TOWARDS THE SECOND MAIN THEOREM ON COMPLEMENTS

YU. G. PROKHOROV AND V. V. SHOKUROV

ABSTRACT. We prove the boundedness of complements modulo two conjectures: Borisov-Alexeev conjecture and effective adjunction for fibre spaces. We discuss the last conjecture and prove it in two particular cases.

CONTENTS

1. INTRODUCTION

This paper completes our previous work [PS01] modulo two conjectures 1.1 and 7.13. The first one relates to Alexeev's, A. and L. Borisov's conjecture:

Conjecture 1.1. *Fix a real number $\varepsilon > 0$. Let $(X, B = \sum b_i B_i)$ be a d-dimensional log canonical pair with nef $-(K_X + B)$, that is, (X, B) is a log semi-Fano variety* (*cf. Definition* 2.5). *Assume also that*

(i) *$K + B$ is ε-lt; and*

(ii) *X is FT that is (X, Θ) is a klt log Fano variety with respect to some boundary Θ.*

The first author was partially supported by grants CRDF-RUM, No. 1-2692-MO-05 and RFBR, No. 05-01-00353-a, 06-01-72017. The second author was partially supported by NSF grant DMS-0400832.

Then X is bounded in the moduli sense, i.e., it belongs to an algebraic family $\mathcal{X}(\varepsilon, d)$.

This conjecture was proved in dimension 2 by V. Alexeev [Ale94] and in toric case by A. Borisov and L. Borisov [BB92] (see also [Nik90], [Bor96], [Bor01], [McK02]).

Remark 1.2. We hope that Conjecture 1.1 can be generalized by weakening condition (ii). For example, we hope that instead of (ii) one can assume that

(ii)′ X is rationally connected, cf. [Zha06].

Recall that a log pair (X, B) is said to be *ε-log terminal* (or simply *ε-lt*) if $\operatorname{totaldiscr}(X, B) > -1 + \varepsilon$, see Definition 2.2 below.

Theorem 1.3 ([Ale94])**.** *Conjecture* 1.1 *holds in dimension two.*

The second conjecture concerns with Adjunction Formula and will be discussed in Section 7.

Our main result is the following.

Theorem 1.4. *Fix a finite subset $\mathfrak{R} \subset [0, 1] \cap \mathbb{Q}$. Let (X, B) be a klt log semi-Fano variety of dimension d such that X is FT and the multiplicities of B are contained in $\Phi(\mathfrak{R})$ (see* 3.2*). Assume the LMMP in dimension $\le d$. Further, assume that Conjectures* 1.1 *and* 7.13 *hold in dimension $\le d$. Then $K + B$ has bounded complements. More precisely, there is a positive integer $n = n(d, \mathfrak{R})$ divisible by denominators of all $r \in \mathfrak{R}$ and such that $K + B$ is n-complemented. Moreover, $K + B$ is nI-complemented for any positive integer I.*

In particular,

$$|-nK - nS - \lfloor (n+1)D \rfloor| \neq \emptyset,$$

where $S := \lfloor B \rfloor$ and $D := B - S$. For $B = 0$, $| - nK| \neq \emptyset$, where n depends only on d.

Note that the last paragraph is an immediate consequence of the first statemet and the definition of complements.

In the case when $K + B$ is numerically trivial our result is stronger. For the definition of 0-pairs we refer to 2.5.

Theorem 1.5 (cf. [Bla95], [Ish00])**.** *Fix a finite subset $\mathfrak{R} \subset [0, 1] \cap \mathbb{Q}$. Let (X, B) be a 0-pair of dimension d such that X is FT and the multiplicities of B are contained in $\Phi(\mathfrak{R})$. Assume the LMMP in dimension $\le d$. Further, assume that Conjectures* 1.1 *and* 7.13 *hold in dimension $\le d$. Then there is a positive integer $n = n(d, \mathfrak{R})$ such that $n(K_X + B) \sim 0$.*

Addendum 1.6. Our proofs show that we do not need Conjecture 1.1 in dimension d in all generality. We need it only for some special value $\varepsilon^o := \varepsilon^o(d, \mathfrak{R}) > 0$ (cf. Corollary 1.7).

We will prove Conjecture 7.13 in §8 in dimension ≤ 3 under additional assumption that the total space is projective and FT.

Corollary 1.7. *Fix a finite rational subset $\mathfrak{R} \subset [0, 1] \cap \mathbb{Q}$ and let I be the least common multiple of denominators of all $r \in \mathfrak{R}$. Let (X, B) be a klt log semi-Fano threefold such that X is FT and the multiplicities of B are contained in $\Phi(\mathfrak{R})$. Assume that Conjecture 1.1 holds in dimension 3 for ε^o as in Addendum 1.6. Then $K + B$ has a bounded n-complement such that $I \mid n$. In particular, there exists a positive integer number n such that $I \mid n$ and*

$$|-nK - nS - \lfloor (n+1)D \rfloor| \neq \emptyset,$$

where $S := \lfloor B \rfloor$ and $D := B - S$; n depends only on $\mathfrak{R}$ and ε^o for Addendum 1.6. For $B = 0$, $|-nK| \neq \emptyset$, where n depends only on ε^o.

Proof. Immediate by Addendum 1.6, Theorem 1.4, and Corollary 8.15. □

Corollary 1.8 (cf. [Sho00])**.** *Fix a finite subset $\mathfrak{R} \subset [0, 1] \cap \mathbb{Q}$ and let I be the least common multiple of denominators of all $r \in \mathfrak{R}$. Let (X, B) be a klt log semi-del Pezzo surface such that X is FT and the multiplicities of B are contained in $\Phi(\mathfrak{R})$. Then $K + B$ has a bounded n-complement such that $I \mid n$. In particular, there exists a positive integer n such that $I \mid n$ and*

$$|-nK - nS - \lfloor (n+1)D \rfloor| \neq \emptyset,$$

where $S := \lfloor B \rfloor$ and $D := B - S$; n depends only on $\mathfrak{R}$. For $B = 0$, $|-nK| \neq \emptyset$, where n is an absolute constant.

Proof. Immediate by Theorems 1.4, 1.3, and 8.1. □

The following corollaries are consequences of our techniques. The proofs will be given in 9.8, 9.9, and 9.10.

Corollary 1.9 (cf. [Ish00])**.** *Fix a finite rational subset $\mathfrak{R} \subset [0, 1] \cap \mathbb{Q}$. Let (X, D) be a three-dimensional 0-pair such that X is FT and the multiplicities of B are contained in $\Phi(\mathfrak{R})$. Assume that (X, D) is not klt. Then there exists a positive integer n such that $n(K + D) \sim 0$; this n depends only on $\mathfrak{R}$.*

Corollary 1.10. *Fix a finite rational subset $\mathfrak{R} \subset [0, 1] \cap \mathbb{Q}$ and let I be the least common multiple of denominators of all $r \in \mathfrak{R}$. Let (X, B) be a klt log semi-Fano threefold such that X is FT and the multiplicities* *of B are contained in $\Phi(\mathfrak{R})$. Then there exists a real number $\bar{\varepsilon}$ such that $K+B$ has a bounded n-complement with $I \mid n$ if there are two divisors E (exceptional or not) with discrepancy $a(E, X, B) \leq -1+\bar{\varepsilon}$; this $\bar{\varepsilon}$ depends only on $\mathfrak{R}$.*

Corollary 1.11 (cf. [Bla95], [Sho00])**.** *Fix a finite rational subset $\mathfrak{R} \subset [0,\, 1] \cap \mathbb{Q}$. Let (X, D) be a two-dimensional 0-pair such that the multiplicities of B are contained in $\Phi(\mathfrak{R})$. Then there exists a positive integer n such that $n(K+D) \sim 0$; this n depends only on $\mathfrak{R}$.*

We give a sketch of the proof of our main results in Section 4. One can see that our proof essentially uses reduction to lower-dimensional global pairs. However it is expected that an improvement of our method can use reduction to local questions in the same dimension. In fact we hope that the hypothesis in our main theorem 1.4 should be the existence of local complements and Conjecture 1.1 for ε-lt Fano varieties (without a boundary), where $\varepsilon \geq \varepsilon^o > 0$, ε^o is a constant depending only on the dimension (cf. Addendum 1.6 and Corollary 1.7). If $\dim X = 2$, we can take $\varepsilon^o = 1/7$. Note also that our main theorem 1.4 is weaker than one can expect. We think that the pair (X, B) can be taken arbitrary log-semi-Fano (possibly not klt and not FT) and possible boundary multiplicities can be taken arbitrary real numbers in $[0, 1]$ (not only in $\Phi(\mathfrak{R})$). The only hypothesis we have to assume is the existence of an $\mathbb{R}$-complement $B^+ \geq B$ (cf. [Sho00]). However the general case needs actually a *finite* set of natural numbers for complements, and there are no such universal number for all complements (cf. [Sho93, Example 5.2.1]).

Acknowledgements. The work was conceived in 2000 when the first author visited the Johns Hopkins University and finished during his stay in Max-Planck-Institut für Mathematik, Bonn in 2006. He would like to thank these institutes for hospitality. Finally both authors are grateful to the referee whose constructive criticism helped us to revise the paper very much.

2. Preliminaries

2.1. **Notation.** All varieties are assumed to be algebraic and defined over an algebraically closed field $\Bbbk$ of characteristic zero. Actually, main results holds for any $\Bbbk$ of characteristic zero not necessarily algebraically closed since they are related to singularities of general members of linear systems (see [Sho93, 5.1]). We use standard terminology and notation of the Log Minimal Model Program (LMMP) [KMM87], [Kol92], [Sho93]. For the definition of complements and their properties we refer to [Sho93], [Sho00], [Pro01] and [PS01]. Recall that a *log pair* (or a *log variety*) is a pair (X, D) consisting of a normal variety X and a *boundary* D, i.e., an $\mathbb{R}$-divisor $D = \sum d_iD_i$ with multiplicities $0 \le d_i \le 1$. As usual K_X denotes the canonical (Weil) divisor of a variety X. Sometimes we will write K instead of K_X if no confusion is likely. Everywhere below $a(E, X, D)$ denotes the discrepancy of E with respect to $K_X + D$. Recall the standard notation:

$$\begin{aligned} \mathrm{discr}(X, D) &= \inf_E\{a(E, X, D) \mid \operatorname{codim} \mathrm{Center}_X(E) \ge 2\}, \\ \mathrm{totaldiscr}(X, D) &= \inf_E\{a(E, X, D) \mid \operatorname{codim} \mathrm{Center}_X(E) \ge 1\}. \end{aligned}$$

In the paper we use the following strong version of ε-log terminal and ε-log canonical property.

Definition 2.2. A log pair (X, B) is said to be *ε-log terminal* (*ε-log canonical*) if $\mathrm{totaldiscr}(X, B) > -1 + \varepsilon$ (resp., $\mathrm{totaldiscr}(X, B) \ge -1 + \varepsilon$).

2.3. Usually we work with $\mathbb{R}$-divisors. An $\mathbb{R}$-divisor is an $\mathbb{R}$-linear combination of prime Weil divisors. An $\mathbb{R}$-linear combination $D = \sum \alpha_iL_i$, where the L_i are integral Cartier divisors is called an $\mathbb{R}$-Cartier divisor. The pull-back f^* of an $\mathbb{R}$-Cartier divisor $D = \sum \alpha_iL_i$ under a morphism $f\colon Y \to X$ is defined as $f^*D := \sum \alpha_i f^*L_i$. Two $\mathbb{R}$-divisors D and D' are said to be $\mathbb{Q}$- (*resp.,* $\mathbb{R}$-)*linearly equivalent* if $D - D'$ is a $\mathbb{Q}$- (resp., $\mathbb{R}$-)linear combination of principal divisors. For a positive integer I, two $\mathbb{R}$-divisors D and D' are said to be *I-linearly equivalent* if $I(D - D')$ is an (integral) principal divisor. The $\mathbb{Q}$-linear (resp., $\mathbb{R}$-linear, I-linear) equivalence is denoted by $\sim_{\mathbb{Q}}$ (resp., $\sim_{\mathbb{R}}$, $\sim_I$). Let $\Phi \subset \mathbb{R}$ and let $D = \sum d_iD_i$ be an $\mathbb{R}$-divisor. We say that $D \in \Phi$ if $d_i \in \Phi$ for all i.

2.4. Let $f\colon X \to Z$ be a morphism of normal varieties. For any $\mathbb{R}$-divisor Δ on Z, define its *divisorial pull-back* $f^\bullet\Delta$ as the closure of the usual pull-back $f^*\Delta$ over $Z \setminus V$, where V is a closed subset of codimension ≥ 2 such that $V \supset \operatorname{Sing} Z$ and f is equidimensional over $Z \setminus V$. Thus each component of $f^\bullet\Delta$ dominates a component of Δ. It is easy to see that the divisorial pull-back $f^\bullet\Delta$ does not depend on the choice of V. Note however that in general $f^\bullet$ does not coincide with the usual pull-back f^* of $\mathbb{R}$-Cartier divisors.

Definition 2.5. Let (X, B) be a log pair of global type (the latter means that X is projective). Then it is said to be

log Fano variety if $K + B$ is lc and $-(K + B)$ is ample;

weak log Fano (WLF) variety if $K+B$ is lc and $-(K+B)$ is nef and big;
log semi-Fano (ls-Fano) variety if $K+B$ is lc and $-(K+B)$ is nef;
0-*log pair* if $K+B$ is lc and numerically trivial*.

In dimension two we usually use the word *del Pezzo* instead of *Fano*.

Lemma-Definition 2.6. *Let X be a normal projective variety. We say that X is* FT (Fano type) *if it satisfies the following equivalent conditions:*

(i) *there is a $\mathbb{Q}$-boundary Ξ such that (X,Ξ) is a klt log Fano;*
(ii) *there is a $\mathbb{Q}$-boundary Ξ such that (X,Ξ) is a klt weak log Fano;*
(iii) *there is a $\mathbb{Q}$-boundary Θ such that (X,Θ) is a klt 0-pair and the components of Θ generate $N^1(X)$;*
(iv) *for any divisor Υ there is a $\mathbb{Q}$-boundary Θ such that (X,Θ) is a klt 0-pair and* $\operatorname{Supp}\Upsilon\subset\operatorname{Supp}\Theta$.

Similarly one can define relative FT and 0-varieties X/Z, and the results below hold for them too.

Proof. Implications (i) $\Longrightarrow$ (iv), (iv) $\Longrightarrow$ (iii), (i) $\Longrightarrow$ (ii) are obvious and (ii) $\Longrightarrow$ (i) follows by Kodaira's lemma (see, e.g., [KMM87, Lemma 0-3-3]). We prove (iii) $\Longrightarrow$ (i). Let (X,Θ) be such as in (iii). Take an ample divisor H such that $\operatorname{Supp}H\subset\operatorname{Supp}\Theta$ and put $\Xi=\Theta-\varepsilon H$, for $0<\varepsilon\ll 1$. Clearly, (X,Ξ) is a klt log Fano. □

Recall that for any (not necessarily effective) $\mathbb{R}$-divisor D on a variety X a D-MMP is a sequence $X=X_1 \dashrightarrow X_N$ of extremal D-negative divisorial contractions and D-flips which terminates on a variety X_N where either the proper transform of D is nef or there exists a D-negative contraction to a lower-dimensional variety (see [Kol92, 2.26]).

Corollary 2.7. *Let X be an FT variety. Assume the LMMP in dimension* $\dim X$. *Then the D-MMP works on X with respect to any $\mathbb{R}$-divisor D.*

Proof. Immediate by Lemma 2.6, (iv). Indeed, in the above notation we may assume that $\operatorname{Supp}D\subset\operatorname{Supp}\Theta$. It remains to note that the D-LMMP is is nothing but the LMMP with respect to $(X,\Theta+\varepsilon D)$ some $0<\varepsilon\ll 1$. □

Lemma 2.8. (i) *Let $f\colon X\to Z$ be a (not necessarily birational) contraction of normal varieties. If X is FT, then so is Z.*

*Such a log pair can be called also a *log Calabi-Yau variety*. However the last notion usually assumes some additional conditions such as $\pi_1(X)=0$ or $q(X)=0$.

(ii) *The FT property is preserved under birational divisorial contractions and flips.*
(iii) *Let (X, D) be an ls-Fano variety such that X is FT. Let $f\colon Y \to X$ be a birational extraction such that $a(E, X, D) < 0$ for every f-exceptional divisor E over X. Then Y is also FT.*

We need the following result of Ambro [Amb05, Th. 0.2] which is a variant of Log Canonical Adjunction (cf. 7.13, [Fuj99]).

Theorem 2.9 ([Amb05, Th. 0.2])**.** *Let (X, D) be a projective klt log pair, let $f\colon X \to Z$ be a contraction, and let L be a $\mathbb{Q}$-Cartier divisor on Z such that*

$$K + D \sim_{\mathbb{Q}} f^*L.$$

Then there exists a $\mathbb{Q}$-Weil divisor D_Z such that (Z, D_Z) is a log variety with Kawamata log terminal singularities and $L \sim_{\mathbb{Q}} K_Z + D_Z$.

Proof of Lemma 2.8. First note that (ii) and the birational case of (i) easily follows from from 2.6 (iii). To prove (i) in the general case we apply Theorem 2.9. Let $\Theta = \sum_i \theta_i\Theta_i$ be a $\mathbb{Q}$-boundary on X whose components generate $N^1(X)$ and such that (X, Θ) is a klt 0-pair. Let A be an ample divisor on Z. By our assumption $f^*A \equiv \sum_i \delta_i\Theta_i$. Take $0 < \delta \ll 1$ and put $\Theta' := \sum_i(\theta_i - \delta\delta_i)\Theta_i$. Clearly, $K + \Theta' \equiv -\delta f^*A$ and (X, Θ') is a klt log semi-Fano variety. By the base point free theorem $K + \Theta' \sim_{\mathbb{Q}} -\delta f^*A$. Now by Theorem 2.9 there is a $\mathbb{Q}$-boundary Θ_Z such that (Z, Θ_Z) is klt and $K_Z + \Theta_Z \sim_{\mathbb{Q}} -\delta A$. Hence (Z, Θ_Z) is a klt log Fano variety. This proves (i).

Now we prove (iii). Let Ξ be a boundary such that (X, Ξ) is a klt log Fano. Let D_Y and Ξ_Y be proper transforms of D and Ξ, respectively. Then (Y, D_Y) is an ls-Fano, (Y, Ξ_Y) is klt and $-(K_Y + \Xi_Y)$ is nef and big. However Ξ_Y is not necessarily a boundary. To improve the situation we put $\Xi' := (1 - \varepsilon)D_Y + \varepsilon\Xi_Y$ for small positive ε. Then (Y, Ξ') is a klt weak log Fano. □

Definition 2.10. Let X be a normal variety and let D be an $\mathbb{R}$-divisor on X. Then a $\mathbb{Q}$-*complement* of $K_X + D$ is a log divisor $K_X + D'$ such that $D' \geq D$, $K_X + D'$ is lc and $n(K_X + D') \sim 0$ for some positive integer n.

Now let $D = S + B$, where B and S have no common components, S is an effective integral divisor and $\lfloor B \rfloor \leq 0$. Then we say that $K_X + D$ is n-*complemented*, if there is a $\mathbb{Q}$-divisor D^+ such that

(i) $n(K_X + D^+) \sim 0$ (in particular, nD^+ is integral divisor);
(ii) $K_X + D^+$ is lc;
(iii) $nD^+ \geq nS + \lfloor (n + 1)B \rfloor$.

In this situation, $K_X + D^+$ is called an *n-complement* of $K_X + D$.

Note that an n-complement is not necessarily a $\mathbb{Q}$-complement (cf. Lemma 3.3).

Remark 2.11. Under (i) and (ii) of 2.10, the condition (iii) follows from the inequality $D^+ \geq D$. Indeed, write $D = \sum d_i D_i$ and $D^+ = \sum d_i^+ D_i$. We may assume that $d_i^+ < 1$. Then we have

$$nd_i^+ \geq \lfloor nd_i^+ \rfloor = \lfloor (n+1)d_i^+ \rfloor \geq \lfloor (n+1)d_i \rfloor .$$

Corollary 2.12. *Let D^+ be an n-complement of D such that $D^+ \geq D$. Then D^+ is also an nI-complement of D for any positive integer I.*

For basic properties of complements we refer to [Sho93, §5] and [Pro01], see also §3.

2.13. Fix a class of (relative) log pairs $(\mathcal{X}/\mathcal{Z} \ni o, \mathcal{B})$, where o is a point on each $Z \in \mathcal{Z}$. We say that this class has *bounded complements* if there is a constant Const such that for any log pair $(X/Z, B) \in (\mathcal{X}/\mathcal{Z}, \mathcal{B})$ the log divisor $K + B$ is n-complemented near the fibre over o for some $n \leq \mathrm{Const}$.

2.14. Notation. Let X be a normal d-dimensional variety and let $\mathscr{B} = \sum_{i=1}^r B_i$ be any reduced divisor on X. Recall that $Z_{d-1}(X)$ usually denotes the group of Weil divisors on X. Consider the vector space $\mathfrak{D}_{\mathscr{B}}$ of all $\mathbb{R}$-divisors supported in $\mathscr{B}$:

$$\mathfrak{D}_{\mathscr{B}} := \left\{ D \in Z_{d-1}(X) \otimes \mathbb{R} \mid \operatorname{Supp} D \subset \mathscr{B} \right\} = \sum_{i=1}^{r} \mathbb{R} \cdot B_i.$$

As usual, define a norm in $\mathfrak{D}_{\mathscr{B}}$ by

$$\|B\| = \max(|b_1|, \dots, |b_r|),$$

where $B = \sum_{i=1}^r b_i B_i \in \mathfrak{D}_{\mathscr{B}}$. For any $\mathbb{R}$-divisor $B = \sum_{i=1}^r b_i B_i$, put $\mathfrak{D}_B := \mathfrak{D}_{\operatorname{Supp} B}$.

3. Hyperstandard multiplicities

Recall that *standard multiplicities* $1 - 1/m$ naturally appear as multiplicities in the divisorial adjunction formula $(K_X + S)|_S = K_S + \mathrm{Diff}_S$ (see [Sho93, §3], [Kol92, Ch. 16]). Considering the adjunction formula for fibre spaces and adjunction for higher codimensional subvarieties one needs to introduce a bigger class of multiplicities.

