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STRAIGHT PROJECTIVE-METRIC SPACES WITH CENTERS

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ABSTRACT. It is proved that a straight projective-metric space has an open set of centers, if and only if it is either the hyperbolic or a Minkowskian geometry. It is also shown that if a straight projective-metric space has some finitely many well-placed centers, then it is either the hyperbolic or a Minkowskian geometry.

1. Introduction

Let $(\mathcal{M}, d)$ be a metric space given in a set $\mathcal{M}$ with the metric d. If $\mathcal{M}$ is a projective space $\mathbb{P}^n$ or an affine space $\mathbb{R}^n \subset \mathbb{P}^n$ or a proper open convex subset of $\mathbb{R}^n$ for some $n \in \mathbb{N}$, and the metric d is complete, continuous with respect to the standard topology of $\mathbb{P}^n$, and the geodesic lines of d are exactly the non-empty intersection of $\mathcal{M}$ with the straight lines, then the metric d is called *projective*.

If $\mathcal{M} = \mathbb{P}^n$ and the geodesic lines of d are isometric with a Euclidean circle, or $\mathcal{M} \subseteq \mathbb{R}^n$ and the geodesic lines of d are isometric with a Euclidean straight line, then $(\mathcal{M}, d)$ is called a *projective-metric space* of dimension n (see [1, p. 115] and [8, p. 188])

Such projective-metric spaces are called of *elliptic, parabolic or hyperbolic type* according to whether $\mathcal{M}$ is $\mathbb{P}^n$, $\mathbb{R}^n$, or a proper convex subset of $\mathbb{R}^n$. The projective-metric spaces of the latter two types are called *straight* [2, p. 1].

A *metric point reflection* $\rho_{d;O}$ of a projective-metric space $(\mathcal{M}, d)$ is a non-identical, involutive d-isometry of $\mathcal{M}$ onto $\mathcal{M}$ such that it fixes point O and keeps every geodesic line passing through O. A *center* of the projective-metric space $(\mathcal{M}, d)$ is a point $O \in \mathcal{M}$, where there exists a metric point reflection $\rho_{d;O}$. If every point of a projective-metric space is a center, then it is said to be *symmetric*.

What are the symmetric projective-metric spaces?

Working with G-spaces[1], Busemann proved in [2] that a symmetric G-space of elliptic type is elliptic [2, (49.5)][2], and a symmetric G-space of dimension 2 is either elliptic, or hyperbolic, or Minkowskian [2, (52.8)].

In this paper we complement Busemann's results for straight projective-metric spaces by proving *directly* in *every dimension* (Theorem 3.1 and Theorem 3.2) that a straight projective-metric space has a non-empty open set of centers if and

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[1]Projective-metric spaces are the Desarguesian G-spaces [8, p. 188].

[2]"There is no similar theorem for straight G-spaces" as Busemann proves on [2, p. 346].

only if it is a Minkowskian or the hyperbolic geometry, respectively. Further, we show (Theorems 4.3, 4.4 and 4.5) that a straight projective-metric space has some finitely many well-placed centers if and only if it is a Minkowskian or the hyperbolic geometry, respectively.

2. Notations and preliminaries

Points of $\mathbb{R}^n$ are denoted as $A, B, \dots$, vectors are $\overrightarrow{AB}$ or $\boldsymbol{a}, \boldsymbol{b}, \dots$. Latter notations are also used for points if the origin is fixed. Open segment with endpoints A and B is denoted by $\overline{AB}$ and the line through A and B is denoted by AB. The Euclidean scalar product is $\langle\cdot,\cdot\rangle$.

The *affine ratio* $(A, B; C)$ of the collinear points A, B and $C \neq B$ is defined by $(A, B; C)\overrightarrow{BC} = \overrightarrow{AC}$. The *affine cross ratio* of the collinear points A, B, $C \neq B$, and $D \neq A$ is $(A, B; C, D) = (A, B; C)/(A, B; D)$ [1, page 243]. The *affine point reflection* $\bar{\rho}_O$ at point O is defined by $(X, \bar{\rho}_O(X); O) = -1$ for every point $X \neq O$ and by $\bar{\rho}_O(O) = O$.

According to [7, p. 64], a point O is called a *projective center* of the set $\mathcal{S} \subseteq \mathbb{P}^n$, if there is a projectivity ϖ such that $\varpi(O)$ is the affine center of $\varpi(\mathcal{S})$.

Fix a point O in the convex open bounded domain $\mathcal{D} \subseteq \mathbb{R}^n$. We define O^* as the locus of every point P which is the harmonic conjugate[3] of O with respect to points A and B, where $\{A, B\} = \partial\mathcal{D} \cap OP$. It is easy to see that a point O is a projective center of $\mathcal{D}$ if and only if O^* is a straight line that does not intersect $\mathcal{D}$ [7, p. 64].

If a projective-metric space $(\mathcal{M}, d)$ is given, we denote the geodesic line on the projective line ℓ by $\tilde{\ell}$, i.e. $\tilde{\ell} = \ell \cap \mathcal{M}$.

2.1. Projectively invariant metrics on projective lines. The following easy, perhaps folkloric results are provided here for the sake of completeness, and because the author could not find a really good reference for them.

