Representation spaces of the Jordan plane

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Abstract

We investigate relations between the properties of an algebra and its varieties of finite-dimensional module structures, on the example of the Jordan plane $R = k\langle x, y\rangle/(xy - yx - y^2)$.

Complete description of irreducible components of the representation variety $mod(R, n)$ obtained for any dimension n, it is shown that the variety is equidimensional.

The influence of the property of the non-commutative Koszul (Golod-Shafarevich) complex to be a DG-algebra resolution of an algebra (NCCI), on the structure of representation spaces is studied. It is shown that the Jordan plane provides a new example of RCI (representational complete intersection).

Key words: Representation spaces, irreducible components, Golod-Shafarevich complex, NCCI(noncommutative complete intersections), RCI(representational complete intersections)

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Contents

1 Introduction

We investigate how properties of a quadratic algebra are reflected in the properties of its finite dimensional representations. Our main example is the prominent since the foundation of noncommutative geometry [4],[34] quadratic algebra $R = k\langle x, y\rangle/(xy - yx - y^2)$, known as the Jordan plane. According to the Artin-Schelter classification [4] there are three types of algebras of global dimension two: the usual quantum plane $k\langle x, y\rangle/(xy - qyx)$, for $q \neq 0$ and the Jordan plane are regular algebras, and one more, which is not: $k\langle x, y\rangle/(xy)$,

The Jordan plane appeared also in many different contexts in mathematics and physics [24, 22, 3, 2, 1, 19, 6]. To mention a few connections to other important objects, let us note that R is a subalgebra of the first Weyl algebra A_1. The latter has no finite dimensional representations, but R turns out to have quite a rich structure of them. Category of finite dimensional modules over R contains, for example, as a full subcategory mod $GP(n, 2)$, where $GP(n, 2)$ is the Gelfand–Ponomarev algebra [12] with the nilpotency degrees of variables x and y, n and 2 respectively.

We prove that the space of n-dimensional representations of the Jordan plane, which we call the *Jordan variety*, is equidimensional and irreducible components are parametrised by partitions of n. The description of irreducible components is given for any n. This result is obtained on the basis of a lucky choice of stratification of the representation space. The stratum $\mathcal{U}(\mathcal{P})$ is defined as those pairs of matrices, where the matrix Y (the image of the variable y under the representation) has the fixed Jordan form. Closures of chosen in such a way strata provide a complete list of irreducible components of the corresponding variety. As a consequence, irreducible components parametrised by partitions of n, defined by the block structure of the Jordan form of Y, the eigenvalues of Y are not involved, since in any representation this matrix is nilpotent.

In each stratum there is a vector bundle structure, such that the dimension of the base is m and the dimension of the fiber is $n^2 - m$, so the dimension of each stratum is n^2. Since irreducible components of the variety are precisely closures of these strata, we can see that variety of n-dimensional representations is equidimensional for any n.

Thus the following theorem is proved in the paper.

Theorem 12.1 *Any irreducible component K_j of the representation variety $mod(R, n)$ of the Jordan plane contains only one stratum $U_{\mathcal{P}}$ from the stratification related to the Jordan normal form of Y, and is the closure of this stratum.*

The number of irreducible components of the variety $mod(R, n)$ is equal to the number of the partitions of n.

The variety $mod(R, n)$ is equidimensional and the dimension of components is equal to n^2.

The *generic* situation in this variety is provided by the component corresponding to the full Jordan block Y. We study it in details, especially since the further results on the structure of representation variety for R show an exceptional role of such strata, for example, they turn out to be the only building blocks in the analogue of the Krull-Remark-Schmidt decomposition theorem on the level of irreducible components. The following holds.

Corollary *Only the irreducible component $K_{(n)} = \overline{\mathcal{U}_{(n)}}$ which is the closure of the stratum corresponding to the trivial partition of n (the full block Y) contains an open dense subset consisting of indecomposable modules.*

Another evidence of the special role of this 'full block' stratum presented in section 10 where we estimate dimensions of image algebras. In section 10 we prove an analogue of the Gerstenhaber–Taussky–Motzkin theorem [13], [16], [25] on the dimension of algebras generated by two commuting matrices. We show that the dimension of image algebras of representations of R does not exceed $n(n+2)/4$ for even n and $(n+1)^2/4$ for odd n, and this estimate is attained in the stratum related the full Jordan block Y.

In the section 12 we perform a study of the generic strata in terms of the properties of algebras-images. Namely, for algebras-images from the strata with full-block Y, we clarify, using the Ringel classification of local complete algebras [29], that they are all *tame* for $n \leqslant 4$ and *wild* for $n \geqslant 5$. The main tool in this section is to solve the problem of finding defining relations for algebra-image of finite dimensional representation. When it is done, and we have precisely written relations, we compare them with the Ringel list of local complete algebras, to determine whether they are tame or wild.

Referring to results due to Kontsevich [20], concerning the existence of the *algebraic* bundle on the generic subvariety in the space of all pairs of $n \times n$ matrices $\mathcal{M}_n$; to the well-known results on the *commuting variety*, where on the generic subvariety (consisting of pairs of matrices, one of which is nonderogatory) there is a vector bundle with the base, consisting of the set of nonderogatory matrices, of dimension n^2, and the n-dimensional fiber, consisting of all polynomials on the matrix; we guess that in any closed subvariety of $\mathcal{M}_n$ given by a (homogeneous) quadratic matrix equation, in general position there is a natural structure of algebraic (or sometimes even a vector) bundle.

Another main subject of our attention is to investigate, how certain properties of Golod-Shafarevich (noncommutative Koszul) complex, like property of algebra of being NCCI (*noncommutative complete intersection*) could be reflected on the level of representation varieties. The algebras which are NCCI fall into the class of Golod-Shafarevich algebras, which is defined in [EF] by E.Zelmanov and have been studied in [31], [9], [32],[33],[35] etc. We ensure that Jordan plane is a NCCI and derive from the previous results on representation spaces that (for infinitely many n), spaces of n-dimensional representations of Jordan plane are complete intersections. By this we establish a new example of RCI (*representational complete intersection*), the notion introduced by Ginzburg and Etingof in [10]. In this paper the main source of examples of RCIs came from the preprojective algebras of finite quivers. The general interrelation between those two properties also was clarified, namely it was shown that RCI implies NCCI (in the above sense), so the restriction on algebra formulated in terms of properties of representation spaces (RCI)is quite strong.

Let us note, that sections 1-9 can be considered as a preparatory part of the paper, where we establish various properties of Jordan plane itself and algebras-images of finite dimensional representations. These include facts that R is Koszul, multiplication formulas in canonical basis, description of prime, primitive ideals and automorphisms, the fact that R is residually finite dimensional, as well as description of irreducible modules, some properties of indecomposable modules, description of all modules in terms of $\mathcal{B}-Toeplitz$ matrices. These facts were established in our earlier work [18], and will serve here as a basis for obtaining further results.

2 Structural properties of the images of representations and quivers

We consider here the case when k is an algebraically closed field of characteristic zero. Let $\rho : R \to \mathrm{End}(k^n)$ be an arbitrary finite dimensional representation of R, denote by $A_{\rho,n} = \rho(R)$ an image of R in the endomorphism ring. We will write also A_n or A when it is clear from the context which ρ and n we mean.

We derive in this section some main structural properties of algebras $A_{n,\rho}$. They all are basic algebras, the image of y in any such algebra is nilpotent, complete system of orthogonal idempotents in A is defined by the set of different eigenvalues of $\rho(x)$, etc.

Lemmata below describe the structural properties of image algebras for R.

The following fact certainly allows many different proofs, we present here the shortest we know.

Lemma 2.1 *Let $Y = \rho_n(y)$. Then the matrix Y is nilpotent.*

P r o o f. Suppose that the matrix Y is not nilpotent and hence has a nonzero eigenvalue z. We take the eigenprojector P on the eigenspace $E_z = ker(Y - zI)^n$, corresponding to this eigenvalue. It is obviously commutes with $Y : PY = YP$ and is an idempotent operator: $P^2 = P$. Hence multiplying our relation $XY - YX = Y^2$ from the right and from the left hand side by P and using the above two facts, we can observe that operators $X' = PXP$ and $Y' = PYP$ also satisfy the same relation: $X'Y' - Y'X' = Y'^2$. Taking into account that Y' has only one nonzero eigenvalue z, we see that traces of right and left hand sides of the relation can not coincide. Indeed, $tr(X'Y' - Y'X') = 0$ and $trY'^2 = z^2 dim E_z \neq 0$. This contradiction completes the proof. □

Let us prove also a little bit more general fact of linear algebra.

Lemma 2.2 *Let X, Y be $n \times n$ matrices over an algebraically closed field k of characteristic zero. Assume that the commutator $Z = XY - YX$ commutes with Y. Then Z is nilpotent.*

P r o o f. Assume the contrary. Then Z has a non-zero eigenvalue $z \in k$. Let

$$L = \ker (Z - zI)^n \quad \text{and} \quad N = \mathrm{Im}(Z - zI)^n.$$

The subspace L is known as a main subspace for Z corresponding to the eigenvalue z. Clearly $L \neq \{0\}$. It is well-known that the whole space k^n can be split to a direct sum $L \oplus N$ of Z-invariant linear subspaces L and N. Due to $ZY = YZ$, the subspaces L and N are also invariant for Y.

Consider the linear projection P along N onto L. Since L and N are invariant under both Y and Z, we have $ZP = PZ$ and $YP = PY$. Multiplying the equality $Z = XY - YX$ by P from the left and from the right hand side and using the equalities $ZP = PZ$, $YP = PY$ and $P^2 = P$, we get

$$ZP = PXPY - YPXP.$$

Since ZP vanishes on N and $ZP|_L$ has only one eigenvalue z, then after restriction to L, we have $\operatorname{tr} ZP = z \dim L$. On the other hand $\operatorname{tr} PXPY = \operatorname{tr} YPXP$ since the trace of a product of two matrices does not depend on the order of the product. Thus, the last display implies that $z \dim L = 0$, which is not possible since $z \neq 0$ and $\dim L > 0$. □

Coming back to the case of the Jordan plane, we have further

Lemma 2.3 *Let $X = \rho_n(x)$ and $\{\lambda_1, ..., \lambda_r\} = \mathrm{Spec} X$. Then the matrix $S = S(X) = (X - \lambda_1 I) \ldots (X - \lambda_r I)$ is nilpotent.*

Proof. Note that $\operatorname{Spec} p(X) = p(\operatorname{Spec} X)$ for any polynomial p. $\operatorname{Spec} X$ in our case is $\{\lambda_1, \ldots, \lambda_r\}$ and hence $\operatorname{Spec} S = \{0\}$. Therefore the matrix S is nilpotent. □

Let $J(A) = J$ be the Jacobson radical of the algebra $A_{\rho,n}$.

Lemma 2.4 *Any nilpotent element of the algebra $A = \rho(R)$ belongs to the radical $J(A)$.*

Proof. We will use here the feature of an algebra A that it has the presentation as a quotient of the free algebra, containing our main relation. Namely, it has a presentation: $A = k\langle x, y | xy - yx = y^2, R_A \rangle$, where $R_A \subset k\langle x, y\rangle$ is the set of additional relations specific for the given image algebra. Thus we can think of elements in A as of polynomials in two variables (subject to some relations). Let $Q(x)$ be a polynomial on one variable $Q(x) \in k[x]$ and $Q(X) \in A$ be a nilpotent element with the degree of nilpotency N: $Q^N = 0$. We show first that $Q \in J(A)$. We have to check that for any polynomial $a \in k\langle x, y\rangle$, $1 - a(X,Y)Q(X)$ is invertible. It suffices to verify that $a(X,Y)Q(X)$ is nilpotent. By Lemma 2.1 Y is nilpotent. Denote by m the degree of nilpotency of Y: $Y^m = 0$. Let us verify that $(a(X,Y)Q(X))^{mN} = 0$. Present $a(X,Y)$ as $u(X) + Yb(X,Y)$. If then we consider a word of length not less then mN on letters $\alpha = u(X)Q(X)$ and $\beta = Yb(X,Y)Q(X)$ then we can see that it is equal to zero. Indeed, if there are at least m letters β then using the relation $XY - YX = Y^2$ to commute the variables one can rewrite the word as a sum of words having a subword Y^m. Otherwise our word has the subword $\alpha^N = u(X)^N Q(X)^N = 0$. Thus, $Q(X) \in J(A)$.

Note now that if we have an arbitrary nilpotent polynomial $G(X,Y)$, we can separate the terms containing Y: $G(X,Y) = Q(X) + YH(X,Y)$. To obtain nilpotency of any element $a(X,Y)G(X,Y)$ it suffices to verify nilpotency of $a(X,Y)Q(X)$, which was already proven, because the relation $[X,Y] = Y^2$ allows to commute with Y, preserving (or increasing) the degree of Y. □

Since $A = \rho(R)$ is finite dimensional, we have:

Corollary 2.5 *The Jacobson radical of $A = \rho(R)$ consists precisely of all nilpotent elements.*

In particular,
Corollary 2.6 *Let $Y = \rho(y)$. Then $Y \in J(A)$.*

Let us formulate here another property of the radical, which will be on use later on.

