Hodge correlators II

A.B. Goncharov

To Pierre Deligne for his 65th birthday

Contents

1 Introduction

1.1 Summary

Let X be a compact Kahler manifold of dimension n. Denote by $\mathcal{CC}_\bullet(X)$ the cyclic homology complex of the reduced cohomology algebra of X. We define a linear map, the *Hodge correlator map*:

$$\mathrm{Cor}_{\mathcal{H}} : H_0\Big(\mathcal{CC}_\bullet(X) \otimes H_{2n}(X)[-2]\Big) \longrightarrow \mathbb{C}. \quad (1)$$

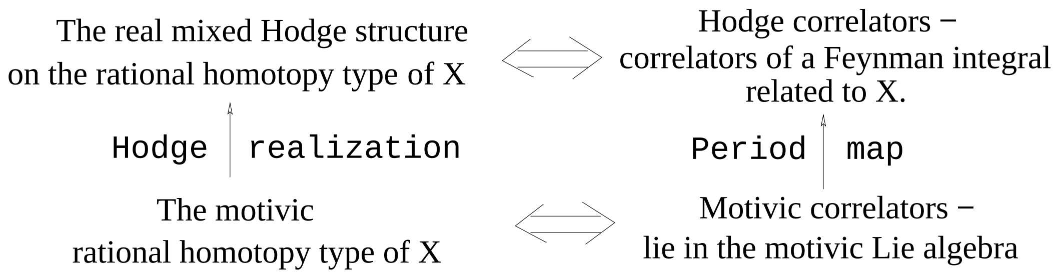
Evaluating this map on cycles we get complex numbers, called *Hodge correlators.*

We introduce a Feynman integral related to X. We show that its correlators, defined via the standard perturbative series expansion procedure, are well defined, and identify them with the Hodge correlators. We show that the Hodge correlator map defines a functorial real mixed Hodge structure of the rational homotopy type of X. The Hodge correlators are the homotopy periods of X.

The category of real mixed Hodge structures (MHS) is canonically identified with the category of representations of the Hodge Galois group. We show that the Hodge correlator map provides an explicit action of the Hodge Galois group on the formal neighborhood of the trivial local system on X. It encodes a functorial homotopy action of the Hodge Galois group on the rational homotopy type of X, describing the real MHS on the latter.

A real MHS on the rational homotopy type of a complex algebraic variety was defined, by different methods, by J. Morgan [M] and R. Hain [H]. Our construction should give the same real MHS. We prove this for the real MHS on the pronilpotent completion of the fundamental group of X.

Now let X be a regular projective algebraic variety over a field k. Assuming the motivic formalism, we define *motivic correlators of* X. They lie in the motivic Lie coalgebra of k. Given an embedding $k \hookrightarrow \mathbb{C}$, the periods of the Hodge realization of the motivic correlators are the Hodge correlators. The motivic correlators come together with an explicit formula for their coproduct in the motivic Lie coalgebra of k. This is one of the main advantages of presenting homotopy periods of $X(\mathbb{C})$ as periods of the motivic correlators. Using this one can perform arithmetic analysis of the homotopy periods of varieties defined over number fields. Summarising:



A similar picture for a smooth but not necessarily compact complex curve X was obtained in [G1]. In that case there are natural harmonic representatives for $\mathrm{gr}^W H^*(X)$. This allows to avoid discussion of DG objects, and makes the story simpler.

The results of this paper admit a generalization where X is replaced by an arbitrary regular complex algebraic variety, which we hope to discuss elsewhere.

In the rest of the Introduction we give a rather detailed account of our constructions and results. Complete details are available in the main body of the paper.

1.2 Hodge correlators for a compact Kahler manifold

Given a graded vector space V, denote by $\mathcal{C}_V$ the cyclic tensor envelope of V:

$$\mathcal{C}_V := \oplus_{m=0}^{\infty}\Big(\otimes^m V\Big)_{\mathbb{Z}/m\mathbb{Z}}$$

where the subscript $\mathbb{Z}/m\mathbb{Z}$ denotes the coinvariants of the cyclic shift

$$v_1 \otimes v_2 \otimes ... \otimes v_m \longmapsto (-1)^s v_m \otimes v_1 \otimes ... \otimes v_{m-1}, \qquad s = \deg(v_m)\sum_{i=1}^{m-1}\deg(v_i).$$

We denote by $(v_1 \otimes v_2 \otimes ... \otimes v_m)_{\mathcal{C}}$ the projection of the element $v_1 \otimes v_2 \otimes ... \otimes v_m$ to $\mathcal{C}_V$. Let

$$\overline{\mathrm{H}}_*(X) := \oplus_{i=1}^{2n-1}\mathrm{H}_i(X),$$

be the reduced cohomology algebra of X. Let us shift it to the right by one:

$$\mathbb{H}_* := \overline{\mathrm{H}}_*(X)[-1].$$

It sits in the degrees $[-(2n-1), -1]$. We also need the dual object:

$$\mathbb{H}^* := \overline{\mathrm{H}}^*(X)[1], \qquad \overline{\mathrm{H}}^*(X) := \oplus_{i=1}^{2n-1}\mathrm{H}^i(X).$$

The spaces $\mathcal{C}_{\mathbb{H}^*}$ and $\mathcal{C}_{\mathbb{H}_*}$ are graded dual to each other.
Finally, let us introduce the shifted by two fundamental cohomology class of X:

$$\mathcal{H} := H^{2n}(X)[2]. \tag{2}$$

Let $\mathcal{H}^\vee$ is its dual.

Choose a base point $a \in X$. Choose a splitting of the de Rham complex of X, bigraded in the usual way, into an arbitrary subspace $\mathcal{H}ar_X$ isomorphically projecting onto the cohomology of X ("harmonic forms") and its orthogonal complement. We take the δ-function δ_a at the point $a \in X$ as a representative of the fundamental class.

Our first goal is to define a degree zero linear map, a precursor of the Hodge correlator map:

$$\mathrm{Cor}^*_{\mathcal{H},a} : \mathcal{C}_{\mathbb{H}^*} \otimes \mathcal{H}^\vee \longrightarrow \mathbb{C}. \tag{3}$$

We need a Green current $G_a(x,y)$ on $X \times X$. It satisfies the differential equation

$$(2\pi i)^{-1}\overline{\partial}\partial G_a(x,y) = \delta_\Delta - P_{\mathrm{Har}} \tag{4}$$

where δ_Δ is the δ-function of the diagonal, and P_{Har} is the Schwarz kernel of the projector onto the space of harmonic forms, realized by an (n,n)-form on $X \times X$. Given a basis $\{\alpha_i\}$ in the space of "harmonic forms" of dimensions $\neq 0, 2n$, and the dual basis $\{\alpha_i^\vee\}$, we have

$$P_{\mathrm{Har}} = \delta_a \otimes 1 + 1 \otimes \delta_a + \sum \alpha_i^\vee \otimes \alpha_i, \qquad \int_X \alpha_i \wedge \alpha_j^\vee = \delta_{ij}.$$

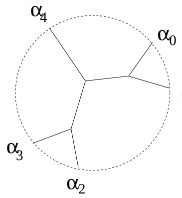


Figure 1: A plane trivalent tree decorated by harmonic forms α_i.

The two currents on the right hand side of (4) represent the same cohomology class, so the equation has a solution by the $\overline{\partial}\partial$-lemma. Let us choose such a solution.

Let us pick a cyclic tensor product of harmonic forms

$$W = (\alpha_0 \otimes \ldots \otimes \alpha_m)_{\mathcal{C}}.$$

Take a plane trivalent tree T decorated by W, see Fig 1. Let us cook up a top degree current on

$$X^{\{\text{internal vertices of } T\}}. \tag{5}$$

Let $\mathcal{A}^*_M[-1]$ be the de Rham complex of M, shifted by 1 to the right. There is a degree zero linear map

$$\xi : S^*(\mathcal{A}^*_M[-1]) \to \mathcal{A}^*_M[-1], \qquad \varphi_0 \cdot \ldots \cdot \varphi_m \longmapsto \mathrm{Sym}_{m+1}\Big(\varphi_0 \wedge d^{\mathbb{C}}\varphi_1 \wedge \ldots \wedge d^{\mathbb{C}}\varphi_m\Big). \tag{6}$$

Take a decorating harmonic form α_i assigned to an external edge E_i of the tree T. Put the form α_i to the copy of X assigned to the internal vertex of the edge E_i. Apply the operator ξ to the wedge product of the Green currents assigned to the edges of T. Finally, multiply the obtained currents on (5) with an appropriate sign, discussed in Section 2.3. One shows that we get a current. Integrating it over (5) we get a number assigned to T. Taking the sum over all trees T, we get a complex number $\mathrm{Cor}^*_{\mathcal{H},a}(W)$. Altogether, we get the map (3). One checks that its degree is zero.

The $\cup$-product in $\mathrm{H}^*(X)$, followed by the projection to $\overline{\mathrm{H}}^*(X)$, induces an algebra structure on $\overline{\mathrm{H}}^*(X)$. There is a differential δ on $\mathcal{C}_{\mathbb{H}^*}$, provided by products of neighbors in a cyclic word:

$$\delta(\overline{\alpha}_0 \otimes \ldots \otimes \overline{\alpha}_m)_{\mathcal{C}} = \mathrm{Cycle}_{m+1}(-1)^{|\alpha_m|}(\overline{\alpha}_0 \otimes \ldots \otimes \overline{\alpha}_{m-2} \otimes \overline{\alpha_{m-1} \cup \alpha_m})_{\mathcal{C}}.$$

Here Cycle_{m+1} means the sum of cyclic shifts, $\overline{\alpha} \in \mathbb{H}^*$ is the shifted by one element $\alpha \in \overline{\mathrm{H}}^*(X)$, and $|\overline{\alpha}|$ is its degree. The complex $(\mathcal{C}_{\mathbb{H}^*}, \delta)$ is nothing else but the *cyclic homology complex* for the algebra $\overline{\mathrm{H}}^*(X)$.

There is a subspace of *shuffle relations* in $\mathcal{C}_{\mathbb{H}_*}$ generated by the elements

$$\sum_{\sigma \in \Sigma_{p,q}} \pm(v_0 \otimes v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(p+q)})_{\mathcal{C}}, \qquad p, q \geq 1,$$

where the sum is over all (p,q)-shuffles, and the signs are given by the standard sign rule. Set

$$\mathcal{CL}ie^{\vee}_{\mathbb{H}^*} := \frac{\mathcal{C}_{\mathbb{H}^*}}{\text{Shuffle relations}}; \qquad \mathcal{CL}ie_{\mathbb{H}_*} := \text{the dual of } \mathcal{CL}ie^{\vee}_{\mathbb{H}^*}.$$

Let $\mathcal{L}ie_{\mathbb{H}_*}$ be the free graded Lie algebra generated by $\mathbb{H}_*$. The space $\mathcal{CL}ie_{\mathbb{H}_*}$ is the projection to the cyclic envelope of the space $\mathcal{L}ie_{\mathbb{H}_*} \otimes \mathbb{H}_*$.

Theorem 1.1 *The map (3) induces a well defined linear map, called the* Hodge correlator map*:*

$$\mathrm{Cor}^*_{\mathcal{H},a} : H^0_\delta\Big(\mathcal{C}\mathcal{L}ie^\vee_{\mathbb{H}^*} \otimes \mathcal{H}^\vee\Big) \longrightarrow \mathbb{C}. \qquad (7)$$

– It does not depend on the choices involved in the definition of the map (3).

We prove this in Section 5.2, as a consequence of an equivalent Theorem 4.1 proved there.

Now let X be the set of complex points of an algebraic variety $X_{/\mathbb{Q}}$ over $\mathbb{Q}$. Then the rational de Rham cohomology $H^*_{\mathrm{DR}}(X_{/\mathbb{Q}}, \mathbb{Q})$ induce a $\mathbb{Q}$-rational structure on the space on the left in (7). So we get a collection of numbers, the images of the elements of this $\mathbb{Q}$-vector space under the Hodge correlator map. We call them *the real periods of the homotopy type of* X.

Dualising map (7) we get a homology class, *the Hodge correlator class*:

$$\mathbf{H}^*_{X,a} \in H_0^\delta\Big(\mathcal{C}\mathcal{L}ie_{\mathbb{H}_*} \otimes \mathcal{H}\Big). \qquad (8)$$

Describing special derivations. Let $\delta : \mathbb{H}_* \longrightarrow \Lambda^2\mathbb{H}_*$ be the degree 1 map dualising the product on $\overline{\mathrm{H}}^*(X)$. See Section 3.2 for a discussion of signs. It gives rise to a differential δ of the free Lie algebra $\mathcal{L}ie_{\mathbb{H}_*}$, providing it with a DG Lie algebra structure.

The image of the shifted by one fundamental class $\mathrm{H}_{2n}(X)[-1]$ under the map dualising the commutative product map $\mathrm{H}^*(X) \otimes \mathrm{H}^*(X) \to \mathrm{H}^{2n}(X)$ provides a special element $\mathrm{S} \in \mathcal{L}ie_{\mathbb{H}_*}$, better understood as a map

$$\mathrm{S} : \mathrm{H}_{2n}(X)[-1] \longrightarrow [\mathbb{H}_*(X), \mathbb{H}_*(X)] \subset \mathcal{L}ie_{\mathbb{H}_*}.$$

Definition 1.2 *A derivation of the Lie subalgebra* $\mathcal{L}ie_{\mathbb{H}_*}$ *is special if it kills the element* S. *We denote by* $\mathrm{Der}^S(\mathcal{L}ie_{\mathbb{H}_*})$ *the Lie algebra of all special derivations of the Lie algebra* $\mathcal{L}ie_{\mathbb{H}_*}$.

The commutator with the differential δ on $\mathcal{L}ie_{\mathbb{H}_*}$ provides the Lie algebra $\mathrm{Der}^S\mathcal{L}ie_{\mathbb{H}_*}$ with a DG Lie algebra structure. In Section 3.2 we show the following

Proposition 1.3 *(i) There is a natural isomorphism*

$$\theta : \mathcal{C}\mathcal{L}ie_{\mathbb{H}_*} \otimes \mathcal{H} \stackrel{\sim}{\longrightarrow} \mathrm{Der}^S\mathcal{L}ie_{\mathbb{H}_*}. \qquad (9)$$

(ii) There is a Lie algebra structure on $\mathcal{C}\mathcal{L}ie_{\mathbb{H}_*} \otimes \mathcal{H}$ *which, together with the differential* δ*, makes it into a DG Lie algebra. The isomorphism (9) is an isomorphism of DG Lie algebras.*

We prove it in Section 3.2. In particular, there is an isomorphism of Lie algebras

$$\theta_0 : H_0^\delta\Big(\mathcal{C}\mathcal{L}ie_{\mathbb{H}_*} \otimes \mathcal{H}\Big) \stackrel{\sim}{\longrightarrow} H_0^\delta\Big(\mathrm{Der}^S\mathcal{L}ie_{\mathbb{H}_*}\Big). \qquad (10)$$

It is crucial for our definition of motivic correlators.

Here is a sketch of constructions leading to a proof of Proposition 1.3. First, given an element $F \otimes \mathcal{H} \in \mathcal{C}_{\mathbb{H}_*} \otimes \mathcal{H}$, there is a derivation $\theta_{F\otimes\mathcal{H}}$ of the tensor algebra $\mathrm{T}_{\mathbb{H}_*}$ acting on the generators as follows:

$$\theta_{F\otimes\mathcal{H}} : p \longmapsto \sum_q \frac{\partial F}{\partial q}\langle q \cap \mathcal{H} \cap p\rangle.$$

Here $\frac{\partial F}{\partial q}$ is the non-commutative partial derivative, e.g.

$$\frac{\partial}{\partial q_1}(q_1q_2q_1q_3)_{\mathcal{C}} = q_2q_1q_3 + q_3q_1q_2,$$

the sum is over a basis $\{q\}$ in $\mathbb{H}_*$, and $\langle q \cap \mathcal{H} \cap p \rangle$ is the iterated cap product with the fundamental class $\mathcal{H}$. Second, the condition $F \in \mathcal{CL}ie_{\mathbb{H}_*} \otimes \mathcal{H}$ just means that $\theta_{F\otimes\mathcal{H}}$ preserves the Lie subalgebra $\mathcal{L}ie_{\mathbb{H}_*}$. Finally, $\theta_{F\otimes\mathcal{H}}(\mathrm{S}) = 0$, i.e. it is a special derivation.

There is a Lie cobracket on the space $\mathcal{C}_{\mathbb{H}^*} \otimes \mathcal{H}^\vee$, making it into a graded Lie coalgebra, and $(\mathcal{C}_{\mathbb{H}^*} \otimes \mathcal{H}^\vee, \delta)$ into a DG Lie coalgebra. Namely, take a cyclic word, cut it into two pieces, insert the Casimir element $\sum_i \alpha_i^\vee \otimes \alpha_i$ and take the sum over all cuts with appropriate signs, as on Fig 2. Thus the dual $\mathcal{C}_{\mathbb{H}_*} \otimes \mathcal{H}$ is a DG Lie algebra. Furthermore, $\mathcal{CL}ie^\vee_{\mathbb{H}^*} \otimes \mathcal{H}^\vee$ inherits a DG Lie coalgebra structure. So $\mathcal{CL}ie_{\mathbb{H}_*} \otimes \mathcal{H}$ is a DG Lie algebra. Therefore $H^0_\delta\Big(\mathcal{CL}ie^\vee_{\mathbb{H}^*} \otimes \mathcal{H}^\vee\Big)$ is a Lie coalgebra.