Example 3.1. Let $f\colon X \to Z \ni P$ be a minimal two-dimensional elliptic fibration over a one-dimensional germ (X is smooth). We can write a natural formula $K_X = f^*(K_Z + D_{\mathrm{div}})$, where $D_{\mathrm{div}} = d_P P$ is an effective divisor (cf. 7.2 below). From Kodaira's classification of singular fibres (see [Kod63]) we obtain the following values of d_P:

Type	$m\mathrm{I}_n$	II	III	IV	I_b^*	II^*	III^*	IV^*
d_P	$1-\frac{1}{m}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{5}{6}$	$\frac{3}{4}$	$\frac{2}{3}$

Thus the multiplicities of D_{div} are not necessarily standard.

3.2. Fix a subset $\mathfrak{R} \subset \mathbb{R}_{\geq 0}$. Define

$$\Phi(\mathfrak{R}) := \left\{ 1 - \frac{r}{m} \quad \Big| \quad m \in \mathbb{Z}, \quad m > 0 \quad r \in \mathfrak{R} \right\} \bigcap \Big[0,\, 1\Big].$$

We say that an $\mathbb{R}$-boundary B has *hyperstandard multiplicities* with respect to $\mathfrak{R}$ if $B \in \Phi(\mathfrak{R})$. For example, if $\mathfrak{R} = \{0, 1\}$, then $\Phi(\mathfrak{R})$ is the set of *standard multiplicities*. The set $\mathfrak{R}$ is said to be *rational* if $\mathfrak{R} \subset \mathbb{Q}$. Usually we will assume that $\mathfrak{R}$ is rational and finite. In this case we denote

$$I(\mathfrak{R}) := \mathrm{lcm}\big(\text{denominators of } r \in \mathfrak{R} \setminus \{0\}\big).$$

(3.2.1) Denote by $\mathscr{N}_d(\mathfrak{R})$ the set of all $m \in \mathbb{Z}$, $m > 0$ such that there exists a log semi-Fano variety (X, D) of dimension $\leq d$ satisfying the following properties:

(i) X is FT and $D \in \Phi(\mathfrak{R})$;
(ii) either (X, D) is klt or $K_X + D \equiv 0$;
(iii) $K_X + D$ is m-complemented, $I(\mathfrak{R}) \mid m$, and m is minimal under these conditions.

Since any nef divisor is semiample on FT variety, for any log semi-Fano variety (X, D) satisfying (i) and (ii), there exists some m in (iii). Put

$$N_d = N_d(\mathfrak{R}) := \sup \mathscr{N}_d(\mathfrak{R}), \quad \varepsilon_d = \varepsilon_d(\mathfrak{R}) := 1/(N_d + 2).$$

We expect that $\mathscr{N}_d(\mathfrak{R})$ is bounded whenever $\mathfrak{R}$ is finite and rational, see Theorems 1.4 and 1.5. In particular, $N_d < \infty$ and $\varepsilon_d > 0$. For $\varepsilon \geq 0$, define also the set of *semi-hyperstandard* multiplicities

$$\Phi(\mathfrak{R}, \varepsilon) := \Phi(\mathfrak{R}) \cup [1 - \varepsilon,\, 1].$$

Fix a positive integer n and define the set $\mathscr{P}_n \subset \mathbb{R}$ by

$$\alpha \in \mathscr{P}_n \iff 0 \leq \alpha \leq 1 \quad \text{and} \quad \lfloor (n+1)\alpha \rfloor \geq n\alpha.$$

This set obviously satisfies the following property:

Lemma 3.3. *If $D \in \mathscr{P}_n$ and D^+ is an n-complement, then $D^+ \geq D$.*

Taking 2.12 into account we immediately obtain the following important.

Corollary 3.4. *Let $D \in \mathscr{P}_n$ and let D^+ be an n-complement of D. Then D^+ is an nI-complement of D for any positive integer I.*

Lemma 3.5 (cf. [Sho00, Lemma 2.7])**.** *If $\mathfrak{R} \subset [0, 1] \cap \mathbb{Q}$, $I(\mathfrak{R}) \mid n$, and $0 \leq \varepsilon \leq 1/(n+1)$, then*

$$\mathscr{P}_n \supset \Phi(\mathfrak{R}, \varepsilon).$$

Proof. Let $1 \geq \alpha \in \Phi(\mathfrak{R}, \varepsilon)$. If $\alpha \geq 1 - \varepsilon$, then

$$(n+1)\alpha > n + 1 - \varepsilon(n+1) \geq n.$$

Hence, $\lfloor (n+1)\alpha \rfloor \geq n \geq n\alpha$ and $\alpha \in \mathscr{P}_n$. Thus we may assume that $\alpha \in \Phi(\mathfrak{R})$. It is sufficient to show that

$$\left\lfloor (n+1)\left(1 - \frac{r}{m}\right) \right\rfloor \geq n\left(1 - \frac{r}{m}\right) \tag{3.5.1}$$

for all $r \in \mathfrak{R}$ and $m \in \mathbb{Z}$, $m > 0$. We may assume that $r > 0$. It is clear that (3.5.1) is equivalent to the following inequality

$$(n+1)\left(1 - \frac{r}{m}\right) \geq k \geq n\left(1 - \frac{r}{m}\right), \tag{3.5.2}$$

for some $k \in \mathbb{Z}$ (in fact, $k = \left\lfloor (n+1)\left(1 - \frac{r}{m}\right)\right\rfloor$). By our conditions, $N := nr \in \mathbb{Z}$, $N > 0$. Thus (3.5.2) can be rewritten as follows

$$mn - N + m - r \geq mk \geq mn - N. \tag{3.5.3}$$

Since $m - r \geq m - 1$, inequality (3.5.3) has a solution in $k \in \mathbb{Z}$. This proves the statement. $\square$

Proposition 3.6 ([Pro01, Prop. 4.3.2], [PS01, Prop. 6.1])**.** *Let $f\colon Y \to X$ be a birational contraction and let D be an $\mathbb{R}$-divisor on Y such that*

(i) *$K_Y + D$ is nef over X,*
(ii) *$f_*D \in \mathscr{P}_n$ (in particular, f_*D is a boundary).*

Assume that $K_X + f_*D$ is n-complemented. Then so is $K_Y + D$.

Proposition 3.7 ([Pro01, Prop. 4.4.1], [PS01, Prop. 6.2])**.** *Let $(X/Z \ni o, D = S + B)$ be a log variety. Set $S := \lfloor D \rfloor$ and $B := \{D\}$. Assume that*

(i) *$K_X + D$ is plt;*
(ii) *$-(K_X + D)$ is nef and big over Z;*
(iii) *$S \neq 0$ near $f^{-1}(o)$;*
(iv) *$D \in \mathscr{P}_n$.*

Further, assume that near $f^{-1}(o) \cap S$ there exists an n-complement $K_S + \mathrm{Diff}_S(B)^+$ of $K_S + \mathrm{Diff}_S(B)$. Then near $f^{-1}(o)$ there exists an n-complement $K_X + S + B^+$ of $K_X + S + B$ such that $\mathrm{Diff}_S(B)^+ = \mathrm{Diff}_S(B^+)$.

Adjunction on divisors (cf. [Sho93, Cor. 3.10, Lemma 4.2]**).** Fix a subset $\mathfrak{R} \subset \mathbb{R}_{\geq 0}$. Define also the new set

$$\overline{\mathfrak{R}} := \left\{ r_0 - m \sum_{i=1}^{s} (1 - r_i) \;\middle|\; r_0, \dots, r_s \in \mathfrak{R},\ m \in \mathbb{Z},\ m > 0 \right\} \cap \mathbb{R}_{\geq 0}.$$

It is easy to see that $\overline{\mathfrak{R}} \supset \mathfrak{R}$. For example, if $\mathfrak{R} = \{0, 1\}$, then $\overline{\mathfrak{R}} = \mathfrak{R}$.

Lemma 3.8. (i) *If $\mathfrak{R} \subset [0,\ 1]$, then $\overline{\mathfrak{R}} \subset [0,\ 1]$.*

(ii) *If $\mathfrak{R}$ is finite and rational, then so is $\overline{\mathfrak{R}}$.*

(iii) $I(\mathfrak{R}) = I(\overline{\mathfrak{R}})$.

(iv) *Let $\mathfrak{G} \subset \mathbb{Q}$ be an additive subgroup containing 1 and let $\mathfrak{R} = \mathfrak{R}_{\mathfrak{G}} := \mathfrak{G} \cap [0, 1]$. Then $\overline{\mathfrak{R}} = \mathfrak{R}$.*

(v) *If the ascending chain condition (a.c.c.) holds for the set $\mathfrak{R}$, then it holds for $\overline{\mathfrak{R}}$.*

Proof. (i)-(iv) are obvious. We prove (v). Indeed, let

$$q^{(n)} = r_0^{(n)} - m^{(n)} \sum_{i=1}^{s^{(n)}} (1 - r_i^{(n)}) \in \overline{\mathfrak{R}}$$

be an infinite increasing sequence, where $r_i^{(n)} \in \mathfrak{R}$ and $m^{(n)} \in \mathbb{Z}_{>0}$. By passing to a subsequence, we may assume that $m^{(n)} \sum_{i=1}^{s^{(n)}} (1 - r_i^{(n)}) > 0$, in particular, $s^{(n)} > 0$ for all n. There is a constant $\varepsilon = \varepsilon(\mathfrak{R}) > 0$ such that $1 - r_i^{(n)} > \varepsilon$ whenever $r_i^{(n)} \neq 1$. Thus, $0 \leq q^{(n)} \leq r_0^{(n)} - m^{(n)} s^{(n)} \varepsilon$ and $m^{(n)} s^{(n)} \leq (r_0^{(n)} - q^{(n)})/\varepsilon$. Again by passing to a subsequence, we may assume that $m^{(n)}$ and $s^{(n)}$ are constants: $m^{(n)} = m$, $s^{(n)} = s$. Since the numbers $r_i^{(n)}$ satisfy a.c.c., the sequence

$$q^{(n)} = r_0^{(n)} + m \sum_{i=1}^{s} r_i^{(n)} - ms$$

is not increasing, a contradiction. □

Proposition 3.9. *Let $\mathfrak{R} \subset [0,\ 1]$, $1 \in \mathfrak{R}$, $\varepsilon \in [0,\ 1]$, and let $(X, S + B)$ be a plt log pair, where S is a prime divisor, $B \geq 0$, and $\lfloor B \rfloor = 0$. If $B \in \Phi(\mathfrak{R}, \varepsilon)$, then $\mathrm{Diff}_S(B) \in \Phi(\overline{\mathfrak{R}}, \varepsilon)$.*

Proof. Write $B = \sum b_i B_i$, where the B_i are prime divisors and $b_i \in \Phi(\mathfrak{R}, \varepsilon)$. Let $V \subset S$ be a prime divisor. By [Sho93, Cor. 3.10] the multiplicity d of $\mathrm{Diff}_S(B)$ along V is computed using the following relation:

$$d = 1 - \frac{1}{n} + \frac{1}{n}\sum_{i=0}^{s} k_i b_i = 1 - \frac{\beta}{n},$$

where n, $k_i \in \mathbb{Z}_{\geq 0}$, and $\beta := 1 - \sum k_i b_i$. It is easy to see that $d \geq b_i$ whenever $k_i > 0$. If $b_i \geq 1-\varepsilon$, this implies $d \geq 1-\varepsilon$. Thus we may assume that $b_i \in \Phi(\mathfrak{R})$ whenever $k_i > 0$. Therefore,

$$\beta = 1 - \sum k_i\left(1 - \frac{r_i}{m_i}\right),$$

where $m_i \in \mathbb{Z}_{>0}$, $r_i \in \mathfrak{R}$. Since $(X, S+B)$ is plt, $d < 1$. Hence, $\beta > 0$. If $m_i = 1$ for all i, then

$$\beta = 1 - \sum k_i(1 - r_i) \in \overline{\mathfrak{R}}.$$

So, $d \in \Phi(\overline{\mathfrak{R}})$ in this case. Thus we may assume that $m_0 > 1$. Since $1 - \frac{r_i}{m_i} \geq 1 - \frac{1}{m_i}$, we have $m_1 = \cdots = m_s = 1$ and $k_0 = 1$. Thus,

$$\beta = \frac{r_0}{m_0} - \sum_{i=1}^{s} k_i(1-r_i) = \frac{r_0 - m_0\sum_{i=1}^{s} k_i(1-r_i)}{m_0}$$

and $m_0\beta = r_0 - m_0\sum_{i=1}^{s} k_i(1-r_i) \in \overline{\mathfrak{R}}$. Hence, $d = 1 - \frac{m_0\beta}{m_0 n} \in \Phi(\overline{\mathfrak{R}})$. □

Proposition 3.10. *Let $1 \in \mathfrak{R} \subset [0,1]$ and let (X,B) be a klt log semi-Fano of dimension $\leq d$ such that X is FT. Assume the LMMP in dimension d. If $B \in \Phi(\mathfrak{R}, \varepsilon_d)$, then there is an n-complement $K+B^+$ of $K+B$ for some $n \in \mathscr{N}_d(\mathfrak{R})$. Moreover, $B \in \mathscr{P}_n$, and so $B^+ \geq B$.*

In the proposition we do not assert that $\mathscr{N}_d(\mathfrak{R})$ is finite. However later on we use the proposition in the induction process when the set of indices is finite (cf. the proof of Lemma 4.3 and see 4.7 – 4.10).

Proof. If $\varepsilon_d = 0$, then $\Phi(\mathfrak{R}, \varepsilon_d) = \Phi(\mathfrak{R})$ and there is nothing to prove. So we assume that $\varepsilon_d > 0$. If X is not $\mathbb{Q}$-factorial, we replace X with its small $\mathbb{Q}$-factorial modification. Write $B = \sum b_i B_i$. Consider the new boundary $D = \sum d_i B_i$, where

$$d_i = \begin{cases} b_i & \text{if } b_i < 1-\varepsilon_d, \\ 1-\varepsilon_d & \text{otherwise.}\end{cases}$$

Clearly, $D \in \Phi(\mathfrak{R})$. Since $D \leq B$, there is a klt $\mathbb{Q}$-complement $K+D+\Lambda$ of $K+D$ (by definition, $\Lambda \geq 0$). Run $-(K+D)$-MMP. Since all the birational transformations are $K+D+\Lambda$-crepant, they preserve the klt property of $(X, D+\Lambda)$ and (X,D). Each extremal ray is Λ-negative, and therefore is birational. At the end we get a model $(\bar{X}, \bar{D})$ which is a log semi-Fano variety. By definition, since $\bar{D} \in \Phi(\mathfrak{R})$ and X is FT, there is an n-complement $\bar{D}^+$ of $K_{\bar{X}} + \bar{D}$ for some $n \in \mathscr{N}_d(\mathfrak{R})$. Note that $I(\mathfrak{R}) \mid n$, so $\bar{D} \in \Phi(\mathfrak{R}) \subset \mathscr{P}_n$. By Proposition 3.6 we can pull-back this complement to X and this gives us an n-complement of $K_X + B$. The last assertion follows by Lemmas 3.5 and 3.3. □

4. General reduction

In this section we outline the main reduction step in the proof of our main results 1.4 and 1.5. First we concentrate on the klt case. The non-klt case of 1.5 will be treated in 4.13 and 9.5. Note that in Theorem 1.4 it is sufficient to find only one integer $n = n(d, \mathfrak{R})$ divisible by $I(\mathfrak{R})$ and such that $K + B$ is n-complemented. Other statements immediately follows by Corollary 3.4, Lemma 3.5 and by the definition of complements.

4.1. Setup. Let $(X, B = \sum b_i B_i)$ be a klt log semi-Fano variety of dimension d such that $B \in \Phi(\mathfrak{R})$. In particular, B is a $\mathbb{Q}$-divisor. Assume that X is FT. By induction we may assume that Theorems 1.4 and 1.5 hold in dimension $d-1$. So, by this inductive hypothesis, $\varepsilon_{d-1}(\overline{\mathfrak{R}}) > 0$ whenever $\mathfrak{R} \subset [0,\, 1]$ is finite and rational. Take any $0 < \varepsilon' \le \varepsilon_{d-1}(\overline{\mathfrak{R}})$. Put also $I := I(\mathfrak{R})$.

(4.1.1) First assume that the pair (X, B) is ε'-lt. Then the multiplicities of B are contained in the finite set $\Phi(\mathfrak{R}) \cap [0,\, 1-\varepsilon']$. By Conjecture 1.1 the pair (X, B) is bounded. Hence $(X, \operatorname{Supp} B)$ belongs to an algebraic family and we may assume that the multiplicities of B are fixed. Let $m := nI$. The condition that $K + B$ is m-complemented is equivalent to the following

$$\exists\, \overline{B} \in |-K - \lfloor (m+1)B \rfloor| \text{ such that } \left(X, \frac{1}{m}\left(\lfloor (m+1)B \rfloor + \bar{B}\right)\right) \text{ is lc}$$

(see 2.10, [Sho93, 5.1]). Obviously, the last condition is open in the deformation space of $(X, \operatorname{Supp} B)$. By Proposition 5.4 below and Noetherian induction the log divisor $K + B$ has a bounded nI-complement for some $n \le C(d, \mathfrak{R})$. From now on we assume that (X, B) is not ε'-lt.

4.2. We replace (X, B) with log crepant $\mathbb{Q}$-factorial blowup of all divisors E of discrepancy $a(E, X, B) \le -1 + \varepsilon'$, see [Kol92, 21.6.1]. Condition $B \in \Phi(\mathfrak{R})$ will be replaced with $B \in \Phi(\mathfrak{R}, \varepsilon') \cap \mathbb{Q}$. Note that our new X is again FT by Lemma 2.8. From now on we assume that X is $\mathbb{Q}$-factorial and

$$\operatorname{discr}(X, B) > -1 + \varepsilon'. \tag{4.2.1}$$

(4.2.2) For some $n_0 \gg 0$, the divisor n_0B is integral and the linear system $|-n_0(K+B)|$ is base point free. Let $\overline{B} \in |-n_0(K+B)|$ be a general member. Put $\Theta := B + \frac{1}{n_0}\overline{B}$. By Bertini's theorem $\operatorname{discr}(X, \Theta) = \operatorname{discr}(X, B)$. Thus we have the following

- $K_X + \Theta$ is a klt $\mathbb{Q}$-complement of $K_X + B$,
- $\operatorname{discr}(X, \Theta) \geq 1 + \varepsilon'$, and
- $\Theta - B$ is supported in a movable (possibly trivial) divisor.

Define a new boundary D with $\operatorname{Supp} D = \operatorname{Supp} B$:

$$D := \sum d_i B_i, \qquad \text{where} \quad d_i = \begin{cases} 1 & \text{if} \quad b_i \geq 1 - \varepsilon', \\ b_i & \text{otherwise.} \end{cases} \tag{4.2.3}$$

Here the B_i are components of B. Clearly, $D \in \Phi(\mathfrak{R})$, $D > B$, and by (4.2.1) we have $\lfloor D \rfloor \neq 0$.

Lemma 4.3 (the simplest case of the global-to-local statement)**.** *Fix a finite set $\mathfrak{R} \subset [0, 1] \cap \mathbb{Q}$. Let $(X \ni o, D)$ be the germ of a $\mathbb{Q}$-factorial klt d-dimensional singularity, where $D \in \Phi(\mathfrak{R}, \varepsilon_{d-1}(\overline{\mathfrak{R}}))$. Then there is an n-complement of $K_X + D$ with $n \in \mathscr{N}_{d-1}(\overline{\mathfrak{R}})$.*

Recall that according to our inductive hypothesis, $\mathscr{N}_{d-1}(\overline{\mathfrak{R}})$ is finite.

Proof. Consider a plt blowup $f\colon \tilde{X} \to X$ of (X, D) (see [PS01, Prop. 3.6]). By definition the exceptional locus of f is an irreducible divisor E, the pair $(\tilde{X}, \tilde{D} + E)$ is plt, and $-(K_{\tilde{X}} + \tilde{D} + E)$ is f-ample, where $\tilde{D}$ is the proper transform of D. We can take f so that $f(E) = o$, i.e., E is projective. By Adjunction $-(K_E + \operatorname{Diff}_E(\tilde{D}))$ is ample and $(E, \operatorname{Diff}_E(\tilde{D}))$ is klt. By Proposition 3.9 we have $\operatorname{Diff}_E(\tilde{D}) \in \Phi(\overline{\mathfrak{R}}, \varepsilon_{d-1}(\overline{\mathfrak{R}}))$. Hence there is an n-complement of $K_E + \operatorname{Diff}_E(\tilde{D})$ with $n \in \mathscr{N}_{d-1}(\overline{\mathfrak{R}})$, see Proposition 3.10. This complement can be extended to $\tilde{X}$ by Proposition 3.7. □

Claim 4.4. *The pair (X, D) is lc.*

Proof. By Lemma 4.3 near each point $P \in X$ there is an n-complement $K + B^+$ of $K + B$ with $n \in \mathscr{N}_{d-1}(\overline{\mathfrak{R}})$. By Lemma 3.5, we have $\mathscr{P}_n \supset \Phi(\mathfrak{R}, \varepsilon)$. Hence, by Lemma 3.3, $B^+ \geq B$. On the other hand, nB^+ is integral and for any component of $D - B$, its multiplicity in B is $\geq \varepsilon' > 1/(n+1)$. Hence, $B^+ \geq D$ and so (X, D) is lc near P. □

4.5. Run $-(K+D)$-MMP (anti-MMP). If X is FT, this is possible by Corollary 2.7. Otherwise $K + B \equiv 0$ and $-(K+D)$-MMP coincides with $K + B - \delta(D - B)$-MMP for some small positive δ.

It is clear that property $B \in \Phi(\mathfrak{R}, \varepsilon')$ is preserved on each step. All birational transformations are $(K + \Theta)$-crepant. Therefore $K + \Theta$ is klt on each step. Since $B \leq \Theta$, $K+B$ is also klt. By Claim 4.4 the log canonical property of (X, D) is also preserved and X is FT on each step by 2.8.

