\to\infty} \Lambda(u)\mathrm{e}^{-Lu+\sum_\ell y_\ell} = \omega_+ \mathrm{e}^{i\gamma(L-\widetilde{M}/2)} + \omega_- \mathrm{e}^{i\gamma \widetilde{M}/2} \tag{4.16}$$

has to match (3.25). This implies $\omega_\pm = \pm 1$ and $\widetilde{M} = L$. As before, requiring analyticity of the transfer matrix eigenvalues

$$\Lambda^{(Y=0)}(u) = \prod_{\ell=1}^{L} \sinh(u - y_\ell + i\gamma) \prod_{k=1}^{L} \frac{\sinh\frac{1}{2}\left(u - u_k - \frac{i}{2}\gamma\right)}{\sinh\frac{1}{2}\left(u - u_k + \frac{i}{2}\gamma\right)} - \prod_{\ell=1}^{L} \sinh(u - y_\ell)^L \prod_{i=1}^{L} \frac{\sinh\frac{1}{2}\left(u - u_k + \frac{3i}{2}\gamma\right)}{\sinh\frac{1}{2}\left(u - u_k + \frac{i}{2}\gamma\right)}, \tag{4.17}$$

leads to the Bethe equations determining the parameters u_k in the q function (4.15) for the topological sector $Y = 0$:

$$\prod_{\ell=1}^{L} \frac{\sinh\left(u_k - y_\ell + \frac{i}{2}\gamma\right)}{\sinh\left(u_k - y_\ell - \frac{i}{2}\gamma\right)} = \prod_{j=1}^{L} \frac{\sinh\frac{1}{2}\left(u_k - u_j + i\gamma\right)}{\sinh\frac{1}{2}\left(u_k - u_j - i\gamma\right)}, \quad k = 1,\ldots,L. \tag{4.18}$$

The eigenvalues (4.17) coincide with those of the XXZ spin-1/2 chain with antiperiodic boundary conditions [29,30],

$$\sigma_{L+1}^{\alpha} = K^{-1}\sigma_1^{\alpha}K, \quad K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.19}$$

For small systems we have checked that the Bethe equations (4.13) and (4.18) for the sectors $Y = 2, -1$ and 0, respectively, yield all eigenvalues of the transfer matrix $\tilde{\tau}(u)$ of the anyon model.

V. GROUND STATE AND LOW LYING EXCITATIONS

To find the spectrum of the anyon chain the Bethe equations (4.13), (4.18) have to be solved for the different topological sectors. From Eqs. (4.12) and (4.17) the eigenvalues of the Hamiltonian (3.17) corresponding to a set $\{u_k\}$ of Bethe roots

[3] Note that in the critical disordered regime ($\gamma \in \mathbb{R}$) considered here these states are never ground states of the anyon chain.

are

$$Y=2,-1: E(\{u_k\})=L\cos\gamma-2\sum_{k=1}^{M}\frac{\sin^2\gamma}{\cosh(2u_k)-\cos\gamma},$$
$$Y=0: E(\{u_k\})=L\cos\gamma-\sum_{k=1}^{L}\frac{\sin\gamma\,\sin\frac{\gamma}{2}}{\cosh u_k-\cos\frac{\gamma}{2}}. \quad (5.1)$$

A. Thermodynamic limit

The thermodynamic properties of the system do not depend on the boundary conditions. Since we have identified the spectrum of the D_3 anyon chain with that of the XXZ Heisenberg spin chain with boundary conditions depending on the topological sector the structure of the phase diagram as well as certain bulk properties can be inferred from the corresponding properties of the XXZ chain. For $J^z/|J|\leqslant -1$ ($>+1$) the latter is a gapped ferromagnet (antiferromagnet). The simple fusion path states in the $M=0$ sectors for topological charge $Y=2,-1$ discussed above form the ground states of the anyon model in the former interval and lead to the $\mathbb{Z}_3$-sublattice structure in this phase observed in Ref. [7]. Similarly, the ground state for $J^z/|J|\geqslant 1$ is in the $M=L/2$ sector implying a $\mathbb{Z}_2$-sublattice structure as in the antiferromagnetic XXZ chain.