Lemma 2.1. *Let the function* $h\colon (a, b) \times (a, b) \to \mathbb{R}$ *be such that* $h(x, y) + h(y, z) = h(x, z)$ *and* $h(x, y) = h(\varpi(x), \varpi(y))$ *for every* $x, y, z \in (a, b)$ *and any projectivity* $\varpi\colon (a, b) \to (a, b)$. *If* h *is bounded on an open subset of* $(a, b) \times (a, b)$, *then there exists a constant* $c \in \mathbb{R}$ *such that*

$$h(x, y) = c|\ln(a, b; x, y)|.$$

Proof. Let $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$ and fix the projectivity $\omega\colon x \in (a, b) \mapsto \frac{x-a}{b-x} \in \mathbb{R}_+$. Then the function $f\colon (x, y) \in \mathbb{R}_+ \times \mathbb{R}_+ \mapsto h(\omega^{-1}(x), \omega^{-1}(y)) \in \mathbb{R}$ clearly satisfies $f(x, y) + f(y, z) = f(x, z)$ and $f(x, y) = f(\hat{\varpi}(x), \hat{\varpi}(y))$ for every $x, y, z \in \mathbb{R}_+$ and for any surjective projectivity $\hat{\varpi}\colon \mathbb{R}_+ \to \mathbb{R}_+$, as $\omega^{-1} \circ \hat{\varpi} \circ \omega\colon (a, b) \to (a, b)$ is a surjective projectivity.

[3] Point P satisfies $(A, B; O, P) = -1$.

A projectivity of an affine straight line is an affinity. An affinity of a straight line with a fixed point is a dilation, thus we have $f(x,y) = f(bx,by)$ for every $x,y \in \mathbb{R}_+$ and $b \in \mathbb{R}_+$. Choosing $b = 1/x$ implies $f(x,y) = f(1,y/x)$ for every $x,y \in \mathbb{R}_+$, from which $f(1,y/x) + f(1,z/y) = f(x,y) + f(y,z) = f(x,z) = f(1,z/x)$ follows for every $x,y \in \mathbb{R}_+$, hence $f(1,s) + f(1,t) = f(1,st)$ for every $s,t \in \mathbb{R}_+$.

Let $g(u) = f(1,e^u)$ for every $u \in \mathbb{R}_+$. Then $g(p) + g(q) = g(p+q)$ for every $p,q \in \mathbb{R}_+$, hence, by known properties of Cauchy's functional equation [9], $g(u) = cu$ follows for some $c \in \mathbb{R}$ and every $u \in \mathbb{R}_+$. Thus

$$\begin{aligned} h(x,y) &= f(\omega(x),\omega(y)) = f(1,\omega(y)/\omega(x)) = g(\ln(\omega(y)/\omega(x))) \\ &= c \cdot \ln(\omega(y) : \omega(x)) = c \ln\Big(\frac{y-a}{b-y} : \frac{x-a}{b-x}\Big). \end{aligned}$$

□

Lemma 2.2. *Let $m\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a continuous function that satisfies $m(x,y) = 0$ if and only if $x = y$, fulfills $m(x,y) + m(y,z) = m(x,z)$ if and only if $y \in [x,z]$, and satisfies $m(x,y) = m(\varpi(x),\varpi(y))$ for every surjective projectivity $\varpi\colon \mathbb{R} \to \mathbb{R}$ that has no fixed point. Then there are constants $c_+, c_- \in \mathbb{R}$ such that*

$$m(x,y) = c_{\operatorname{sign}(x-y)}|y-x|.$$

Proof. A projectivity of an affine straight line is an affinity. An affinity of $\mathbb{R}$ has the form $\varpi\colon x \mapsto ax+b$, where $a,b \in \mathbb{R}$. If $a \neq 1$, then $\frac{b}{1-a}$ is a fixpoint of ϖ, therefore, we have $\varpi\colon x \mapsto x+b$. Thus we have $m(x,y) = m(x+b,y+b)$ for every $x,y \in \mathbb{R}$ and $b \in \mathbb{R}$.

Choosing $b = -x$, we obtain $m(x,y) = m(0,y-x)$ for every $x,y \in \mathbb{R}$, from which $m(0,y-x) + m(0,z-y) = m(x,y) + m(y,z) = m(x,z) = m(0,z-x)$, hence $m(0,s) + m(0,t) = m(0,s+t)$ follows for every $s,t \in \mathbb{R}_+$. Let $g(u) = m(0,u)$ for every $u \in \mathbb{R}_+$. Then $g(p) + g(q) = g(p+q)$ for every $p,q \in \mathbb{R}_+$, hence, by known properties of Cauchy's functional equation [9], $g(u) = c_+u$ follows for some $c_+ \in \mathbb{R}$ and every $u \in \mathbb{R}_+$. Thus, $m(x,y) = m(0,y-x) = g(y-x) = c_+(y-x)$ for every $x \le y$.

Now choose $b = -y$ to obtain $m(x,y) = m(x-y,0)$ for every $x,y \in \mathbb{R}$, implying $m(x-y,0) + m(y-z,0) = m(x,y) + m(y,z) = m(x,z) = m(x-z,0)$, hence $m(s,0) + m(t,0) = m(s+t,0)$ for every $s,t \in \mathbb{R}_-$. Let $g(u) = m(u,0)$ for every $u \in \mathbb{R}_-$. Then $g(p)+g(q) = g(p+q)$ for every $p,q \in \mathbb{R}_-$, hence, by known properties of Cauchy's functional equation [9], $g(u) = c_-u$ follows for some $c_- \in \mathbb{R}$ and every $u \in \mathbb{R}_-$. Thus, $m(x,y) = m(y-x,0) = g(x-y) = c_-(x-y)$ for every $y \le x$. □

2.2. Straight projective-metric spaces. The following two (most) important examples are distinguished among the straight projective-metric spaces by the property that an isometry of one geodesic on another or itself is a projectivity [3, II.8.(3)].