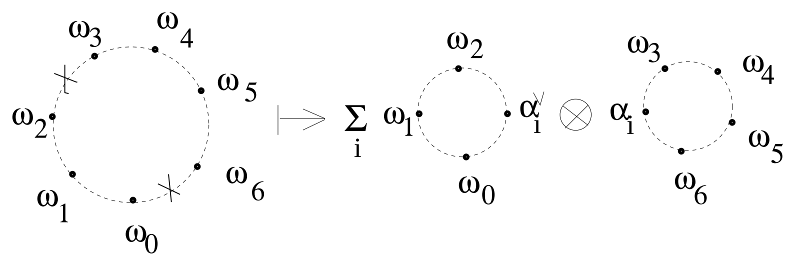


Figure 2: The Lie cobracket on $\mathcal{C}_{\mathbb{H}^*} \otimes \mathcal{H}^\vee$. The even factors $\mathcal{H}^\vee$ are not shown.

Functoriality of the Hodge correlator classes. Given a map $f : X \to Y$ of Kahler manifolds, the map $f_* : H_*(X) \to H_*(Y)$ induces a Lie algebra map

$$f_* : \mathcal{CL}ie_{\mathbb{H}_*(X)} \to \mathcal{CL}ie_{\mathbb{H}_*(Y)}.$$

It commutes with the differentials (Lemma 5.15). Here is our key result:

Theorem 1.4 *For any map $f : X \to Y$ of Kahler manifolds the derivation $\mathbf{H}^*_{Y,f(a)}$ preserves the Lie subalgebra $f_*(\mathcal{L}ie_{\mathbb{H}_*(X)})$, and its restriction to the latter is homotopic to $f_*(\mathbf{H}^*_{X,a})$.*

Theorem 1.1 can be viewed as a special case of Theorem 1.4 when $f : X \to X$ is the identity map, but we use two different splittings/Green currents to define the Hodge correlators on X. We prove Theorem 1.4 in Section 5.3. We prove a more precise Theorem 5.17, calculating the coboundary explicitly.[1]

Remark. Let $(\mathcal{L}_\bullet, \delta)$ be a DG Lie algebra with a differential δ. Denote by $\mathrm{Der}\mathcal{L}_\bullet$ the Lie algebra of derivations of the Lie algebra $\mathcal{L}_\bullet$. The commutator with the differential δ is a differential on the Lie algebra $\mathrm{Der}\mathcal{L}_\bullet$, making it into a DG Lie algebra. Then one has the following elementary Lemma.

[1]There is a different, less computational proof, which we discuss elsewhere. We also hope to give a different proof using the Feynman integral from Section 1.6 and the BV formalism elsewhere.

Lemma 1.5 *The Lie algebra $H_0^\delta\Big(\text{Der}\mathcal{L}_\bullet\Big)$ consists of the derivations of the Lie algebra $\mathcal{L}_\bullet$ preserving the grading, commuting with the differential δ, and defined up to a homotopy.*

The degree zero cycles are the derivations preserving the grading and commuting with the differential. The degree zero boundaries are the differentials homotopic to zero.

Applying the map (10) to the Hodge correlator class (8) class we arrive at an element

$$\mathbf{H}^*_{X,a} \in H_0^\delta\Big(\text{Der}^S \mathcal{L}ie_{\mathbb{H}_*}\Big). \tag{11}$$

In the next two subsections we explain how this element defines a functorial real mixed Hodge structure on the rational homotopy type of compact Kahler manifolds.

1.3 Rational homotopy type DG Lie algebras

Let M be a smooth simply connected manifold, $a \in M$. According to Quillen [Q] and Sullivan [S], the rational homotopy type of M is described by a DG Lie algebra over $\mathbb{Q}$ concentrated in degrees $0, -1, -2, \ldots$. It is well defined up to a quasiisomorphism. There is a similar DG Lie algebra $\text{L}(M,a)$ over $\mathbb{Q}$ in the case when M is not necessarily simply connected. Its zero cohomology $H^0\text{L}(M,a)$ is a Lie algebra, isomorphic to the pronilpotent completion of $\pi_1(M,a)$.

The DG Lie algebra $\text{L}(M,a) \otimes \mathbb{C}$ appears as follows. Let DGCom (respectively DGCoLie) be the category of differential graded commutative algebras (respectively Lie coalgebras) over $\mathbb{Q}$. Then there is a pair of adjoint functors (the Bar construction followed to the projection to the indecomposables, and the Chevalley complex)

$$\mathcal{B} : \text{DGCom} \longrightarrow \text{DGCoLie}, \qquad \mathcal{S} : \text{DGCoLie} \longrightarrow \text{DGCom},$$

Take the de Rham DGCom $\mathcal{A}^*(M)$ over $\mathbb{C}$. Given a point $a \in M$, the evaluation map $\text{ev}_a : \mathcal{A}^*(M) \to \mathbb{C}$ provides a DGCom denoted $\mathcal{A}^*(M,a)$. Applying to it the functor $\mathcal{B} \otimes \mathbb{C}$ we get a DGCoLie quasiisomorphic to the Lie coalgebra dual to $\text{L}(M,a) \otimes \mathbb{C}$.

Suppose now that M is a compact Kahler manifold. Then, thanks to the formality theorem [DGMS], the DG commutative algebra $\mathcal{A}^*(M,a)$ is quasiisomorphic to its cohomology, the reduced cohomology $\widetilde{H}^*(M,\mathbb{C})$ of X. So applying the functor $\mathcal{B}$ to the rational reduced cohomology $\widetilde{H}^*(M,\mathbb{Q})$ we get the minimal model of the rational homotopy type DG Lie algebra. We reserve from now on the notation $\text{L}(M,a)$ to this DG Lie algebra.

Let X be a smooth irreducible projective variety over a field k. Let us choose an embedding $\sigma : k \subset \mathbb{C}$. Then the rational homotopy type DG Lie algebra $\text{L}(X(\mathbb{C}),a)$ is supposed to be an object of motivic origin: One should exist a DG Lie algebra $\text{L}_{\mathcal{M}}(X,a)$ in the yet hypothetical abelian tensor category of mixed motives over k, whose Betti realization is the DG Lie algebra $\text{L}(X(\mathbb{C}),a)$. The Hodge realization $\text{L}_{\text{Hod}}(X,a)$ is known thanks to Morgan [M].

Let $\mathcal{L}ie_{\widetilde{\mathbb{H}}_*}$ be the free graded Lie algebra generated by

$$\widetilde{\mathbb{H}}_* := \text{H}_{>0}(X)[-1],$$

sitting in degrees $[-(2n-1), 0]$. The Lie algebra $\mathcal{L}ie_{\widetilde{\mathbb{H}}_*}$ has DG Lie algebra with the differential dualising the product on $\text{H}^{>0}(X)$. The formality theorem [DGMS] implies:

Theorem 1.6 *The associate graded for the weight filtration* $\mathrm{gr}^W \mathrm{L}(X,a)$ *of the rational homotopy type DG Lie algebra is quasiisomorphic to the DG Lie algebra* $\mathcal{L}ie_{\widetilde{\mathbb{H}}_*}$.

The zero cohomology $H^0\mathrm{L}_{\mathcal{M}}(X,a)$ is supposed to be a Lie algebra isomorphic to the (conjectural) unipotent motivic fundamental group of X. The latter was defined unconditionally for unirational X's over number fields [DG].

1.4 Hodge correlators and a real MHS on the rational homotopy type

The real Hodge Galois group. The category of real mixed Hodge structures is an abelian tensor category [D]. The associate graded for the weight filtration provides a fiber functor $H \longmapsto \mathrm{gr}^W H$ on this category. Therefore it is canonically equivalent to the category of representations of a pro-algebraic group over $\mathbb{R}$, called the Hodge Galois group G_{Hod}. It is isomorphic to a semidirect product of $\mathbb{C}^*_{\mathbb{C}/\mathbb{R}}$ and a prounipotent algebraic group U_{Hod}:

$$0 \longrightarrow U_{\mathrm{Hod}} \longrightarrow G_{\mathrm{Hod}} \longrightarrow \mathbb{C}^*_{\mathbb{C}/\mathbb{R}} \longrightarrow 0, \qquad \mathbb{C}^*_{\mathbb{C}/\mathbb{R}} \hookrightarrow G_{\mathrm{Hod}}.$$

The action of the group $\mathbb{C}^*_{\mathbb{C}/\mathbb{R}}$ provides the Lie algebra of the unipotent group $\mathrm{U}_{\mathrm{Hod}}$ with a structure of a Lie algebra in the category of direct sums of pure real Hodge structures. We denote this Lie algebra by $\mathrm{L}_{\mathrm{Hod}}$. So G_{Hod}-modules are nothing else but $\mathrm{L}_{\mathrm{Hod}}$-modules in the category of real Hodge structures.

According to Deligne [D2], the Lie algebra $\mathrm{L}_{\mathrm{Hod}}$ is generated by certain generators $g_{p,q}$, $p,q \geq 1$, satisfying the only relation $\overline{g}_{p,q} = -g_{q,p}$. In Section 4 of [G1] we defined a different set of generators, denoted below by $G^*_{p,q}$ and called the canonical generators – they behave nicer in families.[2] In the mixed Tate case (i.e. $p = q$) they essentially coincide with the ones defined in [L]. So a real mixed Hodge structure is nothing else but the following data:

- a real Hodge structure V, i.e. a real vector space V, whose complexification $V_{\mathbb{C}}$ is bigraded,
- a collection of imaginary operators $\{G^*_{p,q}\}$ on $V_{\mathbb{C}}$, $p,q > 0$, of the bidegree by $(-p,-q)$, such that $\overline{G}^*_{p,q} = -G^*_{q,p}$.

Indeed, a representation of $\mathbb{C}^*_{\mathbb{C}/\mathbb{R}}$ in V is the same thing as a real Hodge structure in V, and the operators $G^*_{p,q}$ are the images of the canonical generators of $\mathrm{L}_{\mathrm{Hod}}$. We encode a collection of the operators $\{G^*_{p,q}\}$ by a single operator

$$G^* = \sum_{p,q>0} G^*_{p,q}.$$

The Hodge Galois group action on the rational homotopy type DG Lie algebra.

Theorem-Construction 1.7 *There is a canonical map*

$$\mathrm{L}_{\mathrm{Hod}} \longrightarrow H_0^{\delta}\Big(\mathrm{Der}^S \mathcal{L}ie_{\mathbb{H}_*}\Big), \qquad G^* \longmapsto \mathbb{H}^*_{X,a}, \tag{12}$$

The action of the pure Hodge group $\mathbb{C}^*_{\mathbb{C}/\mathbb{R}}$ *is provided by the real Hodge structure on* $H^*(X)$.

[2] There are, in fact, two natural normalizations of the generators, denoted by $G^*_{p,q}$ and $G_{p,q}$ – they differ by certain binomial coefficients depending on p,q. Both play a role in our story. We use $*$ in the objects related to the $G_{p,q}$-generators, e.g. $\mathrm{Cor}^*_{\mathcal{H}}$. The Hodge correlators related to the generators $G^*_{p,q}$ are defined by using the operator ω, see Section 2.1, instead of the operator ξ in (6).

This leads to a real MHS on $H^\delta_* \mathrm{L}(X,a)$ as follows.

Definition 1.8 *A derivation of the graded Lie algebra $\mathcal{L}ie_{\widetilde{\mathbb{H}}_*}$ is special if it kills both the generator $H_{2n}(X)[-1]$ and the element $\mathrm{S} = \delta(H_{2n}(X)[-1])$.*

Denote by $\mathrm{Der}^S \mathcal{L}ie_{\widetilde{\mathbb{H}}_*}$ the Lie algebra of all special derivations. The inclusion $\mathbb{H}_* \hookrightarrow \widetilde{\mathbb{H}}_*$ provides an isomorphism of graded Lie algebras

$$\mathrm{Der}^S \mathcal{L}ie_{\mathbb{H}_*} \stackrel{\sim}{\longrightarrow} \mathrm{Der}^S \mathcal{L}ie_{\widetilde{\mathbb{H}}_*}. \tag{13}$$

One easily checks that the differentials preserve the subalgebras of special derivations. So the map (13) is an isomorphism of DG Lie algebras. Therefore it induces an isomorphism

$$H^\delta_0\Big(\mathrm{Der}^S \mathcal{L}ie_{\mathbb{H}_*}\Big) = H^\delta_0\Big(\mathrm{Der}^S \mathcal{L}ie_{\widetilde{\mathbb{H}}_*}\Big).$$

The element $\mathbb{H}^*_{X,a}$ lives in the Lie algebra on the left. We transform it to an element $\widetilde{\mathbb{H}}^*_{X,a}$ of the Lie algebra on the right. So we get a map

$$\mathrm{L}_{\mathrm{Hod}} \longrightarrow H^\delta_0\Big(\mathrm{Der}^S \mathcal{L}ie_{\widetilde{\mathbb{H}}_*}\Big), \qquad G^* \longmapsto \widetilde{\mathbb{H}}^*_{X,a}, \tag{14}$$

The Lie algebra $H^\delta_0\Big(\mathrm{Der}^S \mathcal{L}ie_{\widetilde{\mathbb{H}}_*}\Big)$ acts by derivations of $\mathcal{L}ie_{\widetilde{\mathbb{H}}_*}$, defined up to a homotopy. Finally, $\mathcal{L}ie_{\widetilde{\mathbb{H}}_*}$ is isomorphic to $\mathrm{gr}^W \mathrm{L}(X,a)$ by Theorem 1.6.

Summarising, we defined a real MHS on $\mathrm{L}(X,a)$. When the point a varies, we get a variation of real MHS. The proof of this is completely similar to the one given in [G1] in the case of curves, and thus is omitted.

In particular, we get a real MHS on the pronilpotent completion $\pi_1^{\mathrm{nil}}(X,a)$.

Theorem 1.9 *Let X be a complex regular projective variety. Then the real MHS on $\pi_1^{\mathrm{nil}}(X,a)$ provided by the Hodge correlators coincides with the classical one.*

Proof. We proved in [G1] that this is true for any smooth curve X. The Hodge correlator real MHS on $\pi_1^{\mathrm{nil}}(X,a)$ is functorial by Theorem 1.4. By the Lefschetz theorem there exists a plane section Y of X such that $\pi_1(Y)$ surjects onto $\pi_1(X)$. This implies Theorem 1.9.

The real MHS on $\pi_1^{\mathrm{nil}}(X,a)$ can be described via Chen's iterated integrals. Theorem 1.9 tells that the Hodge correlators provide the same real periods as iterated integrals. It would be interesting to relate the Hodge correlators to the iterated integrals directly. I know how to do this only in the simplest cases.

1.5 Motivic correlators for a regular projective variety

Motivic Galois groups. One expects that there exists an abelian tensor category $\mathcal{MM}_k$ of mixed motives over a field k. Each object M of this category should carry a canonical weight filtration $W_\bullet$. The functor

$$\omega : \mathcal{MM}_k \stackrel{?}{\longrightarrow} \mathcal{PM}_k, \qquad M \longmapsto \mathrm{gr}^W M$$

should be a fiber functor with values in the semisimple tensor category $\mathcal{PM}_k$ of pure motives. The algebraic group scheme of all automorphisms of this fiber functor is called the motivic Galois group $G_{\rm Mot}$. Thanks to the Tannakian formalism, the functor ω then provides a canonical equivalence between the category of all mixed motives and the category of representations of $G_{\rm Mot}$ in the category of pure motives. So the motivic Galois group $G_{\rm Mot}$ contains all the information telling us how the mixed motives are obtained from the pure ones.

The motivic Galois group $G_{\rm Mot}$ is a prounipotent algebraic group in the tensor category of pure motives. We denote by $\mathrm{L}_{\rm Mot}$ its Lie algebra, and by $\mathcal{L}_{\rm Mot}$ the dual Lie coalgebra – see, say, [G4] for more details.

Motivic correlators. The motivic Galois group should act by symmetries of the rational motivic homotopy type of a variety X. The latter, however, is well defined only up to a quasi-isomorphism. Here is a precise statement:

One should exist a homomorphism of Lie algebras in the category of pure motives

$$\mathrm{L}_{\rm Mot} \stackrel{?}{\longrightarrow} H_0^\delta\Big(\mathrm{Der}(\mathrm{gr}^W \mathrm{L}_{\mathcal{M}}(X,a))\Big). \tag{15}$$

The $\mathbb{R}$-Hodge realization of the motivic Lie algebra $\mathrm{L}_{\rm Mot}$ is the Hodge Lie algebra $\mathrm{L}_{\rm Hod}$. So the $\mathbb{R}$-Hodge realization of the map (15) should give the map (12).