For $-1<J^z/|J|\leqslant 1$ (or $0\leqslant\gamma<\pi$ in the present notation) the XXZ model is in its critical regime and known to be conformally invariant. This observation confirms the conclusion on the location of the phase boundaries reached from the numerical analysis of the anyon chain in Ref. [7]. The ground state of the spin chain in the critical phase is in the zero magnetization sector $M=L/2$. The energy density has been obtained within the root density formalism applied to the Bethe equations of the XXZ spin-1/2 chain (with $J=+1$) yielding [31]

$$\epsilon_\infty=\lim_{L\to\infty}\frac{E_0(L)}{L}=\cos\gamma-\frac{\sin\gamma}{\gamma}\int_{-\infty}^{\infty}du \times\frac{\sin\gamma}{\cosh(\pi u/\gamma)(\cosh(2u)-\cos\gamma)}. \quad (5.2)$$

The elementary excitations have a linear dispersion with Fermi velocity

$$v_F=\pi\frac{\sin\gamma}{\gamma}. \quad (5.3)$$

In the spectrum of the D_3 anyon chain with $-1<J^z\leqslant 1$ (and $J=+1$) these massless excitations are realized near momenta $k\pi/2$, $k=0,1,2,3$. This is the $\mathbb{Z}_4$ critical region identified in Ref. [7]. Similar to the spin chain the spectrum of the anyon Hamiltonian is invariant under the change $J\to -J$ for lattice sizes L being multiples of four, i.e.,

$$\mathrm{spec}(\tilde{H}(J,J^z))=\mathrm{spec}(\tilde{H}(-J,J^z)). \quad (5.4)$$

The corresponding unitary transformation relating the two Hamiltonians changes the momentum such that gapless excitations for $J=-1$ exist at 0, π only, characteristic for the $\mathbb{Z}_2$ critical region of the D_3 anyon chain.

B. Finite size spectrum and conformal field theory of the Heisenberg spin chain

The low energy excitations of the periodic spin chain (2.1) for $-1<J^z/|J|\leqslant 1$ are described by a conformal field theory with central charge $c=1$ [32,33]. The operator content of this CFT can be identified from the finite size scaling of the energies of the ground state and excitations through [34–36]

$$E_{h\bar{h}}(L)-L\epsilon_\infty=-\frac{\pi v_F}{6L}c+\frac{2\pi v_F}{L}(h+\bar{h}+R_{h\bar{h}}(L)). \quad (5.5)$$

Here $(h,\bar{h})$ are the conformal weights of primary operators of the CFT, and $R_{h\bar{h}}(L)$ subsumes the subleading corrections to scaling of the corresponding energy level from which the irrelevant perturbations of the conformal fixed point Hamiltonian present in the lattice model can be identified.

For periodic boundary conditions the low energy spectrum of the XXZ chain is that of a Gaussian model, i.e., a free boson with compactification radius $r=r_G(\gamma)=\sqrt{(\pi-\gamma)/(2\pi)}\in(0,1/\sqrt{2}]$ for exchange constants $(J,J^z)=(\pm 1,\cos\gamma)$ in (2.1) [37]. The scaling dimensions of the primary operators, highest weights of two commuting $U(1)$ Kac-Moody algebras, are

$$X_{mn}(r)=h_{mn}(r)+\bar{h}_{mn}(r)=r^2m^2+\frac{n^2}{4r^2}. \quad (5.6)$$

The quantum number m is the $U(1)$ charge $\frac{1}{2}\sum\sigma_j^z$ in the corresponding state, and the vorticity n is related to its momentum. For twisted boundary conditions with a diagonal unimodular twist matrix in (4.19) parameterized by an angle ϕ the scaling dimensions (5.6) are modified to $X_{m,n+\phi/\pi}$. In addition there exists a marginal operator in the sector $m=0$ with anomalous dimension $X=2$ for all γ. Its presence results in the continuous line $0\leqslant\gamma<\pi$ of critical points. The partition function of the theory satisfies the duality condition $Z(r)=Z(1/2r)$ as a consequence of the invariance of the spectrum (5.6) under the simultaneous interchanges $m\leftrightarrow n$ and $r\leftrightarrow 1/2r$.

In the case of the antidiagonal twisted boundary conditions (4.19) the $O(2)$ bulk symmetry of the XXZ spin chain is broken up to a $\mathbb{Z}_2\otimes\mathbb{Z}_2$ with the factors being generated by rotation around the z axis by π and a global spin flip, respectively. The low-energy spectrum is that of a $U(1)$-twisted KacMoody algebra without conserved charge. In this case the conformal weights are [38]

$$(h,\bar{h})_{k_1k_2}=\left(\frac{(4k_1+1)^2}{16},\frac{(4k_2+1)^2}{16}\right),\quad k_1,k_2\in\mathbb{Z} \quad (5.7)$$

independent of the anisotropy.