2.2.1. Minkowski geometry. Given an open, strictly convex, bounded domain $\mathcal{I} \subset \mathbb{R}^n$, the *indicatrix*, that is (centrally) symmetric to the origin, the function

$d_{\mathcal{I}}\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$d_{\mathcal{I}}(\boldsymbol{x}, \boldsymbol{y}) = \inf\{\lambda > 0 : (\boldsymbol{y} - \boldsymbol{x})/\lambda \in \mathcal{I}\}$$

is a metric on $\mathbb{R}^n$ [1, VI.48], and is called *Minkowski metric* on $\mathbb{R}^n$. The projective-metric spaces of type $(\mathbb{R}^n, d_{\mathcal{I}})$ are all called *Minkowski geometry.* It is the Euclidean geometry if and only if $\mathcal{I}$ is an ellipsoid [1, (48.7)].

2.2.2. Hilbert geometry. Given an open, strictly convex, bounded domain $\mathcal{I} \subset \mathbb{R}^n$, that does not contain two coplanar non-collinear segments, the function $d_{\mathcal{I}}\colon \mathcal{I} \times \mathcal{I} \to \mathbb{R}$ defined by

$$d_{\mathcal{I}}(A, B) = \begin{cases} 0, & \text{if } A = B, \\ \frac{1}{2}\big|\ln(A, B; C, D)\big|, & \text{if } A \neq B, \text{ where } \overline{CD} = \mathcal{I} \cap AB, \end{cases}$$

is a projective metric on $\mathcal{I}$ [1, VI.50], and is called the *Hilbert metric* on $\mathcal{I}$. The projective-metric space $(\mathcal{I}, d_{\mathcal{I}})$ is called *Hilbert geometry* given in $\mathcal{I}$. It is the hyperbolic geometry if and only if $\mathcal{I}$ is an ellipsoid [1, (50.2)].

2.3. Isometries and metric point reflections. Although some of the statements here are valid more generally, we confine ourselves here to straight projective-metric spaces $(\mathcal{M}, d)$.

An isometry keeps the geodesic lines, therefore, it is the restriction of a collineation [4, Theorem 3.1.]. A collineation is, by Staudt's theorem [5, (ii) Fundamental theorem of projective geometry, p. 30], a projective map of $\mathbb{P}^n$, so we obtain that

isometries are restrictions of projective maps. (2.1) ⟨7⟩

If there exists a metric point reflection $\rho_{d;O}$, then O is the metric midpoint of the geodesic segment $\overline{P\rho_{d;O}P} \subset \widetilde{P\rho_{d;O}P}$ for any point $P \neq O$, hence we have that

there is at most one metric point reflection at every point. (2.2) ⟨5, 6⟩

The easy formal proof of the following statement is left to the reader.

If O is a center, and ϖ is a projective map of $\mathbb{P}^n$, then $\varpi(O)$ is a center of the projective-metric space $(\varpi(\mathcal{M}), d_{\varpi})$, where $d_{\varpi}(\varpi(P), \varpi(Q)) = d(P, Q)$ for every point $P, Q \in \mathcal{M}$. (2.3) ⟨7, 8⟩

Lemma 2.3. *We have $\rho_{d;\rho_{d;O}Q} = \rho_{d;O} \circ \rho_{d;Q} \circ \rho_{d;O}$.*

Proof. The map $\imath := \rho_{d;O} \circ \rho_{d;Q} \circ \rho_{d;O}$ is clearly a non-trivial isometry, and fixes point $Q' := \rho_{d;O}Q$, because $\imath Q' = \imath\rho_{d;O}Q = \rho_{d;O}\rho_{d;Q}Q = \rho_{d;O}Q = Q'$. Further, it satisfies $\imath^2 = (\rho_{d;O} \circ \rho_{d;Q} \circ \rho_{d;O}) \circ (\rho_{d;O} \circ \rho_{d;Q} \circ \rho_{d;O}) = \mathrm{id}$.

Assume that points $A', B' \in \mathcal{M}$ are such that $Q' \in \widetilde{A'B'}$. Let $A = \rho_{d;O}A'$ and $B = \rho_{d;O}B'$. Then $Q = \rho_{d;O}Q' \in \rho_{d;O}\big(\widetilde{A'B'}\big) = \widetilde{AB}$, hence $\imath\big(\widetilde{A'B'}\big) = \widetilde{\imath A'\imath B'} = \rho_{d;O} \circ \rho_{d;Q}\big(\widetilde{AB}\big) = \rho_{d;O}\big(\widetilde{BA}\big) = \widetilde{B'A'}$, i.e. $\imath$ keeps every geodesic passing through Q'.

Thus, $\imath$ is non-trivial, isometric, fixes Q', involutive, and keeps the geodesic lines passing through Q', therefore, by (2.2), it is the metric point reflection $\rho_{d;Q'}$. □

Lemma 2.4. *The set of the centers is closed.*

Proof. Let O_n be a sequence of centers of the projective-metric space $(\mathcal{M}, d)$ converging to O_∞. Then we have the sequence of points $P_n = \rho_{d;O_n}(P)$ for any point $P \in \mathcal{M}$.

From $d(P_n, O_n) = d(P, O_n)$, $O_n \to O_\infty$ and the triangle inequality it follows that $d(P_n, O_\infty) \le d(P_n, O_n) + d(O_n, O_\infty) = d(P, O_n) + d(O_n, O_\infty) \le d(P, O_\infty) + d(O_\infty, O_n) + d(O_n, O_\infty) \le d(P, O_\infty) + \varepsilon$ for any $\varepsilon > 0$ if $n \in \mathbb{N}$ is big enough. Thus, the sequence of points P_n is bounded, hence it has congestion points.