Denote by $\mathcal{L}ie_{\mathrm{Mot}(\mathbb{H}_*)}$ the free graded Lie algebra generated by the reduced homology motive of X shifted to the right by 1. It is a DG Lie algebra. Similarly, there is a slightly bigger DG Lie algebra $\mathcal{L}ie_{\mathrm{Mot}(\widetilde{\mathbb{H}}_*)}$. Theorem 1.6 tells that one should exist a canonical isomorphism

$$\mathcal{L}ie_{\mathrm{Mot}(\widetilde{\mathbb{H}}_*)} \stackrel{?}{=} \mathrm{gr}^W \mathrm{L}_{\mathcal{M}}(X,a). \tag{16}$$

Just like in Definition 1.8, one defines special derivations of the Lie algebra on the left, and hence on the right of (16). The motivic Galois Lie algebra should act by special derivations of the Lie algebra $\mathrm{gr}^W \mathrm{L}^{\rm Mot}(X,a)$. There is a canonical isomorphism

$$\mathrm{Der}^S \mathcal{L}ie_{\mathrm{Mot}(\mathbb{H}_*)} \stackrel{\sim}{\longrightarrow} \mathrm{Der}^S \mathcal{L}ie_{\mathrm{Mot}(\widetilde{\mathbb{H}}_*)}. \tag{17}$$

Therefore the map (15) plus isomorphisms (17) and (16) provide a canonical map of Lie algebras

$$\mathrm{L}_{\rm Mot} \stackrel{?}{\longrightarrow} H_0^\delta\Big(\mathrm{Der}^S \mathcal{L}ie_{\mathrm{Mot}(\mathbb{H}_*)}\Big). \tag{18}$$

Dualizing the map (18) and using the motivic version of isomorphism (10) we get a morphism of Lie coalgebras, called the *motivic correlator map* for X:

$$\mathrm{Cor}_{\rm Mot} : H^0_\delta\Big(\mathcal{CL}ie^\vee_{\mathrm{Mot}(\mathbb{H}_*)} \otimes \mathcal{H}^\vee\Big) \stackrel{?}{\longrightarrow} \mathcal{L}_{\rm Mot}. \tag{19}$$

Its source is easily defined in terms of the cohomology motive of X. The target is remarkable: it carries all the information how the mixed motives are glued from the pure ones. The cobracket of an element in $\mathcal{L}_{\rm Mot}$ contains all the information about the complexity of the element. Therefore, since (19) is a Lie coalgebra map, and the cobracket in the left Lie coalgebra is very transparent, see Fig 2, we get a full control of the arithmetic complexity of motivic correlators.

Relating motivic and Hodge correlators. The hypothetical motivic Lie coalgebra $\mathcal{L}_{\rm Mot}$ is an inductive limit of objects in the semisimple category $\mathcal{PM}_k$ of pure motives over k. One can decompose it into the isotipical components:

$$\mathcal{L}_{\rm Mot} \stackrel{?}{=} \bigoplus_{M\in {\rm Iso}(\mathcal{PM}_k)} \mathcal{L}_M \otimes M^\vee, \qquad \mathcal{L}_M = {\rm Hom}_{\mathcal{PM}_k}(M^\vee, \mathcal{L}_{\rm Mot}).$$

Here M runs through the isomorphism classes of simple objects in the category $\mathcal{PM}_k$, $M^\vee$ is the dual motive, and $\mathcal{L}_M$ is a $\mathbb{Q}$-vector space.

One can do this in the Hodge realization. Namely, denote by $\mathcal{L}_{\rm Hod}$ the $\mathbb{R}$-Hodge Lie coalgebra. Then there is a decomposition into the isotipical components:

$$\mathcal{L}_{\rm Hod} = \bigoplus_{H\in {\rm Iso}(\mathcal{P}{\rm Hod}_{\mathbb{R}})} \mathcal{L}_H \otimes H^\vee.$$

Proposition 1.10 *There is a canonical linear map, called the real period map:*

$$\mathrm{P}_{\mathbb{R}}: \mathcal{L}_{\rm Hod} \longrightarrow i\mathbb{R}.$$

The Hodge and motivic correlator are compatible as follows.

Proposition 1.11 *Let X be a regular projective variety over $\mathbb{R}$. Then the composition*

$$H^0_\delta\Big(\mathcal{CL}ie^\vee_{{\rm Hod}(\mathbb{H}_*)} \otimes \mathcal{H}^\vee\Big) \stackrel{{\rm Cor}_{\rm Hod}}{\longrightarrow} \mathcal{L}_{\rm Hod} \stackrel{\mathrm{P}_{\mathbb{R}}}{\longrightarrow} i\mathbb{R}.$$

coincides with the Hodge correlator map ${\rm Cor}_{\mathcal{H}}$.

Therefore, given a regular projective variety X over $\mathbb{Q}$, and assuming the existence the motivic correlator map ${\rm Cor}_{\rm Mot}$, the Hodge correlator map coincides with the composition

$$H^0_\delta\Big(\mathcal{CL}ie^\vee_{{\rm Mot}(\mathbb{H}_*)} \otimes \mathcal{H}^\vee\Big) \stackrel{{\rm Cor}_{\rm Mot}}{\longrightarrow} \mathcal{L}_{\rm Mot} \stackrel{r_{\rm Hod}}{\longrightarrow} \mathcal{L}_{\rm Hod} \stackrel{\mathrm{P}_{\mathbb{R}}}{\longrightarrow} i\mathbb{R}.$$

Here $r_{\rm Hod}$ is the $\mathbb{R}$-Hodge realization functor.

1.6 A Feynman integral for Hodge correlators

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two DG Lie algebras. Let us forget for a moment about their differentials, treating them as graded Lie superalgebras. Then there is an affine $\mathbb{G}_m$-superscheme[3] of graded Lie superalgebra maps $\mathcal{H}om_{\rm Lie}(\mathcal{L}_1, \mathcal{L}_2)$. Its points with values in a graded supercommutative algebra A are defined by

$$\mathcal{H}om_{\rm Lie}(\mathcal{L}_1, \mathcal{L}_2)(A) := {\rm Hom}_{\rm Lie}(\mathcal{L}_1, \mathcal{L}_2 \otimes A).$$

Denote by $\mathcal{L}ie_V$ the free graded Lie superalgebra generated by a finite dimensional graded vector space V. Let $\mathcal{G}$ be a graded Lie superalgebra. Then there is an isomorphism of affine $\mathbb{G}_m$-superschemes

$$\mathcal{H}om_{\rm Lie}(\mathcal{L}ie_{V^*}, \mathcal{G}) = V \otimes \mathcal{G}. \tag{20}$$

[3]That is, an affine superscheme equipped with an action of the algebraic group $\mathbb{G}_m$

Suppose now that $\mathcal{L}ie_{V^*}$ has a differential Δ providing it with a DG Lie algebra structure. Since Δ is an odd (degree 1) infinitesimal automorphism of the Lie superalgebra $\mathcal{L}ie_{V^*}$, it is transformed by isomorphism (20) to an infinitesimal automorphism of the affine $\mathbb{G}_m$-superscheme $V \otimes \mathcal{G}$, that is to a homological vector field v_Δ on $V \otimes \mathcal{G}$, making it into an affine DG scheme – see Section 4.3 for background.

Furthermore, let $\mathcal{D}$ be a derivation of the graded Lie algebra $\mathcal{L}ie_V$ commuting with Δ. Then isomorphism (20) transforms it to a homological vector field $v_\mathcal{D}$ commuting with v_Δ.

Example. Let $(\mathcal{L}, d)$ be a DG Lie algebra. Then $\mathcal{L}[1]$ is a DG space with the homological vector field given by the Chern-Simons vector field $Q_{\rm CS}$ (see Section 4.2):

$$\dot{\alpha} = Q_{\rm CS}(\alpha) = d\alpha + \frac{1}{2}[\alpha, \alpha].$$

In particular, given a DG commutative algebra A and a Lie algebra $\mathcal{G}$, there is a DG Lie algebra $A \otimes \mathcal{G}$ with the Lie bracket $[a_1 \otimes g_1, a_2 \otimes g_2] := a_1 a_2 \otimes [g_1, g_2]$, and hence a DG scheme $A[1] \otimes \mathcal{G}$.

The functor $\mathcal{B}$: DGCom $\to$ DGColie takes an $A \in$ DGCom to the cofree Lie coalgebra CoLie$(A[1])$ cogenerated by the graded space $A[1]$, with a certain differential Δ.

The functor $\mathcal{B}$ is characterised by the following universality property:

- One has a functorial isomorphism of the affine DG schemes:

$$\mathcal{H}om_{\rm CoLie}(\mathcal{G}^*, \mathcal{B}(A)) = A[1] \otimes \mathcal{G}. \tag{21}$$

In other words, the functorial isomorphism (21) transforms the differential Δ on $\mathcal{B}(A)$ to the Chern-Simons homological vector field $Q_{\rm CS}$ on $A[1] \otimes \mathcal{G}$.

Applying (21) to the algebra $A = \overline{\rm H}^*(X)$, we get an isomorphism of DG schemes

$$\mathcal{H}om_{\rm Lie}(\mathcal{L}ie_{\mathbb{H}_*}, \mathcal{G}) = \mathbb{H}^*(X) \otimes \mathcal{G} =: \mathcal{G}_H[1]. \tag{22}$$

The differential δ on the DG Lie algebra $\mathcal{L}ie_{\mathbb{H}_*}$ is transformed to the Chern-Simons homological vector field Q_{CS} on $\mathcal{G}_H[1]$.

Let $\mathcal{G}$ be a Lie algebra with a non-degenerate scalar product $Q(*, *)$. For example one can take the Lie algebra ${\rm Mat}_N$ of $N \times N$ matrices with $Q(A, B) := {\rm Tr}(AB)$.

The Hodge correlator $\mathbf{H}^*_{X,a}$, see (11), is a derivation of $\mathcal{L}ie_{\mathbb{H}_*}$ commuting with δ and defined up to a homotopy. So it is transformed by isomorphism (22) to a homological vector field ${\rm Q_{Hod}}$ on $\mathcal{G}_H[1]$, commuting with the Chern-Simons vector field ${\rm Q_{CS}}$ and well defined up to commutators $[{\rm Q_{CS}}, R]$. Our next goal is to show that we get a Hamiltonian vector field, whose Hamiltonian is given by a Feynman integral introduced below.

Given $\mathcal{G}$-valued differential forms φ_i on X, we define a differential form

$$\langle \varphi_1, \varphi_2, \varphi_3 \rangle := Q(\varphi_1, [\varphi_2, \varphi_3]).$$

Choose a splitting of the bigraded de Rham complex "harmonic" forms $\mathcal{H}ar_X$ and its orthogonal complement. Let Ker $d^{\mathbb{C}} \subset \mathcal{A}^*(X)$ be the kernel of $d^{\mathbb{C}}$. Let us define, using the splitting, a function on the functional space

$$\text{Ker } d^{\mathbb{C}} \otimes \mathcal{G}[1].$$

Let $\psi \in \text{Ker } d^{\mathbb{C}} \otimes \mathcal{G}[1]$. Write $\text{Ker } d^{\mathbb{C}} = \mathcal{H}ar_X \oplus \text{Im } d^{\mathbb{C}}$, and

$$\psi = \psi_0 + \alpha, \quad \psi_0 \in \text{Im } d^{\mathbb{C}} \otimes \mathcal{G}[1], \quad \alpha \in \mathcal{H}ar_X \otimes \mathcal{G}[1].$$

Choose φ such that $d^{\mathbb{C}}\varphi = \psi_0$. Then there is an action:

$$S(\psi) = \int_X \frac{1}{2}(\varphi, d\psi) + \frac{1}{6}\langle \varphi, \psi, \psi \rangle, \qquad d^{\mathbb{C}}\varphi = \psi_0. \tag{23}$$

Observe that $S(\psi)$ is independent of the choice of φ. Indeed, $\int_X \langle d^{\mathbb{C}}\eta, d\psi \rangle = \int_X \langle d^{\mathbb{C}}\eta, \psi, \psi \rangle = 0$ since $d^{\mathbb{C}}\psi = 0$. Notice that $d^{\mathbb{C}}$ (Lagrangian in (23)) is very close to the Chern-Simons Lagrangian:

$$d^{\mathbb{C}}\Big(\frac{1}{2}(\varphi, d\psi) + \frac{1}{6}\langle \varphi, \psi, \psi \rangle\Big) = \frac{1}{2}\langle \psi_0, d\psi \rangle + \frac{1}{6}\langle \psi_0, \psi, \psi \rangle$$

We would like to "integrate" $e^{iS(\psi)}$ along the fibers of the natural projection provided by the splitting of the de Rham complex:

$$\text{Ker } d^{\mathbb{C}} \otimes \mathcal{G}[1] \longrightarrow H^*(X) \otimes \mathcal{G}[1], \qquad \psi = \psi_0 + \alpha \longmapsto \alpha.$$

However the quadratic part $(\varphi, d\psi)$ of the Lagrangian is highly degenerate. Precisely, by the classical Hodge theory there exists a subspace $F \subset \mathcal{A}_X^{*,*}$ such that there is an isomorphism of bicomplexes

$$\mathcal{A}_X^{*,*} = \mathcal{H}ar_X \quad \bigoplus \quad \begin{array}{ccc} d^{\mathbb{C}}F & \xrightarrow{d} & d^{\mathbb{C}}dF \\ d^{\mathbb{C}} \uparrow & & d^{\mathbb{C}} \uparrow \\ F & \xrightarrow{d} & dF \end{array} \tag{24}$$

Here all arrows in the square are isomorphisms. Then

$$\text{Ker } d^{\mathbb{C}} = \mathcal{H}ar_X \oplus d^{\mathbb{C}}F \oplus d^{\mathbb{C}}dF,$$

and kernel of the quadratic form $(\varphi, d\psi)$ is the subspace $\mathcal{H}ar_X \oplus d^{\mathbb{C}}dF$. The form is non-degenerate on $d^{\mathbb{C}}F$. We integrate over $\psi \in d^{\mathbb{C}}F$, i.e. over a "a quarter F of the de Rham complex", getting a function, the Hodge correlator, which depends on the decomposition (24):

$$\widetilde{\text{Cor}}^*_{\mathcal{H}}(\alpha) := \int e^{iS(\alpha + d^{\mathbb{C}}\varphi)}\mathcal{D}\varphi, \qquad \varphi \in F. \tag{25}$$

Elaborating

$$S(\alpha + d^{\mathbb{C}}\varphi) := \int_X \frac{1}{2}(\varphi, dd^{\mathbb{C}}\varphi) + \frac{1}{6}\langle \varphi, d^{\mathbb{C}}\varphi, d^{\mathbb{C}}\varphi \rangle + \frac{1}{3}\langle \alpha, \varphi, d^{\mathbb{C}}\varphi \rangle + \frac{1}{6}\langle \alpha, \alpha, \varphi \rangle,$$

we use the perturbative series expansion on the tree level to make sense of the integral, treating α as a small parameter – see Section 4.4. It involves only plane trivalent trees decorated by the harmonic form α, i.e. it amounts to minimizing the action:

$$\widetilde{\text{Cor}}^*_{\mathcal{H}}(\alpha) = \text{Min}_{\varphi \in F} S(\alpha + d^{\mathbb{C}}\varphi).$$

The space $\mathcal{G}_{\text{H}}[1]$ has an even symplectic structure

$$\omega(\overline{h}_1 \otimes g_1, \overline{h}_2 \otimes g_2) := (-1)^{|h_1|} Q(g_1, g_2) \int_X h_1 \wedge h_2.$$

So function (25) gives rise to a Hamiltonian vector field on $\mathcal{G}_{\rm H}[1]$:

$$\widetilde{\rm Q}^*_{\rm Hod} \in S^{>0}(\mathcal{G}_{\rm H}[1]^\vee) \otimes \mathcal{G}_{\rm H}[1]. \quad (26)$$

Here on the right stands the reduced Chevalley cochain complex of the graded Lie algebra $\mathcal{G}_{\rm H}$ with the adjoint coefficients. It has a differential δ given by the commutator with $Q_{\rm CS}$.

Theorem 1.12 *i) The element (26) provides a well defined, functorial cohomology class*

$$\mathrm{Q}^*_{\rm Hod} \in H^\delta_0\Big({\rm Der}\mathcal{G}_{\rm H}\Big). \quad (27)$$

ii) The class (27) is the image of the class $\mathbf{H}^*_{X,a}$ under the isomorphism (22).

The part i) means that the cohomology class (26) does not depend on the choice of the subspaces F and $\mathcal{H}ar_X$, as well as on the choice of a Green current involved in its perturbative series expansion definition. The part i) is deduced from ii) and Theorem 1.1. The part ii) is obtained by comparing the correlators from Sections 2.3/4.1 with the ones from Section 4.5.