Corollary 2.7 *The Jacobson radical of $A = \rho(R)$ consists of all polynomials on $X = \rho(x)$ and $Y = \rho(y)$ without constant term if and only if X is nilpotent in A.*

P r o o f. In one direction this is trivial. We should ensure only that if $X^N = 0$ then $p(X,Y)^{2N} = 0$ for any polynomial p such that $p(0,0) = 0$, using the relation $XY - YX = Y^2$, which is an easy check.□

Lemma 2.8 *Let $A_{\rho,n}$ be as above, and $X = \rho_n(x)$, $Y = \rho_n(y)$ be its generators. Then A/J is a commutative one-generated ring $k[x]/S(x)$, where $S(x) = (x - \lambda_1)\dots(x - \lambda_k)$, and $\lambda_1, \dots, \lambda_k$ are all different eigenvalues of the matrix X.*

P r o o f. From Corollary 2.6 we can see that A/J is an algebra with one generator x: $A/J \simeq k[x]/I$. We are going to find now an element which generates the ideal I.

First of all by lemmas 2.3 and 2.4 $S \in J(A)$, hence $S(x) = (x-\lambda_1)\dots(x-\lambda_r) \in I$. Let us show now that S divides any element in I. If some polynomial $p \in k[x]$ does not vanish in some eigenvalue λ of X then $p(X) \notin J(A)$. Indeed, the matrix $p(X)$ has a non-zero eigenvalue, than $p(\lambda) \neq 0$ and hence $I - \frac{1}{p(\lambda)}p(X)$ is non-invertible. Therefore $p(X) \notin J(A)$. Thus, $S(x)$ is the generator of I. This finishes the proof. □

Corollary 2.9 *The system $e_i = p_i(X)/p_i(\lambda_i)$, where*

$$p_i(X) = (X - \lambda_1 I)\dots(\widehat{X - \lambda_i I})\dots(X - \lambda_r I)$$

and λ_i are different eigenvalues of $X = \rho(x)$ is a complete system of orthogonal idempotents of A/J.

P r o o f. Orthogonality of e_i is clear from the presentation of A/J as $k[x]/\mathrm{id}(S)$ proven in Lemma 2.8. □

Proposition 2.10 *For any finite dimensional representation ρ the semisimple part of A_ρ is a product of a finite number of copies of the field k:*

$$A/J = \prod_{i=1}^{r} k_i,$$

where r is the number of different eigenvalues of the matrix $X = \rho(x)$.

P r o o f. We shall construct an isomorphism between A/J and $\prod_{i=1}^{r} k_i$ using the system e_i, $i = 1, \ldots, r$ of idempotents from Corollary 2.9. Clearly e_i form a basis of A/J as a linear space over k. From the presentation of A/J as a quotient $k[x]/\mathrm{id}(S)$ given in lemma 2.8 it is clear that the dimension of A/J is equal to the degree of polynomial $S(x)$, which coincides with the number of different eigenvalues of the matrix X. Since idempotents e_i are orthogonal, they are linearly independent and therefore form a basis of A/J. The multiplication of two arbitrary elements $a, b \in A/J$, $a = a_1e_1 + \ldots + a_re_r$, $b = b_1e_1 + \ldots + b_re_r$ is given by the formula $ab = a_1b_1e_1 + \ldots + a_rb_re_r$ due to orthogonality of the idempotents e_i. Hence the map $a \mapsto (a_1, \ldots, a_r)$ is the desired isomorphism of A/J and $\prod_{i=1}^{r} k_i$. □

Since all images are basic algebras we can associate to each of them a *quiver* in a conventional way (see, for example, [15], [17]).

The vertices will correspond to the idempotents e_i or, by Corollary 2.9, equivalently, to the different eigenvalues of matrix X. The number of arrows from vertex e_i to the vertex e_j is the $\dim_k e_i(J/J^2)e_j$. There are a finite number of such quivers in fixed dimension n (the number of vertices bounded by n, the number of arrows between any two vertices roughly by n^2).

We can define an equivalence relation on representations of algebra R using quivers of their images.

Definition 2.11 *Two representations ρ_1 and ρ_2 of the algebra R are quiver-equivalent $\rho_1 \sim_Q \rho_2$ if quivers associated to algebras $\rho_1(R)$ and $\rho_2(R)$ coincide.*

This will lead to a rough classification of representations by means of quivers of their images.

Toy example. As an example let us clarify the question on how many quiver-equivalence classes appear in the family of representations

$$\mathcal{U}_{(n)} = \{(X, Y) \in \mathrm{mod}\,(R, n) | \mathrm{rk}\, Y = n - 1\}$$

and which quivers are realized.

Proposition 2.12 *For any $n \geqslant 3$ families of representations $\mathcal{U}_{(n)}$ belong to one quiver-equivalence class. Corresponding quiver consists of one vertex and two loops.*

Let us ensure first the following lemma.

Lemma 2.13 *If in the representation $\rho : R \to A$, $X = \rho(x)$ has only one eigenvalue λ, then the corresponding quiver Q_A has one vertex and number of loops is a dimension of the vector space $\mathrm{Span}_k\{\bar{X} - \lambda I, \bar{Y}\}$, which does not exceed 2. Here $\bar{Y} = \varphi Y$ and $\bar{X} = \varphi X$ for $\varphi : A \to A/J^2$.*

P r o o f. Due to the description of idempotents above, in the case of one eigenvalue the only idempotent is unit. Hence we have to calculate $\dim_k J/J^2$, where $J = Jac(A)$. Since $X - \lambda I$ satisfies the same relation as X, we could apply corollary 2.7 and result immediately follows. □

P r o o f. (of Proposition 2.12) This will directly follow from Lemma 2.13, after we show in section 6 that X has only one eigenvalue in the family $\mathcal{U}_{(n)}$ and take into account that when we have full block Y, the dimension of the linear space $\mathrm{Span}_k\{\bar{X} - \lambda I, \bar{Y}\}$ can not be smaller then 2. □

3 Jordan Calculus

Here we shall prove lemmata containing formulas for multiplication in the canonical linear basis of the Jordan plane, it will be on use for various purposes later on.

3.1 Gröbner basis of the ideal and a linear basis of algebra

The basis of our algebra as a vector space over k consists of the monomials $y^k x^l$, $k, l = 0, 1, \ldots$. Let us ensure this in a canonical way.

For this we remind the definition of a Gröbner basis of an ideal and the method of construction of a linear basis of an algebra given by relations, based on the Gröbner basis technique. Using this canonical method it could be easily shown that, for example, some Sklyanin algebras enjoys a PBW property. This was proved in [27], the arguments there are very interesting in their own right, but quite involved.

Let $A = k\langle X\rangle/I$. The first essential step is to fix an ordering on the semigroup $ß = \langle X\rangle$. We fix some linear ordering in the set of variables X and extend it to an *admissible* ordering on ß, i.e. extend it in a way to satisfy the conditions:

1) if $u, v, w \in ß$ and $u < v$ then $uw < vw$ and $wu < wv$;

2) the descending chain condition (d.c.c.): there is no infinite properly descending chain of elements of ß.

We shall use the *degree-lexicographical* ordering in the semigroup ß, namely for arbitrary $u = x_{i_1} \ldots x_{i_n}, v = x_{j_1} \ldots x_{j_k} \in ß$ we say $u > v$, when either $\deg u > \deg v$ or $\deg u = \deg v$ and for some l : $x_{i_l} > x_{j_l}$ and $x_{i_m} = x_{j_m}$ for any $m < l$. This ordering is admissible.

Denote by $\bar{f}$ the highest term of polynomial $f \in A = k\langle X\rangle$ with respect to the above order.

Definition 4.2. Subset $G \in I, I \lhd k\langle X\rangle$ is a *Gröbner basis* of an ideal I if the set of highest terms of elements of G generates the ideal of highest terms of $I : id\{\bar{G}\} = \bar{I}$.

Definition 4.3. We say that a monomial $u \in \langle X\rangle$ is *normal* if it does not contain as a submonomial any highest term of an element of the ideal I.

From these two definitions it is clear that normal monomial is a monomial which does not contain any highest term of an element of Gröbner basis of the ideal I. If Gröbner basis turns out to be finite then the set of normal words is constructible.

In the case when an ideal I of defining relations for A has a finite Gröbner basis, the algebra called *standardly finitely presented(s.f.p.)*.

It is an easy, but useful fact that $\langle X \rangle$ is isomorphic to the direct sum $I \oplus \langle N \rangle_k$ as a linear space over k, where $\langle N \rangle_k$ is the linear span of the set of normal monomials from $\langle X \rangle$ with respect to the ideal I. Hence given a Gröbner basis G of an ideal I, we can construct a linear basis of an algebra $A = \langle X \rangle / I$ as a set of normal (with respect to I) monomials, at least in case when A is s.f.p.

As a consequence we immediately get the following

Lemma 3.1 *The system of monomials $y^n x^m$ form a basis of algebra R as a vector space over k.*

P r o o f. With respect to the ordering $x > y$ on the set of generators, and corresponding degree-lexicographic ordering on monomials $\langle X \rangle$, the relation $xy - yx - y^2$ forms a Gröbner basis of the ideal generated by itself. This follows from the Bergman Diamond Lemma [5], which says, that the set of generators of an ideal form a Gröbner basis if all ambiguities between highest terms of this set are solvable. We say that two monomials $u, v \in \langle X \rangle$ form an *ambiguity*, if there exists a monomial $w \in \langle X \rangle$, such that $w = au = bv$ for some $a, b \in \langle X \rangle$. In our case the only highest term, the monomial xy does not form any ambiguity with itself, so the set of ambiguities is empty.

3.2 Multiplication formulas in the Jordan plane

We say that an element is in the *normal form*, if it is presented as a linear combination of normal monomials.

After we have a linear basis of normal monomials we should know how to multiply them to get again an element in the normal form.

Now we are going to prove the following lemmata, where we express precisely the normal forms of some products.

Lemma 3.2 *The normal form of the monomial xy^n in algebra R is the following:*

$xy^n = y^n x + ny^{n+1}$.

P r o o f. This can be proven by induction on n. The case $n = 1$ is just the defining relation of our algebra. Suppose $n > 1$ and the equality $xy^{n-1} = y^{n-1}x + (n-1)y^n$ holds. Multiplying it by y from the right and reducing the result by the relation $xy - yx = y^2$, we obtain

$$xy^n = y^{n-1}xy + (n-1)y^{n+1} = y^n x + y^{n+1} + (n-1)y^{n+1} = y^n x + ny^{n+1}.$$

The proof is now complete. □

Lemma 3.3 *The normal form of the monomial $x^n y$ in algebra R is the following:*

$$x^n y = \sum_{k=1}^{n+1} \alpha_{k,n} y^k x^{n-k+1}, \text{ where } \alpha_{k,n} = n!/(n-k+1)! \text{ for } k = 1, ..., n+1.$$

P r o o f. We are going to prove this formula inductively using the previous lemma. As a matter of fact we shall obtain recurrent formulas for $\alpha_{k,n}$. In the case $n = 1$ the relation $xy - yx = y^2$ implies the desired formula with $\alpha_{1,1} = \alpha_{2,1} = 1$. Suppose n is a positive integer and there exist positive integers $\alpha_{k,n}$, $k = 1, \ldots, n+1$ such that $x^n y = \sum_{k=1}^{n+1} \alpha_{k,n} y^k x^{n-k+1}$. Multiplying the latter equality by x from the left and using lemma 3.2 we obtain

$$x^{n+1} y = \sum_{k=1}^{n+1} \alpha_{k,n} x y^k x^{n-k+1} = \sum_{k=1}^{n+1} \alpha_{k,n} y^k x^{n-k+2} + \sum_{k=1}^{n+1} \alpha_{k,n} k y^{k+1} x^{n-k+1}.$$

Rewriting the second term as $\sum_{k=1}^{n+2} \alpha_{k-1,n}(k-1) y^k x^{n-k+2}$ (here we assume that $\alpha_{0,n} = 0$), we arrive to

$$x^{n+1} y = \sum_{k=1}^{n+2} \alpha_{k,n+1} y^k x^{n-k+2},$$

where $\alpha_{k,n+1} = \alpha_{k,n} + (k-1)\alpha_{k-1,n}$ for $k = 1, \ldots, n+1$ and $\alpha_{n+2,n+1} = (n+1)\alpha_{n+1,n}$.

Let us prove now the formula for $\alpha_{k,n}$. For $n = 1$ it is true since $\alpha_{1,1} = \alpha_{1,2} = 1$. Then we use inductive argument. Suppose the formula is true for n. We are going to apply the recurrent formula appeared above:

$$\alpha_{k,n+1} = \alpha_{k,n} + (k-1)\alpha_{k-1,n} = \frac{n!}{(n-k+1)!} + (k-1)\frac{n!}{(n-k+2)!} = \frac{(n+1)!}{(n-k+2)!}$$

and the formula is verified for $1 \leqslant k \leqslant n+1$. For $k = n+2$, we have $\alpha_{n+2,n+1} = (n+1)\alpha_{n+1,n} = (n+1)n! = (n+1)!$. This completes the proof. □

3.3 Koszulity of the Jordan plane

In the section 3.2 we have seen that relation $xy - yx - y^2$ forms a Gröbner basis with respect to the ordering $x > y$. This means, we can apply to this presentation of the algebra the Priddy criterion of Koszulity [28], which asserts that if an algebra has a presentation by a quadratic Gröbner basis, then it is Koszul. This leads us to

Proposition 3.4 *The Jordan plane $R = k\langle x, y \,|\, xy - yx - y^2 \rangle$ is Koszul.*

In section 14 this fact will follow also from more general statement related to the properties of noncommutative Koszul (Golod-Shafarevich) complex, but this argument provides a simple constructive and direct proof of Koszulity.