If P_∞ is a congestion point of P_n, then

$$d(P, O_\infty) + d(O_\infty, P_\infty) = \lim_{n\to\infty} (d(P, O_n) + d(O_n, P_n)) = \lim_{n\to\infty} d(P, P_n) = d(P, P_\infty)$$

proves that $O_\infty \in \widetilde{PP_\infty}$, and $d(P, O_\infty) = \lim_{n\to\infty}(d(P, O_n) = \lim_{n\to\infty}(d(P_n, O_n) = d(P_\infty, O_\infty)$ proves that O_∞ is the metric midpoint of the segment $\overline{PP_\infty}$. Thus, $P_\infty = \rho_{d;O_\infty}(P)$, hence the Lemma. □

Two point reflections define an isometry defined by $\tau_{PQ} := \rho_{d;P} \circ \rho_{d;Q}$. We call such isometries *translations*.

Lemma 2.5. *For any three collinear points O, P, Q*

(1) $O \in \overline{PQ}$ *if and only if* $O \in \overline{\rho_{d;P}(O)\rho_{d;Q}(O)}$, *and*
(2) $d(\tau_{PQ}(O), O) = 2d(P, Q)$.

Proof. To prove (1) we need only to observe that the points P and Q are on the same side of O as the points $\rho_{d;P}(O)$ and $\rho_{d;Q}(O)$, respectively.

For (2) we let $\delta := d(\tau_{PQ}(O), O) = d(\rho_{d;P}(\rho_{d;Q}(O)), O) = d(\rho_{d;Q}(O), \rho_{d;P}(O))$, and consider three cases:

(a) if $O \in \overline{PQ}$, then $O \in \overline{\rho_{d;P}(O)\rho_{d;Q}(O)}$, hence
$\delta = d(\rho_{d;Q}(O), O) + d(O, \rho_{d;P}(O)) = 2d(Q, O) + d(O, P) = 2d(P, Q)$;
(b) if $P \in \overline{OQ}$, then $\rho_{d;P}(O) \in \overline{O\rho_{d;Q}(O)}$, hence
$\delta = d(\rho_{d;Q}(O), O) - d(O, \rho_{d;P}(O)) = 2d(Q, O) - d(O, P) = 2d(P, Q)$;
(c) if $Q \in \overline{OP}$, then $\rho_{d;Q}(O) \in \overline{O\rho_{d;P}(O)}$, hence
$\delta = d(\rho_{d;P}(O), O) - d(O, \rho_{d;Q}(O)) = 2d(P, O) - d(O, Q) = 2d(P, Q)$. □

Lemma 2.6. *Assume that every point of the geodesic line $\tilde{\ell}$ is a center. Then every isometry of $\tilde{\ell}$ is a restriction of τ_{PQ} or $\rho_{d;P}$, where P, Q are any points on $\tilde{\ell}$.*

Proof. By definition we have an isometry $\imath\colon \tilde{\ell} \to \mathbb{R}$.

If $\jmath$ is an isometry on $\tilde{\ell}$, then $\imath \circ \jmath \circ \imath^{-1}$ is an isometry on $\mathbb{R}$. Every isometry on $\mathbb{R}$ has the form of either $x \mapsto a + x$ or $x \mapsto a - x$ for some $a \in \mathbb{R}$, so we have for a fixed $a \in \mathbb{R}$ either $\imath(\jmath(\imath^{-1}(x))) = a + x$ or $\imath(\jmath(\imath^{-1}(x))) = a - x$ for every $x \in \mathbb{R}$.

Thus, every isometry $\jmath$ on $\tilde{\ell}$ is either $\jmath(P) = \imath^{-1}(a+\imath(P))$ or $\jmath(P) = \imath^{-1}(a-\imath(P))$, for some $a \in \mathbb{R}$.

If $\jmath(\cdot) = \imath^{-1}(a+\imath(\cdot))$, then $d(\jmath(P), P) = |\imath(\jmath(P)) - \imath(P)| = |a+\imath(P) - \imath(P)| = a$, hence, by Lemma 2.5(2), $\jmath = \tau_{QR}$, where $Q, R \in \tilde{\ell}$ and $d(Q, R) = a/2$.

If $\jmath(\cdot) = \imath^{-1}(a - \imath(\cdot))$, then we have a point $O \in \tilde{\ell}$ such that $\imath(O) = a/2$, and $d(\jmath(P), O) = |\imath(\jmath(P)) - \imath(O)| = |a/2 - \imath(P)| = |\imath(P) - \imath(O)| = d(P, O)$, as well as $d(\jmath(P), P) = |\imath(\jmath(P)) - \imath(P)| = 2|a/2 - \imath(P)| = 2d(P, O)$, hence, by (2.2), $\jmath = \rho_O$. □

3. Open set of centers

Firstly, we note the well-known fact that

Minkowski geometries and the hyperbolic geometry are symmetric. (3.1) ⟨6, 7⟩

Theorem 3.1. *The set of the centers of a projective-metric space of parabolic type contains a non-empty open set of centers if and only if it is Minkowskian geometry.*

Proof. By (3.1) and Lemma 2.3, we only need to prove that if every point of a projective-metric space of parabolic type is a center, then it is a Minkowskian geometry.

First, we prove that

if O is a center of a projective-metric space of parabolic type, then the metric point reflection ρ_O is the affine point reflection $\bar{\rho}_O$. (3.2) ⟨6, 11⟩

Let the straight line ℓ avoid O and let $\ell' = \rho_O(\ell)$. As ρ_O keeps the straight lines containing O, every straight line l through O and a point P of ℓ coincides $\rho_O l$. As all these lines are in the common plane $\mathbb{R}^2_{O,\ell}$ of O and ℓ, we conclude that ℓ and ℓ' are in $\mathbb{R}^2_{O,\ell}$.