We view the class $Q^*_{\rm Hod}$ as a vector field on the DG space $\mathcal{G}_H[1]$ – we call it the *Hodge vector field*. The part i) of Theorem 1.12 just means that it commutes with the Chern-Simons field Q_{CS}, and is well defined up to commutators with Q_{CS}. Let ε be an odd element of degree 1, so $\varepsilon^2 = 0$. Then the part i) of Theorem 1.12 is equivalent to the

Theorem 1.13 *The vector field $\mathrm{Q}_{CS} + \varepsilon Q^*_{\rm Hod}$ is a homological vector field on the DG space $\mathcal{G}_H[1]$, well defined up to commutators $[\mathrm{Q}_{CS}, *]$.*

So a real mixed Hodge structure on the rational homotopy type of X is the same thing as a deformation of the DG space $\mathcal{G}_H[1]$ over the odd line ${\rm Spec}\mathbb{R}[\varepsilon]$, *functorial in $\mathcal{G}$, considered up to an isomorphism.*

The Hodge Galois group and complex local systems An $\mathbb{R}$-mixed Hodge structure on the pronilpotent fundamental group $\pi_1^{\rm nil}(X,x)$ tell us that there is an extension of Lie algebras, where $\mathcal{G}_{\rm Hod}$ is the Lie algebra of the Hodge Galois group $G_{\rm Hod}$:

$$0 \longrightarrow \pi_1^{\rm nil}(X,x) \longrightarrow \pi_1^{\rm Hod}(X,x) \longrightarrow \mathcal{G}_{\rm Hod} \longrightarrow 0. \quad (28)$$

A base point $x \in X$ provides a splitting $s_x : \mathcal{G}_{\rm Hod} \to \pi_1^{\rm Hod}(X,x)$, and hence an action of $\mathcal{G}_{\rm Hod}$ on the Lie algebra $\pi_1^{\rm nil}(X,x)$, and thus a real MHS on $\pi_1^{\rm nil}(X,x)$.

It follows that the Lie algebra $\mathcal{G}_{\rm Hod}$ acts on representations of the Lie algebra $\pi_1^{\rm nil}(X,x)$. The latter are automatically nilpotent, and form the formal neighborhood of the trivial local system. So $\mathcal{G}_{\rm Hod}$ acts on the formal neighborhood of the trivial local system. In Section 4.6 we describe this action explicitly by using the class (27).

Acknowledgments. I am very grateful to Maxim Kontsevich for useful discussions.

A part of this work was done during my visits to the IHES at the Summer of 2007 and MPI(Bonn) at the Summer of 2008. I am grateful to these institutions for the hospitality and support. I was supported by the NSF grant DMS-0653721. I am grateful to the referee for useful comments.

2 Hodge correlators for compact Kahler manifolds

2.1 The polydifferential operator ω

In this subsection we recall some key definitions from [G1]. The proofs can be found in *loc. cit.*.

Let φ_0, ..., φ_m be smooth forms on a complex manifold M. We define a form $\omega(\varphi_0, ..., \varphi_m)$ as follows. Given forms f_i of degrees $\deg(f_i)$ and a function $F(f_1, \dots, f_m)$ set

$$\mathrm{Sym}_m F(f_1, \dots, f_m) := \sum_{\sigma \in \Sigma_m} \mathrm{sgn}_{\sigma; f_1, \dots, f_m} F(f_{\sigma(1)}, \dots, f_{\sigma(m)}), \tag{29}$$

where the sign of the transposition of f_1, f_2 is $(-1)^{(\deg(f_1)+1)(\deg(f_2)+1)}$, and the sign of a permutation written as a product of transpositions is the product of the signs of the corresponding transpositions. Observe that this is nothing else but the symmetrization of the elements $\overline{f}_i \in \mathcal{A}^*_M[-1]$ of the shifted de Rham complex of M assigned to the elements f_i. Set

$$\omega(\varphi_0, ..., \varphi_m) := \frac{1}{(m+1)!} \mathrm{Sym}_{m+1} \Big(\sum_{k=0}^{m} (-1)^k \varphi_0 \wedge \partial\varphi_1 \wedge ... \wedge \partial\varphi_k \wedge \overline{\partial}\varphi_{k+1} \wedge ... \wedge \overline{\partial}\varphi_m \Big). \tag{30}$$

There is a degree zero linear map

$$\omega : \mathrm{Sym}^m \Big(\mathcal{A}^\bullet_M[-1] \Big) \longrightarrow \mathcal{A}^\bullet_M[-1], \qquad \omega : \varphi_0 \wedge ... \wedge \varphi_m \longmapsto \omega(\varphi_0, ..., \varphi_m).$$

Its key property is the following:

$$d\omega(\varphi_0, \dots, \varphi_m) = (-1)^m \partial\varphi_0 \wedge \dots \wedge \partial\varphi_m + \overline{\partial}\varphi_0 \wedge \dots \wedge \overline{\partial}\varphi_m + \tag{31}$$

$$\frac{1}{m!} \mathrm{Sym}_{m+1} \Big((-1)^{\deg(\varphi_0)} \overline{\partial}\partial\varphi_0 \wedge \omega(\varphi_1, \dots, \varphi_m) \Big). \tag{32}$$

Set $d^{\mathbb{C}} := \partial - \overline{\partial}$. Consider the following differential forms:

$$\xi(\varphi_0, \dots, \varphi_m) := \frac{1}{(m+1)!} \mathrm{Sym}_{m+1} \Big(\varphi_0 \wedge d^{\mathbb{C}}\varphi_1 \wedge \dots \wedge d^{\mathbb{C}}\varphi_m \Big) = \tag{33}$$

$$\frac{1}{(m+1)!} \mathrm{Sym}_{m+1} \Big(\sum_{k=0}^{m} (-1)^k \binom{m}{k} \varphi_0 \wedge \partial\varphi_1 \wedge \dots \wedge \partial\varphi_k \wedge \overline{\partial}\varphi_{k+1} \wedge \dots \wedge \overline{\partial}\varphi_m \Big).$$

$$\eta(\varphi_0, \dots, \varphi_m) := \frac{1}{(m+1)!} \mathrm{Sym}_{m+1} \Big(d^{\mathbb{C}}\varphi_0 \wedge \dots \wedge d^{\mathbb{C}}\varphi_m \Big) = \tag{34}$$

$$\frac{1}{(m+1)!} \mathrm{Sym}_{m+1} \Big(\sum_{k=0}^{m} (-1)^k \binom{m+1}{k+1} \partial\varphi_0 \wedge \dots \wedge \partial\varphi_k \wedge \overline{\partial}\varphi_{k+1} \wedge \dots \wedge \overline{\partial}\varphi_m \Big).$$

Every summand in the form η appears with the coefficient ± 1. We have

$$d^{\mathbb{C}}\xi(\varphi_0, \dots, \varphi_m) = \eta(\varphi_0, \dots, \varphi_m). \tag{35}$$

2.2 Green currents

Let M be a compact Kahler manifold. Denote by $(\mathcal{A}_M^{*,*}, \partial, \overline{\partial})$ the Dolbeault bicomplex on M. Recall the Hodge decomposition $H^*(M, \mathbb{C}) = \oplus_{p,q} H^{p,q}(M)$. Choose a representative for each $(p.q)$-cohomology class of M. We call such a datum *a harmonic splitting of the Dolbeaut bicomplex*, and the representatives the *harmonic representatives.*

Green current of a cycle. Let M be a smooth complex projective variety, and $Z \subset M$ a complex algebraic cycle of codimension d_Z in M. Then integration over Z is a (d_Z, d_Z)-current δ_Z, representing the cohomology class of the cycle Z in the Dolbeault bicomplex $\mathcal{A}_M^{*,*}$.

On the other hand, given a choice of a splitting s of the Dolbeaut bicomplex on M, we can represent the cohomology class of the cycle Z by a smooth differential form Har_Z on M, its harmonic representative.

Definition 2.1 *Let M be a smooth complex projective variety. A Green current G_Z corresponding to an algebraic cycle $Z \subset M$ and a splitting s is a current on M satisfying the equation*

$$(2\pi i)^{-1}\overline{\partial}\partial G_Z = \delta_Z - \mathrm{Har}_Z. \tag{36}$$

Here on the right we have a difference of two (d_Z, d_Z)-currents whose cohomology classes provide the class of Z. Thus their difference is a ∂- and $\overline{\partial}$-closed (d_Z, d_Z)-current. By the $\overline{\partial}\partial$-lemma there exists a solution of the differential equation (36). It is a $(d_Z - 1, d_Z - 1)$-current, well defined up to elements from $\mathrm{Im}\partial + \mathrm{Im}\overline{\partial}$. There is a solution smooth outside of the cycle Z.

Green current of a map. Let $f : M \to N$ be a map of compact Kahler manifolds. Choose splittings of the Dolbeaut complexes on M and N. Denote by Δ_f the graph of the map. It is a cycle in $X \times Y$. Let Har_{Δ_f} be the harmonic representative of the cohomology class of the cycle Δ_f. Denote by δ_{Δ_f} the δ-current defined by the cycle Δ_f. Then a Green current of the map f is a current on $M \times N$ satisfying the equation

$$(2\pi i)^{-1}\overline{\partial}\partial G_Z = \delta_{\Delta_f} - \mathrm{Har}_{\Delta_f}. \tag{37}$$

Green current of the diagonal. Let f be the identity map of M. Let p_1 and p_2 be the projections of $M \times M$ onto the first and second factors. Let Δ_M be the diagonal in $M \times M$. Then δ_{Δ_M} is an (n, n)-current on $M \times M$. Choose a splitting of the Dolbeaut bicomplex on M. Let $\alpha_1, .., \alpha_N$ be the corresponding representatives of a Hodge basis in $\oplus_{0<p+q<2n} H^{p,q}(M, \mathbb{C})$. Denote by $\alpha_1^\vee, .., \alpha_N^\vee$ the representatives for the dual basis: one has $\int_M \alpha_i \wedge \alpha_j^\vee = \delta_{ij}$.

The constant function 1 is a canonical representative of a class in H^0. Let μ be a $2n$-current on M such that $\int_M \mu = 1$. A Green current $G_\mu(x, y)$ is an $(n-1, n-1)$-current on $M \times M$ which satisfies

$$(2\pi i)^{-1}\overline{\partial}\partial G_\mu(x, y) = \delta_{\Delta_M} - \Big(p_1^*\mu + p_2^*\mu + \sum_{k=1}^{N} p_1^*\alpha_k^\vee \wedge p_2^*\alpha_k\Big). \tag{38}$$

and is symmetric with respect to the permutation of the factors: $G_\mu(x, y) = G_\mu(y, x)$. On the right of (38) stands the difference of two (n, n)-currents whose cohomology classes provide the identity map on $H^*(M, \mathbb{C})$. This current is invariant under the permutation of the factors in $X \times X$. So symmetrising any solution to (38) we get a symmetric solution.

There is a specific choice of μ. A point $a \in M$ provides a $2n$-current $\mu := \delta_a$ on M: one has $\langle \delta_a, \varphi \rangle = \varphi(a)$. The corresponding Green current is denoted $G_a(x, y)$. It satisfies the equation

$$(2\pi i)^{-1}\overline{\partial}\partial G_a(x,y) = \delta_{\Delta_M} - \Big(\delta_{\{a\}\times M} + \delta_{M\times\{a\}} + \sum_k p_1^*\alpha_k^\vee \wedge p_2^*\alpha_k\Big). \tag{39}$$

2.3 Hodge correlators

Let X be an n-dimensional smooth compact complex Kahler variety. We start by introducing some notation. First, we use from now on the notation

$$\mathbb{H} \quad \text{for} \quad \mathbb{H}_*(X) \tag{40}$$

We allow the following abuse of notation: we denote sometimes by the same letter α a harmonic form and the corresponding element of $\mathbb{H}^*(X)$. Strictly speaking, the second element is $\alpha \otimes \mathbf{E}$ where $\mathbf{E}$ is a degree -1 element, and thus should have a different notation, $\overline{\alpha}$. Similarly h denotes, depending on the context, either a homology class, or a shifted by 1 homology class. The latter is also denoted by $\overline{h}$ to emphasize the shift of the degree.

We denote by $|\overline{\omega}| := \deg(\omega) - 1$ the degree of a form ω in the shifted de Rham complex $\mathcal{A}^*(X)[1]$, and similarly $|\overline{h}| := \deg(h) + 1$ is the degree in $H_*(X)[-1]$. We reserve the notation $\deg(\omega)$ and $\deg(h)$ for their degrees before the shift.

A $2n$-current μ on X is *admissible* if it is either a smooth volume form of total mass 1 on X, or the δ-current δ_a for some point $a \in X$. Given an admissible $2n$-current μ, we are going to define a linear map, the (precursor of) Hodge correlator map[4]:

$$\mathrm{Cor}_{\mathcal{H},\mu} : \mathcal{C}_{\mathbb{H}^*} \otimes \mathcal{H}^\vee \longrightarrow \mathbb{C}. \tag{41}$$

Let us choose a harmonic splitting of the Dolbeaut bicomplex $\mathcal{A}^{*,*}(X)$. Given homogeneous harmonic forms α_i, consider a cyclic element

$$W = \mathcal{C}(\overline{\alpha}_0 \otimes \overline{\alpha}_1 \otimes \ldots \otimes \overline{\alpha}_m). \tag{42}$$

We use the standard terminology about trees and decorated trees, see Section 2 of [G1]. Let T be a plane trivalent tree decorated by the forms $\alpha_0, \ldots, \alpha_m$. The external vertices of a plane tree have a cyclic order provided by a chosen orientation of the plane, say clockwise. We assume that the cyclic order of the forms α_i is compatible with the cyclic order of the vertices, as on Fig 3. Although the decoration depends on the presentation of W as a cyclic tensor product of the α_i's, constructions below depend only on W. We are going to assign to such a W-decorated tree T a current κ_W on

$$X^{\{\text{internal vertices of } T\}}. \tag{43}$$

Given a finite set $\mathcal{X} = \{x_1, \ldots, x_{|\mathcal{X}|}\}$, there is a $\mathbb{Z}/2\mathbb{Z}$-torsor, the *orientation torsor of* $\mathcal{X}$. Its elements are expressions $x_1 \wedge \ldots \wedge x_{|\mathcal{X}|}$; interchanging two neighbors we change the sign of the expression. The *orientation torsor* or_T of a graph T is the orientation torsor of the set of edges of T. An orientation of the plane induces an orientation of a plane trivalent tree.

[4]We skip the subscript μ whenever possible.

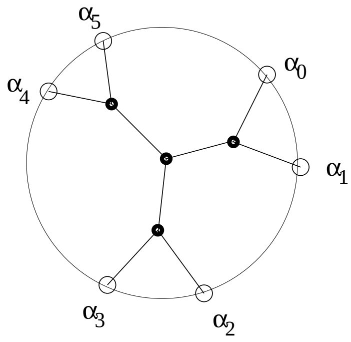


Figure 3: A plane trivalent tree decorated by harmonic forms α_i.

Take an internal edge E of the tree T. There is a projection

$$p_E : X^{\{\text{internal vertices of } T\}} \longrightarrow X^{\{\text{vertices of E}\}} = X \times X. \tag{44}$$

Choose a Green current $G_\mu(x, y)$ corresponding to an admissible $2n$-current μ on X. . Assign to the edge E the Green current $G_\mu(x, y)$ on the right space in (44). Since the Green current is symmetric, the output is well defined. Denote by G_E its pull back by the map p_E^*. Since the map p_E is transversal to the wave front of the Green current, the pull back is well defined.

Let $\{E_0, \ldots, E_{2m}\}$ be the set of edges of T numbered so that the internal edges are the first $m - 2$ of them. Set $k = m - 3$. Let us introduce degree -1 variables $\mathbf{E}_i$ matching the edges E_i. Let E_{α_i} be the external edge of the tree T decorated by the form α_i. Let p_{α_i} be the projection from (43) onto the factor corresponding to the internal vertex of T assigned to the edge E_{α_i}. Abusing notation, we denote below $p_{\alpha_s}^* \alpha_s$ by α_s.

Given homogeneous forms φ_i assigned to the edges E_i, let us introduce an expression

$$\omega\Big((\varphi_0 \wedge \mathbf{E}_0) \wedge \ldots \wedge (\varphi_k \wedge \mathbf{E}_k)\Big) = (-1)^s \omega\Big(\varphi_0 \wedge \ldots \wedge \varphi_k\Big) \bigwedge \mathbf{E}_0 \wedge \ldots \wedge \mathbf{E}_k, \tag{45}$$

where $s = (\deg(\varphi_1)+1)+2(\deg(\varphi_2)+1)+...+k(\deg(\varphi_k)+1)$. Its left hand side can be understood as follows: we apply the operators ∂ and $\overline{\partial}$ to the factors, e.g. $\partial(\varphi_m \wedge \mathbf{E}_m) := \partial\varphi_m \wedge \mathbf{E}_m$ and get the sign $(-1)^s$ by moving $\mathbf{E}_i$ to the right.