4 Automorphisms of the Jordan plane

In this section we intend to describe the group of automorphisms of the Jordan plane R in order to use this later on for constructing examples of tame up to automorphism strata. The automorphism group turns out to be quite small, compared with automorphisms of the first Weyl algebra A_1, which contains R as a subalgebra. Automorphisms of the A_1 were described in [23], the case of an arbitrary Weyl algebra A_n was discussed in [21].

We are going to prove.

Theorem 4.1 *All automorphisms of $R = k\langle x, y | xy - yx = y^2\rangle$ are of the form $x \mapsto \alpha x + p(y)$, $y \mapsto \alpha y$, where $\alpha \in k \setminus \{0\}$ and $p \in k[y]$ is a polynomial on y. Hence the group of automorphisms isomorphic to a semidirect product of an additive group of polynomials $k[y]$ and a multiplicative group of the field k^* : $\mathrm{Aut}(R) \simeq k[y] \rtimes k^*$.*

P r o o f. Key observation for this proof is that in our algebra there exists the minimal ideal with commutative quotient. Namely, the two-sided ideal J generated by y^2.

Lemma 4.2 *If the quotient R/I is commutative then $y^2 \in I$.*

P r o o f. The images of x and y in this quotient commute. Hence

$$I = (x + I)(y + I) - (y + I)(x + I) = xy - yx + I = y^2 + I.$$

Therefore $y^2 \in I$. □

The property of an ideal to be a minimal ideal with commutative quotient is invariant under automorphisms.

Let us denote by $\widetilde{y} = f(x, y)$ the image of y under an automorphism φ. Then the ideal generated by $\widetilde{y}^2$ coincides with the ideal generated by y^2: $J = \mathrm{id}(y^2) = \mathrm{id}(\widetilde{y}^2)$.

Using the property of multiplication in R from lemma 3.3, we can see that two-sided ideal generated by y^2 coincides with the left ideal generated by y^2: $Ry^2R = y^2R$. Indeed, let us present an arbitrary element of Ry^2R in the form $\sum a_i y^2 b_i$, where a_i, $b_i \in R$ are written in the normal form $a_i = \sum \alpha_{k,l} y^k x^l$, $b_i = \sum \beta_{k,l} y^k x^l$. Using the relations from Lemma 3.2, we can pull y^2 to the left through a_i's and get the sum of monomials, which all contain y^2 at the left hand side. Thus, $\sum a_i y^2 b_i = y^2 u$, $u \in R$.

Obviously automorphism maps the one-sided ideal y^2R onto the one-sided ideal $\widetilde{y}^2R$, both of which coincide with $J = \mathrm{id}(y^2) = \mathrm{id}(\widetilde{y}^2)$. From this we obtain a presentation of y^2 as $\widetilde{y}^2 u$ for some $u \in R$. Considering usual degrees of these polynomials (on the set of variables x, y), we get $2 = 2k + l$, where $k = \deg \widetilde{y}$ and $l = \deg u$. Obviously $k \neq 0$. Hence the only possibility is $k = 1$ and $l = 0$.

Thus, $\varphi(y) = \widetilde{y} = \alpha x + \beta y + \gamma$ and $u = c$ for some $\alpha, \beta, \gamma, c \in k$. Substituting these expressions into the equality $y^2 = \widetilde{y}^2 u$, we get $c(\alpha x +$

$\beta y+\gamma)^2 = y^2$. Comparing the coefficients of the normal forms of the right and left hand sides of this equality, we obtain $\alpha = \gamma = 0$, $\beta \neq 0$. Hence $\varphi(y) = \beta y$.

Now we intend to use invertibility of φ. Due to it there exists $\alpha_{ij} \in k$ such that $x = \sum \alpha_{ij}\widetilde{y}^i\widetilde{x}^j$. Substituting $\widetilde{y} = \beta y$, we get $x = \sum_{r=0}^{N} p_r(y)\widetilde{x}^r$, where N is a positive integer, $p_r \in k[y]$ and $p_N \neq 0$. Comparing the degrees on x of the left and right hand sides of the last equality we obtain $1 = kN$, where $k = \deg_x\widetilde{x}$. Hence $k = N = 1$, that is $x = p_0(y) + p_1(y)\widetilde{x}$ and $\widetilde{x} = q_0(y)+q_1(y)x$, where $p_0, p_1, q_0, q_1 \in k[y]$. Substituting $\widetilde{x} = q_0(y)+q_1(y)x$ into $x = p_0(y) + p_1(y)\widetilde{x}$, we obtain $q_1 \in k$, that is $\widetilde{x} = cx + p(y)$ for $c \in k$. One can easily verify that the relation $\widetilde{x}\widetilde{y} - \widetilde{y}\widetilde{x} = \widetilde{y}^2$ is satisfied for $\widetilde{x} = cx + p(y)$, $\widetilde{y} = \beta y$ if and only if $c = \beta$. This gives us the general form of the automorphisms: $\widetilde{x} = cx + p(y)$, $\widetilde{y} = cy, c \neq 0$.

Now we see that the group of automorphisms is a semidirect product of the normal subgroup isomorphic to the additive group of polynomials $k[y]$ and the subgroup isomorphic to the multiplicative group k^*. The precisely written formula for multiplication in $\mathrm{Aut}R$ is the following:

$$\varphi_1\varphi_2 = (p_1(y), c_1)(p_2(y), c_2) = (c_2p_1(y) + p_2(c_1y), c_1c_2)$$

for $\varphi_1, \varphi_2 \in \mathrm{Aut}R$. □

5 Prime and primitive ideals

On our way we describe here also prime ideals of the Jordan plane and point out which of them are primitive. It is quite straightforward.

Theorem 5.1 *All prime ideals of* $R = k\langle x, y | xy - yx - y^2\rangle$ *have a shape* $\mathrm{id}(y)$ *or* $\mathrm{id}(y, x - \alpha)$.

Lemma 5.2 *Any two-sided proper ideal* I *in* R *contains a polynomial in* y.

P r o o f. For each $f \in R \setminus \{0\}$, let $k(f)$ be the x-degree of the normal form of f and $k(I) = \min_{f\in I\setminus\{0\}} k(f)$. Assume that $k(I) > 0$. Let $f \in I$ be such that $k(f) = k(I)$. Since I is a two-sided ideal, $g = [f, y] \in I$. Since f is not a polynomial in y, we have $g \neq 0$. On the other hand, it is easy to see that $k([h, y]) < k(h)$ for each $h \in R$. Hence $k(g) < k(f)$. We have arrived to a contradiction with the equality $k(f) = k(I)$. Thus $k(I) = 0$ and therefore there is $f \in I \setminus \{0\}$, being a polynomial in y.

We shall take into account that in purely differential Ore extensions quotients by prime ideals are domains. So we can substitute primeness with completely primeness in our proof.

Lemma 5.3 *Let* P *be a prime two-sided ideal of the Jordan plane* R. *Then* $y \in P$.

P r o o f. By Lemma 5.2 there is a non-zero polynomial p such that $p(y) \in P$. Since k is algebraically closed, we can assume that $p(y) = \prod(y - \lambda_j)$, $\lambda_j \in k$. If all $y - \lambda_j \notin P$, we have zero divisors in R/P. Thus $y - \lambda_j \in P$ for some $\lambda_j \in k$. Now we shall show that $\lambda_j = 0$. Indeed, since $y - \lambda_j \in P$, $y = \lambda_j$ in R/P. In the quotient R/P we still have the defining relation of R, so $[x, \lambda_j] = \lambda_j^2$ and $\lambda_j^2 = 0$, which implies $\lambda_j = 0$.

P r o o f. (of Theorem 5.1).

Let us notice that the ideal generated by y is prime in R. Indeed, $R/\mathrm{id}(y) = k[x]$ is a domain, which means that $\mathrm{id}(y)$ is completely prime, and by the above remark is prime.

Ideal $\mathrm{id}(y, x-\alpha)$ is also prime due to a similar reason. Indeed, $R/\mathrm{id}(y, x-\alpha) = k$ is a domain. We have to demonstrate that there are no other prime ideals. Let P be a proper prime ideal in R. By Lemma 5.3, $y \in P$. Thus, P can be generated just by y or the ideal P is generated by $\{y\} \cup \mathcal{F}$, where $\mathcal{F}$ is a subset of $k[x]$. Then $R/P = (R/\mathrm{id}(y))/\mathrm{id}(\mathcal{F}) = k[x]/\mathrm{id}(\mathcal{F})$. Since $k[x]$ is a principle ideal domain, we have $\mathrm{id}(\mathcal{F}) = \mathrm{id}(g)$, where $g \in k[x]$. Thus, $R/P = k[x]/\mathrm{id}(g)$. Since R/P must be a domain, we have $g = x - \alpha$ for some $\alpha \in k$. Thus, $P = \mathrm{id}(y, x - \alpha)$.

Now we can easily see which of these ideals are primitive.

Corollary 5.4 *The complete set of primitive ideals in R consists of the ideals* $\mathrm{id}(y, x - \alpha)$, $\alpha \in k$.

6 Irreducible modules, description of all finite dimensional modules

We describe here all irreducible and completely reducible modules over the Jordan plane. For arbitrary module, we give a description of $X = \rho(x)$, in terms of $\mathcal{B} - Toeplitz$ matrices, subject to the Jordan normal form of the matrix $Y = \rho(y)$. This kind of description of the set of all representations will be useful for obtaining results in the stratification of representation variety, we suggest later in the paper.

We will need the following definitions.

Definition 6.1 *Let us remind that upper triangular (rectangular) Toeplitz matrix is a matrix with entries a_{ij} defined only by the difference $i - j$ and it has zeros below the main diagonal (or upper main diagonal in a proper rectangular case).*

Definition 6.2 *We call a matrix corresponding to the partition $\mathcal{P}$ block-upper triangular Toeplitz ($\mathcal{B}$-Toeplitz), if all blocks, defined by the partition, including diagonal blocks, are upper triangular (rectangular) Toeplitz matrices.*

Definition 6.3 *We call a matrix corresponding to the partition $\mathcal{P}$ $\mathcal{J}$-Toeplitz if it is a sum of $\mathcal{B}$-Toeplitz matrix and a matrix with diagonal blocks having the sequence $0, 1, 2, ...$ on the first upper diagonal and zeros elsewhere.*

Theorem 6.4 *The complete set of finite dimensional representations of R (subject to the Jordan normal form of Y) can be described as a set of pairs of matrices (X_n, Y_n), where Y_n is in the Jordan form (with zero eigenvalues) corresponding to the partition $\mathcal{P}$ of n and X_n is a $\mathcal{J}$-Toeplitz matrix defined by $\mathcal{P}$.*

From this theorem immediately follows a precise description of all *irreducible* and *completely reducible* modules.

Corollary 6.5 *A complete set of pairwise non-isomorphic finite dimensional irreducible R-modules is $\{S_a | a \in k\}$, where S_a defined by the following action of X and Y on one-dimensional vector space: $Xu = \alpha u, Yu = 0$.*

All completely reducible representations are given by matrices: $Y_n = (0)$, X_n is a diagonal matrix $\operatorname{diag}(a_1, ..., a_n)$.

P r o o f. Let us describe an arbitrary representation $\rho_n : R \to M_n(k)$ of R, for $n \in \mathbb{N}$. We can assume that the image of one of the generators $Y = \rho_n(y)$ is in the Jordan normal form.

The full Jordan block case.

Let us first find all possible matrices $X = \rho_n(x)$ in the case when Y is just the full Jordan block: $Y = J_n$. We have to find then matrices $X = (a_{ij})$ satisfying the relation $[X, J_n] = J_n^2$. Let $B = [X, J_n] = (b_{ij})$, then $b_{ij} = a_{i+1,j} - a_{i,j-1}$. From the condition $B = J_n^2$ it follows that $b_{ij} = 0$ if $i \neq j-2$ and $b_{ij} = 1$ if $i = j-2$. Here and later on we will use the following numeration of diagonals: main diagonal has number 0, upper diagonals have positive numbers $1, 2, \ldots, n-1$ and lower diagonals have negative numbers $-1, -2, \ldots, -n+1$.

The first condition above means that in the matrix X elements of any diagonal with number $0 \leqslant k \neq 1$ coincide and are zero for $k < 0$. From the second condition it follows that the elements of the first upper diagonal form an arithmetic progression with difference 1: $a+1, \ldots, a+n-1$.

Denote by X_n^0 a matrix with the sequence $0, 1, 2, ...$ on the first upper diagonal and zeros elsewhere. Then our family of representations consists of pairs of matrices $(X_n, Y_n) = (X_n^0 + T, J_n)$, where T is an arbitrary upper triangular Toeplitz matrix.

The case of an arbitrary partition.

Consider now the general case when the Jordan normal form of Y contains several Jordan blocks: $Y = (J_1, ..., J_m)$, corresponding to the partition $\mathcal{P}$.