Assume that ℓ intersects ℓ', i.e. there is a point P in $\ell \cap \ell'$. Then $\rho_O(P)$ is also in $\ell \cap \ell'$ and is different from P as O is the metric midpoint of the segment $\overline{P\rho_O(P)}$, and $d(O, P) > 0$. Thus, ℓ and ℓ' have two different common points, hence $\ell \equiv \ell'$. This is a contradiction as $O \in \overline{P\rho_O(P)} \subset \ell$, but $O \notin \ell$. Thus, ℓ does not intersect ℓ', that, as these straight lines are in their common plane $\mathbb{R}^2_{O,\ell}$, implies that $\ell \parallel \ell'$. So, ρ_O maps every straight line into a parallel straight line.

Let O and A be arbitrary different points. Let B be any point outside their common straight line. By the above observation $AB \parallel \rho_O(A)\rho_O(B)$ and $A\rho_O(B) \parallel \rho_O(A)B$, hence quadrangle $\mathcal{P} := AB\rho_O(A)\rho_O(B)\Box$ is a parallelogram. As O is the intersection of the diagonals of $\mathcal{P}$, it follows that O is the affine midpoint of the segments $\overline{A\rho_O(A)}$. This proves (3.2).

Let A and B be arbitrary different points, and let O be the d-metric midpoint of segment $\overline{AB}$. Then $\rho_O(A) = B$, and by (3.2), O is the affine midpoint of $\overline{AB}$ too.

Thus, the affine midpoint and the d-metric midpoint of any segment coincide which, by [2, (17.9)], implies that d is a Minkowskian metric. □

Theorem 3.2. *The set of the centers of a projective-metric space of hyperbolic type contains a non-empty open set if and only if it is the hyperbolic geometry.*

Proof. By (3.1) and Lemma 2.3, we only need to prove that if every point of a projective-metric space $(\mathcal{M}, d)$ of hyperbolic type is a center, then it is the hyperbolic geometry.

By [1, Lemma 12.1, pp. 226], a bounded open convex set $\mathcal{I}$ in $\mathbb{R}^n$ $(n \geq 2)$ is an ellipsoid if and only if every section of it by any 2-dimensional plane is an ellipse. This means, that we only need to prove the statement in dimension 2.[4]

As it is convex and proper subset of $\mathbb{R}^2$, $\mathcal{M}$ cannot contain two intersecting affine straight line, because otherwise it coincides with the affine plane $\mathbb{R}^2$.

Assume now that $\mathcal{M}$ contains an affine line.

A convex domain in the plane which contains a straight line is either a half plane or a strip bounded by two parallel lines [1, Exercise [17.8]], therefore, $\mathcal{M}$ is either $\mathcal{P}_{(0,\infty)} := \{(x, y) \in \mathbb{R}^2 : 0 < x\}$ or $\mathcal{P}_{(0,1)} := \{(x, y) \in \mathbb{R}^2 : 0 < x < 1\}$ in proper linear coordinatizations of $\mathbb{R}^2$. As the perspective projectivity $\varpi\colon (x, y) \mapsto \big(\frac{x}{x+1}, \frac{y}{x+1}\big)$ maps $\mathcal{P}_{(0,\infty)}$ onto $\mathcal{P}_{(0,1)}$ bijectively, (2.3) immediately implies that it is enough to consider the case $\mathcal{M} = \mathcal{P}_{(0,1)}$.

By Lemma 2.6 the point reflections of $(\mathcal{M}, d)$ restricted onto a line $\tilde{\ell}$ generate every isometry of $\tilde{\ell}$, and, by (2.1), every point reflection of $(\mathcal{M}, d)$ is the restriction of a projective map of the projective plane onto $\mathcal{M}$, hence Lemma 2.2 gives that $d((x, y),(x, z)) = c(x)|z - y|$ for a continuous functions $c\colon (0, 1) \to \mathbb{R}_+$. Function c is a constant, because the point reflection $\rho_{d;(t,0)}$ maps d-isometrically the lines $\ell_x := \{(x, y) : y \in \mathbb{R}\}$ $(x \in (0, 1))$ onto lines ℓ_z, where $\frac{1}{z} = 1 + \big(\frac{1-t}{t}\big)^2 \frac{1-x}{x}$.

In the same way as in the above paragraph, Lemma 2.1 gives

$$d((x, \lambda + \sigma x),(\mu x, \lambda + \mu\sigma x)) = \bar{c}(\lambda, \sigma)\Big|\ln\Big(0, \frac{1}{x}; 1, \mu\Big)\Big| = \bar{c}(\lambda, \sigma)\Big|\ln\frac{1 - \mu x}{\mu(1 - x)}\Big|,$$

where $\bar{c}\colon \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function. Function $\bar{c}$ is a constant, because the point reflection $\rho_{d;O}$, where $O = \big(\frac{\lambda}{2\lambda+\sigma}, \frac{\lambda(\lambda+\sigma)}{2\lambda+\sigma}\big)$, maps the open segment $\overline{(0, \lambda)(1, \lambda + \sigma)}$ onto $\overline{(0, 0)(1, 0)}$ d-isometrically.