Claim 4.5.1. *None of components of $\lfloor D \rfloor$ is contracted.*

Proof. Let $\varphi\colon X \to \bar{X}$ be a $K+D$-positive extremal contraction and let E be the corresponding exceptional divisor. Assume that $E \subset \lfloor D \rfloor$. Put $\bar{D} := \varphi_* D$. Since $K_X + D$ is φ-ample, we can write

$$K_X + D = \varphi^*(K_{\bar{X}} + \bar{D}) - \alpha E, \qquad \alpha > 0.$$

On the other hand, since $(\bar{X}, \bar{D})$ is lc, we have

$$-1 \leq a(E, \bar{X}, \bar{D}) = a(E, X, D) - \alpha = -1 - \alpha < -1,$$

a contradiction. □

Corollary 4.6. *Condition* (4.2.1) *holds on each step of our MMP.*

Proof. Note that all our birational transformations are $(K+\Theta)$-crepant. Hence by (4.2.2), it is sufficient to show that none of the components of Θ of multiplicity $\geq 1-\varepsilon'$ is contracted. Assume that on some step we contract a component B_i of multiplicity $b_i \geq 1-\varepsilon'$. Then by (4.2.3) B_i is a component of $\lfloor D \rfloor$. This contradicts Claim 4.5.1. □

4.7. Reduction. After a number of divisorial contractions and flips

$$X \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_N = Y, \tag{4.7.1}$$

we get a $\mathbb{Q}$-factorial model Y such that either

(4.7.2) there is a non-birational $K_Y + D_Y$-positive extremal contraction $\varphi\colon Y \to Z$ to a lower-dimensional variety Z, or

(4.7.3) $-(K_Y + D_Y)$ is nef.

Here $\Box_Y$ denotes the proper transform of $\Box$ on Y.

Claim 4.8. *In case* (4.7.2)*, Z is a point, i.e., $\rho(Y) = 1$ and $-(K_Y + B_Y)$ is nef.*

Proof. Let $F = \varphi^{-1}(o)$ be a general fibre. Since $\rho(Y/Z) = 1$ and $-(K+B) \equiv \Theta - B \geq 0$, the restriction $-(K+B)|_F$ is nef. It is clear that $B|_F \in \Phi(\mathfrak{R}, \varepsilon')$. Assume that Z is of positive dimension. Then $\dim F < \dim X$. By our inductive hypothesis and Proposition 3.10 there is a bounded n-complement $K_F + B|_F^+$ of $K_F + B|_F$ for some $n \in \mathscr{N}_{d-1}(\mathfrak{R}) \subset \mathscr{N}_{d-1}(\overline{\mathfrak{R}})$. By Lemmas 3.3 and 3.5, we have $B|_F^+ \geq D|_F \geq B|_F$. On the other hand, $(K_X + D)|_F$ is φ-ample, a contradiction. □

4.9. Therefore we have a $\mathbb{Q}$-factorial FT variety Y and two boundaries $B_Y = \sum b_i B_i$ and $D_Y = \sum d_i B_i$ such that $\operatorname{discr}(Y, B_Y) > -1 + \varepsilon'$, $B_Y \in \Phi(\mathfrak{R}, \varepsilon')$, $D_Y \in \Phi(\mathfrak{R})$, $D_Y \geq B_Y$, and $d_i > b_i$ if and only if $d_i = 1$ and $b_i \geq -1 + \varepsilon'$. Moreover, one of the following two cases holds:

(4.9.1) $\rho(Y) = 1$, $K_Y + D_Y$ is ample, and (Y, B_Y) is a klt log semi-Fano variety, or

(4.9.2) (Y, D_Y) is a log semi-Fano variety with $\lfloor D_Y \rfloor \neq 0$. Since $D > B$, this case does not occur if $K + B \equiv 0$.

These two cases will be treated in sections 6 and 9, respectively.

4.10. Outline of the proof of Theorem 1.4. Now we sketch the basic idea in the proof of boundedness in case (4.9.1). By (4.1.1) we may assume that (X, B) is not ε'-lt. Apply constructions of 4.2, 4.5 and 4.7. Recall that on each step of (4.7.1) we contract an extremal ray which is $(K + D)$-positive. By Proposition 3.6 we can pull-back n-complements with $n \in \mathcal{N}_d(\mathfrak{R})$ of $K_Y + D_Y$ to our original X. However it can happen in case $\rho(Y) = 1$ that $K_Y + D_Y$ has no any complements. In this case we will show in Section 6 below that the multiplicities of B_Y are bounded from the above: $b_i < 1 - c$, where $c > 0$. By Claim 4.5.1 divisorial contractions in (4.7.1) do not contracts components of B with multiplicities $b_i \geq 1 - \varepsilon'$. Therefore the multiplicities of B also are bounded from the above. Combining this with $\operatorname{discr}(X, B) > -1 + \varepsilon'$ and Conjecture 1.1 we get that $(X, \operatorname{Supp} B)$ belong to an algebraic family. By Noetherian induction (cf. (4.1.1)) we may assume that $(X, \operatorname{Supp} B)$ is fixed. Finally, by Proposition 5.4 we have that (X, B) has bounded complements.

Case (4.9.2) will be treated in Sect. 9. In fact in this case we study the contraction $f\colon Y \to Z$ given by $-(K + D)$. When Z is a lower-dimensional variety, f is a fibration onto varieties with trivial log canonical divisor. The existence of desired complements can be established inductively, by using an analog of Kodaira's canonical bundle formula (see Conjecture 7.13).

The proof of Theorem 1.5 in case when (X, B) is not klt is based on the following

Lemma 4.11. *Let $(X, B = \sum b_i B_i)$ be a 0-pair of dimension d such that $B \in \Phi(\mathfrak{R}, \varepsilon')$, where $\varepsilon' := \varepsilon_{d-1}(\overline{\mathfrak{R}})$. Assume the LMMP in dimension d. Further, assume either*

(i) *(X, B) is not klt and Theorems 1.4-1.5 hold in dimension $d-1$, or*
(ii) *Theorem 1.5 holds in dimension d.*

Then there exists $\lambda := \lambda(d, \mathfrak{R}) > 0$ *such that either* $b_i = 1$ *or* $b_i \le 1-\lambda$ *for all* b_i.

Proof. The proof is by induction on d. Case $d = 1$ is well-known; see, e.g., Corollary 5.7. If $B = 0$, there is nothing to prove. So we assume that $B > 0$. Assume that, for some component $B_i = B_0$, we have $1 - \lambda < b_0 < 1$.

First consider the case when (X, B) is not klt. Replace (X, B) with its $\mathbb{Q}$-factorial dlt modification so that $\lfloor B \rfloor \ne 0$. If $B_0 \cap \lfloor B \rfloor = \emptyset$, we run $K + B - b_0B_0$-MMP. After several steps we get a 0-pair $(\hat{X}, \hat{B})$ such that $\lfloor \hat{B} \rfloor \ne 0$, $\hat{B}_0 \ne 0$ and one of the following holds:

(a) $\hat{B}_0 \cap \lfloor \hat{B} \rfloor \ne \emptyset$, or
(b) there is an extremal contraction $\hat{f} \colon \hat{X} \to \hat{Z}$ to a lower-dimensional variety such that both $\lfloor \hat{B} \rfloor$ and $\hat{B}_0$ are $\hat{f}$-ample, and $\hat{B}_0 \cap \lfloor \hat{B} \rfloor = \emptyset$. (In particular, $\hat{Z}$ is not a point.)

In the second case we can apply induction hypothesis restricting $\hat{B}$ to a general fibre of $\hat{f}$. In the first case, replacing our original (X, B) with a dlt modification of $(\hat{X}, \hat{B})$, we may assume that $B_0 \cap \lfloor B \rfloor \ne \emptyset$. Let $B_1 \subset \lfloor B \rfloor$ be a component meeting B_0. Then $(B_1, \mathrm{Diff}_{B_1}(B - B_1))$ is a 0-pair with $\mathrm{Diff}_{B_1}(B - B_1) \in \Phi(\overline{\mathfrak{R}}, \varepsilon')$ (see Lemma 3.5). Write $\mathrm{Diff}_{B_1}(B - B_1) = \sum \delta_i \Delta_i$ and let Δ_0 be a component of $B_0 \cap B_1$. Then the multiplicity δ_0 of Δ_0 in $\mathrm{Diff}_{B_1}(B - B_1)$ is computed as follows: $\delta_0 = 1 - 1/r + \sum_l k_l b_l / r$, where r and k_l are non-negative integers and r, $k_0 > 0$ (see [Sho93, Corollary 3.10]). Since $b_0 > 1 - \varepsilon' > 1/2$, we have $k_0 = 1$ and $k_l = 0$ for $l \ne 0$. Hence, $\delta_0 = 1 - 1/r + b_0/r$. By induction we may assume either $\delta_0 = 1$ or $\delta_0 \le 1 - \lambda(d-1, \overline{R})$. Thus we have either $b_0 = 1$ or

$$b_0 \le 1 - r\lambda(d-1, \overline{R}) \le 1 - \lambda(d-1, \overline{R})$$

and we can put $\lambda(d, R) = \lambda(d-1, \overline{R})$ in this case.

Now consider the case when (X, B) is klt. Replace (X, B) with its $\mathbb{Q}$-factorialization and again run $K + B - b_0B_0$-MMP: $(X, B) \dashrightarrow (X', B')$. Clearly at the end we get a B_0'-positive extremal contraction $\varphi \colon (X', B') \to W$ to a lower-dimensional variety W. If W is not a point, we can apply induction restricting B' to a general fibre. Thus replacing (X, B) with (X', B') we may assume that X is $\mathbb{Q}$-factorial, $\rho(X) = 1$ and $(X, B - b_0B_0)$ is a klt log Fano variety. In particular, X is FT. By Theorem 1.5 and Proposition 3.10 the log divisor $K + B$ is n-complemented for some $n \in \mathscr{N}_d(\mathfrak{R})$. For this complement B^+, we have $B^+ \ge B$. Since $K_X + B \equiv 0$, $B^+ = B$. In particular, nB

is integral. Thus we can put $\lambda := 1/\operatorname{lcm}(\mathscr{N}_{d-1}(\mathfrak{R}))$. This proves the statement in case (ii). $\square$

Corollary 4.12. *Notation and assumptions as in Lemma* 4.11. *Let E be any divisor* (*exceptional or not*) *over X. Then either $a(E, X, B) = 1$ or $a(E, X, B) \leq 1 - \lambda$.*

Proof. We can take $\lambda < \varepsilon'$. If $1 - \lambda < a(E, X, B) < 1$, replace (X, B) with a crepant blowup of E, see [Kol92, 21.6.1] and apply Lemma 4.11. $\square$

4.13. Proof of Theorem 1.5 in the case when (X, B) is not klt. Let (X, B) be a 0-pair such that X is FT and $B \in \Phi(\mathfrak{R})$. We assume that (X, B) is not klt. Replace (X, B) with its $\mathbb{Q}$-factorial dlt modification. Then in particular, X is klt. Moreover, $\lfloor B \rfloor \neq 0$ (and $B \in \Phi(\mathfrak{R})$). Let $\lambda(d, \mathfrak{R})$ be as in Lemma 4.11 and let $0 < \lambda < \lambda(d, \mathfrak{R})$. If X is not λ-lt, then for each exceptional divisor E of discrepancy $a(E, X, 0) < -1 + \lambda$ by Corollary 4.12 we have $a(E, X, B) = 1$. Hence as in 4.2 replacing (X, B) with blowup of all such divisors E, see [Kol92, 21.6.1], we get that X is λ-lt and (X, B) is a 0-pair with $B \in \Phi(\mathfrak{R})$ and $\lfloor B \rfloor \neq 0$. Run K-MMP: $X \dashrightarrow X'$ and let B' be the birational transform of B. Since $B \neq 0$, X' admits a K-negative Fano fibration $X' \to Z'$ over a lower-dimensional variety Z'. By our construction X' is λ-lt and (X', B') is a non-klt 0-pair with $B' \in \Phi(\mathfrak{R})$. By Proposition 3.6 we can pull-back n-complements from X' to X if $I(\mathfrak{R}) \mid n$. If Z' is a point, then $\rho(X') = 1$ and X' is a klt Fano variety. In this case, arguing as in 4.1.1 we get that $(X', \operatorname{Supp} B')$ belong to an algebraic family. By Proposition 5.4 and Noetherian induction the log divisor $K + B$ has a bounded nI-complement for some $n \leq C(d, \mathfrak{R})$. Finally, if $\dim Z' > 0$, then we apply Proposition 9.4 below. This will be explained in 9.5. Theorem 1.5 in the non-klt case is proved.

5. Approximation and complements

The following Lemma 5.2 shows that the existence of n-complements is an open condition in the space of all boundaries B with fixed $\operatorname{Supp} B$.

5.1. Notation. Let $\mathscr{B}$ be a finite set of prime divisors B_i. Recall that $\mathfrak{D}_{\mathscr{B}}$ denotes the $\mathbb{R}$-vector space all $\mathbb{R}$-Weil divisors B with $\operatorname{Supp} B = \sum_{B_i \in \mathscr{B}} B_i$, where the B_i are prime divisors. Let

$$\mathfrak{I}_{\mathscr{B}} := \left\{ \sum \beta_i B_i \in \mathfrak{D}_{\mathscr{B}} \mid 0 \leq \beta_i \leq 1,\ \forall i \right\}$$

be the unit cube in $\mathfrak{D}_{\mathscr{B}}$.

Lemma 5.2. *Let (X, B) be a log pair where B is an $\mathbb{R}$-boundary. Assume that $K + B$ is n-complemented. Then there is a constant $\varepsilon = \varepsilon(X, B, n) > 0$ such that $K + B'$ is also n-complemented for any $\mathbb{R}$-boundary $B' \in \mathfrak{D}_B$ with $\|B - B'\| < \varepsilon$.*

Proof. Let $B^+ = B^\sharp + \Lambda$ be an n-complement, where Λ and B have no common components and $B^\sharp \in \mathfrak{D}_B$. Write $B = \sum b_i B_i$, $B' = \sum b'_i B_i$, $B^\sharp = \sum b^+_i B_i$. Take ε so that

$$0 < (n+1)\varepsilon < \min(1 - \{(n+1)b_i\} \mid 1 \le i \le r, \quad b_i < 1).$$

We claim that B^+ is also an n-complement of B' whenever $\|B - B'\| < \varepsilon$. If $b_i = 1$, then obviously $b^+_i = 1$. So, it is sufficient to verify the inequalities $nb^+_i \ge \lfloor (n+1)b'_i \rfloor$ whenever $b^+_i < 1$ and $b_i < 1$. Indeed, in this case,

$$\lfloor (n+1)b'_i \rfloor \le \lfloor (n+1)b_i + (n+1)(b'_i - b_i) \rfloor = \lfloor (n+1)b_i \rfloor \le nb^+_i.$$

(because $(n+1)(b'_i - b_i) < (n+1)\varepsilon < \min(1 - \{(n+1)b_i\})$). This proves the assertion. □

Corollary 5.3. *For any $D \in Z_{d-1}(X)$, the subset*

$$\mathfrak{U}^n_D := \{B \in \mathfrak{I}_D \mid K + B \text{ is } n\text{-complemented}\}$$

is open in $\mathfrak{I}_D$.

Proposition 5.4. *Fix a positive integer I. Let X be an FT variety such that K_X is $\mathbb{Q}$-Cartier and let $B_1, \dots, B_r$ are $\mathbb{Q}$-Cartier divisors on X. Let $\mathscr{B} := \sum_{i=1}^r B_i$. Then for any boundary $B \in \mathfrak{I}_{\mathscr{B}}$ such that $K + B$ is lc and $-(K + B)$ is nef, there is an n-complement of $K + B$ for some $n \le \operatorname{Const}(X, \mathscr{B})$ and $I \mid n$.*

Lemma 5.5. *In notation of Proposition* 5.4 *the following holds.*

(5.5.1) (Effective base point freeness) *There is a positive integer N such that for any integral nef Weil $\mathbb{Q}$-Cartier divisor of the form $mK + \sum m_i B_i$ the linear system $|N(mK + \sum m_i B_i)|$ is base point free.*

Proof. Indeed, we have $\operatorname{Pic}(X) \simeq \mathbb{Z}^\rho$ (see, e.g., [IP99, Prop. 2.1.2]). In the space $\operatorname{Pic}(X) \otimes \mathbb{R} \simeq \mathbb{R}^\rho$ we have a closed convex cone $\overline{\operatorname{NEF}}(X)$, the cone of nef divisors. This cone is dual to the Mori cone $\overline{\operatorname{NE}}(X)$, so it is rational polyhedral and generated by a finite number of semiample Cartier divisors $M_1, \dots, M_s$. Take a positive integer N' so that all the linear systems $|N' M_i|$ are base point free, and $N'K$, $N'B_1, \dots, N'B_r$ are Cartier. Write

$$N'K \sim_{\mathbb{Q}} \sum_{i=1}^s \alpha_{i,0} M_i, \quad N'B_j \sim_{\mathbb{Q}} \sum_{i=1}^s \alpha_{i,j} M_i, \qquad \alpha_{i,j} \in \mathbb{Q}, \qquad \alpha_{i,j} \ge 0.$$

Let N'' be the common multiple of denominators of the $\alpha_{i,j}$. Then

$$N'^2 N'' \left(mK + \sum_{j=1}^{r} m_j B_j \right) \sim N' N'' m \left(\sum_{i=1}^{s} \alpha_{i,0} M_i \right) +$$

$$\sum_{j=1}^{r} N' N'' m_j \left(\sum_{i=1}^{s} \alpha_{i,j} M_i \right) = \sum_{i=1}^{s} \left(m N'' \alpha_{i,0} + \sum_{j=1}^{r} N'' m_j \alpha_{i,j} \right) N' M_i$$

The last (integral) divisor generates a base point free linear system, so we can take $N = N'^2 N''$. □

In the proof of Proposition 5.4 we follow arguments of [Sho00, Example 1.11], see also [Sho93, 5.2].

Proof of Proposition 5.4. Define the set
(5.5.2)

$$\mathscr{M} = \mathscr{M}_{\mathscr{B}} := \{ B \in \mathfrak{I}_{\bar{B}} \quad | \quad K + B \text{ is lc and } -(K+B) \text{ is nef} \}$$

Then $\mathscr{M}$ is a closed compact convex polyhedron in $\mathfrak{I}_{\mathscr{B}}$. It is sufficient to show the existence of some n-complement for any $B \in \mathscr{M}$. Indeed, then $\mathscr{M} \subset \bigcup_{n \in \mathbb{Z}_{>0}} \mathfrak{U}^n_{\mathscr{B}}$. By taking a finite subcovering $\mathscr{M} \subset \bigcup_{n \in \mathcal{S}} \mathfrak{U}^n_{\mathscr{B}}$ we get a finite number of such n.

Assume that there is a boundary $B^o = \sum_{i=1}^{r} b_i^o B_i \in \mathscr{M}$ which has no any complements. By [Cas57, Ch. 1, Th. VII] there is infinite many rational points $(m_1/q, \dots, m_r/q)$ such that

$$\max \left(\left| \frac{m_1}{q} - b_1^o \right|, \dots, \left| \frac{m_r}{q} - b_r^o \right| \right) < \frac{r}{(r+1) q^{1+1/r}} < \frac{1}{q^{1+1/r}}.$$

Denote $b_i := m_i/q$ and $B := \sum b_i B_i$. Thus, $\|B - B^o\| < 1/q^{1+1/r}$. Then our proposition is an easy consequence of the following

Claim 5.6. *For $q \gg 0$ one has*

(5.6.1) $\lfloor (qN+1) b_i^o \rfloor \le qN b_i$ *whenever* $b_i < 1$;

(5.6.2) $B \equiv B^o$ *and* $-(K+B)$ *is nef*; *and*

(5.6.3) $K + B$ *is lc.*

Indeed, by (5.5.1) the linear system $|-qN(K+B)|$ is base point free. Let $F \in |-qN(K+B)|$ be a general member. Then $K + B + \frac{1}{qN} F$ is an qN-complement of $K + B^o$, a contradiction. □

Proof of Claim. By the construction

$$\lfloor (qN+1) b_i^o \rfloor = m_i N + \lfloor b_i^o + qN(b_i^o - b_i) \rfloor$$

Put $c := \max_{b_i^o<1}\{b_1^o, \dots b_r^o\}$. Then for $b_i < 1$ we have $b_i^o < c < 1$ and for $q \gg 0$,

$$b_i^o + qN(b_i^o - b_i) < c + \frac{qN}{q^{1+1/r}} < 1.$$

This proves (5.6.1).

Further, let $L_1, \dots, L_r$ be a finite set of curves generating $N_1(X)$. We have

$$|L_j \cdot (B - B^o)| = \left| \sum_i \frac{m_i}{q}(L_j \cdot B_i) - \sum_i b_i^o (L_j \cdot B_i) \right|$$
$$< \frac{1}{q^{1+1/r}} \sum_i (L_j \cdot B_i).$$

If $q \gg 0$, then the right hand side is $\ll 1/q$ while the left hand side is from the discrete set $\pm L_j \cdot (K + B^o) + \frac{1}{qN}\mathbb{Z}$ (because qB is an integral divisor and by our assumption (5.5.1)). Hence the left hand side is zero and $B \equiv B^o$. This proves (5.6.2).

Finally, we have to show that $K + B$ is lc. Assume the converse. By (5.5.1) the divisor $qN\,(K + B)$ is Cartier. So there is a divisor E of the field $\Bbbk(X)$ such that $a(E, X, B) \le -1 - 1/qN$ and $a(E, X, B^o) \ge -1$. On the other hand, $a(E, X, \sum \beta_i B_i)$ is an affine linear function in β_i:

$$\frac{1}{qN} \le a(E, X, B^o) - a(E, X, B) = \sum c_i(b_i^o - b_i) < \frac{\text{Const}}{q^{1+1/r}}$$

which is a contradiction. □

The following is the first induction step to prove Theorem 1.4.