Observe that (45) does not depend on the order of the edges $E_0, \ldots, E_k$. Set

$$\widetilde{\kappa}_T(W) := \omega\Big((G_{E_0} \wedge \mathbf{E}_0) \wedge \ldots \wedge (G_{E_k} \wedge \mathbf{E}_k)\Big) \bigwedge (\alpha_0 \wedge \mathbf{E}_{\alpha_0}) \wedge \ldots \wedge (\alpha_m \wedge \mathbf{E}_{\alpha_m}). \tag{46}$$

Here $(\alpha_0 \wedge \mathbf{E}_{\alpha_0}) \wedge \ldots \wedge (\alpha_m \wedge \mathbf{E}_{\alpha_m}) := (-1)^t \alpha_0 \wedge \ldots \wedge \alpha_m \bigwedge \mathbf{E}_{\alpha_0} \wedge \ldots \wedge \mathbf{E}_{\alpha_m}$, where the sign is obtained by moving $\mathbf{E}$'s through α's. It is invariant with respect to the cyclic shift of W. So it is determined by the cyclic word W. Therefore $\widetilde{\kappa}_T(W)$ is also determined by the cyclic word W.

Moving the odd variables $\mathbf{E}_s$ in (46) to the right we get

$$\widetilde{\kappa}_T(W) = \pm\omega(G_{E_0} \wedge \ldots \wedge G_{E_k}) \bigwedge \alpha_0 \wedge \ldots \wedge \alpha_m \bigwedge \mathbf{E}_0 \wedge \ldots \wedge \mathbf{E}_{2m}. \tag{47}$$

Now we use an orientation Or_T of the tree T to get rid of the factor $\mathbf{E}_0 \wedge \ldots \wedge \mathbf{E}_{2m}$, setting

$$\kappa_T(W) := \pm\mathrm{sgn}(E_0 \wedge \ldots \wedge E_{2m}) \wedge \omega(G_{E_0} \wedge \ldots \wedge G_{E_k}) \bigwedge \alpha_0 \wedge \ldots \wedge \alpha_m \tag{48}$$

where $\pm$ is the same sign as in (47), and $\mathrm{sgn}(E_0 \wedge \ldots \wedge E_{2m}) \in \{\pm 1\}$ is the difference between the element $E_0 \wedge \ldots \wedge E_{2m} \in \mathrm{or}_T$ and the generator Or_T corresponding to the clockwise orientation of the plane.

Remark. The form $\omega(G_{E_0} \wedge \ldots \wedge G_{E_k}) \bigwedge \alpha_0 \wedge \ldots \wedge \alpha_m$ is of even degree. So one can move the factor $\mathbf{E}_0 \wedge \ldots \wedge \mathbf{E}_{2m}$ in (47) to the left without getting an extra sign.

Lemma 2.2 *$\kappa_T(W)$ is a current on (43).*

Proof. A Green current $G_a(x, y)$ has the same type of singularity at the diagonal and at the cycles $x = a$ and $y = a$. These singularities are integrable since by the general theory of elliptic equations a solution of the equation (39) exists as a current.

More specifically, for the current $\partial G_a(0, z)$ the singularity at $z = 0$ is described by the singularity of the Bochner-Martinelly kernel:

$$\partial G(0, z) \sim \frac{1}{(2\pi i)^n} \frac{\alpha_n(\overline{z}, d\overline{z}) \wedge dz_1 \wedge \ldots \wedge dz_n}{(|z_1|^2 + \ldots + |z_1|^2)^n}, \qquad \alpha_n(\overline{z}, d\overline{z}) = \mathrm{Alt}_n(\overline{z}_1 d\overline{z}_2 \wedge \ldots \wedge d\overline{z}_n).$$

Here $z_1, \ldots, z_n$ are local coordinates near $z = 0$. Writing it in the spherical coordinates, one sees that this form, multiplied by a smooth 1-form, has an integrable singularity. Similar argument is applied to $\overline{\partial} G(0, z)$, and the form $G(0, z)$ is even less singular. This is another way to see that the form $\kappa_T(W)$ is integrable at the generic points of the diagonals $x = y$.

Since diagonals in (43) intersect transversally, the form $\kappa_T(W)$ is integrable on intersections of the diagonals. In particular, it is integrable at subvarieties of the diagonals given by $x = a = y$.

Although the singularity at $x = a$ is integrable, it causes a problem since $\kappa_T(W)$ contains products of several Green forms with the same singularity at $x = a$. In fact a single product like $G_a(s_1, x) \wedge \partial G_a(s_2, x) \wedge \overline{\partial} G_a(s_3, x)$ does have a singularity at $x = a$. We claim that nevertheless the non-integrable singularity disappears after the skew-symmetrization. Indeed, setting $a = 0$, one can write a Green current in local coordinates near $x = 0$ as $G_0(x, y) = B(x) + S(x, y)$ where $B(x)$ is the Bochner-Martinelly type current and $S(x, y)$ is smooth. Applying the skew-symmetric polydifferential operator ω we get

$$\omega\Big(B(x) + S(x, y_1)) \wedge (B(x) + S(x, y_2)) \wedge (B(x) + S(x, y_3)\Big) =$$

$$\mathrm{Alt}_{y_1, y_2, y_3} \omega\Big(B(x) \wedge S(x, y_2) \wedge S(x, y_3)\Big) + \text{a smooth form}.$$

So the claim follows from the integrabilty of the Green current $G_a(x, y)$. The lemma is proved.

Definition 2.3 *Let $p_T : X^{\{internal\ vertices\ of\ T\}} \longrightarrow point$ be the natural projection. The number $\mathrm{Cor}_{\mathcal{H}}(W \otimes \mathcal{H})$ is given by*

$$\mathrm{Cor}_{\mathcal{H}}(W \otimes \mathcal{H}) := p_{T*}\Big(\sum_T \kappa_T(W)\Big),$$

where the sum is over all plane trivalent trees T decorated by W.

Lemma 2.4 *The Hodge correlator map (41) is a linear map of degree zero, provided the number of factors in the cyclic tensor product is bigger then 3.*

Proof. A trivalent graph T with $m+1$ external legs has $m-1$ internal vertices and $m-2$ internal edges. Let W is as in (42). Then we integrate a form of degree $(2n-2)(m-2)+(m-3)+\sum_{i=0}^{m}\deg(\alpha_i)$ over a cycle of dimension $2n(m-1)$. The result can be non-zero only if $(2n-2)(m-2)+(m-3)+\sum_{i=0}^{m}\deg(\alpha_i)=2n(m-1)$, i.e. $\sum_{i=0}^{m}(\deg(\alpha_i)-1)=(2n-2)$. The left hand side here equals $\deg W$. The lemma follows.

Proposition 2.5 *Let $\overline{\alpha}_i \in \mathbb{H}^*_X$. Then for any $p,q\geq 1$ one has*

$$\sum_{\sigma\in\Sigma_{p,q}} \pm \mathrm{Cor}_{\mathcal{H}}(\overline{\alpha}_0\otimes\overline{\alpha}_{\sigma(1)}\otimes\ldots\otimes\overline{\alpha}_{\sigma(p+q)})=0, \tag{49}$$

where the sum is over all (p,q)-shuffles $\sigma\in\Sigma_{p,q}$, and the signs are the standard ones.

The proof is identical to the proof of the shuffle relation in Section 2 of [G1].

The Hodge correlator. Let $\mathcal{C}$ be the operator of weighted projection on the cyclic tensor algebra:

$$\mathcal{C}(x_1\otimes\ldots\otimes x_m):=\frac{1}{|\mathrm{Aut}(W)|}(x_1\otimes\ldots\otimes x_m)_{\mathcal{C}}. \tag{50}$$

where $W=(x_1\otimes\ldots\otimes x_m)_{\mathcal{C}}$, and $|\mathrm{Aut}(W)|$ is the order of its automorphism group.

Given a collection of harmonic differential forms ω_i and homology classes h_i we define the Hodge correlator map of the cyclic tensor product of the expressions $\omega_i|h_i$ as a cyclic product of their homology factors taken with the coefficient given by the Hodge correlator map applied to the cyclic product of the corresponding differential form factors, with the sign computed via the standard sign rule. Here is an example.

$$\mathrm{Cor}_{\mathcal{H}}(\overline{\omega}_1|\overline{h}_1\otimes\overline{\omega}_2|\overline{h}_2\otimes\overline{\omega}_3|\overline{h}_3):= \tag{51}$$

$$(-1)^{|\overline{\omega}_3|(|\overline{h}_2|+|\overline{h}_1|)+|\overline{h}_1||\overline{\omega}_2|}\mathrm{Cor}_{\mathcal{H}}(\overline{\omega}_1\otimes\overline{\omega}_2\otimes\overline{\omega}_3)\cdot(\overline{h}_1\otimes\overline{h}_2\otimes\overline{h}_3)_{\mathcal{C}}=$$

$$(-1)^{|\overline{\omega}_3|(|\overline{h}_2|+|\overline{h}_1|)+|\overline{h}_1||\overline{\omega}_2|}(-1)^{\deg\omega_2}\int_X(\omega_1\wedge\omega_2\wedge\omega_3)\cdot(\overline{h}_1\otimes\overline{h}_2\otimes\overline{h}_3)_{\mathcal{C}}.$$

Since $\mathrm{Cor}_{\mathcal{H}}(\overline{\omega}_1\otimes\ldots\otimes\overline{\omega}_m)$ is graded cyclic invariant, the same is true for $\mathrm{Cor}_{\mathcal{H}}(\overline{\omega}_1|\overline{h}_1\otimes\ldots\otimes\overline{\omega}_m|\overline{h}_m)$. In particular if $\deg(\omega_i)=\deg(h_i)$, then it is plain cyclic invariant.

Since $\overline{\alpha}|h_{\overline{\alpha}}$ is even, properties of the Hodge correlators written in this form are more transparent. For example, the shuffle relations look as follows (all signs are pluses):

$$\sum_{\sigma\in\Sigma_{p,q}}\mathrm{Cor}_{\mathcal{H}}\Big(\overline{\alpha}_0|\overline{h}_{\alpha_0}\otimes\overline{\alpha}_{\sigma(1)}|\overline{h}_{\alpha_{\sigma(1)}}\otimes\ldots\otimes\overline{\alpha}_{\sigma(p+q)}|\overline{h}_{\alpha_{\sigma(p+q)}}\Big)_{\mathcal{C}}=0. \tag{52}$$

Dualising the map (41) we get an element $\mathbf{G}\in\mathcal{C}_{\mathbb{H}}\otimes\mathcal{H}$, the precursor of the Hodge correlator. We write it as follows. Choose a basis in the reduced cohomology $\mathbb{H}^*$ of X. Denote by α_i the corresponding harmonic representatives. Let h_{α_i} be the dual basis of the reduced homology $\mathbb{H}$.

Consider the Casimir element

$$\mathrm{Id}_X:=1+\alpha|h_\alpha+\alpha|h\otimes\alpha|h+\alpha|h\otimes\alpha|h\otimes\alpha|h+\ldots,\qquad \alpha|h:=\sum_s\overline{\alpha}_s|\overline{h}_{\alpha_s}, \tag{53}$$

where the sum is over a basis $\{\alpha_s\}$ of harmonic forms. Applying operator (50), we get the cyclic Casimir element:

$$\mathcal{C}\mathrm{Id}_X := \mathcal{C}\Big(1 + \alpha|h + \alpha|h \otimes \alpha|h + \alpha|h \otimes \alpha|h \otimes \alpha|h + \ldots\Big). \qquad (54)$$

Then one has, where $\mathcal{H}$ is the shifted fundamental cohomology class (2),

$$\mathbf{G} = \mathrm{Cor}_{\mathcal{H}}\Big(\mathcal{C}\mathrm{Id}_X\Big) \otimes \mathcal{H} \in \mathcal{C}_{\mathbb{H}} \otimes \mathcal{H}. \qquad (55)$$

We elaborate this as follows. Let $W = \mathcal{C}(\overline{\alpha}_1 \otimes \ldots \otimes \overline{\alpha}_m)$ be a basis in $\mathcal{C}_{\mathbb{H}}$. Then

$$\mathbf{G} = \sum_W \frac{1}{|\mathrm{Aut}(W)|} \mathrm{Cor}_{\mathcal{H},\mu}(\overline{\alpha}_1|\overline{h}_{\alpha_1} \otimes \ldots \otimes \overline{\alpha}_m|\overline{h}_{\alpha_m}) \otimes \mathcal{H}. \qquad (56)$$

3 Motivic rational homotopy type DG Lie algebras

3.1 Motivic set-up

To describe the known realizations of the motivic DG Lie algebra $\mathrm{L}_{\mathcal{M}}(X, x)$ let us recall the corresponding set-ups (see [G1] for more details).

Let F be a field. We work in one of the following categories $\mathcal{C}$:

i) *Motivic.* The hypothetical abelian category of mixed motives over a field F.
ii) *Hodge.* $F = \mathbb{C}$, and $\mathcal{C}$ is the category of mixed $\mathbb{Q}$- or $\mathbb{R}$-Hodge structures.
iii) *Mixed l-adic.* F is an arbitrary field such that $\mu_{l^\infty} \not\in F$, and $\mathcal{C}$ is the mixed category of l-adic $\mathrm{Gal}(\overline{F}/F)$–modules with a filtration $W_\bullet$ indexed by integers, such that gr^W_n is a pure of weight n.
iv) *Motivic Tate.* F is a number field, $\mathcal{C}$ is the abelian category of mixed Tate motives over F, equipped with the Hodge and l-adic realization functors, c.f. [DG].

The setup i) is conjectural. The other three are well defined. A category $\mathcal{C}$ from the list above is an L–category, where $L = \mathbb{Q}$ in i), iv); $L = \mathbb{Q}$ or $\mathbb{R}$ in ii); and $L = \mathbb{Q}_l$ in iii).

Each of the categories has an invertible object, the *Tate object*, which is denoted, abusing notation, by $\mathbb{Q}(1)$ in all cases. We set $\mathbb{Q}(n) := \mathbb{Q}(1)^{\otimes n}$. Each object in $\mathcal{C}$ carries a canonical weight filtration $W_\bullet$, morphisms in $\mathcal{C}$ are strictly compatible with this filtration. The weight of $\mathbb{Q}(1)$ is -2.

One should have a $\mathcal{C}$-motivic rational homotopy type DG Lie algebra $\mathrm{L}_{\mathcal{C}}(X, x)$ for each of the categories $\mathcal{C}$. It is a DG Lie algebra in the category $\mathcal{C}$, equipped with a weight filtration $W_\bullet$.

Denote by $\mathrm{L}^{(l)}(X, x)$ the l-adic realization of the DG Lie algebra $\mathrm{L}_{\mathcal{M}}(X, x)$. The Lie algebra of the Galois group $\mathrm{Gal}(\overline{F}/F)$ acts by derivations of the graded Lie algebra $\mathrm{L}^{(l)}(X, x)$ commuting with the differential and defined modulo homotopy. We give a linear algebra description of the Lie algebra of all such derivations of the graded Lie algebra $\mathrm{L}^{(l)}(X, x)$.

Example: X is a smooth projective curve. The associated graded $\mathrm{gr}^W \pi_1^{\mathrm{nil}}(X, x)$ of the pronilpotent completion $\pi_1^{\mathrm{nil}}(X, x)$ of $\pi_1(X, x)$ is canonically isomorphic to the quotient of the free, graded by the weight, Lie algebra generated by $H_1(X)$ of weight -1, by the ideal generated by the element $\mathrm{S} = \sum_i [q_i, p_i]$, where (q_i, p_i) is a symplectic basis of $H_1(X)$ for the intersection form. The Lie algebra $\pi_1^{\mathrm{nil}}(X, x)$ is non-canonically isomorphic to $\mathrm{gr}^W \pi_1^{\mathrm{nil}}(X, x)$.

On the other hand, in this case

$$\mathbb{H} = H_2(X)[-1] \oplus H_1(X)[-1].$$

The DG Lie algebra $\mathrm{L}_{\mathcal{M}}(X, x)$ is a free graded Lie algebra generated by $\mathbb{H}$, equipped with the differential

$$\delta : H_2(X)[-1] \longrightarrow \Lambda^2(H_1(X)[-1])$$

dualising to the product map. Since δ is injective, the DG Lie algebra $\mathrm{L}_{\mathcal{M}}(X, x)$ is quasiisomorphic to the Lie algebra generated by $H_1(X)$, sitting in degree 0, with a single relation

$$\delta H_2(X)[-1] = \sum_i [q_i, p_i] = 0.$$

3.2 Cyclic words and derivations

In this subsections we present a version of the non-commutative symplectic geometry of M. Kontsevich [K] for an algebra with a Poincare duality.