By the aboves we have

$$d((x, 0),(s, y)) = \begin{cases} \bar{c}(0, 0)\big|\ln\frac{x(1-s)}{s(1-x)}\big|, & \text{if } x \neq s, \\ c(1/2)|y|, & \text{if } x = s, \end{cases}$$

for every $x, s \in (0, 1)$ and $y \in \mathbb{R}$, hence $c(1/2)|y| = \bar{c}(0, 0)\lim_{x\to s}\big|\ln\frac{x(1-s)}{s(1-x)}\big| = 0$ by the continuity of d. This contradiction proves that a projective-metric space $(\mathcal{P}_{(0,1)}, d)$ cannot be symmetric.

[4] Although this is already proved in [2, (52.8)], we give here a more direct proof.

Assume now that $\mathcal{M}$ contains no affine line.

Then every supporting line ℓ of $\mathcal{M}$ at any point M of $\partial\mathcal{M}$ can intersect $\partial\mathcal{M}$ only in a point, a segment or a ray. Let ℓ^+ be a straight line parallel to ℓ that is in the other side of ℓ than $\mathcal{M}$ is. Now, the projectivity of $\mathbb{P}^2$ that takes the line at infinity to ℓ^+ maps $\mathcal{M}$ to a bounded, convex domain of $\mathbb{R}^2$, so, we can suppose from now on without loss of generality by (2.3), that $\mathcal{M}$ is bounded.

First, we reprove [6, Lemma 1 and Corollary] as

For any inner point O in $\mathcal{M}$, there exist two (maybe ideal) points P and Q in O^* such that PQ does not intersect $\mathcal{M}$. (3.3) ⟨8⟩

There is at least one chord $\overline{AC}$ of $\mathcal{M}$ which is bisected by O. Then, the harmonic conjugate $\hat{P}$ of O with respect to A and B, is on the line at infinity.

If O^* has a further point at infinity, then let $\hat{Q}$ be that point.

If O^* has only $\hat{P}$ at infinity, then O^* is a connected curve, hence it cannot lie completely within the strip formed by the two supporting lines of $\mathcal{M}$ which are parallel to AC because in that case it would intersect $\mathcal{M}$. Thus, a point $\hat{Q}$ of O^* exists outside this strip.

Thus, line $\hat{P}\hat{Q}$ does not intersect $\mathcal{M}$, but intersects O^* in the points $\hat{P}$ and $\hat{Q}$.

Now we prove that

A point $O \in \mathcal{M}$ is a center of $(\mathcal{M}, d)$ if and only if it is a projective center of $\mathcal{M}$, and the metric point reflection ρ_O is $\varpi^{-1} \circ \bar{\rho}_{\varpi O} \circ \varpi$ for a proper projectivity ϖ. (3.4) ⟨8, 11⟩

If O is a projective center of $\mathcal{M}$, then a projectivity ϖ exists such that $\varpi(O)$ is an affine center of $\varpi(\mathcal{M})$. Then $\bar{\rho}_{\varpi(O)}$ is an involutive isometry with respect to $d'(\cdot,\cdot) := d(\varpi(\cdot), \varpi(\cdot))$, that keeps the straight line through $\varpi(O)$ invariant. Thus $\varpi^{-1} \circ \bar{\rho}_{\varpi O} \circ \varpi$ is an involutive isometry with respect to d, that keeps the straight line through $\varpi(O)$ invariant. That is, O is a center of $(\mathcal{M}, d)$.

Assume now that O is a center of $(\mathcal{M}, d)$. By (3.3) we have two (maybe ideal) points P and Q in O^* such that PQ does not intersect $\mathcal{M}$.

Let ϖ be the projectivity that maps line PQ into the ideal line. Then $\varpi(O)$ is the affine midpoint of the chords $\overline{AC} := \varpi(OP) \cap \varpi(\mathcal{M})$ and $\overline{BD} := \varpi(OQ) \cap \varpi(\mathcal{M})$. With this is mind, (2.3) allows us to assume without loss of generality that O is the affine midpoint of two chords. Let these chords be $\overline{AC}$ and $\overline{BD}$.

As $\rho_{d;O}$ and $\bar{\rho}_O$ are both restrictions of their corresponding unique collineations [4, Theorem 3.1.], and these collineations coincide on points A, C, B, D and O three of which are in general position, hence $\rho_{d;O} \equiv \bar{\rho}_O$ follows. This proves (3.4).

As every point of $\mathcal{M}$ is a center of $(\mathcal{M}, d)$, from (3.4) it follows that every point of $\mathcal{M}$ is a projective center, hence [7, Theorem 3.3(a)] gives that $\mathcal{M}$ is an ellipse.

Lemma 2.6, (3.4), and Lemma 2.1 give that $d(X, Y) = c_\ell h(X, Y)$, where h is the Hilbert metric on $\mathcal{M}$ and c_ℓ is a constant.

Consider the different chords $\overline{AB}$ and $\overline{CD}$ of $\mathcal{M}$, where $A, B, C, D \in \partial\mathcal{M}$.

If $\overline{AB}\cap\overline{CD}=\emptyset$, then one of the intersections $\overline{AC}\cap\overline{BD}$ or $\overline{AD}\cap\overline{BC}$ is not empty, and that intersection point O is such that $\rho_{d;O}(\overline{AB})=\overline{CD}$, hence $c_{AB}=c_{CD}$, i.e. $c_\ell=c_{\ell'}$ if $(\ell\cap\ell')\cap\mathcal{M}=\emptyset$. If $\overline{AB}\cap\overline{CD}=\{O\}$, then let $\overline{EF}$ be a chord of $\mathcal{M}$ ($E,F\in\partial\mathcal{M}$) such that it does not intersects the quadrangle $ACBD$. Then $c_{AB}=c_{EF}=c_{CD}$ proves that c_ℓ does not depend on ℓ, hence it is a constant c.