Cut an arbitrary matrix X into the square and rectangular blocks of sizes defined by $\mathcal{P}$. Denote the blocks by A_{ij}, $i, j = \overline{1, m}$.

Then we can describe the structure of the matrix $B = [X, Y]$ in the following way:

$$B = \begin{pmatrix} [A_{11}] & [A_{12}] & & [A_{1m}] \\ & [A_{22}] & & \\ & & \ddots & \\ [A_{m1}] & [A_{m2}] & & [A_{mm}] \end{pmatrix}, \quad \text{where } [A_{ij}] = A_{ij}J_i - J_jA_{ij}.$$

From the condition $B = Y^2$ we have that $[A_{ii}, J_i] = J_i^2$ and hence A_{ii} is the same as in the previous case when Y was just a full Jordan block and $A_{ij}J_i - J_jA_{ij} = 0$ for $i \neq j$. The latter condition means that A_{ij} for $i \neq j$ are upper rectangular Toeplitz matrices. Hence X has a shape of $\mathcal{J}$-Toeplitz matrix.

As a result we have the family of representations described in Theorem 6.4. □

Using arguments analogues to the above we can ensure.

Proposition 6.6 *Let Y be a matrix in the Jordan normal form defined by the partition $\mathcal{P}$, then the centralizer of Y, $C(Y) = \{Z \in Gl_n(k) \mid ZY = YZ\}$ consists of all $\mathcal{B}$-Toeplitz matrices corresponding to $\mathcal{P}$.*

7 R is residually finite dimensional

Let us consider now one of the sequences of representations constructed in the previous section: $\varepsilon_n : R \to \mathrm{End}\, k^n$, defined by $\varepsilon_n(y) = J_n$, $\varepsilon_n(x) = X_n^0$. Note that this sequence is basic in the following sense. All representations corresponding to Y with full Jordan block could be obtained from ε_n by the following automorphism of R, $\varphi : R \longrightarrow R : x \mapsto x + a, y \mapsto y$ where $a \in R$ such that $[a, y] = 0$.

In addition to the conventional equivalence relation on the representations given by simultaneous conjugation of matrices: $\rho' \sim \rho''$ if there exists $g \in GL(n)$ such that $g\rho' g^{-1} = \rho''$ or equivalently, R-modules corresponding to ρ' and ρ'' are isomorphic, we introduce here one more equivalence relation.

Definition 7.1 *We say that two representations of the algebra R are auto-equivalent (equivalent up to automorphism) $\rho' \sim_A \rho''$ if there exists $\varphi \in$* Aut(R) *such that $\rho'\varphi \sim \rho''$.*

So we can state that any full block representation is auto-equivalent to ε_n for appropriate n.

We will prove now that the sequence of representations ε_n asymptotically is faithful.

Start with the calculation of matrices which are image of monomials $y^k x^m$ under the representation ε_n.

Lemma 7.2 *For the representation ε as above the matrix $\varepsilon(y^k x^m)$ has the following shape: there is only one nonzero diagonal, number $k+m$. In the diagonal appears the sequence $p(0), p(1), ..., p(j), ...$ of values of polynomial $p(j) = (k+j)...(k+m+j-1) = \prod_{i=1}^{m}(k+j+i)$ of degree m.*

P r o o f. Image $\varepsilon(x^m)$ of the monomial x^m is a matrix with vector $[1 \cdot 2 \cdot ... \cdot m, 2 \cdot 3 \cdot ... \cdot (m+1), ...]$ on the (upper) diagonal number m in the above numeration and zeros elsewhere. Multiplication by $\varepsilon(y^k)$ acts on matrix by moving up all rows on k steps. We can now see that matrix corresponding to the polynomial $y^k x^m$ can have only one nonzero diagonal, number $m+k$, and vector in this diagonal is the following: $[(k+1)...(m+k), (k+2)...(m+k+1), ...]$.$\square$

Theorem 7.3 *Let ε_n be the sequence of representations of R as above. Then $\cap_{n=0}^{\infty} \ker \varepsilon_n = 0$.*

P r o o f. We are going to show that $\varepsilon_n(f) \neq 0$ for $n \geqslant 2\deg f$. Suppose that for sufficiently large n, $\varepsilon_n(f)$ is zero and get a contradiction. Denote by l the degree of polynomial f, and let $f = f_1 + ... + f_l$ be a decomposition of $f \in R$ on the homogeneous components of degrees $i = 1, ..., l$ respectively. From Lemma 7.2 we know the shape of the matrix which is an image of a monomial $y^k x^m$.

Applying Lemma 7.2 to each homogeneous part of the given polynomial f we get that

$$f_l = \sum_{k+m=l} a_{k,m} y^k x^m = \sum_{r=0}^{l} a_r y^{l-r} x^r$$

is a sum of matrices $\sum_{r=0}^{l} a_r M_r$, where M_r has the vector $[(p(0), ..., p(j)]$:

$$\left(\prod_{i=1}^{r}(l-r+i), \prod_{i=1}^{r}(l-r+i+1), \ldots, \prod_{i=1}^{r}(l-r+i+j), \ldots \right)$$

on the diagonal number l (all other entries are zero). The number on the j-th place of this diagonal is the value in j of the polynomial

$$P(j) = (l-r+j) \cdot ... \cdot (l+j-1)$$

of degree exactly r. Therefore the sum $\sum_{r=0}^{l} a_r M_r$ has a polynomial on j of degree $N = \max\{r : a_r \neq 0\}$ on the diagonal number l. Since any polynomial of degree N has at most N zeros we arrive to a contradiction in the case when lth diagonal has length more than l. Hence for any $n \geqslant 2\deg f$, $\varepsilon_n(f) \neq 0$. □

Let us recall that an algebra R *residually has some property* P means that there exists a system of equivalence relations τ_i on R with trivial intersection, such that in the quotient of R by any τ_i property P holds.

From the Theorem 7.3 we have the following corollary considering equivalence relations modulo ideals $\ker \varepsilon_n$.

Corollary 7.4 *Algebra R is residually finite dimensional.*

8 Indecomposable modules

Lemma 8.1 *Let $M = (X, Y)$ be a (finite dimensional) indecomposable module over $R = k\langle x, y | xy - yx = y^2\rangle$. Then X has a unique eigenvalue.*

P r o o f. Denote by M_λ^X the main eigenspace for X corresponding to its eigenvalue λ: $M_\lambda^X = \bigcup\limits_{k=0}^{\infty} \ker (X - \lambda I)^k$. Obviously $M_\lambda^X = \ker (X - \lambda I)^m$, where m is the maximal size of blocks in the Jordan normal form of X. It is well-known that $M = \bigoplus\limits_i M_{\lambda_i}^X$, where the direct sum is taken over all different eigenvalues λ_i of X. We shall show that $M_{\lambda_i}^X$ are in fact R-submodules.

Let $u \in M_\lambda^X$, that is $(X - \lambda I)^m u = 0$. We calculate $(X - \lambda I)^n Y u$ for arbitrary n. Using the fact that the mapping defined on generators $\varphi(x) = x - \lambda$, $\varphi(y) = y$ extends to an automorphism of R (see 4), we can apply it to the multiplication formula from Lemma 3.3 to get $(x - \lambda)^n y = \sum\limits_{k=1}^{n+1} y^k (x - \lambda)^{n-k+1}$. Taking into account that $Y^l = 0$ for some positive integer l, we can choose N big enough, for example $N \geqslant m + l$, such that

$$(X - \lambda I)^N Y u = \sum_{k=1}^{N+1} \alpha_{k,N} Y^k (X - \lambda I)^{N-k+1} u = 0$$

either due to $(X - \lambda I)^{N-k+1} u = 0$ or due to $Y^k = 0$.

This shows that $Yu \in M_\lambda^X$, that is M_λ^X is invariant with respect to Y. □

As an immediate corollary we have the following.

Theorem 8.2 *Any finite dimensional R-module M decomposes into the direct sum of submodules $M_{\lambda_i}^X$ corresponding to different eigenvalues λ_i of X.*

Corollary 8.3 *Let M be indecomposable module corresponding to the representation $\rho : R \longrightarrow \mathrm{End}(k^n)$, and A_n is the image of this representation. Then A_n is a local algebra, e.i. $A_n/J(A_n) = k$.*

P r o o f. This follows from the above lemma and the fact that any image algebra is basic with semisimple part isomorphic to the sun of r copies of the field k: $\oplus_r k$, where r is a number of different eigenvalues of X, which was proved in proposition 2.10.□

Now using the definition of quiver for the image algebra given in section 2 and Lemma 2.13 we have a complete description of quiver equivalence classes of indecomposable modules.

Corollary 8.4 *Quiver corresponding to the indecomposable module has one vertex. The number of loops is one or two, which is a dimension of the vector space* $\mathrm{Span}_k\{\bar{X} - \lambda I, \bar{Y}\}$, *where* $\bar{X} = \varphi X$, $\bar{Y} = \varphi Y$ *for* $\varphi : A \to A/J^2$.

As another consequence of Theorem 8.2 we can derive an important information on how to glue irreducible modules to get indecomposables. It turns out that it is possible to glue together nontrivially only the copies of the same irreducible module S_a.

Corollary 8.5 *For arbitrary non-isomorphic irreducible modules* S_a, S_b,

$$\mathrm{Ext}^1_R(S_a, S_b) = 0, \text{ if } a \neq b.$$

P r o o f. Indeed, in Corollary 6.5 we derive that irreducible module S_i is one dimensional and given by $X = (a), Y = (0)$, $a \in k$. If $a \neq b$ then for $[M] \in \mathrm{Ext}^1_R(S_a, S_b)$, corresponding X has two different eigenvalues, namely a and b. Then by the above lemma M is decomposable and $[M] = 0$. □

9 Equivalence of some subcategories in mod R

Let us denote by mod $R(\lambda)$ the full subcategory in $\mathrm{mod}\, R$ consisting of modules with the unique eigenvalue λ of X: $\mathrm{mod}\, R(\lambda) = \{M \in \mathrm{mod}\, R | M = M_\lambda(X)\}$. Let us define the functor F_λ on $\mathrm{mod}\, R$, which maps a module M to the module M_λ with the following new action $rm = \varphi_\lambda(r)m$, where φ_λ is an automorphism of R defined by $\varphi_\lambda(x) = x + \lambda$, $\varphi_\lambda(y) = y$. The restriction of F_λ to $\mathrm{mod}\, R(\lambda)$ is an equivalence of categories $F_\lambda : \mathrm{mod}\, R(\lambda) \to \mathrm{mod}\, R(\mu + \lambda)$ for any $\mu \in k$. In particular, we have an equivalence of the categories $\mathrm{mod}\, R(\lambda)$ and $\mathrm{mod}\, R(0)$.

To use this equivalence of categories it is necessary to know that in most cases (but not in all of them), the eigenvalues of the matrix X are just entries of the main diagonal in the standard shape of the matrix described in the Theorem 6.4, more precisely.

Theorem 9.1 *Let in the basis* $\mathcal{E}$ *of the representation vector space,* Y *is in the Jordan normal form, and Jordan blocks have pairwise different sizes:* $n_1, n_2, \dots, n_k$. *Then in the same basis* X *is a* $\mathcal{J}$*-Toeplitz matrix corresponding to the partition* $\mathcal{P} = n_1, n_2, \dots, n_k$ *with numbers* $\lambda_1, \dots, \lambda_k$ *on the diagonals of the main blocks, where* λ_j *are eigenvalues of* X *(not necessarily different).*

P r o o f. Let us first introduce the denotation for the basis $\mathcal{E}$:

$$e^{1,1}, \dots, e^{1,n_1}, e^{2,1}, \dots, e^{2,n_2}, \dots, e^{k,1}, \dots, e^{k,n_k}.$$

Consider the set $\mathcal{A}$ of matrices (in the basis $\mathcal{E}$) such that $A_{(j,l),(j,l)} = c_j$, $1 \leqslant j \leqslant k$, $1 \leqslant l \leqslant n_j$ and $A_{(i,s),(j,l)} = 0$ if $n_j < n_i$, $l > s - n_j$ and if $n_j > n_i$, $l > s$. One can easily verify that $\mathcal{A}$ is an algebra with respect to the matrix multiplication. Let also $\mathcal{D}$ be the subalgebra of diagonal matrices in $\mathcal{A}$ and $\varphi : \mathcal{A} \to \mathcal{D}$ be the natural projection (φ acts by annihilating the off-diagonal part of a matrix).

Looking at the multiplication in $\mathcal{A}$ it is straightforward that φ is an algebra morphism, under the condition that $n_1, ..., n_k$ are pairwise different numbers, that is $\varphi(I) = I$, $\varphi(AB) = \varphi(A)\varphi(B)$ and $\varphi(A + B) = \varphi(A) + \varphi(B)$. It is also easy to check, calculating the powers of the matrix, that if $A \in \mathcal{A}$ and $\varphi(A) = 0$ then the matrix A is nilpotent. Since $\mathcal{J}$-Toeplitz matrices belong to $\mathcal{A}$, it suffices to verify that the eigenvalues of any $A \in \mathcal{A}$ coincide with the eigenvalues of $\varphi(A)$.