C. Conformal field theory of the D_3 anyon chain

Based on our identification of the spectrum of the D_3 anyon chain with subsets of that of the XXZ spin chain with boundary conditions depending on the topological sector we can now deduce the operator content of the low energy field theory for the anyon model: The finite size spectrum of low lying eigenvalues of the Hamiltonian (3.17) in the sectors with topological charge $Y=2$ correspond to primary operators with dimensions (5.6). The selection rules on the

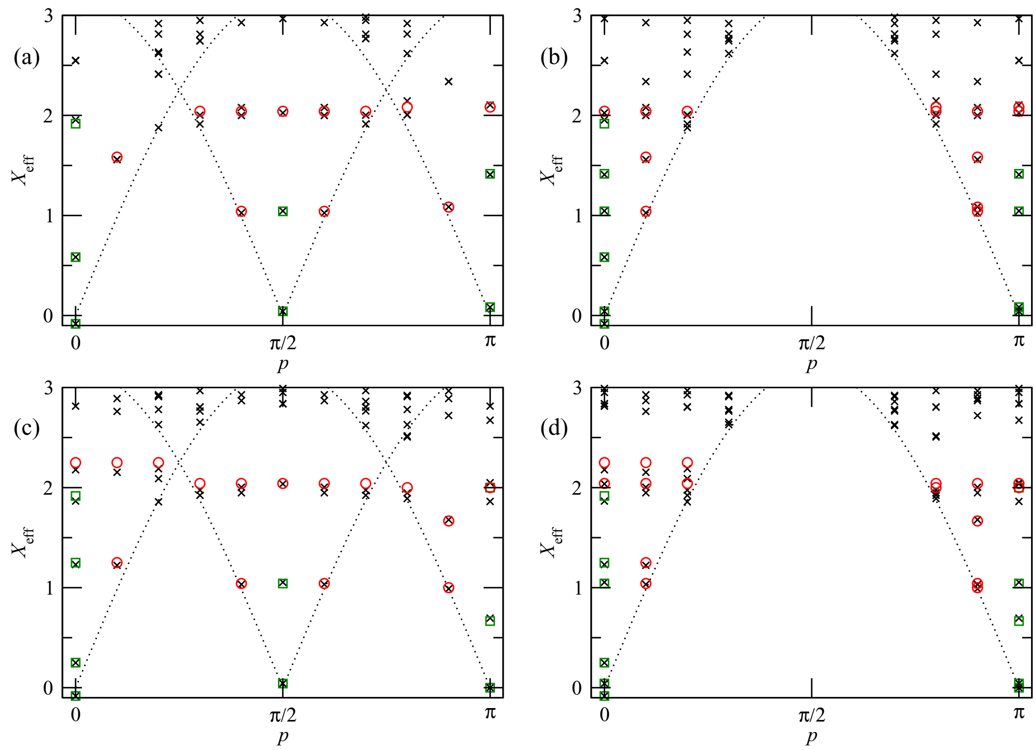


FIG. 3. Effective scaling dimensions $X_{\rm eff} = L(E - L\epsilon_\infty)/(2\pi v_F)$ of the D_3 anyon chain vs momentum of the corresponding excitations at some of the points (5.10) in the critical region. Spectra are shown for coupling constant $\gamma = 2\pi/3$ in the anyon chain corresponding to the $\mathbb{Z}_2$ orbifold CFT with $p = 3$ ($\mathbb{Z}_4$ parafermions) in the $\mathbb{Z}_4$ critical region of the anyon chain with $J = +1$ (a) and the $\mathbb{Z}_2$ critical region, $J = -1$ (b). Bottom panels (c) and (d): similar for $\gamma = \pi/3$ corresponding to the $\mathbb{Z}_2$ orbifold CFT with $p = 6$ (the superconformal minimal model with $c = 1$). Symbols $\times$ are the finite size data for $L = 20$ sites, green squares are the CFT predictions for the scaling dimensions of primary fields, red circles represent some of the descendents. Dotted lines indicate the properly re-scaled quasiparticle dispersion as obtained from the exact solution of the model, i.e., the lower bounds of the continua of collective excitations in the anyon chain.