Corollary 5.7 (One-dimensional case)**.** *Fix a finite set* $\mathfrak{R} \subset [0,\, 1] \cap \mathbb{Q}$ *and a positive integer* I*. Then the set* $\mathscr{N}_1(\mathfrak{R})$ *is finite.*

Proof. Let (X, B) be a one-dimensional log pair satisfying conditions of (3.2.1). Since X is FT, $X \simeq \mathbb{P}^1$. Since $B \in \Phi(\mathfrak{R})$ and $\mathfrak{R}$ is finite, we can write $B = \sum_{i=1}^r b_i B_i$, where $b_i \ge \delta$ for some fixed $\delta > 0$. Thus we may assume that r is fixed and $B_1, \dots, B_r$ are fixed distinct points. Then by Proposition 5.4 we have a desired complements. □

Example 5.8. Let $X \simeq \mathbb{P}^1$. If $\mathfrak{R} = \{0,\, 1\}$, then $I(\mathfrak{R}) = 1$ and $\Phi(\mathfrak{R})$ is the set of standard multiplicities. In this case, it is easy to compute that $\mathscr{N}_1(\mathfrak{R}) = \{1,\, 2,\, 3,\, 4,\, 6\}$ [Sho93, 5.2]. Consider more complicated case when $\mathfrak{R} = \{0,\, \frac{1}{2},\, \frac{2}{3},\, \frac{3}{4},\, \frac{5}{6},\, 1\}$. Then $I = 12$ and one can compute that

$$\mathscr{N}_1(\mathfrak{R}) = 12 \cdot \{1,\, 2,\, 3,\, 4,\, 5,\, 7,\, 8,\, 9,\, 11\}.$$

Indeed, assume that (X, D) has no any $12n$-complements for $n \in \{1, 2, 3, 4, 5, 7, 8, 9, 11\}$. Write $D = \sum_{i=1}^{r} d_i D_i$, where $D_i \neq D_j$ for $i \neq j$. It is clear that the statement about the existence of an n-complement D^+ such that $D^+ \geq D$ is equivalent to the following inequality

$$\sum_i \lceil n d_i \rceil \leq 2n. \tag{5.8.1}$$

Since $d_i = 1 - r_i/m_i$, where $r_i \in \mathfrak{R}$, $m_i \in \mathbb{Z}_{>0}$, we have $d_i \geq 1/6$ for all i. We claim that at least one denominator of d_i does not divide 24. Indeed, otherwise $24D$ is an integral divisor and $D^+ := D + \frac{1}{24}\sum_{j=1}^{k} D_j$ is an 24-complement, where $D_j \in X$ are general points and $k = 24(2 - \deg D)$. Thus we may assume that the denominator of d_1 does not divide 24. Since $d_1 = 1 - r_1/m_1$, where $r_1 \in \mathfrak{R}$, we have $m_1 \geq 3$ and the equality holds only if $r_1 = 2/3$ or $5/6$. In either case, $d_1 \geq 13/18$.

Recall that a log pair (X, D) of global type is said to be *exceptional* if at has at least one $\mathbb{Q}$-complement and any $\mathbb{Q}$-complement is klt. If (X, D) is not exceptional, we can increase d_1 by putting $d_1 = 1$. Then as above $13/18 \leq d_2 \leq 5/6$, so $r = 3$ and $d_3 \leq 5/18$. Now there are only a few possibilities for d_2 and d_3:

d_2	$\frac{13}{18}$	$\frac{3}{4}$	$\frac{7}{9}$	$\frac{19}{24}$	$\frac{4}{5}$	$\frac{13}{16}$	$\frac{5}{6}$
d_3	$\leq \frac{5}{18}$	$\leq \frac{1}{4}$	$\leq \frac{2}{9}$	$\leq \frac{5}{24}$	$\leq \frac{1}{5}$	$\leq \frac{3}{16}$	$\leq \frac{1}{6}$

In all cases $K + D$ has a $12n$ complement for some $n \in \{1, 2, 3, 4, 5\}$. In the exceptional case, there is a finite number of possibilities for $(d_1, \ldots, d_r)$. However the computations are much longer. We omit them.

6. The main theorem: Case $\rho = 1$

6.1. Now we begin to consider case (4.9.1). Thus we assume that (X, B) is klt but not ε'-lt, where we take ε' so that $\varepsilon' N = 1$ for some integer $N \geq N_{d-1}(\overline{\mathfrak{R}}) + 2$. Then obviously, $\varepsilon_{d-1}(\overline{\mathfrak{R}}) \geq \varepsilon' > 0$. Further, assume that applying the general reduction from Section 4 we get a pair (Y, B_Y), where Y is FT and $\mathbb{Q}$-factorial, $\rho(Y) = 1$, the $\mathbb{Q}$-divisor $-(K_Y + B_Y)$ is nef, and

$$\operatorname{discr}(Y, B_Y) > -1 + \varepsilon'.$$

Moreover, $K_Y + D_Y$ is ample and $\lfloor D_Y \rfloor \neq 0$, where D_Y is a boundary satisfying (4.2.3). In particular, $B_Y \neq 0$.

6.2. Assume that the statement of Theorems 1.4 and 1.5 is false in this case. Then there is a sequence of klt log pairs $(X^{(m)}, B^{(m)})$ as in 6.1 and such that complements of $K_{X^{(m)}} + B^{(m)}$ are unbounded. More precisely, for each $K_{X^{(m)}} + B^{(m)}$, let n_m be the minimal positive integer such that $I(\mathfrak{R}) \mid n_m$ and $K_{X^{(m)}} + B^{(m)}$ is n_m-complemented. We assume that the sequence n_m is unbounded. We will derive a contradiction.

6.3. By Corollary 5.7 we have $\dim X^{(m)} \geq 2$. By our hypothesis we have a sequence of birational maps $X^{(m)} \dashrightarrow Y^{(m)}$, where $Y^{(m)}$, $X^{(m)}$, $B^{(m)}$ and $D^{(m)}$ are as above for all m. Recall that by Corollary 4.6
(6.3.1)
$$\operatorname{discr}(Y^{(m)}) \geq \operatorname{discr}(Y^{(m)}, B_Y^{(m)}) \geq \operatorname{discr}(X^{(m)}, B^{(m)}) > -1 + \varepsilon',$$
where $0 < \varepsilon' \leq \varepsilon_{d-1}(\overline{\mathfrak{R}})$. Note that $-(K_{Y^{(m)}} + B^{(m)})$ is nef, so by Conjecture 1.1 the sequence of varieties $Y^{(m)}$ is bounded. By Noetherian induction (cf. (4.1.1)), we may assume that $Y^{(m)}$ is fixed, that is, $Y^{(m)} = Y$.

Let $Y \hookrightarrow \mathbb{P}^N$ be an embedding and let H be a hyperplane section of Y. Note that the multiplicities of $B_Y^{(m)} = \sum b_i^{(m)} B_i^{(m)}$ are bounded from below: $b_i^{(m)} \geq \delta_0 > 0$, where $\delta_0 := \min \Phi(\mathfrak{R}) \setminus \{0\}$. Then, for each $B_i^{(m)}$,
$$\delta_0 H^{d-1} \cdot B_i^{(m)} \leq H^{d-1} \cdot B_Y^{(m)} \leq -H^{d-1} \cdot K_Y.$$
This shows that the degree of $B_i^{(m)}$ is bounded and $B_i^{(m)}$ belongs to an algebraic family. Therefore we may assume that $\operatorname{Supp} B_Y^{(m)}$ is also fixed: $B_i^{(m)} = B_i$.

6.4. Assume that the multiplicities of $B_Y^{(m)}$ are bounded from 1, i.e., $b_i^{(m)} \leq 1 - c$, where $c > 0$. Then we argue as in 4.10. By Claim 4.5.1 divisorial contractions in (4.7.1) do not contract components of $B^{(m)}$ with multiplicities $b_i^{(m)} \geq 1 - \varepsilon'$. Therefore the multiplicities of $B^{(m)}$ on $X^{(m)}$ are also bounded from the 1. Combining this with $\operatorname{discr}(X, B) > -1 + \varepsilon'$ and Conjecture 1.1 we get that $(X, \operatorname{Supp} B)$ belong to an algebraic family. By Noetherian induction (cf. (4.1.1)) we may assume that $(X, \operatorname{Supp} B)$ is fixed. Finally, by Proposition 5.4 we have that (X, B) has a bounded n-complement such that $I(\mathfrak{R}) \mid n$.

Thus by our construction the only possibility is the case below.

6.5. From now on we consider the remaining case when some multiplicity of $B_Y^{(m)}$ is accumulated to 1 and we will derive a contradiction. Since $\operatorname{Supp} B_Y^{(m)}$ does not depend on m, by passing to a subsequence we may assume that the limit $B_Y^{\infty} := \lim\limits_{m\to\infty} B_Y^{(m)}$ exists and $\lfloor B_Y^{\infty} \rfloor \neq 0$.

As above, write $B_Y^\infty = \sum b_i^\infty B_i$. Up to permutations of components we may assume that $b_1^\infty = 1$. It is clear that $-(K_Y + B_Y^\infty)$ is nef and $\operatorname{discr}(Y, B_Y^\infty) \geq -1 + \varepsilon'$. In particular, (Y, B_Y^∞) is plt.

Claim 6.6. *Under the above hypothesis, we have $b_j^\infty \leq 1 - \varepsilon'$ for all $1 < j \leq r$. Moreover, by passing to a subsequence we may assume the following*:

(i) *If $b_j^\infty = 1$, then $j = 1$ and $b_1^{(m)}$ is strictly increasing.*
(ii) *If $b_j^\infty < 1 - \varepsilon'$, then $b_j^{(m)} = b_j^\infty$ is a constant.*
(iii) *If $b_j^\infty = 1 - \varepsilon'$, then $b_j^{(m)}$ is either a constant or strictly decreasing.*

In particular, $B_Y^\infty \in \Phi(\mathfrak{R})$, B_Y^∞ is a $\mathbb{Q}$-boundary, and $D_Y^{(m)} \geq B_Y^\infty$ for $m \gg 0$.

Proof. Since $\rho(Y) = 1$ and Y is $\mathbb{Q}$-factorial, the intersection $B_1 \cap B_j$ on Y is of codimension two and non-empty. For a general hyperplane section $Y \cap H$, by (6.3.1) we have the inequality

$$\operatorname{discr}(Y \cap H, B^\infty \cap H) \geq -1 + \varepsilon'.$$

Thus by Lemma 6.7 below, we have $b_1^\infty + b_j^\infty \leq 2 - \varepsilon'$, i.e., $b_j^\infty \leq 1 - \varepsilon'$ for all $1 < j \leq r$. The rest follows from the fact that the set $\Phi(\mathfrak{R}) \cap [0, 1-\varepsilon']$ is finite. □

Lemma 6.7 (cf. [Sho00, Prop. 5.2], [Pro01, §9])**.** *Let $(S \ni o, \Lambda = \sum \lambda_i \Lambda_i)$ be a log surface germ. Assume that* $\operatorname{discr}(S, \Lambda) \geq -1 + \varepsilon$ *at o for some positive ε. Then $\sum \lambda_i \leq 2 - \varepsilon$.*

Proof. Locally near o there is an étale outside of o Galois cover $\pi \colon S' \to S$ such that S' is smooth. Let $\Lambda' := \pi^*\Lambda$ and $o' := \pi^{-1}(o)$. Then $\operatorname{discr}(S', \Lambda') \geq \operatorname{discr}(S, \Lambda) \geq -1 + \varepsilon$ at o (see, e.g., [Kol92, Proposition 20.3]). Consider the blow up of $o' \in S'$. We get an exceptional divisor E of discrepancy

$$-1 + \varepsilon \leq a(E, S', \Lambda') = 1 - \sum \lambda_i.$$

This gives us the desired inequality. □

Corollary 6.8. *$b_j^\infty = 1 - \varepsilon'$ for some j.*

Proof. Indeed, otherwise $D_Y^{(m)} = B_Y^\infty$ for $m \gg 0$ and $-(K_Y + D_Y^{(m)})$ is nef, a contradiction. □

We claim that the log divisor $K_Y + B_Y^\infty$ is n-complemented, where $n \in \mathscr{N}_{d-1}(\overline{\mathfrak{R}})$. Recall that $b_1^\infty = 1$. Put $B' := B^\infty - B_1$. By the last corollary $B' \neq 0$. By Proposition 3.9 we have $\operatorname{Diff}_{B_1}(B') \in \Phi(\overline{\mathfrak{R}})$. Recall that $-(K_Y+B^\infty)$ is nef. Since (Y,B_Y^∞) is plt, the pair $(B_1,\operatorname{Diff}_{B_1}(B'))$ is klt. Further, $-(K_{B_1}+\operatorname{Diff}_{B_1}(0))$ is ample (because $\rho(Y)=1$), so B_1 is FT. Thus by the inductive hypothesis there is an n-complement $K_{B_1}+\operatorname{Diff}_{B_1}(B')^+$ of $K_{B_1}+\operatorname{Diff}_{B_1}(B')$ for some $n\in\mathscr{N}_{d-1}(\overline{\mathfrak{R}})$. By Lemma 3.5 $B'\in\mathscr{P}_n$. Take a sufficiently small positive δ and let j be such that $b_j^\infty=1-\varepsilon'$. We claim that $B'-\delta B_j\in\mathscr{P}_n$. Indeed, otherwise $(n+1)b_j^\infty$ is an integer. On the other hand,

$$n+1>(n+1)b_j^\infty=(n+1)(1-\varepsilon')\geq n+1-(n+1)/N.$$

where $N\geq N_{d-1}(\overline{\mathfrak{R}})+2\geq n+2$. This is impossible. Thus, $B'-\delta B_j\in\mathscr{P}_n$. Since $-(K_Y+B^\infty-\delta B_j)$ is ample, by Proposition 3.7 the n-complement $K_{B_1}+\operatorname{Diff}_{B_1}(B')^+$ of $K_{B_1}+\operatorname{Diff}_{B_1}(B'-\delta B_j)$ can be extended to Y. So there is an n-complement K_Y+B^+ of $K_Y+B^\infty-\delta B_j$. Write $B^+=\sum b_i^+B_i$. Since $B-\delta B_j\in\mathscr{P}_n$, we have $B^+\geq B^\infty-\delta B_j$. Moreover, since nb_j^+ is an integer and $1\gg\delta>0$, we have $b_j^+\geq b_j^\infty$. Hence $B^+\geq B_Y^\infty$ and B^+ is also an n-complement of K_Y+B^∞ (see Remark 2.11). By Lemma 5.2 K_Y+B^+ is also an n-complement of $K_Y+B^{(m)}$ for $m\gg 0$.

By Lemmas 3.3 and 3.5 we have $B^+\geq B^{(m)}$ for $m\gg 0$. More precisely,

$$b_i^+\quad\begin{cases} =1 & \text{if } b_i^\infty\geq 1-\varepsilon',\\ \geq b_i^{(m)}=b_i^\infty & \text{if } b_i^\infty<1-\varepsilon'.\end{cases}$$

By the construction of D we have $D^{(m)}\leq B^+$. Hence $-(K_Y+D^{(m)})$ is nef, a contradiction. This completes the proof of Theorems 1.4 and 1.5 in case (4.9.1).

7. Effective adjunction

In this section we discuss the adjunction conjecture for fibre spaces. This conjecture can be considered as a generalization of the classical Kodaira canonical bundle formula for canonical bundle, see [Kod63], [Fuj86], [Kaw97], [Kaw98], [Amb99], [Fuj99], [FM00], [Fuj03], [Amb04], [Amb05].

7.1. The set-up. Let $f\colon X\to Z$ be a surjective morphism of normal varieties and let $D=\sum d_iD_i$ be an $\mathbb{R}$-divisor on X such that (X,D) is lc near the generic fibre of f and $K+D$ is $\mathbb{R}$-Cartier over the generic point of any prime divisor $W\subset Z$. In particular, $d_i\leq 1$ whenever $f(D_i)=Z$. Let $d:=\dim X$ and $d':=\dim Z$.

For any divisor $F = \sum \alpha_i F_i$ on X, we decompose F as $F = F^{\mathrm{h}} + F^{\mathrm{v}}$, where

$$F^{\mathrm{h}} := \sum_{f(F_i)=Z} \alpha_i F_i, \qquad F^{\mathrm{v}} := \sum_{f(F_i)\neq Z} \alpha_i F_i.$$

These divisors F^{h} and F^{v} are called the *horizontal* and *vertical* parts of F, respectively.

7.2. Construction. For a prime divisor $W \subset Z$, define a real number c_W as the log canonical threshold over the generic point of W:
(7.2.1)

$$c_W := \sup\{c \mid (X, D + cf^{\bullet}W) \quad \text{is lc over the generic point of } W\}.$$

It is clear that $c_W \in \mathbb{Q}$ whenever D is a $\mathbb{Q}$-divisor. Put $d_W := 1 - c_W$. Then the $\mathbb{R}$-divisor

$$D_{\mathrm{div}} := \sum_W d_W W$$

is called the *divisorial part* of adjunction (or *discriminant* of f) for $K_X + D$. It is easy to see that D_{div} is a divisor, i.e., d_W is zero except for a finite number of prime divisors.

Remark 7.3.
(i) Note that the definition of the discriminant D_{div} is a codimension one construction, so computing D_{div} we can systematically remove codimension two subvarieties in Z and pass to general hyperplane sections $f_H\colon X \cap f^{-1}(H) \to Z \cap H$.
(ii) Let $h\colon X' \to X$ be a birational contraction and let D' be the crepant pull-back of D:

$$K_{X'} + D' = h^*(K_X + D), \quad h_*D' = D.$$

Then $D'_{\mathrm{div}} = D_{\mathrm{div}}$, i.e., the discriminant D_{div} does not depend on the choice of crepant birational model of (X, D) over Z.

The following lemma is an immediate consequence of the definition.

Lemma 7.4. *Notation as in* 7.1.
(i) (effectivity, cf. [Sho93, 3.2]) *If D is boundary over the generic point of any prime divisor $W \subset Z$, then D_{div} effective.*
(ii) (semiadditivity, cf. [Sho93, 3.2]) *Let Δ be an $\mathbb{R}$-divisor on Z and let $D' := D + f^{\bullet}\Delta$. Then $D'_{\mathrm{div}} = D_{\mathrm{div}} + \Delta$.*
(iii) *(X, D) is klt (resp., lc) over the generic point of W if and only if $d_W < 1$ (resp., $d_W \le 1$).*
(iv) *If (X, D) is lc and D is an $\mathbb{R}$- (resp., $\mathbb{Q}$-)boundary, then D_{div} is an $\mathbb{R}$- (resp., $\mathbb{Q}$-)boundary.*

7.5. Construction. From now on assume that f is a contraction, $K_X + D$ is $\mathbb{R}$-Cartier, and $K + D \sim_{\mathbb{R}} f^*L$ for some $\mathbb{R}$-Cartier divisor L on Z. Recall that the latter means that there are real numbers α_j and rational functions $\varphi_j \in \Bbbk(X)$ such that

$$K + D - f^*L = \sum \alpha_j\,(\varphi_j). \tag{7.5.1}$$

Define the *moduli part* D_{mod} of $K_X + D$ by

$$D_{\mathrm{mod}} := L - K_Z - D_{\mathrm{div}}. \tag{7.5.2}$$

Then we have

$$K_X + D = f^*(K_Z + D_{\mathrm{div}} + D_{\mathrm{mod}}) + \sum \alpha_j\,(\varphi_j). \tag{7.5.3}$$

In particular,

$$K_X + D \sim_{\mathbb{R}} f^*(K_Z + D_{\mathrm{div}} + D_{\mathrm{mod}}).$$

Clearly, D_{mod} depends on the choice of representatives of K_X and K_Z, and also on the choice of α_j and φ_j in (7.5.1). Any change of K_X and K_Z and change of α_j and φ_j gives a new D_{mod} which differs from the original one modulo $\mathbb{R}$-linear equivalence.

If $K + D$ is $\mathbb{Q}$-Cartier, the definition of the moduli part is more explicit. By our assumption (7.5.1) there is a positive integer I_0 such that $I_0(K + D)$ is linearly trivial on the generic fibre. Then for some rational function $\psi \in \Bbbk(X)$, the divisor $M := I_0(K+D)+(\psi)$ is vertical (and $\mathbb{Q}$-linearly trivial over Z). Thus,

$$M - I_0 f^*L = (\psi) + \sum I_0\alpha_j\,(\varphi_j), \qquad \alpha_j \in \mathbb{Q}.$$

Rewrite it in a more compact form: $M - I_0 f^*L = \alpha\,(\varphi)$, $\alpha \in \mathbb{Q}$, $\varphi \in \Bbbk(X)$. The function φ vanishes on the generic fibre, hence it is a pull-back of some function $\upsilon \in \Bbbk(Z)$. Replacing L with $L + \frac{\alpha}{I_0}\,(\upsilon)$ we get $M = I_0 f^*L$ and

$$K_X + D - f^*L = \frac{1}{I_0}(\psi), \qquad \psi \in \Bbbk(X). \tag{7.5.4}$$

In other words, $K + D \sim_{I_0} f^*L$. Here L is $\mathbb{Q}$-Cartier. Then again we define the moduli part D_{mod} of $K_X + D$ by (7.5.2), where L is taken to satisfy (7.5.4). In this case, D_{mod} is $\mathbb{Q}$-Cartier and we have

$$K_X + D = f^*(K_Z + D_{\mathrm{div}} + D_{\mathrm{mod}}) + \frac{1}{I_0}\,(\psi). \tag{7.5.5}$$

In particular,

$$K_X + D \sim_{I_0} f^*(K_Z + D_{\mathrm{div}} + D_{\mathrm{mod}}).$$

As above, D_{mod} depends on the choice of representatives of K_X and K_Z, and also on the choice of I_0 and ψ in (7.5.4). Note that I_0 depends only on f and the horizontal part of D. Once these are fixed, we usually will assume that I_0 is a constant. Then any change of K_X, K_Z, and ψ gives a new D_{mod} which differs from the original one modulo I_0-linear equivalence.

Remark 7.5.1. By Lemma 7.4 $(D+f^{\bullet}\Delta)_{\mathrm{mod}} = D_{\mathrm{mod}}$. Roughly speaking this means that "the moduli part depends only on the horizontal part of D".