Let $\mathrm{H}^*(X)$ be the cohomology motive of an n-dimensional regular projective variety X. It is a pure graded commutative algebra with a Poincare duality: $\mathrm{H}^0(X) = \mathbb{Q}(0)$, the trace map provides an isomorphism $\mathrm{H}^{2n}(X) \xrightarrow{\sim} \mathbb{Q}(-n)[-2n]$, and there is a perfect pairing

$$\cup : \mathrm{H}^*(X) \otimes \mathrm{H}^*(X) \longrightarrow \mathrm{H}^{2n}(X)$$

given as product followed by the projection $\mathrm{H}^*(X) \longrightarrow \mathrm{H}^{2n}(X)$. Recall the fundamental class of X shifted by 2:

$$\mathcal{H} := \mathrm{H}^{2n}(X)[2].$$

It is isomorphic to $\mathbb{Q}(-n)[-(2n-2)]$ via the trace map. Thanks to the shift by 2, the Poincare duality is a perfect pairing on $\mathrm{H}^*(X)[1]$ with values in $\mathcal{H}$, and there is a $\cap$-product isomorphism

$$\cap : \mathrm{H}_*(X)[-1] \otimes \mathcal{H} \xrightarrow{\sim} \mathrm{H}^*(X)[1].$$

It induces the reduced $\cap$-product isomorphism

$$\cap : \mathbb{H} \otimes \mathcal{H} \xrightarrow{\sim} \mathbb{H}^*. \quad (57)$$

The canonical pairing $\mathrm{H}_*(X) \otimes \mathrm{H}^*(X) \to \mathbb{Q}(0)$ induces a pairing $\langle *, * \rangle : \mathbb{H} \otimes \mathbb{H}^* \to \mathbb{Q}(0)$. Set

$$\langle * \cap \mathcal{H} \cap * \rangle : \mathbb{H} \otimes \mathcal{H} \otimes \mathbb{H} \longrightarrow \mathbb{Q}(0), \qquad a \otimes \mathcal{H} \otimes b \to \langle a, \mathcal{H} \cap b \rangle, \qquad a, b \in \mathbb{H}. \quad (58)$$

Since $\mathcal{H}$ is even, we can move it freely. The form (58) is a symplectic form on $\mathrm{H}_*(X)[-1]$:

$$\langle a \cap \mathcal{H} \cap b \rangle = -(-1)^{|a||b|} \langle b \cap \mathcal{H} \cap a \rangle.$$

Indeed, X is even dimensional, and it comes from a symmetric bilinear form on $\mathrm{H}_*(X)$.

The cyclic tensor product. Recall the cyclic tensor product $\mathcal{C}_{\mathbb{H}}$ of $\mathbb{H}$. It is a sum of pure objects. We denote by $(v_0 \otimes ... \otimes v_m)_{\mathcal{C}}$ the cyclic tensor product of direct summands v_i of $\mathbb{H}$.

The non-commutative differential. It is a map

$$\mathbb{D} : \mathcal{C}_{\mathbb{H}} \longrightarrow \mathrm{T}_{\mathbb{H}} \otimes \mathbb{H}, \qquad \mathbb{D}(h_0 \otimes ... \otimes h_m)_{\mathcal{C}} := \mathrm{Cyc}_{m+1}\left((h_0 \otimes ... \otimes h_{m-1}) \otimes h_m\right).$$

where Cyc_{m+1} is the the operator of the graded cyclic shift sum. Given a decomposition $\mathbb{H} = \oplus a_i$ into simple objects, we define $\partial_{a_i} F \in \mathrm{T}_{\mathbb{H}}$ via the decomposition

$$\mathbb{D}F = \sum_i \partial_{a_i} F \otimes a_i.$$

We are going to define an action of $\mathcal{C}_{\mathbb{H}} \otimes \mathcal{H}$ by derivations of the DGA $(\mathrm{T}_{\mathbb{H}}, \delta)$, and introduce a DG Lie algebra structure on $\mathcal{C}_{\mathbb{H}} \otimes \mathcal{H}$, making it into an action of a DG Lie algebra on a DGA.

An action of $\mathcal{C}_{\mathbb{H}} \otimes \mathcal{H}$ by derivations of $\mathrm{T}_{\mathbb{H}}$. We are going to define a map

$$\theta : \mathcal{C}_{\mathbb{H}} \otimes \mathcal{H} \longrightarrow \mathrm{Der}(\mathrm{T}_{\mathbb{H}}).$$

Since the algebra $\mathrm{T}_{\mathbb{H}}$ is free, to define its derivation we define it on generators. Given an element $F \otimes \mathcal{H} \in \mathcal{C}_{\mathbb{H}} \otimes \mathcal{H}$, the derivation $\theta_{F\otimes\mathcal{H}}$ acts on $q \in \mathbb{H}$ by

$$\theta_{F\otimes\mathcal{H}} : q \longmapsto \sum_p \partial_p F \otimes \langle p \cap \mathcal{H} \cap q \rangle \in \mathrm{T}_{\mathbb{H}}.$$

Here the sum is over a basis $\{p\}$ in $\mathbb{H}$. Clearly map θ is a degree zero.

A Lie bracket on $\mathcal{C}_{\mathbb{H}} \otimes \mathcal{H}$. Let us define a Lie bracket

$$\{*, *\} : (\mathcal{C}_{\mathbb{H}} \otimes \mathcal{H}) \otimes (\mathcal{C}_{\mathbb{H}} \otimes \mathcal{H}) \longrightarrow \mathcal{C}_{\mathbb{H}} \otimes \mathcal{H},$$

It is handy to have a "left" version of the differential, which differs from $\mathbb{D}F$ only by signs:

$$\mathbb{D}^- : \mathcal{C}_{\mathbb{H}} \longrightarrow \mathbb{H} \otimes \mathrm{T}_{\mathbb{H}}, \qquad (h_0 \otimes ... \otimes h_m)_{\mathcal{C}} := \mathrm{Cyc}_{m+1}\left(h_0 \otimes (h_1 \otimes ... \otimes h_m)\right).$$

It gives rise to "left" partial derivatives $\partial_q^- G$. Set

$$\{F \otimes \mathcal{H}, G \otimes \mathcal{H}\} := \sum_{p,q} \Big(\partial_p F \otimes \langle p \cap \mathcal{H} \cap q \rangle \otimes \partial_q^- G\Big)_{\mathcal{C}} \otimes \mathcal{H}. \tag{59}$$

The sum is over bases $\{p\}$, $\{q\}$ of $\mathbb{H}$. Here is an invariant definition of the bracket $\{*, *\}$. Take

$$\mathbb{D}F \otimes \mathcal{H} \otimes \mathbb{D}^- G \otimes \mathcal{H} \subset \mathrm{T}_{\mathbb{H}} \otimes \mathbb{H} \otimes \mathcal{H} \otimes \mathbb{H} \otimes \mathrm{T}_{\mathbb{H}} \otimes \mathcal{H}.$$

Then we apply map (58) and the product $\mathrm{T}_{\mathbb{H}} \otimes \mathbb{Q}(0) \otimes \mathrm{T}_{\mathbb{H}} \to \mathrm{T}_{\mathbb{H}}$, followed by projection to $\mathcal{C}_{\mathbb{H}}$. Similarly the map $\theta_{F\otimes\mathcal{H}}$ is given as the composition $\mathbb{D}F \otimes \mathcal{H} \otimes \mathbb{H} \hookrightarrow \mathrm{T}_{\mathbb{H}} \otimes \mathbb{H} \otimes \mathcal{H} \otimes \mathbb{H} \to \mathrm{T}_{\mathbb{H}}$.

Proposition 3.1 *The bracket $\{*, *\}$ is a degree zero map. It is skew-symmetric:*

$$\{F \otimes \mathcal{H}, G \otimes \mathcal{H}\} = -(-1)^{|F||G|}\{G \otimes \mathcal{H}, F \otimes \mathcal{H}\}. \tag{60}$$

It satisfies the Jacobi identity. it is dual to the Lie cobracket defined on Fig. 2.

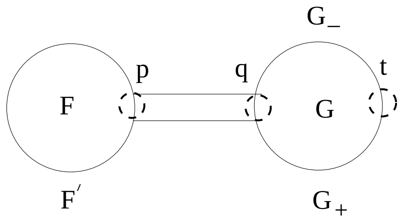


Figure 4: Calculating $\theta_{F\otimes\mathcal{H}} \circ \theta_{G\otimes\mathcal{H}}$.

Proof. The first claim is evident. The second is checked using the fact that $\langle p \cap \mathcal{H} \cap q \rangle$ is a symplectic form. The last is easy to prove directly. We will deduce it from the following

Lemma 3.2 *The map $\theta : \mathcal{C}_{\mathbb{H}} \otimes \mathcal{H} \longrightarrow \mathrm{Der}(\mathrm{T}_{\mathbb{H}})$ is injective and respects the brackets.*

Proof. Let us calculate how the super commutator $[\theta_{F\otimes\mathcal{H}}, \theta_{G\otimes\mathcal{H}}]$ acts on $h \subset \mathbb{H}$. Given certain factors p of F and q, t of G, write (see Fig 4) $F = (F' \otimes p)_{\mathcal{C}}$, and $G = (G_+ q G_- t)_{\mathcal{C}}$. Then

$$\theta_{F\otimes\mathcal{H}} \circ \theta_{G\otimes\mathcal{H}} : h \longmapsto \sum_t (-1)^{|G_+||qG_-t|} \theta_{F\otimes\mathcal{H}}(G_+ q G_-)\langle t \cap \mathcal{H} \cap h \rangle =$$

$$\sum_{t,p,q} (-1)^{|G_+||G_-t|+|F||G_+|} G_+ F' \langle p \cap \mathcal{H} \cap q \rangle G_- \langle t \cap \mathcal{H} \cap h \rangle.$$

Notice that the sign is given by moving G_+ through the remaining factor. We claim that this equals $\theta_{\{F\otimes\mathcal{H}, G\otimes\mathcal{H}\}}(h)$. Indeed, θ for the element $(F'\langle p \cap \mathcal{H} \cap q \rangle G_- t G_+)_{\mathcal{C}}$ maps h to

$$(-1)^{|G_+||Ft|} \sum_t G_+ F' \langle p \cap \mathcal{H} \cap q \rangle G_- \langle t \cap \mathcal{H} \cap h \rangle.$$

The Lemma follows from this. Hence the Proposition is proved.

Corollary 3.3 *The map θ is an injective morphism of graded Lie algebras*

$$\theta : \mathcal{C}_{\mathbb{H}} \otimes \mathcal{H} \longrightarrow \mathrm{Der}(\mathrm{T}_{\mathbb{H}}). \quad (61)$$

The canonical element Δ. Take a plane trivalent tree T_3 with a single internal vertex. Denote

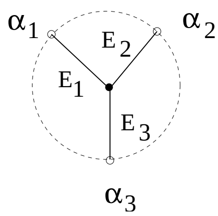


Figure 5: The canonical element Δ.

by $\mathbf{T}$ the orientation super line of this tree. It is a one dimensional space in the degree -3. The clockwise orientation of the plane provides a generator $\mathbf{E}_1 \wedge \mathbf{E}_2 \wedge \mathbf{E}_3 \in \mathbf{T}$. Here $\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3$ are the degree -1 generators assigned to the edges E_1, E_2, E_3 of the tree. There is a map

$$\otimes^3(H^*(X)[1]) \longrightarrow H^{2n}(X) \otimes \mathbf{T} \stackrel{\sim}{=} \mathcal{H}[1].$$

Namely, we assign the shifted by 1 cohomology classes $\alpha_i \otimes \mathbf{E}_i$ to the external vertices of the edges E_i of the tree T_3, and set

$$(\alpha_1 \otimes \mathbf{E}_1) \wedge (\alpha_2 \otimes \mathbf{E}_2) \wedge (\alpha_3 \otimes \mathbf{E}_3) \longmapsto (-1)^{\deg(\alpha_2)} (\alpha_1 \wedge \alpha_2 \wedge \alpha_3)_{H^{2n}(X)} \otimes \mathbf{E}_1 \wedge \mathbf{E}_2 \wedge \mathbf{E}_3.$$

Here $(\alpha)_{H^{2n}(X)}$ denotes the projection of a cohomology class α onto $H^{2n}(X)$. This map is graded cyclic invariant. It gives rise to a map $\mathbb{Q}(0) \longrightarrow \otimes^3(H_*(X)[-1]) \otimes \mathcal{H}[1]$. Killing $H_0(X) \oplus H_{2n}(X)$ and projecting to the cyclic envelope we obtain a canonical injective map

$$\Delta : \mathbb{Q}(0)[-1] \longrightarrow \mathcal{C}(\mathbb{H} \otimes \mathbb{H} \otimes \mathbb{H}) \otimes \mathcal{H} \subset \mathcal{C}_{\mathbb{H}} \otimes \mathcal{H}.$$

We often identify it with its image, also denoted by Δ.

Proposition 3.4 *(i) One has* $\{\Delta, \Delta\} = 0$.
(ii) The map $\delta : * \longmapsto \{\Delta, *\}$ *given by the commutator with* Δ *is a derivation:* $\delta^2 = 0$.
(iii) The bracket $\{*, *\}$ *and the differential* δ *provide* $\mathcal{C}_{\mathbb{H}} \otimes \mathcal{H}$ *with a DG Lie algebra structure.*

Proof. Since $\deg \Delta = 1$, the map δ is of degree 1. Now (i) is equivalent to (ii) thanks to the Jacobi identity. (iii) obviously follows from (ii). The part (i) is easy to check directly. The part (ii) is also follows from the explicit formula for the map δ given in Lemma 3.6 below (whose proof is independent of the proposition). The proposition is proved.

There is a differential $\delta := \theta_\Delta$ on $\mathrm{T}_{\mathbb{H}}$. It obviously provides $\mathrm{T}_{\mathbb{H}}$ with a DGA structure. The differential on $\mathcal{C}_{\mathbb{H}} \otimes \mathcal{H}$ coincides with the one inherited from the differential on $\mathrm{T}_{\mathbb{H}}$.

Lemma 3.5 *The map* θ *is an action of the DG Lie algebra* $\mathcal{C}_{\mathbb{H}} \otimes \mathcal{H}$ *on the DGA* $\mathrm{T}_{\mathbb{H}}$.

Proof. We have to show that $\mathcal{C}_{\mathbb{H}} \otimes \mathcal{H} \otimes \mathrm{T}_{\mathbb{H}} \longrightarrow \mathrm{T}_{\mathbb{H}}$ commutes with δ. This is very similar to, and in fact deduced from the fact that $\{\Delta, \{F \otimes \mathcal{H}, G \otimes \mathcal{H}\}\}$ satisfies Jacobi identity.

Corollary 3.6 *If* $\delta(F \otimes \mathcal{H}) = 0$, *then* $\theta_{F \otimes \mathcal{H}}$ *is a derivation of the graded algebra* $\mathrm{T}_{\mathbb{H}}$ *which commutes with the differential* δ.

The DGA algebra $\mathrm{T}_{\mathbb{H}}$ has a DG Hopf algebra structure with the standard (coshuffle) coproduct ν. The kernel of $\nu - (1 \otimes \mathrm{Id} + \mathrm{Id} \otimes 1)$ is a DG Lie algebra $\mathcal{L}ie_{\mathbb{H}}$. It is the free graded Lie algebra generated by $\mathbb{H}$. The differential δ on $\mathrm{T}_{\mathbb{H}}$ induces a differential on $\mathcal{L}ie_{\mathbb{H}}$.

Recall the space $\mathcal{CL}ie_{\mathbb{H}} \otimes \mathcal{H}$ defined in Section 2.2.

Lemma 3.7 $\mathcal{CL}ie_{\mathbb{H}} \otimes \mathcal{H}$ *is the subspace of the DG Lie algebra* $\mathcal{C}_{\mathbb{H}} \otimes \mathcal{H}$ *preserving* $\mathcal{L}ie_{\mathbb{H}}$. *It is a DG Lie subalgebra.*

Proof. An easy exercise, left to the reader.

Lemma 3.8 *There is an isomorphism of DG Lie algebras*

$$\mathcal{CL}ie_{\mathbb{H}} \otimes \mathcal{H} \stackrel{\sim}{\longrightarrow} \mathrm{Der}^S \mathcal{L}ie_{\mathbb{H}}. \qquad (62)$$

Proof. The space $\mathbb{H}^*$ has a symplectic structure with values in $\mathcal{H}$. So thanks to [K], there is an isomorphism of Lie algebras (62). One easily checks that it commutes with the differentials. The lemma is proved.

Proof of Proposition 1.3. (i) It is equivalent to Lemma 3.8.
(ii) This was done above. The Proposition is proved.

4 Hodge correlator classes and their interpretations

In Section 4.1 we introduce the Hodge correlator class. In Sections 4.2-4.3 we recall some standard background material on cohomology of DG Lie algebras and DG spaces arising from DG Lie algebras. Using this, in Section 4.4 we define versions of the Hodge correlator classes: the *Hodge vector fields*. In Section 4.5 we define Hodge vector fields explicitly as (the sum of) correlators of the Feynman integral from Section 1.6. We interpret them as vector fields on the formal neighborhood of the trivial local system, providing a homotopy action of the Hodge Galois group there.

4.1 The Hodge correlator class

The Hodge correlator $\mathbf{G}$, see (55), depends on the choice of a Green current. To define the latter, we choose a splitting of the Dolbeaut complex into the space of harmonic forms and its orthogonal complement. In particular the volume form μ provides a choice of the harmonic $2n$-form. These choices determine differential equation (38) for the Green current.