$U(1)$ charge $M = L/2 - m$ and the twist angles $\pm 2\pi/3$ in the sector $Y = -1$, imply that levels with

$$2m \in 3\mathbb{Z} \quad \text{and} \quad 3n \in \mathbb{Z} \tag{5.8}$$

($2m = L \mod 2$) appear in the spectrum of the anyon chain. This part of the spectrum is that of a Gaussian model with compactification radius

$$\tilde{r}(\gamma) = 3\, r_G(\gamma) = 3\sqrt{\frac{\pi - \gamma}{2\pi}}. \tag{5.9}$$

In addition, from the low-lying levels in the sector $Y = 0$ we know that the CFT contains operators with scaling dimension 1/8 and 9/8. Together these spectral data lead to the identification of the underlying CFT as the $\mathbb{Z}_2$ orbifolds of a boson compactified on a circle of radius $\tilde{r}(\gamma)$. These CFTs are constructed by projection onto states which are invariant under a symmetry of the model [39,40]: for example, the theory with $\tilde{r}(\gamma = 0) = 3/\sqrt{2}$ is derived from the $SU(2)_1$ Wess-Zumino-Novikov-Witten model—the effective field theory describing the low energy excitations of the isotropic ($\gamma = 0$) Heisenberg chain—by 'modding out' the discrete subgroup D_3 of $SO(3) \subset SU(2)$. The primary fields in the $\mathbb{Z}_2$ orbifold CFT are

(i) the identity corresponding to the ground state, with conformal weights $h_0 = \bar{h}_0 = 0$,

(ii) the $\mathbb{Z}_2$ twist fields σ_i and τ_i, $i = 1,2$, with scaling dimensions $h_\sigma + \bar{h}_\sigma = 1/8$ and $h_\tau + \bar{h}_\tau = 9/8$,

(iii) a field Θ with scaling dimension 2,

and, in addition, the primaries $\tilde{\phi}_{mn}$ with scaling dimensions $X_{mn}(\tilde{r})$ given by (5.6).

For rational values of $\tilde{r}^2$ these CFTs have extended symmetries allowing for a description in terms of a *finite* number of primary fields (admissible highest weight representations with respect to the extended symmetry algebra). For the $\mathbb{Z}_2$ orbifold theories with $c = 1$ and compactification radii $\tilde{r}(\gamma) = \sqrt{p/2}$ (or $1/\sqrt{2p}$ by duality) with p integer these are the identity, the twist fields, the marginal field Θ, and, in addition, two degenerate fields $\Phi^{1,2}$ with conformal weight $h_\Phi = p/4$ and $p - 1$ fields ϕ_λ with conformal weight $h_\lambda = \lambda^2/(4p)$ for $\lambda = 1,2,\ldots,p-1$.

From (5.9) these rational CFTs with $p = 1,\ldots,9$ are realized for coupling constants

$$\gamma(p) = \pi\left(1 - \frac{p}{9}\right), \quad p = 1,\ldots,9. \tag{5.10}$$

Apart from the $SU(2)/D_3$ model for $p = 9$ mentioned above, the D_3 anyon model provides lattice realizations of the Kosterlitz-Thouless theory ($p = 1$), two uncoupled copies of the critical Ising model ($p = 2$), Z_4 parafermions ($p = 3$), the four-state Potts model ($p = 4$), and the superconformal minimal model ($p = 6$), all with central charge $c = 1$ [39]. The finite size spectra of the anyon chain for the $\mathbb{Z}_4$ parafermion

CFT, $p = 3$ or $\gamma = 2\pi/3$, and the superconformal minimal model, $p = 6$ or $\gamma = \pi/3$, are shown in Fig. 3, both for the $\mathbb{Z}_4$ ($J = +1$) and the $\mathbb{Z}_2$ critical case ($J = -1$).