The proof of the theorem is complete. □

4. Finitely many centers

We prove that some finitely many well-placed centers are enough to deduce the symmetry of the straight projective-metric spaces.

Lemma 4.1. *If $d(O,P)/d(O,Q)$ is an irrational number for the collinear centers O,P,Q of a straight projective-metric space $(\mathcal{M},d)$, then every point of the common geodesic $\tilde{\ell}$ of O,P,Q is a center of $(\mathcal{M},d)$.*

Proof. By Lemma 2.4 we need only to prove that the set of centers on $\tilde{\ell}$ is dense.

We may assume without loss of generality that $P\in\overline{OQ}$.

As the projective-metric space is straight, there exists an isometry $\imath$ from $\tilde{\ell}$ to $\mathbb{R}$ such that $\imath(O)=0$, and hence $\imath(P)=d(O,P)=:p$ and $\imath(Q)=d(O,Q)=:q$. By our assumption we have $0<p<q$, and the condition of the lemma gives that p/q is an irrational number. Then, Kronecker's Approximation Theorem [10] gives that for any $x\in\mathbb{R}$ and $\varepsilon>0$ there are $i,j\in\mathbb{Z}$, such that $|ip-jq-x|<\varepsilon$.

Letting $\tau_{OP}:=\rho_{d;P}\circ\rho_{d;O}$ and $\tau_{OQ}:=\rho_{d;Q}\circ\rho_{d;O}$ as before, we obtain by Lemma 2.5(2), that $d(\tau_{OP}(X),X)=2p$ and $d(\tau_{OQ}(X),X)=2q$ for any point $X\in\tilde{\ell}$. Thus, we obtain $\imath\circ\tau_{OP}\circ\imath^{-1}\colon x\mapsto x+2p$ and $\imath\circ\tau_{OQ}\circ\imath^{-1}\colon x\mapsto x+2q$. This means that the set $\mathcal{S}:=\{\tau^i_{OP}(\tau^j_{OQ}(O)) : i,j\in\mathbb{Z}\}$ is dense in $\tilde{\ell}$. However, Lemma 2.3 implies $\tau_{OX}\circ\rho_{d;Y}\circ\tau_{XO}=\rho_{d;\tau_{OX}(Y)}$ for any centers $X,Y\in\tilde{\ell}$, so every point in $\mathcal{S}$ is a center. This proves the Lemma. □

We say that the different points O and P_i,Q_i ($i=1,\dots,k$) form a *pencil* with *tip* O if the points O,P_i,Q_i are collinear for every i. Such a pencil is called *l-dimensional* if the linear space generated by the affine vectors $\overline{OP_i}$ is l-dimensional.

Lemma 4.2. *In a neighborhood of a center O of a straight projective-metric space every point of the affine hyperplane $\mathcal{H}$ spanned by the pencil of centers P_i,Q_i ($i=1,\dots,k$) and tip O is a center, if $d(O,P_i)/d(O,Q_i)$ is irrational for every i.*

Proof. We prove by induction. We consider the n-dimensional straight projective-metric space $(\mathcal{M},d)$.

By Lemma 4.1 we know that all points of the geodesics $\tilde{\ell}_i:=OP_i=P_iQ_i$ ($i=1,\dots,n$) are centers of $(\mathcal{M},d)$.

Assume now that for every l-dimensional pencil of the given type the statement of the lemma is fulfilled.

Let the $(l+1)$-dimensional pencil $\mathcal{P}_{l+1}$ of centers P_i, Q_i and tip O be such that $d(O,P_i)/d(O,Q_i)$ is irrational $(i = 1,\dots,k \le n)$, where we clearly have $k \ge l+1$.

If $k > l+1$, then the pencil of P_i, Q_i and tip O for $i = 1,\dots,k-1$ can be either of dimension $l+1$ or of dimension l. In the former case remove the geodesic $\tilde{\ell}_k := OP_k = P_kQ_k$, and continue this procedure until no removing is possible. This way we can assume that the pencil $\mathcal{P}_{l+1}$ is such that the pencil $\mathcal{P}_l$ of P_i, Q_i and tip O $(i = 1,\dots,k-1)$ is of dimension l.

By the hypothesis of the induction there is a neighborhood $\mathcal{U}_l$ of O in the hyperplane $\mathcal{H}_l$ spanned by the pencil $\mathcal{P}_l$, where every point is a center of the projective-metric space. Further, every point of the geodesic $\tilde{\ell}_k := OP_k = P_kQ_k$ is a center by Lemma 4.1.

Let $\mathcal{O}$ be a suitably small neighborhood of O

Let $\mathcal{H}_l^X$ be the affine subspace spanned by the points of $\rho_X(\mathcal{U}_l)$ for every point $X \in \tilde{\ell}_k \cap \mathcal{O}$. Then every point $P \in \rho_X(\mathcal{U}_l)$ is a center by Lemma 2.3. Let $\mathcal{P}$ be the common plane of ℓ_k and P, and let $Q \in \mathcal{P} \cap \mathcal{U}_l$. Then the geodesic $\widetilde{QP}$ contains at least two centers, namely Q and P.

Let $\mathcal{O}_l^X$ be an open set in $\rho_X(\mathcal{U}_l)$ containing $\rho_X(O)$.

Let the points $P \in \mathcal{O}_l^X$ and $Q \in \rho_X(\mathcal{O}_l^X)$ be such that the geodesic $\widetilde{QP}$ contains a point that is not a center. If there are no such points, then the hypothesis of the induction follows for $l+1$, that proves the statement of the lemma.