First, suppose that λ is not an eigenvalue of A. That is the matrix $A - \lambda I$ is invertible: there exists a matrix $B \in \mathcal{A}$ such that $(A - \lambda I)B = I$. Here we use the fact that if a matrix from a subalgebra of the matrix algebra is invertible, then the inverse belongs to the subalgebra. Then $\varphi((A - \lambda I))\varphi(B) = \varphi((A - \lambda I)B) = \varphi(I) = I$. Therefore λ is not an eigenvalue of $\varphi(A)$. On the other hand, suppose that λ is not an eigenvalue of $\varphi(A)$. Then $\varphi(A) - \lambda I$ is invertible. Clearly

$$A - \lambda I = (\varphi(A) - \lambda I)(I + (\varphi(A) - \lambda I)^{-1}(A - \varphi(A))).$$

Let $B = (\varphi(A) - \lambda I)^{-1}(A - \varphi(A))$. Since φ is a projection, we have that

$$\varphi(B) = \varphi((\varphi(A) - \lambda I)^{-1})(\varphi(A) - \varphi(A)) = 0.$$

As we have already mentioned this means that the matrix B is nilpotent and therefore $I + B$ is invertible. Hence $A - \lambda I = (\varphi(A) - \lambda I)(I + B)$ is invertible as a product of two invertible matrices. Therefore λ is not an eigenvalue of A. Thus, eigenvalues of A and $\varphi(A)$ coincide. This completes the proof. □

10 Analogue of the Gerstenhaber theorem for commuting matrices

In this section we intend to prove an analog of the Gerstenhaber-Taussky-Motzkin theorem (see [13], [25], [16]) on the dimension of images of representations of two generated algebra of commutative polynomials $k[x, y]$. This theorem says that any algebra generated by two matrices $A, B \in M_n(k)$ of size n which commute $AB = BA$ has dimension not exceeding n. It was proved using different means, for example, in [16] one can find arguments, where irreducibility of commuting variety is involved and in [36] purely module theoretic methods were used.

Instead of commutativity we consider the relation $XY - YX = Y^2$ and prove the following

Theorem 10.1 *Let $\rho_n : R \to M_n(k)$ be an arbitrary n-dimensional representation of $R = k\langle x, y | xy - yx = x^2 \rangle$ and $A_n = \rho_n(R)$ be the image algebra. Then the dimension of A_n does not exceed $\frac{n(n+2)}{4}$ for even n and $\frac{(n+1)^2}{4}$ for odd n.*

This estimate is optimal and attained for the image algebra corresponding to the full block Y.

We divide the proof in two lemmas. Start with the second statement of the theorem, that is calculation of the dimension of the image algebras in the full block case.

Let us note first the following

Lemma 10.2 *Let $\rho_n : R \to M_n(k)$ be an n dimensional representation of R, where $Y = \rho(y)$ has a full block Jordan structure. Then ρ_n is auto-equivalent to the fixed representation ε_n, $\varepsilon_n(x) = X_n^0$, $\varepsilon_n(y) = J_n$. For any n the image algebra $A_n = \rho_n(R)$ does not depend on the choice of ρ_n.*

We want to emphasize that the fact we will use here, that Y commutes only with polynomials on Y, is specific for the full block case. We will try to explain why it is so in the course of the proof.

Let us remind that a matrix $Y \in M_n(k)$ called *non-derogatory* if its characteristic polynomial coincides with the minimal polynomial, or if any eigenspace has dimension 1.

It is well-known that

Proposition 10.3 *The matrix Y is non-derogatory iff $C(Y) = Alg(Y)$, where $C(Y)$ is a centralizer of Y in $M_n(k)$ and $Alg(Y)$ — algebra generated by Y.*

Proof. Consider a representation $\rho' \sim \rho$, such that $\rho'(Y) = J_n$ is a full Jordan block. Denote $Y = \rho(y) = \varepsilon(y)$. Since both pairs $\rho'(x)$, $\rho'(y)$ and $\varepsilon(x)$, $\varepsilon(y)$ satisfy the algebra relation, we have $[Y, \rho'(x) - \varepsilon(x)] = 0$. Now, since Y is nilpotent it has one block Jordan structure iff it is non-derogatory. This means that only in case of one block Y we will have that $\rho'(x) - \varepsilon(x) \in C(Y)$ is a polynomial on Y (due to Proposition 10.3). So, if indeed, $Y = J_n$, then $\rho'(x) = \varepsilon(x) = p(Y)$, $p \in k[t]$ and $\rho'(x) = \varepsilon(\varphi(x))$, $\rho'(y) - \varepsilon(\varphi(y))$, where $\varphi(x) = x + p(y), \varphi(y) = y$, that is $\varphi \in \mathrm{Aut}R$. This means that ρ' and ε are auto-equivalent, hence also ρ and ε. From this immediately follows that they have the same image algebras. □

Lemma 10.4 *Let $X, Y \in M_n(k)$ be matrices of size n over the field k, satisfying the relation $XY - YX = Y^2$ and Y has as a Jordan normal form one full block. Denote by A the algebra generated by X and Y. Then for odd n, $\dim A = \frac{(n+1)^2}{4}$ and for even n, $\dim A = \frac{n(n+2)}{4}$.*

P r o o f. In Lemma 7.2 we already have computed the matrices, which are images of monomials $y^k x^m$ under the representation $\varepsilon : (x, y) \mapsto (X^0, J_n)$. Due to the Lemma 10.2 we have only one image algebra for any n. In order to calculate its dimension, let us recall how matrices $\varepsilon(y^k x^m)$ look like and calculate the dimensions of their linear spans.

The matrix $\varepsilon(y^{l-r}x^r)$ on the l-th upper diagonal has a vector $(p(0), p(1), \dots)$, where

$$p(j) = (l - r + j) \dots (l + j - 1) = \prod_{i=1}^{r} (l + j - r + i)$$

and zeros elsewhere. In the j-th place of the l-th diagonal we have a value of a polynomial of degree exactly r. Those diagonals which have number less then the number of elements in it give the impact to the dimension equal to the dimension of the space of polynomials of corresponding degree. When the diagonals become shorter (the number of elements less then the number of the diagonal) then the impact to the dimension of this diagonal equals to the number of the elements in it. Thus, if $n = 2m + 1$, $\dim A = 1 + \dots + m + (m + 1) + m + \dots + 1 = (m + 1)^2 = \frac{(n+1)^2}{4}$. When $n = 2m$, we have $\dim A = 1 + \dots + m + m + \dots + 1 = m(m + 1) = \frac{n(n+1)}{4}$. □

10.1 Maximality of the dimension in the full block case

We know now that any representation of R, which is isomorphic to one with Y in the full block Jordan normal form gives us as an image the same algebra, described in Lemma 10.4 as a certain set of matrices, of dimension $\frac{n(n+2)}{4}$ for even n and $\frac{(n+1)^2}{4}$ for odd n. We intend to prove that this dimension is maximal among dimensions of all image algebras for arbitrary representation, that is the first part of Theorem 10.1.

We start with the proof that this dimension is an upper bound for any image algebra of an indecomposable representation.

A simple preliminary fact we will need is the following.

Lemma 10.5 *Matrices X and Y satisfying the relation $XY - YX = Y^2$ can be by simultaneous conjugation brought to a triangular form.*

P r o o f. Using the defining relation and the fact that Y is nilpotent we can see that any eigenspace of Y is invariant under X. Hence X and Y has joint eigenvector v. Then we consider quotient representation on the space $V/\{v\}$ which has the same property. Continuation of this process supplies us with the basis where both X and Y are triangular. □

Lemma 10.6 *Let $\rho_n : R \to M_n(k)$ be an indecomposable n-dimensional representation of R and $A_n = \rho_n(R)$ be the image algebra. Then the dimension of A_n does not exceed $\frac{n(n+2)}{4}$ for even n and $\frac{(n+1)^2}{4}$ for odd n.*

P r o o f. The algebra $A_n = \{\sum \alpha_{k,m} Y^k X^m\}$ consists now of triangular matrices. Let us present the linear space UT_n of upper triangular $n \times n$

matrices as the direct sum of two subspaces $UT_n = L_1 \oplus L_2$, where L_1 consists of matrices with zeros on upper diagonals with numbers $l, \dots, n$ and L_2 consists of matrices with zeros on upper diagonals with numbers $1, \dots, l-1$, where $l = (n+1)/2$ for odd n and $l = n/2 + 1$ for even n. Let P_j, $j = 1, 2$ be the linear projection in UT_n onto L_j along L_{3-j}. Since A_n is a linear subspace of UT_n, we have that $A_n \subset M_1 + M_2$, where $M_j = P_j(A_n)$. Therefore $\dim A_n \leqslant \dim M_1 + \dim M_2$. The dimension of M_1 clearly does not exceed the dimension of the linear span of those matrices $Y^k X^m$, which do not belong to L_2. Thus,

$$\dim M_1 \leqslant \dim \langle Y^k X^m | k + m < l - 1 \rangle_k.$$

Here we suppose that X (as well as Y) is nilpotent. We can do this because the module is indecomposable. Indeed, lemma 8.1 says that for an indecomposable module X has a unique eigenvalue. This implies that any indecomposable representation is autoequivalent to one with nilpotent X and Y due to the automorphism of R defined by $\varphi_\lambda(x) = x - \lambda$, $\varphi_\lambda(y) = y$. Since autoequivalent representations have the same image algebras we can suppose that X is nilpotent.

Thus the dimension of M_1 does not exceed the number of the pairs (k, m) of non-negative integers such that $k + m < l - 1$, which is equal to $1 + \dots + (l-1)$. On the other hand $\dim M_2 \leqslant \dim L_2$ and the dimension of L_2 does not exceed the total number of entries in the non-zero diagonals.

$$\dim M_2 \leqslant 1 + \dots + (n - l + 1).$$

Taking into account that $\dim A_n \leqslant \dim M_1 + \dim M_2$, we have

$$\dim A_n \leqslant 1 + \dots + (l-1) + 1 + \dots + (n - l + 1).$$

The latter sum equals $\frac{n(n+2)}{4}$ for even n and $\frac{(n+1)^2}{4}$ for odd n. $\square$

After we have proved the estimation for the indecomposable modules, it is easy to see that the same estimate holds for arbitrary module, since the function n^2 is convex.

On the other hand as it was shown in the Lemma 10.4 that this estimate is attained on the algebra $A_n = \varepsilon(R)$ in the case of the full block Y. This completes the proof of Theorem 10.1.

11 Stratification of the Jordan variety

Here we suppose that $k = \mathbb{C}$. Let us consider the variety of R-module structures on k^n and denote it by $mod(R, n)$. Such structures are in 1-1 correspondence to k-algebra homomorphisms $R \to M_n(k)$ (n-dimensional representations), or equivalently to a pairs of matrices (X, Y), $X, Y \in M_n^{(2)}(k)$, satisfying the relation $XY - YX = Y^2$. The group $GL_n(k)$ acts on $mod(R, n)$

by simultaneous conjugation and orbits of this action are exactly the isomorphism classes of n-dimensional R-modules. Denote this orbit of a module M or of a pair of matrices (X, Y) as $\mathcal{O}(M)$ or $\mathcal{O}(X, Y)$ respectively.

Consider the following *stratification* on $mod(R, n)$. Let $\mathcal{U}_{\mathcal{P}}$ be the set of all pairs (X, Y) satisfying the relation, where Y has a fixed Jordan form. Here $\mathcal{P}$ stands for the partition of n, which defines the Jordan form of Y. Clearly $\mathcal{U}_{\mathcal{P}}$ is a union of all orbits where Y has a Jordan form defined by the partition $\mathcal{P}$. We will write $\mathcal{U}_{(n)}$ for the stratum corresponding to the trivial partition $\mathcal{P} = (n)$ or to Y with the full Jordan block.

In this stratum $\mathcal{U}_{\mathcal{P}}$ there is a natural choice of the *vector* bundle structure with the base consisting of Ys with the given Jordan form defined by $\mathcal{P}$. The fiber then will be an affine space of solutions of the equation $XY - YX = YY$ with respect to X, when Y is fixed. The dimension does not depend on the Y, since Ys are all conjugate. It is equal to the dimension of the space of $\mathcal{B} - Toeplitz$ matrices defined by the partition $\mathcal{P}$. If $\mathcal{P} = (n_1 \leqslant n_2 ... \leqslant n_r)$, then the dimension of this fiber is $m = (2r-1)n_1 + (2r-3)n_2 + ... + 3n_{r-1} + n_r$. The dimension of the base is therefore $n^2 - m$. For example, for the strata $\mathcal{U}_{(n)}$ defined by the full block Y, a generic one, the dimensions of the base and the fiber in this vector bundle distributed as $n^2 - n$ and n respectively.

Remark. If one consider the whole space of pairs of matrices $\mathcal{M}_n$, in general position there exists an *algebraic* bundle, described recently in the Kontsevich Arbeitstagung talk [20]. There dimensions of the fiber and the base distributed as $n^2 - 2n - 1$ and $(n+1)^2$ respectively, fibers are not a vector spaces, but still quite nice – they are 'algebraic'. If we try to construct the bundle in our subvariety, in generic situation, on the same principle, where the base consists of polynomials $\det(1 + sX + tY)$, and the fibers of pairs of matrices with this polynomial, we get a bundle with one dimensional base. Indeed, the pairs of matrices (X, Y) with the full block Jordan form of Y are in general position in our variety $mod(R, n)$. Hence in this generic strata X has one eigenvalue and the polynomial, defining the point of the base is $\det(I - aIs + 0It) = (1 + as)^n$, so the base is a one parameter family. The fiber however, is quite complicated: $F_a = \{(X, Y) | X$ with one eigenvalue, Y satisfying $XY - YX = YY\}$. Note that fibers are shifts of each other: $F_b = (X + (b - a)I, Y) = F_a + ((b - a)I, 0)$.