Estimates for the location of the points $p = 1,\ldots,8$ in the critical region have been given in Ref. [7] based on the numerical analysis of the finite size spectrum of the $su(2)_4$ spin-1 anyon chain. In their discussion of the phase diagram the authors of that paper have also linked the transition from the critical regime into the gapped phase with $\mathbb{Z}_2$ sublattice symmetry with the appearance of a marginal operator with zero momentum in the topological sector $Y = 2$ of the ground state for $p = 9$: in fact, it is well known from the analysis of the subleading corrections to scaling $R_{h\bar{h}}(L)$ in (5.5) for the XXZ spin chain that its continuum limit differs from the conformally invariant fixed point Hamiltonian by terms involving an operator from the conformal block of the identity with scaling dimension $X = 4$ and the primary operator with dimension $X = 2\pi/(\pi - \gamma)$ [30,37]. In the $\mathbb{Z}_2$ orbifold CFT with compactification radius $\tilde{r}$ for the anyon chain this is $\tilde{\phi}_{0,6}$ ($\phi_{\lambda=6}$ at the rational points $\tilde{r} = \sqrt{p/2}$ for $p > 6$). This operator has scaling dimension $X_{\lambda=6} = h_{\lambda=6} + \bar{h}_{\lambda=6} = 36/(2p)$, hence is marginal for $p = 9$. The $p = 9$ rational orbifold CFT appears at coupling constant $\gamma = 0$ or $J^z = |J|$ in our parametrization of the anyon chain—in the related spin-1/2 XXZ Heisenberg chain this is the location of the transition between the antiferromagnetic critical (disordered) and massive regimes.

Finally we note that, by duality of the orbifold theory, the full sequence of rational CFTs with integer p appears at compactification radii $1/\sqrt{2p}$ or coupling constants

$$\bar{\gamma}(p) = \pi\left(1 - \frac{1}{9p}\right), \quad p = 1,2,3,\ldots, \tag{5.11}$$

see Fig. 1. These points accumulate near $\gamma = \pi$, where the first order transition into the gapped phase with $\mathbb{Z}_3$ sublattice structure takes place. The identification of the orbifold CFTs at these couplings by numerical studies, however, is difficult as a consequence of the diverging density of low energy states in this regime.

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[1] G. Moore and N. Read, Nonabelions in the fractional quantum Hall effect, Nucl. Phys. B **360**, 362 (1991).

[2] R. Moessner and S. L. Sondhi, An RVB Phase in the Triangular Lattice Quantum Dimer Model, Phys. Rev. Lett. **86**, 1881 (2001).

[3] L. Balents, M. P. A. Fisher, and S. M. Girvin, Fractionalization in an Easy-axis Kagome Antiferromagnet, Phys. Rev. B **65**, 224412 (2002).

[4] A. Kitaev, Anyons in an exactly solved model and beyond, Ann. Phys. (NY) **321**, 2 (2006).

[5] A. Feiguin, S. Trebst, A. W. W. Ludwig, M. Troyer, A. Kitaev, Z. Wang, and M. H. Freedman, Interacting Anyons in Topological Quantum Liquids: The Golden Chain, Phys. Rev. Lett. **98**, 160409 (2007).

[6] S. Trebst, E. Ardonne, A. Feiguin, D. A. Huse, A. W. W. Ludwig, and M. Troyer, Collective States of Interacting Fibonacci Anyons, Phys. Rev. Lett. **101**, 050401 (2008).

[7] C. Gils, E. Ardonne, S. Trebst, D. A. Huse, A. W. W. Ludwig, M. Troyer, and Z. Wang, Anyonic quantum spin chains: Spin-1 generalizations and topological stability, Phys. Rev. B **87**, 235120 (2013).

[8] P. E. Finch, H. Frahm, M. Lewerenz, A. Milsted, and T. J. Osborne, Quantum phases of a chain of strongly interacting anyons, Phys. Rev. B **90**, 081111(R) (2014).

[9] C. Gils, E. Ardonne, S. Trebst, A. W. W. Ludwig, M. Troyer, and Z. Wang, Collective States of Interacting Anyons, Edge States, and the Nucleation of Topological Liquids, Phys. Rev. Lett. **103**, 070401 (2009).

[10] E. Grosfeld and K. Schoutens, Non-Abelian Anyons: When Ising Meets Fibonacci, Phys. Rev. Lett. **103**, 076803 (2009).

[11] F. A. Bais and J. K. Slingerland, Condensate-induced transitions between topologically ordered phases, Phys. Rev. B **79**, 045316 (2009).

[12] G. E. Andrews, R. J. Baxter, and P. J. Forrester, Eight-vertex SOS model and generalized Rogers-Ramanujan-type identities, J. Stat. Phys. **35**, 193 (1984).

[13] V. Pasquier, Etiology of IRF models, Comm. Math. Phys. **118**, 355 (1988).

[14] D. Gepner, Foundations of Rational Quantum Field Theory, I, Preprint CALT-68-1825 (Caltech, 1992) hep-th/9211100.