For convenience of the reader we recall definition of b-divisors and related notions, see [Isk03] for details.

Definition 7.6. Let X be a normal variety. Consider an infinite linear combination $\mathbf{D} := \sum_P d_P P$, where $d_P \in \mathbb{R}$ and P runs through all discrete valuations P of the function field. For any birational model Y of X define the trace of $\mathbf{D}$ on Y as follows $\mathbf{D}_Y := \sum\limits_{\mathrm{codim}_Y P=1} d_P P$. A *b-divisor* is a linear combination $\mathbf{D} = \sum_P d_P P$ such that the trace $\mathbf{D}_Y$ on each birational model Y of X is an $\mathbb{R}$-divisor, i.e., only a finite number of multiplicities of $\mathbf{D}_Y$ are non-zero. In other words, a b-divisor is an element of $\varprojlim \mathrm{Div}_{\mathbb{R}}(Y)$, where Y in the inverse limit runs through all normal birational models $f\colon Y \to X$, $\mathrm{Div}_{\mathbb{R}}(Y)$ is the group of $\mathbb{R}$-divisors of Y, and the map $\mathrm{Div}_{\mathbb{R}}(Y) \to \mathrm{Div}_{\mathbb{R}}(X)$ is the push-forward. Let D be a $\mathbb{R}$-Cartier divisor on X. The *Cartier closure* of D is a b-divisor $\overline{D}$ whose trace on every birational model $f\colon Y \to X$ is f^*D. A b-divisor $\mathbf{D}$ is said to be b-Cartier if there is a model X' and a $\mathbb{R}$-Cartier divisor D' on X' such that $\mathbf{D} = \overline{D'}$. A b-divisor $\mathbf{D}$ is said to be b-nef (resp. b-semiample, b-free) if it is b-Cartier and there is a model X' and a $\mathbb{R}$-Cartier divisor D' on X' such that $\mathbf{D} = \overline{D'}$ and D' is nef (resp. semiample, integral and free).

$\mathbb{Q}$- and $\mathbb{Z}$-versions of b-divisors are defined similarly.

Remark 7.7. Let $g\colon Z' \to Z$ be a birational contraction. Consider the following diagram

$$
\begin{array}{ccc}
X' & \xrightarrow{h} & X \\
{\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow \\
Z' & \xrightarrow{g} & Z
\end{array}
\tag{7.7.1}
$$

where X' is a resolution of the dominant component of $X \times_Z Z'$. Let D' be the crepant pull-back of D that is $K_{X'} + D' = h^*(K_X + D)$ and $h_*D' = D$. By Remark 7.3 we have $g_*D'_{\mathrm{div}} = D_{\mathrm{div}}$. Therefore, the discriminant defines a b-divisor $\mathbf{D}_{\mathrm{div}}$.

For a suitable choice of K'_X, we can write

$$h^*(K_X + D) = K_{X'} + D',$$

Now we fix the choice of K, α_j and φ_j in (7.5.1) (resp. K and ψ in (7.5.4)) and induce them naturally to X'. Then D_{mod} and D'_{mod} are uniquely determined and $g_* D'_{\mathrm{mod}} = D_{\mathrm{mod}}$. This defines a b-divisor $\mathbf{D}_{\mathrm{mod}}$.

We can write

$$K_{Z'} + D'_{\mathrm{div}} + D' = g^*(K_Z + D_{\mathrm{div}} + D_{\mathrm{mod}}) + E,$$

where E is g-exceptional. Since

$$h^* f^*(K_Z + D_{\mathrm{div}} + D_{\mathrm{mod}}) \equiv K_{X'} + D' \equiv f'^*(K_{Z'} + D'_{\mathrm{div}} + D'_{\mathrm{mod}}),$$

we have $E = 0$ (see [Sho93, 1.1]), i.e., g is $(K_Z + D_{\mathrm{div}} + D_{\mathrm{mod}})$-crepant:

$$K_{Z'} + D'_{\mathrm{div}} + D'_{\mathrm{mod}} = g^*(K_Z + D_{\mathrm{div}} + D_{\mathrm{mod}}). \tag{7.7.2}$$

Let us consider some examples.

Example 7.8. Assume that the contraction f is birational. Then by the ramification formula [Sho93, §2], [Kol92, Prop. 20.3] and negativity lemma [Sho93, 1.1] we have $D_{\mathrm{div}} = f_* D$, $K + D = f^*(K_Z + D_{\mathrm{div}})$, and $D_{\mathrm{mod}} = 0$.

Example 7.9. Let $X = Z \times \mathbb{P}^1$ and let f be the natural projection to the first factor. Take very ample divisors $H_1, \dots, H_4$ on Z. Let C be a section and let D_i be a general member of the linear system $|f^* H_i + C|$. Put $D := \frac{1}{2} \sum D_i$. Then $K_X + D$ is $\mathbb{Q}$-linearly trivial over Z. By Bertini's theorem $D + f^* P$ is lc for any point $P \in Z$. Hence $D_{\mathrm{div}} = 0$. On the other hand,

$$K_X + D = f^* K_Z - 2C + \frac{1}{2} f^* \sum H_i + 2C = f^* \left(K_Z + \frac{1}{2} \sum H_i \right).$$

This gives us that $D_{\mathrm{mod}} \sim_{\mathbb{Q}} \frac{1}{2} \sum H_i$.

Example 7.10. Let X be a hyperelliptic surface. Recall that it is constructed as the quotient $X = (E \times C)/G$ of the product of two elliptic curves by a finite group G acting on E and C so that the action of G on E is fixed point free and the action on C has fixed points. Let

$$f \colon X = (E \times C)/G \to \mathbb{P}^1 = C/G$$

be the projection. It is clear that degenerate fibres of f can be only of type $m\mathrm{I}_0$. Using the classification of such possible actions (see, e.g., [BPVdV84, Ch. V, Sect. 5]) we obtain the following cases:

Type		singular fibres	D_{div}
a)	$(2K_X \sim 0)$	$2\mathrm{I}_0$, $2\mathrm{I}_0$, $2\mathrm{I}_0$, $2\mathrm{I}_0$	$\frac{1}{2}P_1 + \frac{1}{2}P_2 + \frac{1}{2}P_3 + \frac{1}{2}P_4$
b)	$(3K_X \sim 0)$	$3\mathrm{I}_0$, $3\mathrm{I}_0$, $3\mathrm{I}_0$	$\frac{2}{3}P_1 + \frac{2}{3}P_2 + \frac{2}{3}P_3$
c)	$(4K_X \sim 0)$	$2\mathrm{I}_0$, $4\mathrm{I}_0$, $4\mathrm{I}_0$	$\frac{1}{2}P_1 + \frac{3}{4}P_2 + \frac{3}{4}P_3$
d)	$(6K_X \sim 0)$	$2\mathrm{I}_0$, $3\mathrm{I}_0$, $6\mathrm{I}_0$	$\frac{1}{2}P_1 + \frac{2}{3}P_2 + \frac{5}{6}P_3$

In all cases the moduli part D_{mod} is trivial.

Assumption 7.11. Under the notation of 7.1 and 7.5 assume additionally that D is a $\mathbb{Q}$-divisor and there is a $\mathbb{Q}$-divisor Θ on X such that $K_X + \Theta$ is $\mathbb{Q}$-linearly trivial over Z and $(F, (1-t)D|_F + t\Theta|_F)$ is a klt log pair for any $0 < t \le 1$, where F is the generic fibre of f. In particular, Θ and D are $\mathbb{Q}$-boundaries near the generic fibre. In this case, both D_{div} and D_{mod} are $\mathbb{Q}$-divisors.

The following result is very important.

Theorem 7.12 ([Amb04])**.** *Notation and assumptions as in* 7.1 *and* 7.5*. Assume additionally that D is a $\mathbb{Q}$-divisor, D is effective near the generic fibre, and (X, D) is klt near the generic fibre. Then we have.*

(i) *The b-divisor $\mathbf{K} + \mathbf{D}_{\mathrm{div}}$ is b-Cartier.*
(ii) *The b-divisor $\mathbf{D}_{\mathrm{mod}}$ is b-nef.*

According (ii) of Theorem 7.12 the b-divisor $\mathbf{D}_{\mathrm{mod}}$ is b-nef for $D \ge 0$ and (X, D) is klt near the generic fibre (see also [Kaw98, Th. 2], [Fuj99]). We expect more.

Conjecture 7.13. *Let notation and assumptions be as in* 7.1 *and* 7.11*. We have*

(7.13.1) (*Log Canonical Adjunction*) $\mathbf{D}_{\mathrm{mod}}$ *is b-semiample.*

(7.13.2) (*Particular Case of Effective Log Abundance Conjecture*) *Let X_η be the generic fibre of f. Then $I_0(K_{X_\eta} + D_\eta) \sim 0$, where I_0 depends only on $\dim X_\eta$ and the multiplicities of D^{h}.*

(7.13.3) (*Effective Adjunction*) $\mathbf{D}_{\mathrm{mod}}$ *is effectively b-semiample, that is, there exists a positive integer I depending only on the dimension of X and the horizontal multiplicities of D (a finite set of rational numbers) such that $I\mathbf{D}_{\mathrm{mod}}$ is very b-semiample, that is, $I\mathbf{D}_{\mathrm{mod}} = \overline{M}$, where M is a base point free divisor on some model Z'/Z.*

Note that by (7.5.5) we may assume that

$$K + D \sim_I f^*(K_Z + D_{\mathrm{div}} + D_{\mathrm{mod}}). \qquad (7.13.4)$$

Remark 7.14. We expect that hypothesis in 7.13 can be weakened as follows.

(7.13.1) It is sufficient to assume that $K+D$ is lc near the generic fibre, the horizontal part D^{h} of D is an $\mathbb{R}$-boundary, $K+D$ is $\mathbb{R}$-Cartier, and $K+D \equiv f^*L$.

(7.13.2) D^{h} is a $\mathbb{Q}$-boundary and $K+D \equiv 0$ near the generic fibre.

(7.13.3) D^{h} is a $\mathbb{Q}$-boundary, $K+D$ is $\mathbb{R}$-Cartier, and $K+D \equiv f^*L$.

This however is not needed for the proof of the main theorem.

Remark 7.15. In the notation of (7.13.2) we have $K_{X_\eta} + D_\eta \sim_{\mathbb{Q}} 0$, where X_η is the generic fibre of f. Assume that

(i) X is FT, and
(ii) LMMP and conjectures 1.1 and 7.13 hold in dimensions $\le \dim X - \dim Z$.

Then the pair (X_η, D_η) satisfies the assumptions of Theorem 1.5 with $\mathfrak{R}$ depending only on horizontal multiplicities of D. Hence $I_0(K_{X_\eta}+D_\eta) \sim 0$, where I_0 depends only on $\dim X_\eta$ and horizontal multiplicities of D. Thus (7.13.2) holds automatically under additional assumptions (i)-(ii).

Example 7.16 (Kodaira formula [Kod63], [Fuj86])**.** Let $f\colon X \to Z$ be a fibration satisfying 7.5 whose generic fibre is an elliptic curve. Then $D^{\mathrm{h}} = 0$, $D = D^{\mathrm{v}}$, and $I_0 = 1$. Thus we can write $K_X + D = f^*L$. The j-invariant defines a rational map $J\colon Z \dashrightarrow \mathbb{C}$. By blowing up Z and X we may assume that both X and Z are smooth and J is a morphism: $J\colon Z \to \mathbb{P}^1$. Let P be a divisor of degree 1 on $\mathbb{P}^1$. Take a positive integer n such that $12n$ is divisible by the multiplicities of all the degenerate fibres of f. In this situation, there is a generalization of the classical Kodaira formula [Fuj86]:

$$12n(K_X + D) = f^*\left(12nK_Z + 12nD_{\mathrm{div}} + nJ^*P\right),$$

We can rewrite it as follows

$$K_X + D = f^*\left(K_Z + D_{\mathrm{div}} + \frac{1}{12}J^*P\right). \tag{7.16.1}$$

Here $D_{\mathrm{mod}} = \frac{1}{12}J^*P$ is semiample and the multiplicities of D_{div} are taken from the table in Example 3.1 if $D = 0$ over such divisors in Z or D is minimal as in Lemma 8.9. (Otherwise to compute D_{div} we can use semiadditivity Lemma 7.4, (ii).)

Example 7.17. Fix a positive integer m. Let $(E, 0)$ be an elliptic curve with fixed group low and let $e_m \in E$ be an m-torsion. Define the action of $\boldsymbol{\mu}_m := \left\{\sqrt[m]{1}\right\}$ on $E \times \mathbb{P}^1$ by

$$\varepsilon(e, z) = (e + e_m, \varepsilon z), \qquad e \in E,\ z \in \mathbb{P}^1,$$

where $\varepsilon \in \boldsymbol{\mu}_m$ is a primitive m-root. The quotient map

$$X := (E \times \mathbb{P}^1)/\boldsymbol{\mu}_m \longrightarrow \mathbb{P}^1/\boldsymbol{\mu}_m \simeq \mathbb{P}^1$$

is an elliptic fibration having exactly two fibres of types $m\mathrm{I}_0$ over points 0 and $\infty \in \mathbb{P}^1$. Using the Kodaira formula one can show that

$$K_X = f^*K_Z + (m-1)F_0 + (m-1)F_\infty,$$

where $F_0 := f^{-1}(0)_{\mathrm{red}}$ and $F_\infty := f^{-1}(\infty)_{\mathrm{red}}$. Hence, in (7.5.4) we have $I_0 = 1$ and

$$L = K_Z + \left(1 - \frac{1}{m}\right) \cdot 0 + \left(1 - \frac{1}{m}\right) \cdot \infty.$$

Clearly,

$$D_{\mathrm{div}} = \left(1 - \frac{1}{m}\right) \cdot 0 + \left(1 - \frac{1}{m}\right) \cdot \infty.$$

Hence $D_{\mathrm{mod}} = 0$, $I = I_0 = 1$, and $K_X = f^*(K_Z + D_{\mathrm{div}})$.

Corollary 7.18 (cf. [Amb04, Th. 3.1])**.** *Let notation and assumptions be as in* 7.1, 7.5, *and* 7.11 (*cf.* 7.14).

(i) *If* $(Z, D_{\mathrm{div}} + D_{\mathrm{mod}})$ *is lc and* $\mathbf{D}_{\mathrm{mod}}$ *is effective, then* (X, D) *is lc.*

(ii) *Assume that* (7.13.1) *holds. If* (X, D) *is lc, then so is* $(Z, D_{\mathrm{div}} + D_{\mathrm{mod}})$ *for a suitable choice of* D_{mod} *in the class of* $\mathbb{Q}$*-linear equivalence* (*respectively* I*-linear equivalence under* (7.13.2)). *Moreover, if* (X, D) *is lc and any lc centre of* (X, D) *dominates* Z*, then* $(Z, D_{\mathrm{div}} + D_{\mathrm{mod}})$ *is klt.*

Proof. For a log resolution $g \colon Z' \to Z$ of the pair (Z, D_{div}), consider base change (7.7.1). Thus $\operatorname{Supp} D'_{\mathrm{div}}$ is a simple normal crossing divisor on Z'.

(i) Put $D_t := (1-t)D + t\Theta$ (see 7.11). Assume that (X, D) is not lc. Then (X, D_t) is also not lc for some $0 < t \ll 1$. Let F be a divisor of discrepancy $a(F, X, D_t) < -1$. Since (X, D_t) is klt near the generic fibre, the centre of F on Z is a proper subvariety. By Theorem 7.12 we can take g so that $\mathbf{K} + (\mathbf{D}_t)_{\mathrm{div}} = \overline{K_{Z'} + (D'_t)_{\mathrm{div}}}$ and $(\mathbf{D}_t)_{\mathrm{mod}} = \overline{(D'_t)_{\mathrm{mod}}}$. Moreover, by [Kol96, Ch. VI, Th. 1.3] we can also take g so that the centre of F on Z' is a prime divisor, say W. Put $(D_t)_Z := (D_t)_{\mathrm{div}} + (D_t)_{\mathrm{mod}}$. By (7.7.2) we have

$$-1 \le a(W, Z, (D_t)_Z) = a(W, Z', (D'_t)_{\mathrm{div}} + (D'_t)_{\mathrm{mod}}) \le a(W, Z', (D'_t)_{\mathrm{div}}).$$

Therefore (X', D'_t) is lc over the generic point of W (see (7.2.1)). In particular, $a(F, X', D'_t) = a(F, X, D_t) \ge -1$, a contradiction.

(ii) By our assumption (7.13.1) $\mathbf{D}_{\mathrm{mod}}$ is b-Cartier, so we can take g so that $\mathbf{D}_{\mathrm{mod}} = \overline{D'_{\mathrm{mod}}}$ and $\mathbf{K} + \mathbf{D}_{\mathrm{div}} = \overline{K_{Z'} + D'_{\mathrm{div}}}$. Moreover by (7.13.1) (respectively by (7.13.2)) we can take g so that D'_{mod} (respectively ID'_{mod}) is semiample (respectively linearly free). By (7.7.2) g is $(K + D_{\mathrm{div}} + D_{\mathrm{mod}})$-crepant. If (Z', D'_{div}) is lc (resp. klt), then replacing D'_{mod} with an effective general representative of the corresponding class of $\mathbb{Q}$-linear equivalence we obtain

$$\operatorname{discr}(Z, D_{\mathrm{div}} + D_{\mathrm{mod}}) = \operatorname{discr}(Z', D'_{\mathrm{div}} + D'_{\mathrm{mod}}) = \operatorname{discr}(Z', D'_{\mathrm{div}}) \geq -1.$$

(resp. > -1). We can suppose also that $\lfloor D_{\mathrm{mod}} \rfloor = 0$. Hence $(Z, D_{\mathrm{div}} + D_{\mathrm{mod}})$ is lc (resp. klt) in this case. Thus we assume that (Z', D'_{div}) is not lc (resp. not klt). Let E be a divisor over Z of discrepancy $a(E, Z', D'_{\mathrm{div}}) \leq -1$. Clearly, we may assume that $\operatorname{Center}_{Z'} E \not\subset \operatorname{Supp} D'_{\mathrm{mod}}$. Then $a(E, Z', D'_{\mathrm{div}} + D'_{\mathrm{mod}}) = a(E, Z', D'_{\mathrm{div}}) \leq -1$. Replacing Z' with its blowup we may assume that E is a prime divisor on Z' (and again $\operatorname{Center}_{Z'} E \not\subset \operatorname{Supp} D'_{\mathrm{mod}}$). Since (X', D') is lc and by (7.2.1), $c_E = 0$, $d_E = 1$, and $a(E, Z', D'_{\mathrm{div}}) = -1$. Then (Z', D'_{div}) is lc. Furthermore, by (7.2.1) the pair $(X', D' + cf'^{\bullet}E)$ is not lc for any $c > 0$. This means that $f^{-1}(\operatorname{Center}_Z(E))$ contains an lc centre. $\square$

The following example shows that the condition $\mathbf{D}_{\mathrm{mod}} \geq 0$ in (i) of Corollary 7.18 cannot be omitted.

Example 7.19. Let $f\colon X \to Z = \mathbb{C}^2$ be a standard conic bundle given by $x^2 + uy^2 + vz^2$ in $\mathbb{P}^2_{x,y,z} \times \mathbb{C}^2_{u,v}$. The linear system $|-nK_X|$ is base point free for $n \geq 1$. Let $H \in |-2K_X|$ be a general member. Now let $\Gamma(t) := \Gamma_1 + t\Gamma_2$, where $\Gamma_1 := \{u = 0\}$ and $\Gamma_2 := \{v = 0\}$. Put $D(t) := \frac{1}{2}H + f^*\Gamma(t)$. Then $2(K + D) = 2f^*\Gamma(t)$ and $D(t)_{\mathrm{div}} = \Gamma(t)$. Since $K_Z = 0$, we have $D_{\mathrm{mod}} = 0$.

For $t = 1$, the log divisor $K_Z + \Gamma(t)$ is lc but $K + D(t) = f^*(K_Z + \Gamma(t))$ is not. Indeed, in the chart $z \neq 0$ there is an isomorphism

$$(X, f^*\Gamma) \simeq (\mathbb{C}^3_{x,y,u}, \{u(x^2 + uy^2) = 0\}). \tag{7.19.1}$$

The explanation of this fact is that the b-divisor $\mathbf{D}_{\mathrm{mod}}$ is non-trivial. To show this we consider the following diagram [Sar80, §2]:

$$\begin{tikzcd}
X' \arrow[r, dashed, "\chi"] \arrow[dd, "f'"] & \tilde{X} \arrow[dr, "h"] & \\
 & & X \arrow[d, "f"] \\
Z' \arrow[rr, "g"] & & Z
\end{tikzcd}$$

where h is the blowup the central fibre $f^{-1}(0)_{\mathrm{red}}$, χ is the simplest flop, g is the blowup of 0, and f' is again a standard conic bundle. Put $t = 1/2$ and let $\tilde{D}$ and D' be the crepant pull-backs of $D := D(t)$ on $\tilde{X}$ and X', respectively. The h-exceptional divisor F appears in $\tilde{D}$ with multiplicity 1/2. Let F' be the proper transform of F on X'. Then $F' = f'^*E$, where E is the g-exceptional divisor. It is easy to see from (7.19.1) that the pair (X, D) is lc but not klt at the generic point of $f^{-1}(0)_{\mathrm{red}}$. So is $(\tilde{X}, \tilde{D})$ at the generic point of the flopping curve. This implies that (X', D') is lc but not klt over the generic point of E. Therefore, $D'_{\mathrm{div}} = E + \Gamma'$, where Γ' is the proper transform of Γ. On Z', we have $K_{Z'} = E$ and $K + D' = f'^*g^*\Gamma$, so $D'_{\mathrm{mod}} = g^*\Gamma - E - D'_{\mathrm{div}} = -\frac{1}{2}E$. Thus $D'_{\mathrm{mod}} \leq 0$ and $2D'_{\mathrm{mod}}$ is free.

8. Two important particular cases of Effective Adjunction

Using a construction and a result of [Kaw97] we prove the following.