11.1 Examples of parametrizable strata

In this section we will give a parametrization (by two parameters) of the stratum $\mathcal{U}_{(n)}$.

Another action involved here is an action of the subgroup of GL_n on those pairs (X, Y), where $Y = J_{\mathcal{P}}$ is in fixed Jordan form. Denote this space by $W_{\mathcal{P}}$. The subgroup which acts there is clearly the centralizer of the given Jordan matrix: $C(J_{\mathcal{P}})$. Orbits of the action of $C(J_{\mathcal{P}})$ on the space $W_{\mathcal{P}}$ are just restrictions of orbits above: $\mathcal{O}_{\mathcal{P}}(X) = \mathcal{O}(X, Y) \cap W_{\mathcal{P}}$.

We sometimes consider instead of action of GL_n on the whole space an action of the centralizer $C(J_{\mathcal{P}})$ on the smaller space $W_{\mathcal{P}}$. While the group is not reductive any more and has a big unipotent part, we act just on the space of matrices and some information easier to get in this setting. It then could be (partially) lifted because of 1-1 correspondence of orbits. More precisely, it could be lifted if we are interested in parametrization, but if we consider, for example, degeneration of orbits the situation may certainly change after their restriction.

What we actually do here is obtaining the parametrization for $W_{(n)}$. Due to 1-1 correspondence between the orbits we then have a parametrization of $\mathcal{U}_{(n)}$.

Let us restrict the orbits even a little further, considering the action of the group $G = C(J_{\mathcal{P}}) \cap SL_n$, where the 1-1 correspondence with the initial orbits will be clearly preserved. In the case $\mathcal{P} = (n)$ the group G can be presented as follows:

$$G = \{I + \alpha_1 Y + \alpha_2 Y^2 + ... + \alpha_{n-1} Y^{n-1}\},$$

due to our description of the centralizer of Y in Proposition 6.6. This group acts on the affine space of the dimension n:

$$W_{(n)} = \{\lambda I + X^0 + c_1 Y + c_2 Y^2 + ... + c_{n-1} Y^{n-1}\}$$

here λ is the eigenvalue of X and X^0 is the matrix with the second diagonal $[0, 1, \dots, (n-1)]$ and zeros elsewhere (defined in section 6).

Let us fix first the eigenvalue: $\lambda = 0$, we get then the space of dimension $n-1$:

$$W'_{(n)} = \{X^0 + c_1 Y + c_2 Y^2 + ... + c_{n-1} Y^{n-1}\}.$$

We intend to calculate now the dimension of the orbit $\mathcal{O}_{(n)}(X, G)$ of X with fixed eigenvalue $\lambda = 0$ under $G-$ action.

Consider the map $\varphi : G \longrightarrow W'_Y$ defined by this action: $\varphi(C) = CXC^{-1}$, then $\operatorname{Im} \varphi = \mathcal{O}_{(n)}(X, G)$. We are going to calculate the rank of Jacobian of this map. We will see that it is constant on G and equals to $n-2$. This tells us that each orbit $\mathcal{O}_{(n)}(X, G)$ is an $n-2$ dimensional manifold and hence there couldn't be more then 2 parameters involved in parametrization of orbits.

11.2 Calculation of the rank of Jacobian

Theorem 11.1 *Let G be an intersection of SL_n with the centralizer of Y. Consider the action of this group on the affine space $W'_Y = \{X^0 + c_1 Y + c_2 Y^2 + ... + c_{n-1} Y^{n-1}\}$ by conjugation. Then the rank of the Jacobian of the map $\varphi : G \longrightarrow W'_Y$ is equal to $n-2$ at any point $C \in G$.*

P r o o f. Clearly $d\varphi(C)(\Delta)$ is the linear part of the map $(C+\Delta)^{-1} X (C+\Delta) - C^{-1} X C$, where

$$C = I + \alpha_1 Y + \alpha_2 Y^2 + ... + \alpha_{n-1} Y^{n-1},$$

$$X = X^0 + c_1Y + c_2Y^2 + ... + c_{n-1}Y^{n-1},$$

$$\Delta = \beta_1 Y + \beta_2 Y^2 + ... + \beta_{n-1}Y^{n-1}.$$

Let us present $(C+\Delta)^{-1}$ in the following way:

$$(C+\Delta)^{-1} = (I + \Delta C^{-1})^{-1}C^{-1} =$$

$$(I - \Delta C^{-1} + \text{lower order terms on } \Delta)C^{-1}.$$

Then

$$(C+\Delta)^{-1}X(C+\Delta) - C^{-1}XC =$$

$$(I - \Delta C^{-1} + \text{lower order terms on } \Delta)C^{-1}X(C+\Delta) - C^{-1}XC =$$

$$-\Delta C^{-2}XC + C^{-1}X\Delta + \text{lower order terms on } \Delta =$$

$$(-\Delta C^{-1}\cdot C^{-1}X + C^{-1}X\cdot \Delta C^{-1})C + \text{lower order terms on } \Delta.$$

Denote $\tilde{\Delta} := \Delta C^{-1}$ and $\tilde{X} := C^{-1}X$. Obviously multiplication by C preserves the rank and rank of linear map $d\varphi(C)(\Delta)$ is equal to the rank of the map $T(\tilde{\Delta}) = [\tilde{X}, \tilde{\Delta}]$.

Here again $\tilde{\Delta}$ has a form

$$\tilde{\Delta} = \gamma_1 Y + \gamma_2 Y^2 + ... + \gamma_{n-1}Y^{n_1}.$$

Let us compute commutator of $\tilde{X}$ with Y^k. Taking into account that C^{-1} is a polynomial on Y, hence commute with Y^k and also the relation in algebra R: $XY^k - Y^kX = kY^{k+1}$. We get $\tilde{X}Y - Y\tilde{X} = C^{-1}XY^k - Y^kC^{-1}X = C^{-1}(XY^k - Y^kX) = C^{-1}kY^{k+1}$. Hence

$$\tilde{X}p(Y) - p(Y)\tilde{X} = C^{-1}Y^2p'(Y)$$

for arbitrary polynomial p. Applying this for the polynomial $\tilde{\Delta}$ we get

$$T(\tilde{\Delta}) = [\tilde{X}, \tilde{\Delta}] = \sum_{k=1}^{n-2} \gamma_k k C^{-1}Y^{k+1},$$

hence this linear map has rank $n-2$. □

From Theorem 11.1 we could deduce the statement concerning parametrization of isoclasses of modules in the stratum $\mathcal{U}_{(n)}$.

We mean by *parametrization* (by m parameters) the existence of m smooth algebraically independent functions which are constant on the orbits and separate them.

Corollary 11.2 *Let $\mathcal{U}_{(n)}$ be the stratum as above. Then the set of isomorphism classes of indecomposable modules from $\mathcal{U}_{(n)}$ could be parameterized by at most two parameters.*

Proof. Directly from Theorem 11.1 applying the theorem on locally flat map [8] to $\varphi : G \longrightarrow W'_{(n)}$ we have that $\mathrm{Im}\varphi = \mathcal{O}_{(n)}(X, G)$ is an $n-2$ dimensional manifold. We have to mention here that this is due to the fact that the image has no self-intersections. This is the case since the preimage of any point P is connected (it is formed just by the solutions of the equation $CX = PC$ for $C \in G$). Hence we can parametrize these orbits lying in the space $W_{(n)}$ of dimension n by at most two parameters. Due to 1-1 correspondence with the whole orbits $\mathcal{O}(X, Y)$ the latter have the same property.□

Proposition 11.3 *Parameters μ and λ are invariant under the action of G on the set of matrices* $\left\{ \begin{pmatrix} \lambda & \mu+1 & & & * \\ & \lambda & \mu+2 & & \\ & & \lambda & \mu+3 & \\ & & & \ddots & \ddots \\ & 0 & & & \lambda & \mu+n-1 \\ & & & & & \lambda \end{pmatrix} \right\}$.

Proof. Direct calculation of ZMZ^{-1} for $Z \in G$ as described above shows that elements in first two diagonals of M will be preserved. □

Hence from Corollary 11.2 and Proposition 11.3 we have the following classification result for representations with the full Jordan block Y.

Theorem 11.4 *Let $P_{\lambda,\mu}$ denotes the pair $(X_{\lambda,\mu}, Y)$, where*

$$X_{\lambda,\mu} = \begin{pmatrix} \lambda & \mu+1 & & & 0 & \\ & \lambda & \mu+2 & & & \\ & & \lambda & \mu+3 & & \\ & & & \ddots & \ddots & \\ & 0 & & & \lambda & \mu+n-1 \\ & & & & & \lambda \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 & & & 0 & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & \ddots & \ddots & \\ & 0 & & & 0 & 1 \\ & & & & & 0 \end{pmatrix}.$$

Every pair $(X, Y) \in \mathcal{U}_{(n)}$ is conjugate to $P_{\lambda,\mu}$ for some λ, μ. No two pairs $P_{\lambda,\mu}$ with different (λ, μ) are conjugate.

Let us mention that number of parameters does not depends of n in this case.

11.3 Examples of tame strata (up to auto-equivalence)

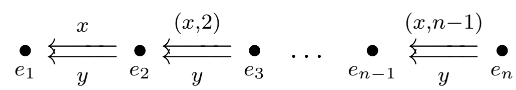
We give here some examples of tame strata in the suggested above stratification related to the Jordan normal form of Y. We show, for example, that the stratum $\mathcal{U}_{(n-1,1)}$ corresponding to the partition $\mathcal{P} = (n-1, 1)$ is tame (but not of finite type) with respect to auto-equivalence relation on modules. The latter was defined in section 7 and its meaning is in gluing together orbits which could be obtained one from another using automorphism of the initial algebra. These examples are quite rare, most strata are wild, as the algebra itself.

Theorem 11.5 *The stratum $\mathcal{U}_{(n)}$ has a finite representation type with respect to auto-equivalence relation on modules.*

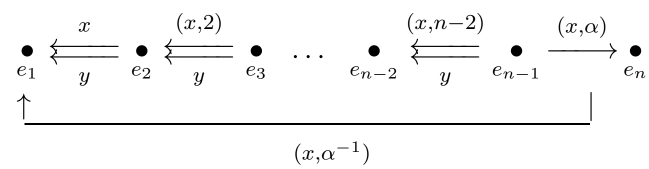
The stratum $\mathcal{U}_{\mathcal{P}}$, for $\mathcal{P} = (n-1, 1)$ is tame, that is parametrizable by one parameter, with respect to auto-equivalence relation.

P r o o f. The proof is analogous to the proof of Theorem 11.4. We present here pictures showing how X and Y act on the basis and where parameters appear:

$\mathcal{P} = (n)$



$\mathcal{P} = (n-1, 1)$



□

12 The full block stratum and Ringel's classification of complete local algebras

Above results suggest to study more carefully the stratum corresponding to the full block Y, since due to Corollary 13.3 they are building blocks in the analogue of the Krull-Remark-Schmidt decomposition theorem on the level of irreducible components. Moreover results on the analogue of Gerstenhaber theorem confirm the exceptional role of this stratum, as the dimension of an image algebra reach its maximum in this stratum. Here we are going to find out when image algebras for this stratum are tame and when they are wild.

We shall show the place of image algebras for representations from $U_{(n)}$, $n \in \mathbb{Z}$ in Ringel's classification of complete local algebras [29] by calculating their defining relations. So the main problem we solve here is the following. Given a finitely presented algebra and its finite dimensional representation. We need to find all defining relations of the image algebra.

As a result we prove that for representations of small dimensions, for $n \leqslant 4$, all image algebras are *tame* and for $n \geqslant 5$ they are *wild* (in the classical sense accepted in the representation theory of finite dimensional algebras).

Recall that a k-algebra A is called *local* if $A = k \oplus \mathrm{Jac}(A)$, where $\mathrm{Jac}(A)$ is the Jacobson radical of A. One can also consider the *completion* of A: $\overline{A} = \varprojlim A/(\mathrm{Jac}(A))^n$. An algebra A is called complete if $A = \overline{A}$.

We have seen in Corollary 8.3, that any indecomposable representation has a local algebra as an image, in particular, representations with full block Y do. Note also that all image algebras are complete, since $\mathrm{Jac}(A)^N = 0$ for N large enough. Indeed, we can use here the Corollary 2.7 which describe the radical, or observe directly that since $A = k \oplus \mathrm{Jac}(A)$ and A consists of polynomials on X^0 and J_n (as an image of one of representations $\varepsilon_n : (x, y) \mapsto (X^0, J_n)$), then $Jac(A)$ consists of those polynomials which have no constant term. The matrices J_n and X^0 are nilpotent of degree n and $n-1$ respectively, hence $\mathrm{Jac}(A)^{2n} = 0$.

Remind also that due to Lemma 10.2 for any n we have only one image algebra for all representations from $\mathcal{U}_{(n)}$.