[15] P. E. Finch and H. Frahm, The $D(D_3)$-anyon chain: Integrable boundary conditions and excitation spectra, New J. Phys. **15**, 053035 (2013).

[16] P. E. Finch, M. Flohr, and H. Frahm, Integrable anyon chains: From fusion rules to face models to effective field theories, Nucl. Phys. B **889**, 299 (2014).

[17] L. Martina, A. Protogenov, and V. Verbus, arXiv:1001.4932.

[18] V. A. Verbus, L. Martina, and A. P. Protogenov, Chain of interacting $SU(2)_4$ anyons and quantum $SU(2)_k \times \overline{SU(2)_k}$ doubles, Theor. Math. Phys. **167**, 843 (2011).

[19] P. E. Finch, From spin to anyon notation: The XXZ Heisenberg model as a D_3 (or $su(2)_4$) anyon chain, J. Phys. A **46**, 055305 (2013).

[20] Ph. Roche, Ocneanu cell calculus and integrable lattice models, Comm. Math. Phys. **127**, 395 (1990).

[21] M. Kohmoto, M. den Nijs, and L. P. Kadanoff, Hamiltonian studies of the $d = 2$ Ashkin-Teller model, Phys. Rev. B **24**, 5229 (1981).

[22] S.-K. Yang, Modular invariant partition function of the Ashkin-Teller model on the critical line and $N = 2$ superconformal invariance, Nucl. Phys. B **285**, 183 (1987).

[23] H. Saleur, Partition functions of the two-dimensional Ashkin-Teller model on the critical line, J. Phys. A **20**, L1127 (1987).

[24] P. di Francesco and J.-B. Zuber, SU(N) lattice integrable models associated with graphs, Nucl. Phys. B **338**, 602 (1990).

[25] A. Klümper and P. A. Pearce, Conformal weights of RSOS lattice models and their fusion hierarchies, Physica A **183**, 304 (1992).

[26] H. Frahm and N. Karaiskos, Inversion identities for inhomogeneous face models, Nucl. Phys. B **887**, 423 (2014).

[27] H. Frahm and N. Karaiskos, Non-Abelian $SU(3)_k$ anyons: Inversion identities for higher rank face models, J. Phys. A **48**, 484001 (2015).
[28] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic Press, London, 1982).
[29] M. T. Batchelor, R. J. Baxter, M. J. O'Rourke, and C. M. Yung, Exact solution and interfacial tension of the six-vertex model with anti-periodic boundary conditions, J. Phys. A **28**, 2759 (1995).
[30] S. Niekamp, T. Wirth, and H. Frahm, The XXZ model with anti-periodic twisted boundary conditions, J. Phys. A: Math. Theor. **42**, 195008 (2009).
[31] C. N. Yang and C. P. Yang, One-dimensional chain of anisotropic spin-spin interactions. II. Properties of the ground-state energy per lattice site for an infinite system, Phys. Rev. **150**, 327 (1966).
[32] L. V. Avdeev and B.-D. Dörfel, Finite-size corrections for the XXX antiferromagnet, J. Phys. A **19**, L13 (1986).
[33] C. J. Hamer, Finite-size corrections for ground states of the XXZ Heisenberg chain, J. Phys. A **19**, 3335 (1986).
[34] H. W. J. Blöte, J. L. Cardy, and M. P. Nightingale, Conformal Invariance, the Central Charge and Universal Finite-Size Amplitudes at Criticality, Phys. Rev. Lett. **56**, 742 (1986).
[35] I. Affleck, Universal Term in the Free Energy at a Critical Point and the Conformal Anomaly, Phys. Rev. Lett. **56**, 746 (1986).
[36] J. L. Cardy, Operator content of two-dimensional conformally invariant theories, Nucl. Phys. B **270**, 186 (1986).
[37] F. C. Alcaraz, M. N. Barber, and M. T. Batchelor, Conformal invariance, the XXZ chain and the operator content of two-dimensional critical systems, Ann. Phys. (NY) **182**, 280 (1988).
[38] F. C. Alcaraz, M. Baake, U. Grimmn, and V. Rittenberg, Operator content of the XXZ chain, J. Phys. A **21**, L117 (1988).
[39] P. H. Ginsparg, Curiosities at $c = 1$, Nucl. Phys. B **295**, 153 (1988).
[40] R. Dijkgraaf, C. Vafa, E. Verlinde, and H. Verlinde, The operator algebra of orbifold models, Comm. Math. Phys. **123**, 485 (1989).