Theorem 8.1. *Conjectures* 7.13 *hold if* $\dim X = \dim Z + 1$.

Remark 8.2. We expect that in this case one can take $I = 12q$, where q is a positive integer such that qD^{h} is an integral divisor.

Proof of Theorem 8.1. We may assume that a general fibre of f is a rational curve (see Example 7.16). Thus the horizontal part D^{h} of D is non-trivial. First we reduce the problem to the case when all components of D^{h} are generically sections. Write $D = \sum d_i D_i$ and take

$$\delta := \min\{d_i \mid D_i \text{ is horizontal and } d_i > 0\}.$$

(we allow components with $d_i = 0$). Let D_i be a horizontal component and let $D_i \to \hat{Z} \xrightarrow{g} Z$ be the Stein factorization of the restriction $f|_{D_i}$. Let $n_i := \deg g$. Let l be a general fibre of f. Since $d_i D_i \cdot l \leq D \cdot l = -K \cdot l = 2$, we have

$$n_i = D_i \cdot l \leq 2/d_i \leq 2/\delta. \tag{8.2.1}$$

Assume that $n_i > 1$. Consider the base change

$$\begin{array}{ccc} \hat{X} & \xrightarrow{h} & X \\ {\scriptstyle \hat{f}}\downarrow & & {\scriptstyle f}\downarrow \\ \hat{Z} & \xrightarrow{g} & Z \end{array}$$

where $\hat{X}$ is the normalization of the dominant component of $X \times_Z \hat{Z}$. Define $\hat{D}$ on $\hat{X}$ by

$$K_{\hat{X}} + \hat{D} = h^*(K_X + D). \tag{8.2.2}$$

More precisely, $\hat{D} = \sum_{i,j} \hat{d}_{i,j}\hat{D}_{i,j}$, where $h(\hat{D}_{i,j}) = D_i$, $1-\hat{d}_{i,j} = r_{i,j}(1-d_i)$, and $r_{i,j}$ is the ramification index along $\hat{D}_{i,j}$. By construction, the ramification locus Λ of h is $\hat{f}$-exceptional, that is $\hat{f}(\Lambda) \neq \hat{Z}$. Therefore, $\hat{D}$ is a boundary near the generic fibre. Similarly, we define $\hat{\Theta}$ as the crepant pull-back of Θ from 7.11. Thus the pair $(\hat{X}, \hat{D})$ satisfies assumptions of 7.1 and 7.11. It follows from (8.2.2) that

$$K_{\hat{X}} + \hat{D} = \hat{f}^* g^* (K_Z + D_{\mathrm{div}} + D_{\mathrm{mod}}).$$

According to [Amb99, Th. 3.2] for the discriminant $\hat{D}_{\mathrm{div}}$ of $\hat{f}$ we have

$$K_{\hat{Z}} + \hat{D}_{\mathrm{div}} = g^*(K_Z + D_{\mathrm{div}}).$$

For a suitable choice of $\hat{D}_{\mathrm{mod}}$ in the class of n_i-linear equivalence, we can write

$$K_{\hat{Z}} + \hat{D}_{\mathrm{div}} + \hat{D}_{\mathrm{mod}} = g^*(K_Z + D_{\mathrm{div}} + D_{\mathrm{mod}}).$$

Therefore, $\hat{D}_{\mathrm{mod}} = g^* D_{\mathrm{mod}}$. If $\hat{I}\hat{D}_{\mathrm{mod}}$ is free for some positive integer $\hat{I}$, then so is $n_i\hat{I}D_{\mathrm{mod}}$. Thus we have proved the following.

Claim 8.3. *Assume that Conjecture* 7.13 *holds for* $\hat{f} \colon \hat{X} \to \hat{Z}$ *with constant* $\hat{I}$. *Then this conjecture holds for* $f \colon X \to Z$ *with* $I := n_i\hat{I}$.

Note that the restriction $\hat{f}|_{h^{-1}(D_i)} \colon h^{-1}(D_i) \to \hat{Z}$ is generically finite of degree n_i. Moreover, $h^{-1}(D_i)$ has a component which is a section over the generic point. Applying Claim 8.3 several times and taking (8.2.1) into account we obtain the desired reduction to the case when all the horizontal D_i's with $d_i > 0$ are generically sections.

8.4. Further by making a birational base change and by blowing up X we can get the situation when

(i) Z and X are smooth,
(ii) the D_i's are regular disjointed sections,
(iii) the morphism f is smooth outside of a simple normal crossing divisor $\Xi \subset Z$,
(iv) $f^{-1}(\Xi) \cup \operatorname{Supp} D$ is also a simple normal crossing divisor.

Let n be the number of horizontal components of D. Note that we allow sections with multiplicities $d_i = 0$ on this step.

Let $\mathcal{M}_n$ be the moduli space of n-pointed stable rational curves, let $f_n \colon \mathcal{U}_n \to \mathcal{M}_n$ be the corresponding universal family, and let $\mathcal{P}_1, \dots, \mathcal{P}_n$ be sections of f_n which correspond to the marked points (see [Knu83]). It is known that both $\mathcal{M}_n$ and $\mathcal{U}_n$ are smooth and projective. Take $d_i \in [0, 1]$ so that $\sum d_i = 2$ and put $\mathcal{D} := \sum d_i\mathcal{P}_i$. Then $K_{\mathcal{U}_n} + \mathcal{D}$ is trivial on the general fibre. However, $K_{\mathcal{U}_n}+\mathcal{D}$ is not numerically trivial everywhere over $\mathcal{M}_n$, moreover, it is not nef everywhere over $\mathcal{M}_n$:

Theorem 8.5 (see [Kee92], [Kaw97]). (i) *There exist a smooth projective variety $\bar{\mathcal{U}}_n$, a $\mathbb{P}^1$-bundle $\bar{f}_n\colon \bar{\mathcal{U}}_n \to \mathcal{M}_n$, and a sequence of blowups (blowdowns) with smooth centres*

$$\sigma\colon \mathcal{U}_n = \mathcal{U}^1 \to \mathcal{U}^2 \to \cdots \to \mathcal{U}^{n-2} = \bar{\mathcal{U}}_n.$$

(ii) *For $\bar{\mathcal{D}} := \sigma_*\mathcal{D}$, the (discrepancy) divisor*

$$\mathcal{F} := K_{\mathcal{U}_n} + \mathcal{D} - \sigma^*(K_{\bar{\mathcal{U}}_n} + \bar{\mathcal{D}})$$

is effective and essentially exceptional *on $\mathcal{M}_n$.*

(iii) *There exists a semiample $\mathbb{Q}$-divisor $\mathcal{L}$ on $\mathcal{M}_n$ such that*

$$K_{\bar{\mathcal{U}}_n} + \bar{\mathcal{D}} = \bar{f}_n^*(K_{\mathcal{M}_n} + \mathcal{L}).$$

Therefore,

$$K_{\mathcal{U}_n} + \mathcal{D} - \mathcal{F} = f_n^*(K_{\mathcal{M}_n} + \mathcal{L}).$$

Recall that for any contraction $\varphi\colon Y \to Y'$, a divisor G on Y is said to be *essentially exceptional* over Y' if for any prime divisor P on Y', the support of the divisorial pull-back $\varphi^\bullet P$ is not contained in $\operatorname{Supp} G$.

Corollary 8.6. *In the above notation we have*

$$(\mathcal{D} - \mathcal{F})_{\mathrm{div}} = 0, \qquad (\mathcal{D} - \mathcal{F})_{\mathrm{mod}} = \mathcal{L}.$$

Moreover, the proof of Theorem 8.1 implies that the b-divisor $\mathbf{G}$, the b-divisor of the moduli part of $\mathcal{D} - \mathcal{F}$, stabilizes on $\mathcal{M}_n$, that is, $\mathbf{G} = \overline{(\mathcal{D} - \mathcal{F})_{\mathrm{mod}}}$.

Proof. See Example 8.10 below. □

Since the horizontal components of D are sections, $(X/Z, D^{\mathrm{h}})$ is generically an n-pointed stable curve [Knu83]. Hence we have the induced rational maps

$$\begin{tikzcd} X \arrow[r, dashed, "\beta"] \arrow[d, "f"] & \mathcal{U}_n \arrow[d, "f_n"] \\ Z \arrow[r, dashed, "\phi"] & \mathcal{M}_n \end{tikzcd}$$

so that $f_n \circ \beta = \phi \circ f$ and $\beta(D_i) \subset \mathcal{P}_i$. Let $\Xi \subset Z$ as above (see 8.4, (iii)). Thus f is a smooth morphism over $Z \setminus \Xi$. Replacing X and Z with its birational models and D with its crepant pull-back we may assume additionally to 8.4 that β and ϕ are regular morphisms.

Now take the $\mathcal{D} = \sum d_i \mathcal{P}_i$ so that it corresponds to the horizontal part $D^{\mathrm{h}} = \sum_{f(D_i)=Z} d_i D_i$. Consider the following commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\mu"] \arrow[rr, bend left, "\beta"] \arrow[ddr, "f"'] & \hat{X} \arrow[r, "\psi"] \arrow[dd, "\hat{f}"] & \mathcal{U}_n \arrow[dd, "f_n"] \arrow[dr, bend left, "\sigma"] & \\
 & & & \bar{\mathcal{U}}_n \arrow[dl, bend left, "\bar{f}_n"] \\
 & Z \arrow[r, "\phi"] & \mathcal{M}_n &
\end{tikzcd}$$

where $\hat{X} := Z \times_{\mathcal{M}_n} \mathcal{U}_n$.

8.7. Since the fibres of f_n are stable curves, near every point $u \in \mathcal{U}_n$ the morphism f_n is either smooth or in a suitable local analytic coordinates is given by

$$(u_1,\, u_2, \ldots, u_{n-2}) \longmapsto (u_1 u_2,\, u_2, \ldots, u_{n-2}).$$

Then easy local computations show that $\hat{X}$ is normal and has only canonical singularities [Kaw97]. Moreover, the pair $(\hat{X}, \hat{D}^{\mathrm{h}} = \psi^* \mathcal{D})$ is canonical because f_n is a smooth morphism near $\operatorname{Supp} \mathcal{D}$.

We have

$$K_X + D = f^*(K_Z + D_{\mathrm{div}} + D_{\mathrm{mod}}).$$

Put $\hat{D} := \mu_* D$. Then $K_X + D = \mu^*(K_{\hat{X}} + \hat{D})$, so

$$\hat{D}_{\mathrm{div}} = D_{\mathrm{div}}, \qquad \hat{D}_{\mathrm{mod}} = D_{\mathrm{mod}},$$

$$K_{\hat{X}} + \hat{D} = \hat{f}^*(K_Z + D_{\mathrm{div}} + D_{\mathrm{mod}}).$$

8.8. Let $\varphi\colon Y \to Y'$ be any contraction, where $\dim Y' \geq 1$. We introduce $G^{\perp} = G - \varphi^{\bullet} G_{\neg}$, where $G_{\neg}$ is taken so that the vertical part $(G^{\perp})^{\mathrm{v}}$ of $G^{\perp}$ is essentially exceptional and $G_{\neg}$ is maximal with this property. In particular, $(G^{\perp})^{\mathrm{v}} \leq 0$ over an open subset $U' \subset Y'$ such that $\operatorname{codim}(Y' \setminus U') \geq 2$. Note that our construction of $G_{\neg}$ and $G^{\perp}$ is in codimension one over Y', i.e., to find $G_{\neg}$ and $G^{\perp}$ we may replace Y' with $Y' \setminus W$, where W is a closed subset of codimension ≥ 2.

Lemma 8.9. *Let $\varphi\colon Y \to Y'$ be a contraction and let G be an $\mathbb{R}$-divisor on Y. Assume that* $\dim Y' \geq 1$. *Assume that* $(Y/Y', G)$ *satisfies conditions* 7.1. *The following are equivalent:*

(i) $G^{\mathrm{v}} - \varphi^{\bullet} G_{\mathrm{div}}$ *is essentially exceptional,*
(ii) $G_{\mathrm{div}} = G_{\neg}$,

(iii) $(G^{\perp})_{\mathrm{div}} = 0$.

Proof. Implications (ii) $\Longleftrightarrow$ (iii) $\Longrightarrow$ (i) follows by definition of G_{div} and semiadditivity (Lemma 7.4). Let us prove (i) $\Longrightarrow$ (ii). Assume that $G^{\mathrm{v}} - \varphi^{\bullet} G_{\mathrm{div}}$ is essentially exceptional. Then by definition $G_{\mathrm{div}} \le G_{\neg}$. On the other hand, for any prime divisor $P \subset Y'$, the multiplicity of $G^{\perp}$ along some component of $\varphi^{\bullet} P$ is equal to 0. Hence the log canonical threshold of $(K + G^{\perp}, \varphi^{\bullet} P)$ over the generic point of P is ≤ 1. So by definition of the divisorial part and Lemma 7.4 we have $0 \le (G^{\perp})_{\mathrm{div}} = G_{\mathrm{div}} - G_{\neg}$. □

Example 8.10. Clearly, for $f_n \colon \mathcal{U}_n \to \mathcal{M}_n$, the discrepancy divisor $\mathcal{F}$ is essentially exceptional. Hence, $(\mathcal{D} - \mathcal{F})_{\mathrm{div}} \ge 0$. On the other hand, by construction every fibre of f_n is reduced. Hence, for every prime divisor $W \subset Z$, the divisorial pull-back $f_n^{\bullet} W$ is reduced and $\mathrm{Supp}(f_n^{\bullet} W + \mathcal{D})$ is a simple normal crossing divisor over the generic point of W. This implies that $c_W \ge 1$ and so $(\mathcal{D} - \mathcal{F})_{\mathrm{div}} = 0$.

Proof of Theorem 8.1 (*continued*). It is sufficient to show that $D_{\mathrm{mod}} = \phi^* \mathcal{L} = \phi^* (\mathcal{D} - \mathcal{F})_{\mathrm{mod}}$ (we replace Z with its blowup if necessary). Then the b-divisor $\mathbf{D}_{\mathrm{mod}}$ automatically stabilizes on Z, i.e., $\mathbf{D}_{\mathrm{mod}} = \overline{D_{\mathrm{mod}}}$. In this situation D_{mod} is effectively semiample because $N\mathcal{L}$ is an integral base point free divisor for some N which depends only on n. Since and ϕ is a regular morphism, to show $D_{\mathrm{mod}} = \phi^* \mathcal{L}$ we will freely replace Z with an open subset $U \subset Z$ such that $\mathrm{codim}(Z \setminus U) \ge 2$. Thus all the statements below are valid over codimension one over Z. In particular, we may assume that $D_{\mathrm{mod}} = (D^{\perp})_{\mathrm{mod}}$. Replacing D with $D^{\perp}$ we may assume that $D_{\neg} = 0$ (we replace Z with U as above). Thus $D^{\mathrm{v}} \le 0$ and D^{v} is essentially exceptional. In particular, $D_{\mathrm{div}} \ge 0$.

On the other hand, by construction the fibres $(\hat{f}^*(z), \hat{D}^{\mathrm{h}} = \psi^* \mathcal{D})$, $z \in Z$ are stable (reduced) curves. In particular, they are slc (semi log canonical [KSB88, §4], [Kol92, Ch. 12]). By the inversion of adjunction [Sho93, §3], [Kol92, Ch. 16-17] for every prime divisor $W \subset Z$ and generic hyperplane sections $H_1, \dots, H_{\dim Z - 1}$ the pair $(\hat{X}, \hat{D}^{\mathrm{h}} + \hat{f}^{\bullet} W + \hat{f}^{\bullet} H_1 + \dots + \hat{f}^{\bullet} H_{\dim Z - 1})$ is lc. Since $\hat{D}^{\mathrm{v}} \le 0$, so is the pair $(\hat{X}, \hat{D} + \hat{f}^{\bullet} W)$. This implies that $c_W \ge 1$ and so $D_{\mathrm{div}} = \hat{D}_{\mathrm{div}} = 0$.

We claim that $\hat{D}^{\mathrm{v}} = \mu_* D^{\mathrm{v}}$ is essentially exceptional. Indeed, otherwise $\hat{D}^{\mathrm{v}}$ is strictly negative over the generic point of some prime divisor $W \subset Z$, i.e., μ contracts all the components E_i of $f^{\bullet} W$ of multiplicity 0. By 8.7 the pair $(\hat{X}, \hat{D} + \varepsilon \hat{f}^{\bullet} W)$ is canonical over the generic point of W for some small positive ε. On the other hand, for the discrepancy of E_i we have $a(E_i, \hat{X}, \hat{D} + \varepsilon \hat{f}^{\bullet} W) = a(E_i, X, D + \varepsilon f^{\bullet} W) = -\varepsilon$. The contradiction proves our claim.

For relative canonical divisors we have

$$K_{\hat{X}/Z} = \psi^* K_{\mathcal{U}_n/\mathcal{M}_n}$$

(see, e.g., [Har77, Ch. II, Prop. 8.10]). Taking $\hat{D}^{\mathrm{h}} = \psi^*\mathcal{D}$ into account we obtain

$$K_{\hat{X}/Z} + \hat{D}^{\mathrm{h}} - \psi^*\mathcal{F} = \psi^*(K_{\mathcal{U}_n/\mathcal{M}_n} + \mathcal{D} - \mathcal{F}) = \psi^* f_n^*\mathcal{L} = \hat{f}^*\phi^*\mathcal{L}.$$

Hence,

$$-\hat{D}^{\mathrm{v}} - \psi^*\mathcal{F} \sim_{\mathbb{R}} K_{\hat{X}} + \hat{D}^{\mathrm{h}} - \psi^*\mathcal{F} \sim_{\mathbb{R}} \hat{f}^*\phi^*\mathcal{L} + \hat{f}^*K_Z$$

over Z, i.e., $\hat{D}^{\mathrm{v}} + \psi^*\mathcal{F}$ is $\mathbb{R}$-linearly trivial over Z.

Since $\psi^*\mathcal{F}$ is also essentially exceptional over Z, by Lemma 8.11 below we have $\hat{D}^{\mathrm{v}} = -\psi^*\mathcal{F}$ and

$$\hat{f}^* D_{\mathrm{mod}} = \hat{f}^*(D_{\mathrm{mod}} + D_{\mathrm{div}}) = K_{\hat{X}/Z} + \hat{D} = \hat{f}^*\phi^*\mathcal{L}.$$

This gives us $D_{\mathrm{mod}} = \phi^*\mathcal{L} = \phi^*(\mathcal{D} - \mathcal{F})_{\mathrm{mod}}$. Therefore D_{mod} is effectively semiample. This proves Theorem 8.1. □

Lemma 8.11 (cf. [Pro03, Lemma 1.6])**.** *Let $\varphi\colon Y \to Y'$ be a contraction with $\dim Y' \geq 1$ and let A, B be essentially exceptional over Y' divisors on Y such that $A \equiv B$ over Y' and A, $B \leq 0$ (both conditions are over codimension one over Y'). Then $A = B$ over codimension one over Y'.*

Proof. The statement is well-known in the birational case (see [Sho93, §1.1]), so we assume that $\dim Y' < \dim Y$. As in [Pro03, Lemma 1.6], replacing Y' with its general hyperplane section $H' \subset Y'$ and Y with $\varphi^{-1}(H')$ we may assume that $\dim \varphi(\operatorname{Supp} A) = 0$ and $\dim \varphi(\operatorname{Supp} B) \geq 0$. The essential exceptionality of A and B is preserved.

We may also assume that Y' is a sufficiently small affine neighbourhood of some fixed point $o \in Y'$ (and $\varphi(\operatorname{Supp} A) = o$). Further, all the conditions of lemma are preserved if we replace Y with its general hyperplane section H. If $\dim Y' > 1$, then we can reduce our situation to the case $\dim Y = \dim Y'$. Then the statement of the lemma follows by [Sho93, §1.1] and from the existence of the Stein factorization. Finally, consider the case $\dim Y' = 1$ (here we may assume that $\dim Y = 2$ and φ has connected fibres). By the Zariski lemma $A = B + a\varphi^* o$ for some $a \in \mathbb{Q}$. Since A and B are essentially exceptional and ≤ 0, $a = 0$. □

□

Example 8.12. Assume that all the components $D_1, \ldots, D_r$ of D^{h} are sections. If $r = 3$, then since $\mathcal{M}_3$ is a point, we have $D_{\mathrm{mod}} = 0$. For $r = 4$ the situation is more complicated: $\mathcal{M}_4 \simeq \mathbb{P}^1$, $\mathcal{U}_4$ is a del Pezzo surface of degree 5, and $f_4\colon \mathcal{U}_4 \to \mathcal{M}_4 = \mathbb{P}^1$ is a conic bundle with three degenerate fibres. Each component of degenerate fibre meets exactly two components of $\mathcal{D}$. Hence $\bar{\mathcal{D}}$ is a normal crossing divisor. It is easy to see that σ contracts a component of a degenerate fibre which meets $\mathcal{D}_i$ and $\mathcal{D}_j$ with $d_i+d_j \le 1$. Clearly, $\bar{\mathcal{U}}_4 \simeq \mathbb{F}_e$ is a rational ruled surface, $e = 0$ or 1. We can write $\bar{\mathcal{D}}_i \sim \Sigma + a_i F$, where Σ is the minimal section and F is a fibre of $\bar{\mathcal{U}}_4 = \mathbb{F}_e \to \mathbb{P}^1$. Up to permutation we may assume that $\bar{\mathcal{D}}_i \neq \Sigma$ for $i = 2, 3, 4$. Taking $\sum d_i = 2$ into account we get

$$K_{\bar{\mathcal{U}}_4}+\bar{\mathcal{D}} \sim -2\Sigma-(2+e)F+\sum d_i(\Sigma+a_iF) = \left(\sum d_ia_i - e\right) F+\bar{f}_n^* K_{\mathcal{M}_4}.$$

Therefore,

$$\deg \mathcal{L} = \sum d_ia_i - e \ge e\sum d_i - ed_1 - e \ge 0.$$

8.13. Now we consider the case when the base variety Z is a curve.

Proposition 8.14. *Assume Conjectures* 1.1 *and* 7.13 *in dimensions* $\le d-1$ *and LMMP in dimension* $\le d$. *If X is FT* (*and projective*) *variety of dimension d, then Conjecture* 7.13 *holds in dimension d.*

Corollary 8.15. *Conjecture* 7.13 *holds true in the following cases*:
(i) $\dim X = \dim Z + 1$,
(ii) $\dim X = 3$ *and X is FT.*

Proof. Immediate by Theorem 8.1 and Proposition 8.14. □

The rest of this section is devoted to the proof of Proposition 8.14. Thus from now on and through the end of this section we assume that the base variety Z is a curve. First we note that $Z \simeq \mathbb{P}^1$ because X is FT.