Theorem 4.1 *(i) The Hodge correlator* $\mathbf{G}$ *is a degree* 0 *element of* $\mathcal{CL}ie_{\mathbb{H}} \otimes \mathcal{H}$.

(ii) One has $\delta\mathbf{G} = 0$.

(iii) Altering either a splitting of the Dolbeaut complex or a Green current we get an element $\widetilde{\mathbf{G}}$*, which differs from* $\mathbf{G}$ *by a coboundary: there is an element* $\mathbf{B} \in \mathcal{CL}ie_{\mathbb{H}} \otimes \mathcal{H}$ *such that*

$$\widetilde{\mathbf{G}} - \mathbf{G} = \delta\mathbf{B}.$$

Thus there is a well defined δ*-cohomology class of* $\mathbf{G}$*, called the* Hodge correlator class*:*

$$\mathbf{H}_X \in H_0^{\delta}(\mathcal{CL}ie_{\mathbb{H}} \otimes \mathcal{H}).$$

We prove this Theorem in Section 5.2.

Theorem 4.1 and Corollary 3.6 plus Lemma 3.7 immediately imply Proposition **??**, that is:

The element $\mathbf{G}$ *provides a derivation of the graded Lie algebra* $\mathcal{L}ie_{\mathbb{H}}$ *commuting with the differential* δ *in* $\mathcal{L}ie_{\mathbb{H}}$*. Its homotopy class depends only on the Hodge correlator class* $\mathbf{H}_X$*.*

4.2 Derivations and cohomology of DG Lie algebras

Let $(\mathcal{G}, \delta)$ be a DG Lie algebra, i.e. a Lie algebra in the tensor category of complexes. There are two slightly different versions of the (DG) Lie algebra cohomology.

1. The standard cochain complex of $\mathcal{G}$ with coefficients in a DG $\mathcal{G}$-module M is given by

$$C^*(\mathcal{G}, M) := S^*(\mathcal{G}[1]^\vee) \otimes M.$$

Here the grading is provided by the gradings on $\mathcal{G}$ and M, and the differential is the sum of the Chevalley differential δ_{Ch} and the differentials induced by the differentials on $\mathcal{G}$ and M. The cohomology of this complex is denoted by $\mathbb{H}^*(\mathcal{G}, M)$.

Let $\mathcal{G}_i$ be DG Lie algebras, and M_i is DG Lie modules over $\mathcal{G}_i$, $i = 1, 2$. Given a morphism of DG Lie algebras $\varphi : \mathcal{G}_1 \to \mathcal{G}_2$ and a φ-equivariant morphism $\psi : M_1 \to M_2$ we get maps

$$\mathbb{H}^*(\mathcal{G}_2, M_2) \xrightarrow{\varphi} \mathbb{H}^*(\mathcal{G}_1, M_2) \xleftarrow{\psi} \mathbb{H}^*(\mathcal{G}_1, M_1). \tag{63}$$

If φ and ψ are quasiisomorphisms, then maps (63) are also isomorphisms. This is proved by a spectral sequence argument applied to maps of the bicomplexes induced by φ and ψ.

Motivation. Let $\mathcal{G}$ be a Lie algebra. Denote by $\mathrm{Der}^{\mathrm{Inn}}(\mathcal{G})$ the Lie algebra of inner derivations of $\mathcal{G}$. Then the Lie algebra of outer derivations of $\mathcal{G}$ is given by

$$\mathrm{Der}^{\mathrm{Out}}(\mathcal{G}) := \frac{\mathrm{Der}(\mathcal{G})}{\mathrm{Der}^{\mathrm{Inn}}(\mathcal{G})} \stackrel{\sim}{=} H^1(\mathcal{G}, \mathcal{G}).$$

Here $\mathcal{G}$ acts on itself by the adjoint action.

Now let $(\mathcal{G}, \delta)$ be a DG Lie algebra which, considered as a graded Lie algebra, is a free non-abelian graded Lie algebra. Then $\mathbb{H}^1(\mathcal{G}, \mathcal{G})$ can be presented as follows:

The group $H^1(\mathcal{G}, \mathcal{G})$ is the only non-zero Lie algebra cohomology group. It is graded space, and the differential δ provides a differential Δ on $H^1(\mathcal{G}, \mathcal{G})$. The spectral sequence argument tells that $\mathbb{H}^i(\mathcal{G}, \mathcal{G}) = H^{i-1}_\Delta\Big(H^1(\mathcal{G}, \mathcal{G})\Big)$. It follows that

$$\mathbb{H}^1(\mathcal{G}, \mathcal{G}) = H^0_\Delta\Big(H^1(\mathcal{G}, \mathcal{G})\Big) =$$

$$\frac{\text{outer derivations of the graded Lie algebra } \mathcal{G} \text{ commuting with the differential } \delta}{\text{derivations homotopic to zero}}.$$

2. We need a cohomology group responsible for derivations rather then outer derivations.

Let us define the reduced cochain complex of a DG Lie algebra $\mathcal{G}$ with coefficients in a DG $\mathcal{G}$-module M as

$$\widetilde{C}^*(\mathcal{G}, M) := S^{>0}(\mathcal{G}[1]^\vee) \otimes M.$$

It is quasiisomorphic to the complex $\mathrm{Cone}\Big(S^*(\mathcal{G}[1]^\vee) \otimes M \longrightarrow M\Big)$. Here the map is induced by the augmentation map $S^*(\mathcal{G}[1]^\vee) \to \mathbb{Q}$. The reduced cohomology $\widetilde{\mathbb{H}}^*(\mathcal{G}, M)$ are the cohomology of this complex.

Given $\mathcal{G}_i$-modules M_i, $i = 1, 2$, a morphism of DG Lie algebras $\varphi : \mathcal{G}_1 \to \mathcal{G}_2$, and a φ-equivariant morphism $\psi : M_1 \to M_2$ we get morphisms

$$\widetilde{\mathbb{H}}^*(\mathcal{G}_2, M_2) \xrightarrow{\varphi} \widetilde{\mathbb{H}}^*(\mathcal{G}_1, M_2) \xleftarrow{\psi} \widetilde{\mathbb{H}}^*(\mathcal{G}_1, M_1). \tag{64}$$

If φ and ψ are quasiisomorphisms, the same is true for the morphisms (64).

Let us assume that $\mathcal{G}$, considered as a graded Lie algebra, is a free non-abelian graded Lie algebra. Then $\widetilde{H}^1(\mathcal{G}, \mathcal{G})$ is the only non-zero cohomology group, identified with the derivations of $\mathcal{G}$. It is a graded space, the differential δ provides a differential Δ on $\widetilde{H}^1(\mathcal{G}, \mathcal{G})$, and

$$\widetilde{\mathbb{H}}^i(\mathcal{G}, \mathcal{G}) = H^{i-1}_\Delta\Big(\widetilde{H}^1(\mathcal{G}, \mathcal{G})\Big).$$

It follows that

$$\widetilde{\mathbb{H}}^1(\mathcal{G}, \mathcal{G}) = H^0_\Delta\Big(\widetilde{H}^1(\mathcal{G}, \mathcal{G})\Big) =$$

$$\frac{\text{derivations of the graded Lie algebra } \mathcal{G} \text{ commuting with the differential } \delta}{\text{derivations homotopic to zero}}.$$

4.3 The DG space $\mathcal{G}[1]$

The category of affine DG spaces is the opposite to the category of graded commutative algebras equipped with a degree one derivation v with $v^2 = 0$. Such a v is called a *homological vector field* on a DG space.

Let $\mathcal{G}$ be a DG Lie algebra over a field k. Then $\mathcal{G}[1]$ has a structure of a DG space with the algebra of functions

$$\mathcal{O}_{\mathcal{G}[1]} := S^*(\mathcal{G}[1]^\vee) = C^*(\mathcal{G}, k). \tag{65}$$

It is identified with the standard Chevalley complex of the DG Lie algebra $\mathcal{G}$. Its differential provides a homological vector field on $\mathcal{G}[1]$.

Alternatively, the homological vector field on $\mathcal{G}[1]$ is also known as the *Chern-Simons vector field* Q_{CS}. It is a quadratic vector field given by

$$\dot{\alpha} = Q_{CS}(\alpha) := d\alpha + \frac{1}{2}[\alpha, \alpha]. \tag{66}$$

It is an odd vector field: the commutator $[\alpha, \alpha]$ vanishes if α is even. One has

$$Q^2_{CS} = \frac{1}{2}[Q_{CS}, Q_{CS}] = 0. \tag{67}$$

Indeed, Q_{CS} is a sum of linear and quadratic terms, $Q_1 + Q_2$. Then $Q_2 \circ Q_2 = 0$ by Jacobi identity, $Q_1 \circ Q_1 - d^2 = 0$, and $Q_2 \circ Q_1 + Q_1 \circ Q_2 = 0$ is equivalent to the Leibniz rule for d.

The Lie algebra $\widetilde{\text{Vect}}_{\mathcal{G}[1]}$ of formal vector fields on $\mathcal{G}[1]$ vanishing at the origin is a Lie subalgebra of derivations of the graded commutative algebra (65). Since the latter algebra is free, a derivation is determined by its action on the generators, and we have

$$\widetilde{\text{Vect}}_{\mathcal{G}[1]} = \widetilde{C}^*(\mathcal{G}, \mathcal{G})[1].$$

The graded Lie algebra structure on $\widetilde{C}^*(\mathcal{G}, \mathcal{G})[1]$ is given by the commutator of vector fields. Since Q_{CS} is a homological vector field, we get a differential

$$D := [Q_{CS}, *], \qquad D^2 = 0.$$

One has

$$\widetilde{\mathbb{H}}^{*+1}(\mathcal{G},\mathcal{G}) = H_D^*(\widetilde{\mathrm{Vect}}_{\mathcal{G}[1]}) = H_D^{*+1}(\widetilde{C}^*(\mathcal{G},\mathcal{G})).$$

In particular,

$$\widetilde{\mathbb{H}}^1(\mathcal{G},\mathcal{G}) = H_D^0\Big(\widetilde{\mathrm{Vect}}_{\mathcal{G}[1]}\Big) =$$

$$\frac{\text{Vector fields on } \mathcal{G}[1], \text{ vanishing at 0, commuting with the Chern-Simons vector field } Q_{CS}}{\text{Commutators with } Q_{CS}}.$$

We identify these classes with the isomorphism classes of deformations of the DG space $\mathcal{G}[1]$ over the one dimensional odd line $\mathrm{Spec}(\mathbb{Z}[\varepsilon])$: a degree zero vector field Q commuting with Q_{CS} provides a homological vector field $Q_{CS} + \varepsilon Q$, and vice versa.

Assume now that the DG Lie algebra $\mathcal{G}$ has an even invariant non-degenerate scalar product $Q(*,*)$. Then the space $\mathcal{G}[1]$ has an even Poisson structure provided by the bivector corresponding to the induced scalar product in $\mathcal{G}^\vee$. The vector field Q_{CS} is a Hamiltonian vector field with the Hamiltonian given by the Chern-Simons functional, a cubic polynomial function on $\mathcal{G}[1]$:

$$CS(\alpha) := \frac{1}{2}(\alpha, d\alpha) + \frac{1}{6}(\alpha, [\alpha,\alpha]).$$

One has $\{CS, CS\} = 0$ in agreement with (67).

Example. Let X be a compact Kähler manifold, and $\mathcal{G}$ a Lie algebra with an invariant non-degenerate scalar product $Q(*,*)$. Then $\mathcal{A}^*(X) \otimes \mathcal{G}$ is a DG Lie algebra with a scalar product of degree $2-2n$, where $n = \dim_{\mathbb{C}} X$:

$$(\alpha,\beta) := (-1)^{\deg(\alpha)} \int_X Q(\alpha \wedge \beta),$$

and $\mathcal{A}^*(X)[1] \otimes \mathcal{G}$ is a DG space with an even symplectic structure.

4.4 Hodge vector fields

Let X be a compact Kähler manifold. The Formality Theorem [DGMS] asserts that assigning to a cohomology class of X its harmonic representative, we get a quasiisomorphism of the DG commutative bigraded algebras $H_X^{*,*} \longrightarrow \mathcal{A}^{*,*}(X)$.

Denote by $\overline{\mathcal{A}}_X^{*,*}$ the quotient of $\mathcal{A}^{*,*}(X)$ by the space spanned by the constants and the Kahler volume form, which are the harmonic representatives $H_X^0 \oplus H_X^{2n}$. So the Formality Theorem implies

Corollary 4.2 *The DG commutative bigraded algebras* $\overline{\mathcal{A}}_X^{*,*}$ *and* $\overline{H}_X^{*,*}$ *are quasiisomorphic.*

Let $\mathcal{G}$ be a Lie algebra. The bigraded DG Com's in Corollary 4.2 provide bigraded DG Lie algebras

$$\mathcal{G}_{\mathrm{A}} := \overline{\mathcal{A}}_X^{*,*} \otimes \mathcal{G}, \qquad \mathcal{G}_{\mathrm{H}} := \overline{H}_X^{*,*} \otimes \mathcal{G}.$$

We assume that $\mathcal{G}$ is a complexification of a real Lie algebra. Since both DG Com's from Corollary 4.2 have real structures, there is a complex involution σ acting on these DG Lie algebras, and hence on their cohomology.

Below we assume that $\mathcal{G}$ is a Lie algebra with non-degenerate invariant scalar product $Q(*,*)$.

Given a point $a \in X$, and using the isomorphism of DG spaces (22), we transform the Hodge correlator class into a cohomology class

$$\mathrm{Q}_{\mathrm{Hod}} = \mathrm{Q}_{X,\mathcal{G},a} \in \widetilde{\mathbb{H}}^1(\mathcal{G}_H, \mathcal{G}_{\mathrm{H}}), \qquad \sigma(\mathrm{Q}_{\mathrm{Hod}}) = -\mathrm{Q}_{\mathrm{Hod}}.$$

Theorem 1.4 implies that it is functorial: A map $f : X \to Y$ of Kahler manifolds gives rise to a DG Lie algebra map $f^* : \mathcal{G}_{\mathrm{H}^*(Y)} \longrightarrow \mathcal{G}_{\mathrm{H}^*(X)}$, which intertwines, up to a homotopy, derivations $\mathrm{Q}_{X,\mathcal{G},a}$ and $\mathrm{Q}_{Y,\mathcal{G},a}$:

$$f^* \circ \mathrm{Q}_{Y,\mathcal{G},f(a)} \quad \text{is homotopic to} \quad \mathrm{Q}_{X,\mathcal{G},a} \circ f^*.$$

Corollary 4.2 implies that the DG Lie algebras $\mathcal{G}_{\mathrm{A}}$ and $\mathcal{G}_{\mathrm{H}}$ are quasiisomorphic. So there is an isomorphism

$$\widetilde{\mathbb{H}}^1(\mathcal{G}_{\mathrm{H}}, \mathcal{G}_{\mathrm{H}}) = \widetilde{\mathbb{H}}^1(\mathcal{G}_{\mathrm{A}}, \mathcal{G}_{\mathrm{A}}).$$

This leads to

Theorem 4.3 *Given a point $a \in X$, there is a functorial cohomology class*

$$\mathbb{V}_{\mathrm{Hod}} \in \widetilde{\mathbb{H}}^1(\mathcal{G}_{\mathrm{A}}, \mathcal{G}_{\mathrm{A}}) \stackrel{\sim}{=} H^0_{\Delta}\Big(\widetilde{H}^1(\mathcal{G}_{\mathrm{A}}, \mathcal{G}_{\mathrm{A}})\Big). \tag{68}$$

Remark. The action of the group $\mathbb{C}^*_{\mathbb{C}/\mathbb{R}}$ on the Dolbeault bicomplex of X providing the Hodge bigrading gives rise to an action on $\widetilde{H}^1(\mathcal{G}_{\mathrm{A}}, \mathcal{G}_{\mathrm{A}})$. It does not commute with the differential δ. Thus it is not the action of the pure Hodge Galois group.

4.5 Constructing the Hodge vector fields

Below we give an explicit construction of the Hodge vector fields $\mathrm{Q}_{\mathrm{Hod}}$.

Let us show that, given a point $a \in X$, a version of the Hodge correlator construction provides a degree zero linear map:

$$\mathrm{S}^*(\mathcal{G}_{\mathrm{H}}[1]) \otimes H_{2n}(X)[-2] \longrightarrow \mathbb{C}. \tag{69}$$

Observe that $H_{2n}(X)$ is in degree $-2n$, and is canonically isomorphic $\mathbb{C}$ since X is oriented.

Choose a Green current $G_a(x, y)$ on $X \times X$. Take a plane trivalent tree T. Decorate its external vertices by harmonic forms $\alpha_0, \ldots, \alpha_m$. Let $Q' \in S^2\mathcal{G} \subset \mathcal{G} \otimes \mathcal{G}$ be the element provided by the form on $\mathcal{G}$. We assign to every internal edge E of T the element $G_a(x, y) \otimes Q'$ in the space of $\mathcal{G}$-valued distributions on $X^{\{\text{vertices of } E\}}$. Using the polydifferential operator ω and, at every internal vertex, the canonical trilinear form

$$\langle l_1, l_2, l_3\rangle = Q(l_1, [l_2, l_3]) : \Lambda^3\mathcal{G} \to \mathbb{C},$$

we cook up a differential form on $X^{\{\text{internal vertices of } T\}}$. It provides a current there. Integrating it, and taking then the sum over all plane trivalent trees, we get the map (69).