As $\rho_{.}(\cdot)$ is continuous in its subscript and $\mathcal{O}_l^X$ is open, there is a (small) $\varepsilon > 0$ such that $\rho_Y(\mathcal{U}_l)$ intersects $\widetilde{QP}$ in a point P_Y if $Y \in \mathcal{Y} := \{Y \in \tilde{\ell}_k : d(X,Y) < \varepsilon\}$. Observe that P_Y depends on Y continuously, hence it either runs over a closed open segment $\mathcal{S}$ or it is a fixed point P.

As there is a point on $\widetilde{QP}$ that is not a center, the ratio $d(Q,P)/d(Q,P_Y)$ is rational for every P_Y by Lemma 4.2, hence P_Y is a fixed point. Moreover, $P_Y \equiv P$, because $P_X = P$.

Thus, every point Z of the open triangle $\mathcal{Z}$ spanned by $\mathcal{Y}$ and P is a center, hence every point of the geodesics $\widetilde{QZ}$ is a center. If $Z \to P$ in $\mathcal{Z}$, the geodesic $\widetilde{QZ}$ tends to $\widetilde{QP}$, and therefore, every point of $\widetilde{QP}$ is a center by Lemma 2.4. This is a contradiction, hence every point of every geodesic $\widetilde{QP}$ is a center, if $P \in \mathcal{O}_l^X$. This proves the hypothesis of the induction, hence the statement of the lemma. □

The following result can be seen as a specific generalization of [2, (51.5)].

Theorem 4.3. *The set of the centers of an n-dimensional straight projective-metric space $(\mathcal{M}, d)$ contains an n-dimensional pencil of points P_i, Q_i $(i = 1,\dots,k \ge n)$ and tip O such that $d(O,P_i)/d(O,Q_i)$ is irrational for every i if and only if it is either a Minkowskian or the hyperbolic geometry.*

Proof. As every point of a Minkowskian or the hyperbolic geometry is a center, we need only to prove the reverse statement of the theorem.

Assume that the set of the centers of $(\mathcal{M}, d)$ contains an n-dimensional pencil of points P_i, Q_i $(i = 1, \ldots, k \geq n)$ and tip O.

By Lemma 4.2, this assumption implies that the set of the centers of $(\mathcal{M}, d)$ contains a neighborhood of O, which by theorems 3.1 and 3.2 proves the desired result. □

For projective-metric spaces of parabolic type or of hyperbolic type containing no affine line we need less centers to deduce that the metric is Minkowskian or hyperbolic.

Theorem 4.4. *The set of the centers of an n-dimensional projective-metric space of parabolic type contains $n+1$ affinely independent point and an additional one affinely independent from the others over the rational numbers if and only if it is a Minkowski geometry.*

Proof. As every point of any Minkowski geometry is a center, we need only to prove the reverse statement of the theorem.

By (3.2), if O is a center, then $\rho_O \equiv \bar{\rho}_O$. The product of any two affine point reflections is an affine translation, so Kronecker's Approximation Theorem [10] gives that the centers generated by repeated applications of the metric point reflections, form a dense set in $\mathbb{R}^n$.

Then Lemma 2.4 and Theorem 4.3 imply the statement of the Theorem. □

Theorem 4.5. *The set of the centers of an n-dimensional projective-metric space of hyperbolic type with no affine line inside contains an $(n-1)$-dimensional pencil of points P_i, Q_i $(i = 1, \ldots, k \geq n-1)$ and tip O such that $d(O, P_i)/d(O, Q_i)$ is irrational for every i if and only if it is the hyperbolic geometry.*

Proof. As every point of the hyperbolic geometry is a center, we need only to prove the reverse statement of the theorem.

Assume that the set of the centers of the n-dimensional projective-metric space $(\mathcal{M}, d)$ of hyperbolic type with no affine line inside contains an $(n-1)$-dimensional pencil of points P_i, Q_i $(i = 1, \ldots, k \geq n-1)$ and tip O.

By Lemma 4.2, this assumption implies that the set of the centers of $(\mathcal{M}, d)$ contains a neighborhood of O in an $(n-1)$-dimensional hyperplane $\mathcal{H}$. By Lemma 2.3 this means that every point of $\mathcal{M} \cap \mathcal{H}$ is a center, which, by (3.4), means that every point of $\mathcal{M} \cap \mathcal{H}$ is a projective center of $\mathcal{M}$. According to [7, Theorem 3.3(a)], this implies that $\mathcal{M}$ is an ellipsoid, hence the theorem. □

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[9] `https://en.wikipedia.org/wiki/Cauchy's_functional_equation` ⟨3⟩

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Added after publication:

In

H. Busemann and B. B. Phadke, A general version of Beltrami's theorem in the large, *Pacific J. Math.*, **115**:2(1984), 299–315.

the following is written on page 310:

“ Main Theorem. *A locally desarguesian simply connected chord space R is either defined in all of S^n or is an arbitrary open convex set of an open hemisphere of S^n (considered as A^n).* ”

“ Theorem. *A simply connected locally desarguesian and locally symmetric G-space is Minkowskian, hyperbolic or spherical.*

We indicate the proof briefly. Locally symmetric G-spaces which generalize locally symmetric Riemann and Finsler spaces are defined, see [11], as G-spaces in which a positive continuous $\beta(p)$ exists such that each $S(p, \beta(p))$ is symmetric in p. In [10, (4.2), (4.4)] we proved that a locally symmetric globally desarguesian G-space is Minkowskian or hyperbolic and that a locally symmetric spherelike G-space is spherical. These results when combined with the Main Theorem give the theorem stated above. ”

where

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