Lemma 8.16. *Fix a positive integer N. Let $f\colon X \to Z \ni o$ be a contraction to a curve germ and let D be an $\mathbb{R}$-divisor on X. Let D^{h} be the horizontal part of D. Assume that*
(i) $\dim X \le d$ *and X is FT over Z,*
(ii) D^{h} *is a $\mathbb{Q}$-boundary and ND^{h} is integral,*
(iii) $K_X + D$ *is lc and numerically trivial over Z.*
Assume LMMP in dimension $\le d$. Further assume that the statement of Theorem 1.4 *holds in dimensions $\le d-1$. Then there is an n-complement $K + D^+$ of $K + D$ near $f^{-1}(o)$ such that $N \mid n$, $n \le \mathrm{Const}(N, \dim X)$, and $a(E, X, D^+) = -1$ for some divisor E with* $\mathrm{Center}_Z\, E = o$.

Proof. Take a finite set $\mathfrak{R} \subset [0, 1] \cap \mathbb{Q}$ and a positive integer I so that $D^{\mathrm{h}} \in \mathfrak{R}$, $I(\mathfrak{R}) \mid I$, and $N \mid I$. Replacing D with $D + \alpha f^* o$

we may assume that (X, D) is maximally lc. Next replacing (X, D) with its suitable blowup we may assume that X is $\mathbb{Q}$-factorial and the fibre $f^{-1}(o)$ has a component, say F, of multiplicity 1 in D. Run $-F$-MMP over Z. This preserves the $\mathbb{Q}$-factoriality and lc property of $K + D$. Clearly, F is not contracted. On each step, the contraction is birational. So at the end we get a model with irreducible central fibre: $f^{-1}(o)_{\mathrm{red}} = F$. Then $D \in \Phi(\mathfrak{R})$. Applying D^{h}-MMP over Z, we may assume that D^{h} is nef over Z. We will show that $K + D$ is n-complemented for some $n \in \mathscr{N}_{d-1}(\overline{\mathfrak{R}})$. Then by Proposition 3.6 we can pull-back complements to our original X. Note that the f-vertical part of D coincides with F, so it is numerically trivial over Z. Since X is FT over Z, $-K_X$ is big over Z. Therefore $D \equiv D^{\mathrm{h}}$ is nef and big over Z. Now apply construction of [PS01, §3] to (X, D) over Z. There are two cases:

(I) (X, F) is plt,
(II) (X, F) is lc but not plt (recall that $F \le D$).

Consider, for example, the second case (the first case is much easier and can be treated in a similar way). First we define an auxiliary boundary to localise a suitable divisor of discrepancy -1. By Kodaira's lemma, for some effective $D^{\mho}$, the divisor $D - D^{\mho}$ is ample. Put $D_{\varepsilon,\alpha} := (1-\varepsilon)D + \alpha D^{\mho}$. Then $K_X + D_{\varepsilon,\alpha} \equiv -\varepsilon D + \alpha D^{\mho}$. So $(X, D_{\varepsilon,\alpha})$ is a klt log Fano over Z for $0 < \alpha \ll \varepsilon \ll 1$. Take $\beta = \beta(\varepsilon, \alpha)$ so that $(X, D_{\varepsilon,\alpha} + \beta F)$ is maximally lc and put $G_{\varepsilon,\alpha} := D_{\varepsilon,\alpha} + \beta F$. Thus $(X, G_{\varepsilon,\alpha})$ is a lc (but not klt) log Fano over Z.

Let $g\colon \widehat{X} \to X$ be an *inductive blowup* of $(X, G_{\varepsilon,\alpha})$ [PS01, Proposition 3.6]. By definition $\widehat{X}$ is $\mathbb{Q}$-factorial, $\rho(\widehat{X}/X) = 1$, the g-exceptional locus is a prime divisor E of discrepancy $a(E, X, G_{\varepsilon,\alpha}) = -1$, the pair $(\widehat{X}, E)$ is plt, and $-(K_{\widehat{X}} + E)$ is ample over X. Since $(X, G_{\varepsilon,\alpha} - \gamma F)$ is klt for $\gamma > 0$, $\mathrm{Center}_Z(E) = o$. Note that, by construction, E is not exceptional on some fixed log resolution of $(X, \operatorname{Supp} G_{\varepsilon,\alpha})$. Hence we may assume that E and g do not depend on ε and α if $0 < \varepsilon \ll 1$. In particular, $a(E, X, D) = -1$.

By (iii) of Lemma 2.8 $\widehat{X}$ is FT over Z. Let $\widehat{D}$ and $\widehat{G}_{\varepsilon,\alpha}$ be proper transforms on $\widehat{X}$ of D and $G_{\varepsilon,\alpha}$, respectively. Then

$$\begin{aligned} 0 \equiv \quad & g^*(K_X + D) && = K_{\widehat{X}} + \widehat{D} + E, \\ & g^*(K_X + G_{\varepsilon,\alpha}) && = K_{\widehat{X}} + \widehat{G}_{\varepsilon,\alpha} + E, \end{aligned}$$

where $-(K_X + G_{\varepsilon,\alpha})$ is ample over Z. Run $-(K_{\widehat{X}} + E)$-MMP starting from $\widehat{X}$ over Z:

$$\begin{tikzcd}
& \widehat{X} \arrow[dl, "g"'] \arrow[dr, dashed] & \\
X \arrow[dr, "f"] & & \overline{X} \arrow[dl, "\bar{f}"'] \\
& Z &
\end{tikzcd}$$

Since $-(K_{\widehat{X}} + E) \equiv \widehat{D}$, we can contract only components of $\widehat{D}$. At the end we get a model $(\overline{X}, \overline{D} + \overline{E})$ such that $-(K_{\overline{X}} + \overline{E})$ is nef and big over Z, $K_{\overline{X}} + \overline{E} + \overline{D} \equiv 0$, and $(\overline{X}, \overline{E} + \overline{D})$ is lc.

We claim that the plt property of $K_{\widehat{X}} + E$ is preserved under this LMMP. Indeed, for $0 < t \ll 1$, the log divisor $K_{\widehat{X}} + (1-t)\widehat{G}_{\varepsilon,\alpha} + E$ is a convex linear combination of log divisors $K_{\widehat{X}} + \widehat{G}_{\varepsilon,\alpha} + E$ and $K_{\widehat{X}} + E$. The first divisor is anti-nef and is trivial only on one extremal ray R, the ray generated by fibres of g. The second one is strictly negative on R. Since $\widehat{X}$ is FT over Z, the Mori cone $\overline{\mathrm{NE}}(\widehat{X}/Z)$ is polyhedral. Therefore $K_{\widehat{X}} + (1-t)\widehat{G}_{\varepsilon,\alpha} + E$ is anti-ample (and plt) for $0 < t \ll 1$. By the base point free theorem there is a boundary $M \geq (1-t)\widehat{G}_{\varepsilon,\alpha} + E$ such that $(\widehat{X}, M)$ is a plt 0-pair. Since E is not contracted, this property is preserved under our LMMP. Hence $(\overline{X}, \overline{M})$ is plt and so is $(\overline{X}, \overline{E})$. This proves our claim. In particular, $\overline{E}$ is normal and FT.

Take $\delta := 1/m$, $m \in \mathbb{Z}$, $m \gg 0$. For any such δ, the pair $(\overline{X}, (1-\delta)\overline{D} + \overline{E})$ is plt and $-(K_{\overline{X}} + (1-\delta)\overline{D} + \overline{E})$ is nef and big over Z. By our inductive hypothesis there is an n-complement $K_{\overline{E}} + \mathrm{Diff}_{\overline{E}}(\overline{D})^+$ of $K_{\overline{E}} + \mathrm{Diff}_{\overline{E}}(\overline{D})$ with $n \in \mathscr{N}_{d-1}(\mathfrak{R})$. Clearly, this is also an n-complement of $K_{\overline{E}} + \mathrm{Diff}_{\overline{E}}((1-\delta)\overline{D})$. Note that nD is integral. We claim that $(1-\delta)\overline{D} \in \mathscr{P}_n$. Indeed, the vertical multiplicities of $(1-\delta)\overline{D}$ are contained in $\Phi(\mathfrak{R})$. Let d_i be the multiplicity of a horizontal component of $\overline{D}$. Then $nd_i \in \mathbb{Z}$. If $d_i = 1$, then obviously $(1-\delta)d_i \in \mathscr{P}_n$. So we assume that $d_i < 1$. Then $\lfloor (n+1)d_i \rfloor = nd_i$ and $\lfloor (n+1)(d_i - \delta) \rfloor = nd_i \geq n(d_i - \delta)$ for $\delta \ll 1$. This proves our claim. Now the same arguments as in [PS01, §3] shows that $K_X + (1-\delta)D$ is n-complemented near $f^{-1}(o)$. Since $D \in \mathscr{P}_n$, there is an n-complement $K_X + D^+$ of $K_X + D$ near $f^{-1}(o)$ and moreover, $a(E, X, D^+) = -1$. □

Corollary 8.17. *Notation as in Proposition* 8.14. *The multiplicities of* D_{mod} *are contained in a finite set.*

Proof. Consider a local n-complement D^+ of $K+D$ near $f^{-1}(o)$. Then $n(K_Z + D^+_{\mathrm{div}} + D^+_{\mathrm{mod}})$ is integral at o. By construction, (X, D^+) has a centre of log canonical singularities contained in $f^{-1}(o)$. Hence $D^+_{\mathrm{div}} = 0$. By semiadditivity (see Lemma 7.4) we have $D^+_{\mathrm{mod}} = D_{\mathrm{mod}}$. Thus nD_{mod} is integral at o. □

Proof of Proposition 8.14. The statement of (7.13.1) follows by Theorem 7.12 (cf. [Kaw98]). Indeed, for any $0 < t < 1$ we put $D_t := (1-t)D + t\Theta$, where Θ is such as in 7.11. Then by Theorem 7.12 $(D_t)_{\mathrm{mod}}$ is semiample. Hence so is D_{mod}.

Assertion (7.13.2) follows by Theorem 1.5 (in lower dimension).

Finally for (7.13.3) we note that by Corollary 8.17 ID_{mod} is integral and base point free for a bounded I because $Z \simeq \mathbb{P}^1$. □

Remark 8.18. It is possible that Proposition 8.14 can be proved by using results of [FM00], [Fuj03]. In fact, in these papers the authors write down the canonical bundle formula (for arbitrary $\dim Z$) in the following form (we change notation a little):

$$b(K+D) = f^*(bK_Z + L^{log,ss}_{X/Z}) + \sum_P s^D_P f^*P + B^D.$$

Here $D_{\mathrm{div}} = \frac{1}{b}\sum_P s^D_P P$, $D_{\mathrm{mod}} = \frac{1}{b} L^{log,ss}_{X/Z}$, and $\operatorname{codim} f(B^D) \geq 2$, so the term B^D is zero in our situation. Under the additional assumption that D is a boundary it is proved that the denominators of D_{mod} are bounded (and D_{mod} is semiample because it is nef on $Z = \mathbb{P}^1$), see [FM00, Theorem 4.5], [Fuj03, Theorem 5.11]. This should imply our Proposition 8.14. We however do not know how to avoid the effectivity condition of D.

9. The main theorem: Case $-(K+D)$ is nef

In this section we prove Theorem 1.4 in case (4.9.2) and Theorem 1.5 in the case when (X, B) is not klt. Thus we apply reduction from §4 and replace (X, B) with (Y, B_Y) and put $D := D_Y$. The idea of the proof is to consider the contraction $f\colon X \to Z$ given by $-(K+D)$ and use Effective Adjunction to pull-back complements from Z. In practice, there are several technical issues which do not allow us to weaken the last assumption in Theorem 1.4, that is, we cannot omit the klt condition when $K + B \not\equiv 0$. Roughly speaking the inductive step work if the following two conditions hold:

(i) $0 < \dim Z < \dim X$, and
(ii) the pair $(Z, D_{\mathrm{div}} + D_{\mathrm{mod}})$ satisfies assumptions of Theorem 1.4.

The main technical step of the proof is Proposition 9.4. The proof is given in 9.5 and 9.6.

9.1. Setup. Let (X, D) be an lc log pair and let $f\colon X \to Z$ be a contraction such that $K + D \sim_{\mathbb{Q}} f^*L$ for some L and X is FT. Further, assume the LMMP in dimension $d := \dim X$. Our proof uses induction by d. So we also assume that Theorems 1.4 and 1.5 hold true for all X of dimension $< d$.

By Lemma 4.11 we have the following.

Corollary 9.2. *In notation of* 9.1 *assume that* $\dim Z > 0$. *Fix a finite rational set* $\mathfrak{R} \subset [0,1]$ *and let* $D \in \Phi(\mathfrak{R})$. *Then the multiplicities of horizontal components of* D *are contained into a finite subset* $M \subset \Phi(\mathfrak{R})$, *where* M *depends only on* $\dim X$ *and* $\mathfrak{R}$.

Proof. Restrict D to a general fibre and apply Lemma 4.11. □

Now we verify that under certain assumptions and conjectures the hyperstand multiplicities transforms to hyperstandard ones after adjunction.

For a subset $\mathfrak{R} \subset [0,1]$, denote

$$\mathfrak{R}(n) := \left(\overline{\mathfrak{R}} + \frac{1}{n}\mathbb{Z}\right) \cap [0,1], \qquad \mathfrak{R}' := \bigcup_{n \in \mathscr{N}_{d-1}(\overline{\mathfrak{R}})} \mathfrak{R}(n) \subset [0,\, 1].$$

These sets are rational and finite whenever so is $\mathfrak{R}$.

Proposition 9.3. *In notation of* 9.1, *fix a finite rational set* $\mathfrak{R} \subset [0,1]$.

(i) *If* $D \in \Phi(\mathfrak{R})$, *then* $D_{\mathrm{div}} \in \Phi(\mathfrak{R}')$.

(ii) *If* $D \in \Phi(\mathfrak{R}, \varepsilon_{d-1})$, *then* $D_{\mathrm{div}} \in \Phi(\mathfrak{R}', \varepsilon_{d-1}) \subset \Phi(\mathfrak{R}', \varepsilon_{d-2})$.

Proof. By taking general hyperplane sections we may assume that Z is a curve. Furthermore, we may assume that X is $\mathbb{Q}$-factorial. Fix a point $o \in Z$. Let d_o be the multiplicity of o in D_{div}. Then $d_o = 1 - c_o$, where c_o is computed by (7.2.1). It is sufficient to show that $d_o \in \Phi(\mathfrak{R}(n)) \cup [1 - \varepsilon_{d-1},\, 1]$ for any point $o \in Z$ and some $n \in \mathscr{N}_{d-1}(\overline{\mathfrak{R}})$. Clearly, we can consider X and Z small neighbourhoods of $f^{-1}(o)$ and o, respectively. We also may assume that $c_o > 0$, so $f^{-1}(o)$ does not contain any centres of log canonical singularities of (X, D). By our assumptions in 9.1 and Lemma 8.16 there is an n-complement $K_X + D^+$ of $K_X + D$ near $f^{-1}(o)$ with $n \in \mathscr{N}_{d-1}(\overline{\mathfrak{R}})$ and moreover, $a(E, X, D^+) = -1$.

Now we show that $d_o \in \Phi(\mathfrak{R}(n)) \cup [1 - \varepsilon_{d-1},\, 1]$. By Lemma 3.5 $D \in \mathscr{P}_n$. Hence, $D^+ \geq D$, i.e., $D^+ = D + D'$, where $D' \geq 0$. Let $F \subset f^{-1}(o)$ be a reduced irreducible component. Since $K_X + D$ is

$\mathbb{R}$-linearly trivial over Z, D' is vertical and $D' = c_o f^*P$. Let d_F and μ be multiplicities of F in D and f^*o, respectively (μ is a positive integer). Since $(X, D+D')$ is lc and $n(D+D')$ is an integral divisor, the multiplicity of F in $D+D'$ has the form k/n, where $k \in \mathbb{Z}$, $1 \le k \le n$. Then $k/n = d_F + c_o\mu$ and

$$c_o = \frac{1}{\mu}\left(\frac{k}{n} - d_F\right), \qquad d_o = 1 - \frac{1}{\mu}\left(\frac{k}{n} - d_F\right).$$

Consider two cases.

a) $d_F \in \Phi(\mathfrak{R})$, so $d_F = 1 - r/m$ ($r \in \mathfrak{R}$, $m \in \mathbb{Z}$, $m > 0$). Then we can write

$$d_o = 1 - \frac{km + rn - nm}{nm\mu} = 1 - \frac{r'}{m\mu} < 1,$$

where

$$0 \le r' = r + \frac{km}{n} - m = \frac{km + rn - nm}{n} \le \frac{nm + rn - nm}{n} \le 1.$$

Therefore, $d_o \in \Phi(\mathfrak{R}(n))$, where $0 \le r' = r + \frac{km}{n} - m \le 1$. This proves, in particular, (i).

b) $d_F > 1 - \varepsilon_{d-1}$. In this case,

$$1 > d_o = 1 - \frac{1}{\mu}\left(\frac{k}{n} - d_F\right) > 1 - \frac{1}{\mu}\left(\frac{k}{n} - 1 + \varepsilon_{d-1}\right) > 1 - \varepsilon_{d-1}.$$

This finishes the proof of (ii). $\square$

Proposition 9.4. *Fix a finite rational subset $\mathfrak{R} \subset [0, 1]$ and a positive integer I divisible by $I(\mathfrak{R})$. Let (X, D) be a log semi-Fano variety of dimension d such that X is $\mathbb{Q}$-factorial FT and $D \in \Phi(\mathfrak{R})$. Assume that there is a $(K+D)$-trivial contraction $f\colon X \to Z$ with $0 < \dim Z < d$. Fix the choice of I_0 and ψ in* 7.5 *so that $\mathbf{D}_{\mathrm{mod}}$ is effective. We take I so that I_0 divides I. Assume the LMMP in dimension d. Further, assume that Conjectures* 1.1 *and* 7.13 *hold in dimension $d-1$ and d, respectively. If $K_Z + D_{\mathrm{div}} + D_{\mathrm{mod}}$ is Im-complemented, then so is $K_X + D$.*

Proof. Put $D_Z := D_{\mathrm{div}} + D_{\mathrm{mod}}$. Apply (i) of Conjecture 7.13 to (X, D). We obtain

$$K + D = f^*(K_Z + D_Z),$$

and (Z, D_Z) is lc, where $D_{\mathrm{div}} \in \Phi(\mathfrak{R}')$. By (7.13.3) $I'' D_{\mathrm{mod}}$ is integral for some bounded I''. Thus replacing $\mathfrak{R}'$ with $\mathfrak{R}' \cup \{1/I'', 2/I'', \dots, (I''-1)/I''\}$ we may also assume that $D_{\mathrm{mod}} \in \Phi(\mathfrak{R}')$. Then $D_Z \in \Phi(\mathfrak{R}')$. Furthermore, by Lemma 2.8 Z is FT and by the construction, $-(K_Z + D_Z)$ is nef. By our inductive hypothesis $K_Z + D_Z$ has bounded complements.

Let $K_Z + D_Z^+$ be an n-complement of $K_Z + D_Z$ such that $I \mid n$. Then $D_Z^+ \geq D_Z$ (see Lemmas 3.3 and 3.5). Put $H_Z := D_Z^+ - D_Z$ and $D^+ := D + f^*H_Z$. Write $D^+ = \sum d_i^+ D_i$. By the above, $d_i^+ \geq d_i$. We claim that $K + D^+$ is an n-complement of $K + D$. Indeed, since $K + D \sim_I f^*(K_Z + D_Z)$, we have

$$\begin{aligned} n(K + D^+) =& n(K + D + f^*H_Z) = \\ & (n/I)I(K + D) + (n/I)If^*H_Z \sim \\ & (n/I)f^*I(K_Z + D_Z) + (n/I)f^*IH_Z = \\ & (n/I)f^*I(K_Z + D_Z^+) = f^*n(K_Z + D_Z^+) \sim f^*0 = 0. \end{aligned}$$

Thus, $n(K + D^+) \sim 0$. Further, since nd_i^+ is a nonnegative integer and $d_i^+ \geq d_i$, the inequality

$$nd_i^+ = \lfloor (n+1)d_i^+ \rfloor \geq \lfloor (n+1)d_i \rfloor$$

holds for every i such that $0 \leq d_i < 1$. Finally, by Corollary 7.18 the log divisor $K + D^+ = f^*(K_Z + D_Z)$ is lc. This proves our proposition. □

9.5. Proof of Theorem 1.5 in the case when (X, B) is not klt (continued). To finish the proof Theorem 1.5 in the non-klt case we have to consider the following situation (see 4.13). (X', B') is a non-klt 0-pair such that $B' \in \Phi(\mathfrak{R})$, X' is λ-lt and X' is FT, where λ depends only on $\mathfrak{R}$ and the dimension of X'. Moreover, there is a Fano fibration $X' \to Z'$ with $0 < \dim Z' < \dim X'$. The disired bounded $nI(\mathfrak{R})$-complements exist by Proposition 9.4 and inductive hypothesis.

9.6. Proof of Theorem 1.4 in Case (4.9.2). To finish our proof of the main theorem we have to consider the case when (X, B) is klt and general reduction from Section 4 leads to case (4.9.2), i.e., $-(K_Y + D_Y)$ is nef. Replace (X, B) with (Y, B_Y) and put $D := D_Y$. Recall that in this situation X is FT and $B \in \Phi(\mathfrak{R}, \varepsilon')$, where $0 < \varepsilon' \leq \varepsilon_{d-1}(\overline{\mathfrak{R}})$. By (4.2.2) there is a boundary $\Theta \geq B$ such that (X, Θ) is a klt 0-pair. For the boundary D defined by (4.2.3) we also have $D \in \Phi(\mathfrak{R})$ and $\lfloor D \rfloor \neq 0$ by (4.2.1). All these properties are preserved under birational transformations in 4.5. By our assumption at the end we have case (4.9.2), i.e., $-(K + D)$ is nef (and semiample). Therefore it is sufficient to prove the following.