Theorem 12.1 *The image algebra A_n of a representation $\rho_n \in \mathcal{U}_{(n)}$ is wild for any $n \geqslant 5$. It has a quotient isomorphic to the wild algebra given by relations $y^2, yx - xy, x^2y, x^3$ from the Ringel's list of minimal wild local complete algebras. The image algebras A_1, A_2 and A_3 are tame.*

P r o o f. We intend to show that for n big enough, the algebra A_n has a quotient isomorphic to the algebra $W = \langle x, y | y^2 = yx - xy = x^2y = x^3 = 0\rangle$, which is number c) in the Ringel's list of minimal wild local complete algebras [29].

The algebra W is 5-dimensional. Let us consider the ideal J in A_n generated by the relations above on the image matrices X and Y. This ideal has obviously codimension not exceeding 5. We intend to show that J has codimension exactly 5 and therefore W should be isomorphic to A/J.

Let us look at the ideal J, which is generated by $\{Y^2, X^2Y, X^3, XY - YX\}$. First, since $XY - YX = Y^2$, J is generated by $\{Y^2, X^2Y, X^3\}$. It is easy to see that Y^2 has zeros on first two diagonals and the vector $\mathbf{1} = (1, \ldots, 1)$ on the third one, X^2Y, X^3 have zeros on the first three diagonals. An arbitrary element of A_n has the constant vector $c\mathbf{1}$ for some $c \in k$ on the main diagonal. Hence we see that a general element of the ideal J has zeros on the first two diagonals and the constant sequence $c\mathbf{1}$ on the third one. Taking into account that A_n comprises the upper triangular matrices that have values of a polynomial of degree at most m on m-th diagonal (Lemma 7.2), we see that the main diagonal gives an impact of 1 to the codimension of J, the first diagonal gives an impact of 2 to the codimension of J if the length of this diagonal is at least 2 (that is $n \geqslant 3$) and the second diagonal, — an impact of 2, if the length of this diagonal is at least 3 (that is $n \geqslant 5$). Thus, $\dim A/J \geqslant 5$ if $n \geqslant 5$. This completes the proof in the case $n \geqslant 5$.

Tameness of A_1 and A_2 is obvious. For A_3 the statement follows from the dimension reason: $\dim A_3 = 4$, it is less then the dimensions of all 2-generated algebras from the Ringel's list of minimal wild algebras. Since his theorem (theorem 1.4 in [29]) states that any local complete algebra is either tame of has a quotient from the list, A_3 can not be wild by dimension reasons. Hence A_3 is tame. □

Let us consider now the case $n = 4$.

Theorem 12.2 *Let $\rho_4 \in \mathcal{U}_{(4)}$ be a four dimensional representation of the algebra R. Then the image algebra $A_4 = \rho_4(R)$ is given by the relations $k\langle x, y | x^2 = -2xy, xy = yx + y^2, x^3 = 0\rangle$ and is tame.*

P r o o f. We intend to show that no one of the algebras from the Ringel's list of minimal wild algebras can be obtained as a quotient of A_4. After that using the Ringel's theorem, we will be able to conclude that it is tame. Suppose that there exists an ideal I of A_4 such that A_4/I is isomorphic to W_j for some $j = 1, 2, 3, 4$, where

$$W_1 = k\langle u, v | u^2, uv - \mu vu \ (\mu \neq 0), v^2u, v^3\rangle,$$
$$W_2 = k\langle u, v | u^2, uv, v^2u, v^3\rangle,$$
$$W_3 = k\langle u, v | u^2, vu, uv^2, v^3\rangle,$$
$$W_4 = k\langle u, v | u^2 - v^2, vu\rangle.$$

Since all W_j are 5-dimensional and A_4 is 6-dimensional, the ideal I should be one-dimensional. Due to our knowledge on the matrix structure of the algebra A_4, we can see that there is only one one-dimensional ideal I_4 in A_4 and that I_4 consists of the matrices with at most one non-zero entry being in the upper right corner of the matrix:

$$I_4 = \left\{ \begin{pmatrix} 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

After factorization by this ideal we get a 5-dimensional algebra given by relations

$$\overline{A}_4 = A_4/I = k\langle x, y | x^2 = -2xy, xy = yx + y^2, x^3 = 0, y^3 = 0\rangle.$$

The question now is whether this algebra is isomorphic to one of the algebras from the above list. Suppose that there exists an isomorphism $\varphi_j : W_j \to \overline{A}_4$ for some $j \in \{1, 2, 3, 4\}$. Denote $\varphi_j(u) = f_j$ and $\varphi_j(v) = g_j$.

First, let us mention that f_j and g_j have zero free terms: $f_j^{(0)} = g_j^{(0)} = 0$ because the equalities $\varphi_j(u^2) = f_j^2 = 0$ and $\varphi_j(v^3) = g_j^3$ imply $(f_j^{(0)})^2 = (g_j^{(0)})^3 = 0$ and therefore $f_j^{(0)} = g_j^{(0)} = 0$ if $j = 1, 2, 3$ and the equalities $\varphi_4(u^2 - v^2) = f_4^2 - g_4^2 = 0$ and $\varphi_4(uv) = f_4 g_4 = 0$ imply $(f_4^{(0)})^2 = (g_4^{(0)})^2$ and $f_4^{(0)} g_4^{(0)} = 0$ and therefore $f_4^{(0)} = g_4^{(0)} = 0$.

The second observation is that the terms of degree 3 and more are zero in A_4. Therefore we can present the polynomials f_j and g_j as the sum of their linear and quadratic (on x and y) parts. So, let $f_j = f_j^{(1)} + f_j^{(2)}$ and $g_j = g_j^{(1)} + g_j^{(2)}$, where

$$f_j^{(1)} = ax + by, \quad g_j^{(1)} = \alpha x + \beta y, \quad f_j^{(2)} = cyx + dy^2, \quad g_j^{(2)} = \gamma yx + \delta y^2.$$

In order to get entire linear part of the algebra A_4 in the range of φ_j we need to have

$$\det \begin{vmatrix} a & b \\ \alpha & \beta \end{vmatrix} \neq 0. \tag{1}$$

For any $j = 1, 2, 3, 4$ we are going to obtain a contradiction of the last condition with the equations on a, b, α, β coming from the relations of the algebra W_j.

For instance, consider the case $j = 2$. From $0 = u^2 = f_j^2 = (f_j^{(1)})^2 = 2(ab - a^2)yx + (b^2 + ab - 2a^2)y^2$ we get $2a(b - a) = 0$ and $b^2 + ab - 2a^2 = 0$. The first equation gives us that either $a = 0$ or $a = b$. In the case $a = 0$ the second equation implies $b = 0$ and the equality $a = b = 0$ already contradicts (1). Another solution is $a = b \neq 0$. From $0 = uv = f_j g_j = f^{(1)} g^{(1)} = (ax + by)(\alpha x + \beta y)$, substituting $a = b$, we get $0 = a(x + y)(\alpha x + \beta y) = a(\beta - \alpha)(yx + 2y^2)$. Hence $\beta = \alpha$, which together with the equality $a = b$ contradicts (1).

In the other three cases one can get a contradiction with (1) along the same lines, which completes the proof. □

Combining theorems 12.1 and 12.2 we get

Theorem 12.3 *The image algebra A_n of a representation $\rho_n \in \mathcal{U}_{(n)}$ is wild for any $n \geqslant 5$ and tame for $n \leqslant 4$.*

13 Irreducible components of the representation space

We are still considering the variety $mod(R, n)$ of R-module structures on k^n, with $k = \mathbb{C}$, as it was set in previous section.

In this section we show (in the basis of obtained above facts) that the Jordan plane can serve as an example of an algebra for which all irreducible components of the representation variety $mod(R, n)$ could be described for any n. The importance of such examples was emphasized in [7] and it is mentioned there that known cases are restricted by algebras of finite representation type (i.e., there are only finitely many isomorphism classes of indecomposable R modules) and one example of infinite representation type in [30]. There is some similarity between the algebra generated by the pair of nilpotent matrices annihilating each other considered in [30] and the Jordan plane, but while in the case of [30] variables x and y act 'independently', there is much more interaction in the case of the Jordan plane.

In previous section we suggested stratification of the representation space $mod(R, n)$ related to the Jordan normal form of Y and denote by $\mathcal{U}_{\mathcal{P}}$ the strata corresponding to the partition $\mathcal{P}$ of n. These strata are going to play a key role in the description of irreducible components.

We are going to show now that each irreducible component of the variety $mod(R,n)$ contains only one stratum and is the closure of this stratum. The number of irreducible components is therefore equal to the number of partitions of n.

Theorem 13.1 *Any irreducible component K_j of the representation variety $mod(R,n)$ of Jordan plane contains only one stratum $U_{\mathcal{P}}$ from the stratification related to the Jordan normal form of Y, and is the closure of this stratum.*

The number of irreducible components of the variety $mod(R,n)$ is equal to the number of the partitions of n.

Proof. One can see that $\mathcal{U}_{\mathcal{P}}$ for any $\mathcal{P}$ is a connected analytic submanifold inside the variety $mod(R,n)$. Now we will use the fact that if a connected analytic manifold $U \subset V$ is contained in the union of varieties V_i ($V = \cup V_i$), then it should be contained in one of them: $U \subset V_j$ (see for example [26], ch 1). Applying this to the decomposition of $V = mod(R,n)$ into an irreducible components we get that each irreducible component contains the whole stratum whenever the stratum touches the irreducible component.

Now we shall show that the stratum $\mathcal{U}_{\mathcal{P}}$ can not be contained in an intersection of two different irreducible components K_i and K_j. In order to do this, let us calculate the dimension of $\mathcal{U}_{\mathcal{P}}$ (as a manifold). It turns out that it does not depend on $\mathcal{P}$ and always equals to the dimensions of K_i and K_j.

Lemma 13.2 *For any partition $\mathcal{P}$ of n, the stratum $\mathcal{U}_{\mathcal{P}}$ has dimension n^2.*

Proof. Note that the manifold $\mathcal{U}_{\mathcal{P}}$ carries the natural vector bundle structure with the base $\mathcal{B}_{\mathcal{P}}$ being the space of matrices Y with the Jordan normal form $J_{\mathcal{P}}$ and the fiber $\mathcal{F}_Y = \{X : XY - YX = Y^2\}$. Dimension of the base is equal to $n^2 - \dim C(J_{\mathcal{P}})$, where $C(J_{\mathcal{P}})$ is a centralizer of $J_{\mathcal{P}}$. The fiber is a shift of the space of matrices commuting with $J_{\mathcal{P}}$, so its dimension is equal to the dimension of $C(J_{\mathcal{P}})$.

Thus we obtain

$$\dim \mathcal{U}_{\mathcal{P}} = \dim C(J_{\mathcal{P}}) + (n^2 - \dim C(J_{\mathcal{P}})) = n^2$$

for any partition $\mathcal{P}$. □

Suppose now that $\mathcal{U}_{\mathcal{P}} \subset K_i \cap K_j$. Then

$$n^2 = \dim \mathcal{U}_{\mathcal{P}} \leqslant \dim K_i \cap K_j \leqslant \dim K_j \leqslant n^2 = \dim U$$

for $j \in \{1,2\}$. Hence $\dim K_i \cap K_j = \dim K_i = \dim K_j$, which contradicts irreducibility of K_j.

Hence we have a picture, where all our strata sitting inside irreducible components and are separated by them.

This ensures at the same time, that the number of irreducible components does not exceed the number of the partitions.

Due to the above, for any partition $n = n_1 + \ldots + n_m$, we can take the irreducible component C_j of the variety $mod(R, n_j)$, containing the strata $\mathcal{U}_{(n_j)}$. Now from those C_j for $1 \leqslant j \leqslant m$ using Theorem 1.2 in [7] we are going to construct an irreducible component $K = \overline{C_1 \oplus \ldots \oplus C_m}$ of the variety $mod(R, n)$. Here $\overline{C_1 \oplus \ldots \oplus C_m}$ has the following meaning. All the modules $M_1 \oplus \ldots \oplus M_m$, $M_i \in C_i$ together with all the elements of their $GL_n(\mathbb{C})$-orbits gives us $C_1 \oplus \ldots \oplus C_m$ and the bar stands for the Zariski closure. This gives us an irreducible component provided $\operatorname{ext}(C_i, C_j) = 0$ for $i \neq j$, where $\operatorname{ext}(C_i, C_j) = \min\{\dim \operatorname{Ext}^1(M, N) : M \in C_i,\ N \in C_j\}$. We can verify the equality $\operatorname{ext}(C_i, C_j) = 0$ in our case by taking $M = (X_1, Y_1) \in C_i$ and $N = (X_2, Y_2) \in C_j$ in such a way that matrices X_1 and X_2 have different eigenvalues. Then by Lemma 8.1 an extension E of M by N should be decomposable, since it is proved there that indecomposable modules always have only one eigenvalue for X. Surely decomposition will be on M and N, since we can find a basis, where X splits into main eigenspaces corresponding to its two different eigenvalues. But as Theorem 8.2 says, these main eigenspaces are also invariant for Y and we get the splitting $E = M \oplus N$. Thus $\operatorname{Ext}^1(M, N) = 0$ and therefore $\operatorname{ext}(C_i, C_j) = 0$ for $i \neq j$ in our situation and we can apply Theorem 1.2 from [7].