Proposition 9.7. *Fix a finite rational subset $\mathfrak{R} \subset [0, 1]$. Let $(X, D = \sum d_i D_i)$ is a d-dimensional log semi-Fano variety such that*

(i) *$D \in \Phi(\mathfrak{R})$, (X, D) is not klt and X is FT,*
(ii) *there is boundary $B = \sum b_i D_i \leq D$ such that either $b_i = d_i < 1 - \varepsilon'$ or $b_i \geq 1 - \varepsilon'$ and $d_i = 1$, where $0 < \varepsilon' \leq \varepsilon_{d-1}(\overline{\mathfrak{R}})$,*

(iii) (X, Θ) *is a klt* 0*-pair for some* $\Theta \geq B$.

Assume the LMMP in dimension d. Further, assume that Conjectures 1.1 *and* 7.13 *hold in dimension d. Then $K + D$ has a bounded n-complement such that $I(\mathfrak{R}) \mid n$.*

The idea of the proof is to reduce the problem to Proposition 9.4 by considering the contraction $f\colon X \to Z$ given by $-(K + D)$. But here two technical difficulties arise. First it may happen that the divisor $-(K + D)$ is big and then f is birational. In this case one can try to extend complements from $\lfloor D \rfloor$ but the pair (X, D) is not necessarily plt and the inductive step (Proposition 3.7) does not work. We have to make some perturbations and birational transformations. Second to apply inductive hypothesis to $(Z, D_{\mathrm{div}} + D_{\mathrm{mod}})$ we have to check if this pair satisfies conditions of Theorem 1.4. In particular, we have to check the klt property of $(Z, D_{\mathrm{div}} + D_{\mathrm{mod}})$. By Corollary 7.18 this holds if any lc centre of (X, D) dominates Z. Otherwise we again need some additional work.

Proof. Note that we may replace B with $B_t := tB + (1 - t)D$ for $0 < t < 1$. This preserves all our conditions (i)-(iii). Indeed, (i) and (ii) are obvious. For (iii), we note that $(X, D^{\diamondsuit})$ is a 0-pair for some $D^{\diamondsuit} \geq D$ (because $-(K + D)$ is semiample). Hence one can replace Θ with $\Theta_t := t\Theta + (1 - t)D^{\diamondsuit}$.

Let $\mu\colon (\tilde{X}, \tilde{D}) \to (X, D)$ be a dlt modification of (X, D). By definition, μ is a $K + D$-crepant birational extraction such that $\tilde{X}$ is $\mathbb{Q}$-factorial, the pair $(\tilde{X}, \tilde{D})$ is dlt, and each μ-exceptional divisor E has discrepancy $a(E, X, D) = -1$ (see, e.g., [Kol92, 21.6.1], [Pro01, 3.1.3]). In particular, $\tilde{D} \in \Phi(\mathfrak{R})$ and $\tilde{X}$ is FT by Lemma 2.8. Let $\tilde{B}$ be the crepant pull-back of B. One can take t so that the multiplicities in $\tilde{B}$ of μ-exceptional divisors are $\geq 1 - \varepsilon_{d-1}$. Thus for the pair $(\tilde{X}, \tilde{D})$ conditions (i)-(iii) hold. Therefore, we may replace (X, D) with $(\tilde{X}, \tilde{D})$ (and B, Θ with their crepant pull-backs).

Let $f\colon X \to Z$ be the contraction given by $-(K + D)$. By Theorem 1.5 (see 4.13 and 9.5) we may assume that $\dim Z > 0$. We apply induction by $N := \dim X - \dim Z$.

First, consider the case $N = 0$. Then $-(K + D)$ is big. We will show that $K + D$ is n-complemented for some $n \in \mathscr{N}_{d-1}(\overline{\mathfrak{R}})$.

Fix $n_0 \gg 0$, and let $\delta := 1/n_0$. Then $D_\delta := D - \delta \lfloor D \rfloor \in \Phi(\mathfrak{R})$. It is sufficient to show that $K_X + D_\delta$ is n-complemented for some $n \in \mathscr{N}_{d-1}(\overline{\mathfrak{R}})$. We will apply a variant of [PS01, Th. 5.1] with hyperstandard multiplicities. To do this, we run $-(K + D_\delta)$-MMP over Z. Clearly, this is equivalent $\lfloor D \rfloor$-MMP over Z. This process preserve the $\mathbb{Q}$-factoriality and lc (but not dlt) property of $K+D$. At the end we get a model (X',D') such that $-(K_{X'}+D'_\delta)$ is nef over Z. Since X' is FT, the Mori cone $\overline{\mathrm{NE}}(X')$ is rational polyhedral. Taking our condition $0<\delta\ll 1$ into account we get that $-(K_{X'}+D'_\delta)$ is nef. Since

$$-(K_{X'}+D'_\delta)=-(K_{X'}+D')+\delta\lfloor D'\rfloor,$$

where $\lfloor D'\rfloor$ is effective, $-(K_{X'}+D'_\delta)$ is also big. Note that (X',D') is lc but not klt. By our assumptions,

$$D'_\delta=(1-\delta)D'+\delta B'\le(1-\delta)D'+\delta\Theta'$$

and $(X',(1-\delta)D'+\delta\Theta')$ is klt. Therefore so is (X',D'_δ). Now we apply [PS01, Th. 5.1] with $\Phi=\Phi(\mathfrak{R})$. This says that we can extend complements from some (possibly exceptional) divisor. By Proposition 3.9 the multiplicities of the corresponding different are contained in $\overline{\mathfrak{R}}$. We obtain an n-complement of $K_{X'}+D'_\delta$ for some $n\in\mathscr{N}_{d-1}(\overline{\mathfrak{R}})$. By Proposition 3.6 we can pull-back this complement to X (we use the inclusion $D'_\delta\in\Phi(\mathfrak{R})\subset\mathscr{P}_n$).

Now assume that Proposition 9.7 holds for all $N'<N$. Run $\lfloor D\rfloor$-MMP over Z. After some flips and divisorial contractions we get a model on which $\lfloor D\rfloor$ is nef over Z. Since X is FT, the Mori cone $\overline{\mathrm{NE}}(X)$ is rational polyhedral. Hence $-(K+D-\delta\lfloor D\rfloor)$ is nef for $0<\delta\ll 1$. As above, put $D_\delta:=D-\delta\lfloor D\rfloor$. We can take $\delta=1/n_0$, $n_0\gg 0$ and then $D_\delta\in\Phi(\mathfrak{R})$. On the other hand, $D_\delta\le(1-\delta)D+\delta B$ for some $\delta>0$. Therefore, (X,D_δ) is klt. Now let $f^\flat\colon X\to Z^\flat$ be the contraction given by $-(K+D_\delta)$. Since $-(K+D_\delta)=-(K+D)+\delta\lfloor D\rfloor$, there is decomposition $f\colon X\xrightarrow{f^\flat}Z^\flat\longrightarrow Z$.

If $\dim Z^\flat=0$, then $Z^\flat=Z$ is a point, a contradiction. If $\dim Z^\flat<\dim X$, then by Corollary 7.18 $(Z^\flat,(D_\delta)_{\mathrm{div}}+(D_\delta)_{\mathrm{mod}})$ is a klt log semi-Fano variety. We can apply Proposition 9.4 to the contraction $X\to Z^\flat$ and obtain a bounded complement of $K+D_\delta$. Clearly, this will be a complement of $K+D$.

Therefore, we may assume that $-(K+D_\delta)$ is big, $f^\flat$ is birational, and so $\lfloor D\rfloor$ is big over Z. In particular, the horizontal part $\lfloor D\rfloor^{\mathrm{h}}$ of $\lfloor D\rfloor$ in non-trivial.

Replace (X,D) with its dlt modification. Assume that $\lfloor D\rfloor^{\mathrm{h}}\ne\lfloor D\rfloor$. As above, run $\lfloor D\rfloor^{\mathrm{h}}$-MMP over Z. For $0<\delta\ll 1$, the divisor $-(K+D-\delta\lfloor D\rfloor^{\mathrm{h}})$ will be nef. Moreover, it is big over Z. Therefore, $-(K+D-\delta\lfloor D\rfloor^{\mathrm{h}})$ defines a contraction $f'\colon X\to Z'$ with $\dim Z'>\dim Z$. By our inductive hypothesis there is a bounded complement.

It remains to consider the case when $\lfloor D \rfloor^{\mathrm{h}} = \lfloor D \rfloor$. Then any lc centre of (X, D) dominates Z. By Corollary 7.18 and Proposition 9.4 there is a bounded complement of $K + D$.

□

This finishes the proof of Theorem 1.4. Corollaries 1.7 and 1.8 immediately follows by this theorem, Corollary 8.15, and [Ale94].

9.8. Proof of Corollary 1.11. Replacing (X, D) with its log terminal modification we may assume that (X, D) is dlt. If $D = 0$, we have $nK \sim 0$ for some $n \le 21$ by [Bla95]. Thus we assume that $D \neq 0$. Run K-MMP. We can pull-back complements by Proposition 3.6. The end result is a K-negative extremal contraction $(X', D') \to Z$ with $\dim Z \le 1$. If Z is a curve, then either $Z \simeq \mathbb{P}^1$ or Z is an elliptic curve. In both cases we apply Proposition 9.4 (the FT property of X is not needed). Otherwise X' is a klt log del Pezzo surface with $\rho(X') = 1$. In particular, X' is FT. In this case the assertion follows by Theorem 1.5.

9.9. Proof of Corollary 1.9. First we construct a crepant dlt model $(\bar{X}, \bar{D})$ of (X, D) such that each component of $\bar{D}$ meets $\lfloor \bar{D} \rfloor$. Replacing (X, D) with its log terminal modification we may assume that (X, D) is dlt, X is $\mathbb{Q}$-factorial, and $\lfloor D \rfloor \neq 0$. If $\lfloor D \rfloor = D$, we put $(\bar{X}, \bar{D}) = (X, D)$. Otherwise, run $K + D - \lfloor D \rfloor$-MMP. Note that none of connected components of $\lfloor D \rfloor$ is contracted. Moreover, the number of connected components of $\lfloor D \rfloor$ remains the same (cf. [Kol92, Prop. 12.3.2], [Sho93, Th. 6.9]). At the end we get an extremal contraction $(X', D') \to Z$ with $\dim Z \le 2$. If Z is not a point, we can apply Proposition 9.4. Otherwise, $\rho(X') = 1$, $\lfloor D' \rfloor$ is connected, and each component of D' meets $\lfloor D' \rfloor$. The same holds on a log terminal modification $(\bar{X}, \bar{D})$ of (X', D') because X is $\mathbb{Q}$-factorial.

By Corollary 1.11, for each component $\bar{D}_i \subset \lfloor \bar{D} \rfloor$, the log divisor $K_{\bar{D}_i} + \mathrm{Diff}_{\bar{D}_i}(\bar{D} - \bar{D}_i)$ has bounded complements, i.e., there is $n_0 = n_0(\mathfrak{R})$ such that $n_0(K_{\bar{D}_i} + \mathrm{Diff}_{\bar{D}_i}(\bar{D} - \bar{D}_i)) \sim 0$. Thus we may assume that $n_0(K_{\bar{X}} + \bar{D})|_{\lfloor \bar{D} \rfloor} \sim 0$. Recall that the multiplicities of $\mathrm{Diff}_{\bar{D}_i}(\bar{D} - \bar{D}_i) = \sum \delta_j \Delta_j$ are computed by the formula $\delta_j = 1 - 1/m_j + (\sum_l k_l d_l)/m_j$, where $m_j, k_l \in \mathbb{Z}$, $m_j > 0$, $k_l \ge 0$, d_l are multiplicities of $\bar{D}$, and $\sum_l k_l d_l \le 1$. Since $d_l \in \Phi(\mathfrak{R})$, there is only a finite number of possibilities for d_l with $k_l \neq 0$. Further, since $n_0 \delta_j \in \mathbb{Z}$, there is only a finite number of possibilities for m_j. Thus we can take $n_1 = n_1(\mathfrak{R})$ such that $n_1 \bar{D}$ is an integral divisor and $n_1(K_{\bar{X}} + \bar{D})|_{\lfloor \bar{D} \rfloor} \sim 0$. Since $\bar{X}$ is FT, there is an integer n_2 such that

$n_1n_2(K_{\bar{X}}+\bar{D})\sim 0$ on $\bar{X}$. This defines a cyclic étale over $\lfloor\bar{D}\rfloor$ cover $\pi\colon \hat{X}\to\bar{X}$. Let $\hat{D}:=\pi^*\bar{D}$. Then $(\hat{X},\hat{D})$ is a 0-pair such that $\lfloor\hat{D}\rfloor$ has at least n_2 connected components. On the other hand, the number of connected componets of a 0-pair is at most two (see [Fuj00, 2.1], cf. [Sho93, 6.9]). Thus, $n(K_{\bar{X}}+\bar{D})\sim 0$, where $n=2n_1$. This proves our corollary.

9.10. Proof of Corollary 1.10. In notation of 4.1, take $0<\bar{\varepsilon}<\varepsilon_2(\overline{\mathfrak{R}})/2$. We may assume that (X,B) is such as in 4.2, so there are (at least) two components B_1 and B_2 of multiplicities $b_i\geq 1-\bar{\varepsilon}$ in B. Then by Lemma 6.7 and (4.2.1) components B_1, B_2 do not meet each other and by Corollary 4.6 this holds on each step of the LMMP as in 4.5. Therefore, we cannot get a model with $\rho=1$. In particular, case (4.9.1) is impossible.

Consider case (4.9.2). If the divisor $-(K_Y+D_Y)$ is big, we can argue as in the proof of Proposition 9.7. Then we do not need Conjecture 1.1. If $K_Y+D_Y\equiv 0$, we can use Corollary 1.9; it is sufficient to have only one divisor E (exceptional or not) with $a(E,X,D)\leq -1+\bar{\varepsilon}$. In other cases we use induction to actual fibrations (Proposition 9.4), that is, with the fibres and the base of dimension ≥ 1 and by our assumptions with dimensions ≤ 2.

References

[Ale94] Valery Alexeev, *Boundedness and K^2 for log surfaces*, Internat. J. Math. **5** (1994), no. 6, 779–810. MR MR1298994 (95k:14048)

[Amb99] Florin Ambro, *The adjunction conjecture and its applications*, PhD thesis, The Johns Hopkins University, http://arXiv.org:math/9903060, 1999.

[Amb04] Florin Ambro, *Shokurov's boundary property*, J. Differential Geom. **67** (2004), no. 2, 229–255. MR MR2153078

[Amb05] ______, *The moduli b-divisor of an lc-trivial fibration*, Compos. Math. **141** (2005), no. 2, 385–403. MR MR2134273

[BB92] A. A. Borisov and L. A. Borisov, *Singular toric Fano three-folds*, Mat. Sb. **183** (1992), no. 2, 134–141. MR MR1166957 (93i:14034)

[Bla95] R. Blache, *The structure of l.c. surfaces of Kodaira dimension zero. I*, J. Algebraic Geom. **4** (1995), no. 1, 137–179.

[Bor96] Alexandr Borisov, *Boundedness theorem for Fano log-threefolds*, J. Algebraic Geom. **5** (1996), no. 1, 119–133. MR MR1358037 (96m:14058)

[Bor01] ______, *Boundedness of Fano threefolds with log-terminal singularities of given index*, J. Math. Sci. Univ. Tokyo **8** (2001), no. 2, 329–342. MR MR1837167 (2002d:14060)

[BPVdV84] W. Barth, C. Peters, and A. Van de Ven, *Compact complex surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in

Mathematics and Related Areas (3)], vol. 4, Springer-Verlag, Berlin, 1984. MR MR749574 (86c:32026)

[Cas57] J. W. S. Cassels, *An introduction to Diophantine approximation*, Cambridge Tracts in Mathematics and Mathematical Physics, No. 45, Cambridge University Press, New York, 1957. MR MR0087708 (19,396h)

[FM00] Osamu Fujino and Shigefumi Mori, *A canonical bundle formula*, J. Differential Geom. **56** (2000), no. 1, 167–188. MR MR1863025 (2002h:14091)

[Fuj86] Takao Fujita, *Zariski decomposition and canonical rings of elliptic threefolds*, J. Math. Soc. Japan **38** (1986), no. 1, 19–37. MR MR816221 (87e:14036)

[Fuj99] Osamu Fujino, *Applications of Kawamata's positivity theorem*, Proc. Japan Acad. Ser. A Math. Sci. **75** (1999), no. 6, 75–79. MR MR1712648 (2000f:14089)

[Fuj00] ———, *Abundance theorem for semi log canonical threefolds*, Duke Math. J. **102** (2000), no. 3, 513–532. MR MR1756108 (2001c:14032)

[Fuj03] O. Fujino, *Higher direct images of log canonical divisors and positivity theorems*, math.AG/0302073, 2003.

[Har77] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52. MR MR0463157 (57 #3116)

[IP99] V. A. Iskovskikh and Yu. G. Prokhorov, *Fano varieties. Algebraic geometry. V.*, Encyclopaedia Math. Sci., vol. 47, Springer, Berlin, 1999.

[Ish00] Shihoko Ishii, *The global indices of log Calabi-Yau varieties – A supplement to Fujino's paper: The indices of log canonical singularities –*, math.AG/0003060, 2000.

[Isk03] V. A. Iskovskikh, *b-divisors and Shokurov functional algebras*, Tr. Mat. Inst. Steklova **240** (2003), no. Biratsion. Geom. Linein. Sist. Konechno Porozhdennye Algebry, 8–20. MR MR1993746 (2004g:14019)

[Kaw97] Yujiro Kawamata, *Subadjunction of log canonical divisors for a subvariety of codimension* 2, Birational algebraic geometry (Baltimore, MD, 1996), Contemp. Math., vol. 207, Amer. Math. Soc., Providence, RI, 1997, pp. 79–88. MR MR1462926 (99a:14024)

[Kaw98] ———, *Subadjunction of log canonical divisors. II*, Amer. J. Math. **120** (1998), no. 5, 893–899. MR MR1646046 (2000d:14020)

[Kee92] Sean Keel, *Intersection theory of moduli space of stable n-pointed curves of genus zero*, Trans. Amer. Math. Soc. **330** (1992), no. 2, 545–574. MR MR1034665 (92f:14003)

[KMM87] Yujiro Kawamata, Katsumi Matsuda, and Kenji Matsuki, *Introduction to the minimal model problem*, Algebraic geometry, Sendai, 1985, Adv. Stud. Pure Math., vol. 10, North-Holland, Amsterdam, 1987, pp. 283–360. MR MR946243 (89e:14015)

[Knu83] Finn F. Knudsen, *The projectivity of the moduli space of stable curves. II. The stacks $M_{g,n}$*, Math. Scand. **52** (1983), no. 2, 161–199. MR MR702953 (85d:14038a)

[Kod63] K. Kodaira, *On compact analytic surfaces. II, III*, Ann. of Math. (2) 77 (1963), 563–626; ibid. **78** (1963), 1–40. MR MR0184257 (32 #1730)

[Kol92] J. Kollár (ed.), *Flips and abundance for algebraic threefolds*, Société Mathématique de France, Paris, 1992, Papers from the Second Summer Seminar on Algebraic Geometry held at the University of Utah, Salt Lake City, Utah, August 1991, Astérisque No. 211 (1992). MR MR1225842 (94f:14013)

[Kol96] János Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 32, Springer-Verlag, Berlin, 1996. MR MR1440180 (98c:14001)

[KSB88] J. Kollár and N. I. Shepherd-Barron, *Threefolds and deformations of surface singularities*, Invent. Math. **91** (1988), no. 2, 299–338. MR MR922803 (88m:14022)

[McK02] James McKernan, *Boundedness of log terminal Fano pairs of bounded index*, math.AG/0205214, 2002.

[Nik90] V. V. Nikulin, *Del Pezzo surfaces with log-terminal singularities. III*, Math. USSR-Izv. **35** (1990), no. 3, 657–675.

[Pro01] Yu. G. Prokhorov, *Lectures on complements on log surfaces*, MSJ Memoirs, vol. 10, Mathematical Society of Japan, Tokyo, 2001. MR MR1830440 (2002e:14027)

[Pro03] ———, *On Zariski decomposition problem*, Proc. Steklov Inst. Math. **240** (2003), 37–65.

[PS01] Yu. G. Prokhorov and V. V. Shokurov, *The first main theorem on complements: from global to local*, Izvestiya Math. Russian Acad. Sci. **65** (2001), no. 6, 1169–1196.

[Sar80] V. G. Sarkisov, *Birational automorphisms of conic bundles*, Izv. Akad. Nauk SSSR Ser. Mat. **44** (1980), no. 4, 918–945, 974. MR MR587343 (82g:14035)

[Sho93] V. V. Shokurov, *3-fold log flips. Appendix by Yujiro Kawamata: The minimal discrepancy coefficients of terminal singularities in dimension three*, Russ. Acad. Sci., Izv., Math. **40** (1993), no. 1, 95–202.

[Sho00] ———, *Complements on surfaces*, J. Math. Sci. (New York) **102** (2000), no. 2, 3876–3932, Algebraic geometry, 10. MR MR1794169 (2002c:14030)

[Zha06] Qi Zhang, *Rational connectedness of log $\mathbb{Q}$-Fano varieties.*, J. Reine Angew. Math. **590** (2006), 131–142.

Yu. G. Prokhorov: Department of Algebra, Faculty of Mathematics, Moscow State University, Moscow 117234, Russia

E-mail address: `prokhoro@mech.math.msu.su`

V. V. Shokurov: The Johns Hopkins University, Department of Mathematics, Baltimore, Maryland 21218, USA

Steklov Mathematical Institute, Russian Academy of Sciences, Gubkina str. 8, 119991, Moscow, Russia

E-mail address: `shokurov@math.jhu.edu`