But the irreducible component $C_{\mathcal{P}} = \overline{C_1 \oplus \ldots \oplus C_m}$, ($C_j$ is an irreducible component of the variety $mod(R, n_j)$, containing the strata $\mathcal{U}_{(n_j)}$), we have just got, uniquely determines its summands $C_1, \ldots, C_m$ according to Theorem 1.1 from [7]. Thus the partition $n = n_1 + \ldots + n_m$ corresponding to their dimensions is also uniquely determined by $C_{\mathcal{P}}$. Hence this component contains the strata $\mathcal{U}_{\mathcal{P}}$ with $\mathcal{P} = (n_1, \ldots, n_m)$ of $mod(R, n)$ and there are at least as many different irreducible components of $mod(R, n)$ as the number of partitions of n.

Taking into account also the above fact that strata are separated by irreducible components, we see that any irreducible component has inside exactly one strata and is its Zariski closure. This completes the description of irreducible components of $mod(R, n)$. □

After we have this theorem, we can denote irreducible components by $K_{\mathcal{P}}$, for any partition $\mathcal{P}$ of n, where $K_{\mathcal{P}}$ is the closure of the corresponding strata $\mathcal{U}_{\mathcal{P}}$.

We also can answer the question in which irreducible components module in general position is indecomposable.

Corollary 13.3 *Only the irreducible component $K_{(n)} = \overline{\mathcal{U}_{(n)}}$ which is the closure of the stratum corresponding to the trivial partition of n (the full block Y) contains an open dense subset consisting of indecomposable modules.*

P r o o f. Suppose our irreducible component K is the closure of a stratum related to a non-trivial partition $n = n_1 + \ldots + n_m$, $m > 1$. Then there are modules in this component $M = (X, Y)$, where X has different eigenvalues.

Hence the Zariski closed set of modules $M = (X, Y)$ in our component for which X has only one eigenvalue, has non-empty complement. The latter complement Ω is a non-empty Zariski open subset of the irreducible variety K and therefore Ω is dense in K. Since by Lemma 8.1 the modules that correspond to X having different eigenvalues are all decomposable, we see that $M \in K$ in general position is decomposable. □

14 NCCI and RCI

In this section we are going to discuss properties of the noncommutative Koszul (Golod-Shafarevich) complex and their implications for the structure of the representation space. Namely, we consider situation when the Golod-Shafarevich complex provides a DG resolution of the algebra. We show that the Jordan plane is an example illustrating the fact that RCI (representational complete intersections), introduced by Ginsburg and Etingof in [10] always have this property of the Golod-Shafarevich complex.

Let us present first a construction, appeared in [14], of the complex associated to an algebra, presented by generators and relations. This complex could be considered as an analogue of the Koszul complex in commutative algebra. Let $A = k\langle x_1, ..., x_d\rangle/\{f_1, ..., f_m\}$. Denote the free algebra $k\langle x_1, ..., x_d\rangle$ by R_d, and an ideal in R_d generated by polynomials $f_1, ..., f_m$ by I. We suppose that $f_1, ..., f_m$ are homogeneous, hence A inherits the grading by the degree on variables $x_1, ..., x_d$ form the free algebra R_d.

Let us denote now by $M = k\langle x_1, ..., x_d, u_1, ..., u_m\rangle$, where variables u_i are in one to one correspondence with polynomials $f_1, ..., f_m$. Linear basis of M over k consists obviously of monomials

$$a_0 u_{i_1} a_1 u_{i_2} ... a_{k-1} u_{i_k} a_k,$$

where a_i are monomials from R_d. *Homological degree* of such a monomial is its degree on variables u_i. Obviously, if $M^{(k)}$ is a linear combination of monomials of homological degree k, M becoming a graded algebra: $M = \sum\limits_{k=0}^{\infty} M^{(k)}$.

We can define the following differential on it. Let $\delta(x_i) = 0, \delta(u_i) = f_i$, and it is extended as a graded derivation, i.e. by the rule:

$$\delta(vw) = \delta(v)w + (-1)^{\deg v} v\delta w.$$

Defined in such a way differential have the following formula:

$$\delta(a_0 u_{i_1} a_1 u_{i_2} ... a_{k-1} u_{i_k} a_k) = \sum_{l=1}^{k} (-1)^l a_0 u_{i_1} ... a_{l-1} f_{i_l} a_l ... u_{i_k} a_k.$$

It is easy to see that $\delta^2 = 0$, and δ drops the homological degree by one, so we have a complex of R_d-modules

$$0 \overset{\delta}{\leftarrow} M^{(0)} \overset{\delta}{\leftarrow} M^{(1)} \overset{\delta}{\leftarrow} M^{(2)} \overset{\delta}{\leftarrow} ...$$

It is clear that $H_0(M) = A$, so we can rewrite it as an A - resolution, which is in general not acyclic:

$$0 \leftarrow A \leftarrow M^{(0)} \overset{\delta}{\leftarrow} M^{(1)} \overset{\delta}{\leftarrow} M^{(2)} \overset{\delta}{\leftarrow} ...$$

it is exact in terms A and $M^{(0)}$, but not in the other terms.

This complex we denote by $Sh_\bullet(A)$.

Definition 14.1 *In case the complex $Sh_\bullet(A)$ is a DG-algebra resolution of A, that is all $H^{(i)}(M) = 0$, for $i \geqslant 1$, we say that A is a noncommutative complete intersection (NCCI).*

Now we give two other equivalent definitions of NCCI [14].

Theorem 14.2 *The following are equivalent:*

(i) *The noncommutative Koszul complex $Sh_\bullet A$ is a DG algebra resolution of A, that is*

$H_0(Sh_\bullet A, \delta) = A$, $H_n(Sh_\bullet A, \delta) = 0$, $n > 0$.

(ii) *The Hilbert series of A coincides with the Golod-Shafarevich series:*

$H_A = (1 - nt + dt^2)^{-1}$.

(iii) *For any right A-module M,*

$\mathrm{Ext}^{\mathrm{i}}_{\mathrm{A}}(\mathrm{A}/\mathrm{A}_+, \mathrm{M}) = 0$, *for* $i \geqslant 3, (\mathrm{gl.dim} A \leqslant 2)$.

In particular, from the condition (iii) one can see, that NCCI implies Koszulity: put $M = k_A$.

This provides us with another way to ensure Koszulity of the Jordan plane R. As we noticed in Lemma **??**, the defining relations of R form a Gröbner basis. Using this it is easy to calculate the Hilbert series of R: $H_R = (1 - t)^{-2}$, so by (ii) it is a NCCI, hence Koszul.

Now we turn to the properties of representation spaces, which should be a reflection of NCCI. The notion of *representational complete intersection* (RCI) was introduced by Ginzburg and Etingof [10]

Definition 14.3 *An algebra is an RCI if for infinitely many n the spaces of n-dimensional representations of that algebra are a (commutative) complete intersections.*

The main source of examples of RCI is provided by the preprojective algebras of finite quivers. It was also shown in the above paper that RCI implies NCCI. Answering the question of Etingof [11] we provide here one more example of RCI (which is not a preprojective algebra).

Theorem 14.4 *The Jordan plane $R = k\langle x, y | xy - yx = y^2 \rangle$ is a RCI.*

P r o o f. We have proved in Lemma13.2 and Theorem13.1 that the representation variety $mod(R, n)$ of the Jordan plane R is an equidimensional variety with each irreducible component having the dimension n^2. This means that the dimension of the variety is equal to the dimension of the whole space of pairs of matrices, $2n^2$, minus the dimension of the space of defining relations, which is n^2, so all those varieties are complete intersections in the classical (commutative) sense.

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References

[1] I. Antoniou, N. Iyudu, R.Wisbauer *On Serre's problem for RIT Algebra,* Communications in Algebra, **31** (12)(2003), 6037–6050.

[2] I. Antoniou, N. Iyudu, *Poincare–Hilbert series, PI and Noetherianity of the enveloping of the RIT Lie Algebra,* Communications in Algebra, 2001, vol. 29, no. 9, pp. 4183–4196.

[3] I. Antoniou, *Internal time and irreversibility of relativistic dynamical systems,* Ph.D. thesis, ULB, Brussels, 1988.

[4] M. Artin, W.F. Schelter, *Graded algebras of global dimension 3,* Adv. in Math. 66(1987), 171–216.

[5] G.Bergman, it The diamond lemma for ring theory,

Adv. in Math. 29 (1978), no. 2, 178218. 16-02

[6] P. Cameron, N. Iyudu, *Graphs of relations and Hilbert series*, Journal of Symbolic Computation, V42, no.11-12, 1066-1078, 2007.

[7] B. Crawley-Boevey, J. Schröer, *Irreducible components of varieties of modules.*, J. Reine Angew. Math., 553 (2002), 201–220.

[8] B. A. Dubrovin, A. T. Fomenko, S. P. Novikov, *Modern geometry—methods and applications. Part II. The geometry and topology of manifolds.* Graduate Texts in Mathematics, 104, Springer-Verlag, New York, 1985, 430 pp.

[9] M.Ershov, *Golod-Shafarevich groups: a survey*, arXiv:1206.0490v1, 2012, 51pp.

[10] P.Etingof, V.Ginzburg, *Noncommutative complete intersections and matrix integrals* arXiv:math/0603272v3, 2007

[11] P.Etingof *Privat communication during the Newton Institute activity on the Algebraic Lie Theory*, 2009

[12] I. M. Gelfand, V. A. Ponomarev, *Indecomposable representations of the Lorenz Group,* Russian Math. Surveys 23 (1968), 1-58.

[13] M. Gerstenhaber, *On dominance and varieties of commuting matrices*, Ann. of Math.(2) 73 (1961) 324–348.

[14] Golod E., Shafarevich I.: On the class field tower [Russian]. Izv. Akad. Nauk SSSR Ser. Mat. **28**, 261–272 (1964)

[15] Ed. Green, *An introduction to noncommutative Gröbner bases,* in: Lecture Notes in Pure and Applied Mathematics, V. 151, 1994, 167-190.

[16] R. Guralnick, *A note on commuting pairs of matrices*, Linear and Multilinear Algebra 31 (1992), no. 1-4, 71–75.

[17] M. Hazewinkel, N. Gubareni, V. V. Kirichenko, *Algebras, rings and modules. Vol. 1. Mathematics and its Applications*, 575. Kluwer Academic Publishers, Dordrecht, 2004, 380pp.

[18] N. Iyudu, *Classification of finite dimensional representations of one noncommutative quadratic algebra*, Max-Planck-Institut für Mathematik Preprint Series, 20 (2005), p.1-20.

[19] N. Iyudu, R. Wisbauer, *Non-trivial stably free modules over crossed products* , J.Phys.A 42(2009) 335209, 11p.

[20] M. Kontsevich, *Noncommutative identities*, Mathematische Arbeitstagung 2011, Max-Planck-Institut für Mathematik, p.1-9

[21] M. Kontsevich, *Automorphisms of Weyl algebra*, Max-Planck-Institut für Mathematik Preprint Series, 60a (2005), Arbeitstagung 2005, p.1-8.

[22] S. Korenskii, *Representations of the Quantum group* $SL_J(2)$, Rus.Math.Surv. 1991, V.46, N6, p.211-212.

[23] L. Makar-Limanov, *On automorphisms of Weyl algebra*, Bull.Sac.Math.France, 112(1984), N3, 359-363.

[24] Yu. I. Manin, *Quantum groups and noncommutative geometry,* CRM.

[25] T. Motzkin, O. Taussky-Todd, *Pairs of matrices with property L. II*, Trans. AMS 80(1955), 387-401.

[26] D. Mumford, *Red book of varieties and schemes*, LNM 1358, Springer, 1999

[27] A. Odesskii, *Sklyanin's elliptic algebras*, Func.Anal.Appl., 23(1989), N3, p. 207-214.

[28] S.Priddy, *Koszul resolutions*, Trans. Amer. Math. Soc. 152 (1970), 3960.

[29] C. M. Ringel, *The representation type of local algebras,* LNM 488, Springer.

[30] J. Schröer, *Varieties of pairs of nilpotent matrices annihilating each other,* Comment. Math. Helv., 79 (2004), 396-426.

[31] A.Smoktunowicz, *A note on Golod-Shafarevich algebras,* arxiv1207.6503, 2012

[32] A.Smoktunowicz, *Growth of modules over generic Golod-Shafarevich algebras,* Asian-European J. Math., 02, 279 (2009)

[33] L.Bartholdi, A.Smoktunowicz, *Images of Golod-Shafarevich algebras with small growth,* arxiv1108.4264, 2011

[34] J.T.Stafford, M. van den Bergh, *Noncommutative curves and noncommutative surfaces*, Bull. Amer. Math. Soc. (N.S.) 38 (2001), no. 2, 171–216

[35] T.Voden, *Subalgebras of Golod-Shafarevich algebras,* Int. J. Algebra Comput., 19, 423 (2009).

[36] A. Wadsworth, *The algebra generated by two commuting matrices,* Linear and Multilinear Algebra 27 (1990), no. 3, 159–162.

[37] Zelmanov E. *Some open problems in the theory of infinite dimensional algebras.* J. Korean Math. Soc. **44**, 1185-1195 (2007)