One checks that the map (69) is a degree zero map. It provides a function on the space $\mathcal{G}_{\mathrm{H}}[1]$: its value on an $\mathcal{G}$-valued harmonic form α is obtained by evaluating the map (69) on $\alpha \otimes \ldots \otimes \alpha$. It is a formal power series on the space $\mathcal{G}_{\mathrm{H}}[1]$, i.e. a formal sum of its homogeneous components. Since $\mathcal{G}_{\mathrm{H}}[1]$ is a Poisson space, it gives rise to a formal Hamiltonian vector field, a *Hodge vector field*:

$$\mathrm{Q}_{\mathrm{Hod}} \in S^*(\mathcal{G}_{\mathrm{H}}[1]^{\vee}) \otimes \mathcal{G}_{\mathrm{H}}[1]. \tag{70}$$

Theorem 1.12ii) plus Theorem 4.1 imply that the Hodge vector field $\mathrm{Q_{Hod}}$ commutes with the vector field Q_{CS}. Furthermore, a different choice of the splitting of the Dolbeaut complex and the Green current results in a Hodge vector field $\widetilde{\mathrm{Q}}_{\mathrm{Hod}}$ such that $\widetilde{\mathrm{Q}}_{\mathrm{Hod}} - \mathrm{Q_{Hod}} \in \mathrm{Im}[Q_{CS}, *]$. Functoriality follows similarly from Theorem 1.4.

The cohomology $\widetilde{\mathbb{H}}^1(\mathcal{G}_H, \mathcal{G}_H)$ are bigraded by the Hodge bigrading.

Lemma 4.4 *The bidegree (p,q) component of $\mathrm{Q_{Hod}}$ is a polynomial vector field on $\mathcal{G}_{\mathrm{H}}[1]$. It can be non-zero only if $p, q > 0$.*

Proof. Let (p_i, q_i) be the Hodge bidegree of a form decorating i-th an external vertex of a trivalent tree T. Then it contributes to the (p,q)-component of $\mathrm{Q_{Hod}}$, where $p = \sum_{i=0}^m p_i - n$ and $q = \sum_{i=0}^m q_i - n$. Since $p_i, q_i \geq 0$ and $p_i + q_i = 2n$, for a given (p,q) the number $m+1$ of vertices of the tree T is bounded from above. So the (p,q)-component is given by a finite sum.

4.6 The Hodge Galois group action on close to trivial local systems

Theorem 4.3 provides a formal vector field $\mathbb{V}_{\mathrm{Hod}}$ on the DG space $\mathcal{G}_{\mathcal{A}^*(X,a)}$. It is well defined up to a homotopy, commutes with the Chern Simons vector field Q_{CS}, and covariant, up to homotopy, for the maps of Kahler manifolds.

Below we show that the vector fields $\mathbb{V}_{\mathrm{Hod}}$ give rise to a functorial vector field $\mathbb{V}_{X,\mathcal{G},a}$ on the formal neighborhood of the trivial G-local system in the space of complex local systems on X. It provides an action of the Lie algebra $\mathrm{L_{Hod}}$: the generator G acts by the vector field $\mathbb{V}_{X,\mathcal{G},a}$.

Given a homological vector field Q on a DG space V, let $V^Q \subset V$ be the set of its fixed points. The tangent space T_fV at a point f is a graded vector space. The linearization of Q at a point $f \in V^Q$ provides a linear operator $Q = Q_{(f)}$ in T_fV, described as follows. The vector fields vanishing at a point form a Lie subalgebra. So the value $[Q, Q']_f$ of the commutator at f depends only on the value Q'_f of Q' at f. Set $Q_{(f)}(v) := [Q, Q']_f$ where $Q'_f = v$. Since $Q^2 = 0$, $Q_{(f)}$ is a differential in T_fV. We denote it simply by Q.

Let H be a vector field commuting with Q. Then at a Q-fixed point f one has $Q(H_f) = 0$. Since $[Q, R]_f = Q(R_f)$, adding to H a commutator $[Q, R]$ with Q and restricting the result to $f \in V^Q$ we do not change the cohomology class of H_f in $H^*_Q(T_fV)$.

In our case Q is the Chern-Simons vector field Q_{CS} on the DG space $\mathcal{G}_{\mathrm{A}}[1]$, and $H = \mathbb{V}_{\mathrm{Hod}}$. The set $\mathrm{Con}_{\mathcal{F}}$ of fixed points of the vector field Q_{CS} on $\mathcal{G}_{\mathrm{A}}[1]$ is the set of flat DG connections. The subset $\mathrm{Con}^1_{\mathcal{F}}$ of $\mathcal{G}$-valued 1-forms $\alpha \in \mathrm{Con}_{\mathcal{F}}$ is identified with the flat connections $d + \alpha$. The flat connection $d + \alpha$ provides a local system $\mathcal{V}_\alpha$. The complex given by the operator Q_{CS} acting on the tangent space to $\alpha \in \mathrm{Con}^1_{\mathcal{F}}$ is identified with the de Rham complex of the local system $\mathcal{V}_\alpha$. Thus one has

$$H^*_{Q_{CS}}(T_\alpha \mathcal{G}_{\mathrm{A}}[1]) = H^*(X, \mathcal{V}_\alpha). \tag{71}$$

Proposition 4.5 *The vector field $\mathbb{V}_{\mathrm{Hod}}$ provides a flow on the formal neighborhood of the trivial local system on X.*

Proof. Thanks to (71), the formal vector field $\mathbb{V}_{\mathrm{Hod}}$ restricted to a flat connection $d+\alpha$ give rise to a cohomology class of the corresponding local system: $\mathbb{V}^\alpha_{\mathrm{Hod}} \in H^1(X, \mathcal{V}_\alpha)$. One easily sees that the class $\mathbb{V}^\alpha_{\mathrm{Hod}}$ does not change under the action of the gauge transformations.

The tangent space to a local system $\mathcal{V}_\alpha$ in the space of all $\mathcal{G}$-local systems on X is identified with $H^1(X, \mathcal{V}_\alpha)$. So we get a vector field $\mathbb{V}_{X,\mathcal{G},a}$ on the formal neighborhood of a trivial local system on X. It is functorial for the maps $f : Y \to X$ of Kahler manifolds: $f^*\mathbb{V}_{X,\mathcal{G},f(a)} = \mathbb{V}_{Y,\mathcal{G},a}$.

4.7 Generalizations: Hodge correlators for local systems

Let $\mathcal{E}$ be a real polarized variation of Hodge structures over a compact Kahler manifold X. Consider the smooth de Rham complex of the local system of the endomorphisms of $\mathcal{E}$

$$\mathcal{G}_{\mathcal{E}} := \mathcal{A}^{*,*}(X) \otimes \mathrm{End}\mathcal{E}.$$

It has two standard differentials D and $D^{\mathbb{C}}$. The $dd^{\mathbb{C}}$-lemma holds for these differentials.

The complex $\mathcal{G}_{\mathcal{E}}$ has a natural bigraded DG Lie algebra structure. The Hodge vector field $\mathrm{Q}_{\mathrm{Hod}}$ has a straitforward analog in this context:

Theorem 4.6 *1. Given a point $a \in X$, there is a cohomology class*

$$\mathrm{Q}_{\mathcal{E},a} \in \widetilde{\mathbb{H}}^1(\mathcal{G}_{\mathcal{E}}, \mathcal{G}_{\mathcal{E}}), \qquad \sigma(\mathrm{Q}_{\mathcal{E},a}) = -\mathrm{Q}_{\mathcal{E},a}.$$

2. It is functorial: A map $f : X \to Y$ of Kahler manifolds gives rise to a DG Lie algebra map $f^* : \mathcal{G}_{\mathcal{E}} \longrightarrow \mathcal{G}_{f^*\mathcal{E}}$, which intertwines, up to a homotopy, derivations $\mathrm{Q}_{\mathcal{E},f(a)}$ and $\mathrm{Q}_{f^*\mathcal{E},a}$:

$$f^* \circ \mathrm{Q}_{\mathcal{E},f(a)} \quad \textit{is homotopic to} \quad \mathrm{Q}_{f^*\mathcal{E},a} \circ f^*.$$

There is an explicit construction of the class $\mathrm{Q}_{\mathcal{E},a}$ generalizing the one from Section 4.5.

5 Functoriality of Hodge correlator classes

In this Section we prove main properties of the Hodge correlators for compact Kahler manifolds. They tell that the Hodge correlator cohomology class is well defined (i.e. does not depend on the choices made in its definition) and functorial. In Section 5.1 we collect some technical results. The reader can skip them, and return when needed.

5.1 Preliminary results

A formula for δ. Let $\omega \in H^k$. Define $s_\omega \in H_{2n-k}$ by the Poincare duality operator:

$$s_\omega := H_{2n} \cap \omega, \qquad \langle s_\omega, \alpha\rangle := \langle H_{2n}, \omega \cup \alpha\rangle = \int_X \omega \cup \alpha.$$

Recall $(H^{2n} \cap h) \cup \beta := H^{2n} \cdot (h, \beta)$ and $\langle h_1 \cap H^{2n} \cap h_2\rangle := (h_1, H^{2n} \cap h_2)$. One has $\langle H^{2n} \cap s_\omega\rangle = \omega$, and hence $\langle \mathcal{H} \cap \overline{s}_\omega\rangle = \overline{\omega}$. We use dual bases $\{\alpha_s\}$ and $\{h_{\alpha_s}\}$, so that $(h_{\alpha_s}, \alpha_{s'}) = \delta_{ss'}$ and notation $\alpha|h := \sum_s \overline{\alpha}_s|\overline{h}_{\alpha_s}$, as in Section 2. Set

$$\delta(\overline{\alpha}|\overline{h}) := (-1)^{|\overline{\alpha}|}\overline{\alpha}|\delta\overline{h}. \tag{72}$$

Lemma 5.1 *One has*

$$\delta(\alpha|h) = -\alpha|h \otimes \alpha|h. \tag{73}$$

Equivalently,

$$\delta : \overline{s}_\omega \longmapsto \int_X \alpha|h \otimes \alpha|h \otimes \overline{\omega}. \tag{74}$$

Proof. One has

$$\delta : x \longmapsto \sum_{\alpha_3} \int_X (\alpha|h \otimes \alpha|h \otimes \overline{\alpha}_3) \cdot \langle \overline{h}_{\alpha_3} \cap \mathcal{H} \cap x \rangle. \tag{75}$$

Furthermore, $\langle \overline{a} \cap \mathcal{H} \cap \overline{b} \rangle := (-1)^{\deg(a)} \langle a \cap H^{2n}(X) \cap b \rangle$. Since

$$\sum_{\alpha_3} \overline{\alpha}_3 \cdot \langle \overline{h}_{\alpha_3} \cap \mathcal{H} \cap \overline{s}_\omega \rangle = \overline{\alpha}_3 \cdot \langle \overline{h}_{\alpha_3}, \overline{\omega} \rangle = \overline{\omega}.$$

we get (74). Since $h_{\alpha_k^\vee} = s_{\alpha_k}$, we see that (74) is equivalent to (73). Indeed,

$$\sum_s (-1)^{|\overline{\alpha}_s^\vee|} \overline{\alpha}_s^\vee | \delta h_{\overline{\alpha}_s^\vee} \stackrel{(74)}{=} \sum_s (-1)^{|\overline{\alpha}_s^\vee|} \overline{\alpha}_s^\vee \int_X \overline{\alpha}_s \otimes \alpha|h \otimes \alpha|h = -\alpha|h \otimes \alpha|h.$$

The lemma is proved.

Given a tree T, denote by T^0 the tree obtained by removing all external edges of T.

We extend the definition of the Hodge correlators to the case when the forms α_i may be arbitrary forms by applying the same recipe as in Section 2.

Two useful integrals. Let T be a plane trivalent tree decorated by $\alpha_1 \otimes \ldots \otimes \alpha_m$, no assumptions made about the forms α_i. Let us assign to internal edges E of the tree T some forms/currents g_E on $X^{\text{vert(E)}}$. Consider the following integrals assigned to this data: [5]

$$\eta_T(\alpha_1 \otimes \ldots \otimes \alpha_m) := \int_{X^{\text{vert}(T^0)}} d^{\mathbb{C}} g_1 \wedge \ldots \wedge d^{\mathbb{C}} g_{2m-1} \wedge \alpha_1 \wedge \ldots \wedge \alpha_m, \tag{76}$$

$$\xi_T(\alpha_1 \otimes \ldots \otimes \alpha_m) := \text{Sym}_{g_1, \ldots, g_{2m-1}} \int_{X^{\text{vert}(T^0)}} g_1 \wedge d^{\mathbb{C}} g_2 \ldots \wedge d^{\mathbb{C}} g_{2m-1} \wedge \alpha_1 \wedge \ldots \wedge \alpha_m. \tag{77}$$

We assume that products of currents g_i make sense. Write $a \sim_{\mathbb{Q}^*} b$ if $a = \lambda b$ for $\lambda \in \mathbb{Q}^*$. Then

$$\kappa_T(d^{\mathbb{C}}\beta \otimes \alpha_1 \otimes \ldots \otimes \alpha_k) \sim_{\mathbb{Q}^*} \eta_T(\beta \otimes \alpha_1 \otimes \ldots \otimes \alpha_k) \qquad \text{assuming } d^{\mathbb{C}}\alpha_i = 0, \tag{78}$$

where we assume that the forms α_i are homogeneous with respect to the Hodge bidegree (p, q).

Lemma 5.2 *If $d^{\mathbb{C}}\alpha_i = 0$ for all i and $m > 3$, then $\eta_T(\alpha_1 \otimes \ldots \otimes \alpha_m) = 0$.*

Proof. The integrand is $d^{\mathbb{C}}$ of the form $g_1 \wedge d^{\mathbb{C}} g_2 \wedge \ldots \wedge d^{\mathbb{C}} g_{2m-1} \wedge \alpha_1 \wedge \ldots \wedge \alpha_m$.

Denote by p_1, p_2 the projections of X^2 to the factors.

Lemma 5.3 *a) Let T be a plane trivalent tree decorated by $W = \mathcal{C}(\alpha_1 \otimes \ldots \otimes \alpha_m)$, $d^{\mathbb{C}}\alpha_i = 0$. We assign to an internal edge E of T a decomposable current $p_1^*\omega \wedge p_2^*\beta$, where $d^{\mathbb{C}}\omega = 0$, and to the other internal edges F of T some currents g_F. We assume that the corresponding form $\kappa_T(W)$, see (48), is a current. Then its integral is zero unless the following condition holds:*

E is an external edge of T^0, and ω labels an external vertex of E in T^0, as on Fig. 6. (79)

[5] Here and later on the symmetrization Sym is taken with respect to the shifted by 1 degree $|g| := \deg(g) + 1$

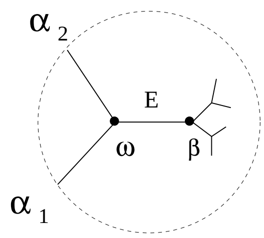


Figure 6: Only such diagrams can contribute.

b) Let us assume (79). Denote by T_β and T_ω the two trees obtained from T by cutting the edge E, so that if T_ω is decorated by $\omega \wedge \alpha_1 \wedge \alpha_2$ (see Fig. 7). The integral is proportional to

$$\int_X \alpha_1 \wedge \alpha_2 \wedge \omega \ \cdot \ \int_X \kappa_{T'_E}(d^{\mathbb{C}}\beta \otimes \alpha_3 \otimes \ldots \otimes \alpha_m), \tag{80}$$

unless E is the only internal edge of T, in which case the contribution is

$$\int_X \alpha_1 \wedge \alpha_2 \wedge \omega \ \cdot \ \int_X (\beta \wedge \alpha_3 \wedge \alpha_4). \tag{81}$$

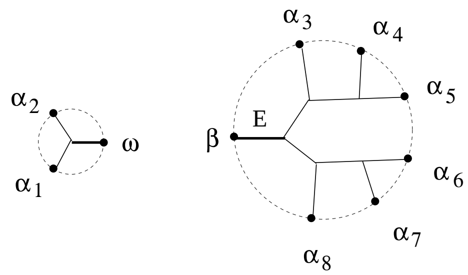


Figure 7: A possible non-zero contribution of a decomposable current $p_1^*\omega \wedge p_2^*\beta$.

Proof. The external edges of the trees T_β and T_ω inherit decorations ω and β or $d^{\mathbb{C}}\beta$. If $T^0 = E$, we get integral (81).

Assume that the tree T^0 has more then one edge. Then we may assume that E is decorated by ω and $d^{\mathbb{C}}\beta$. Indeed, there is a unique internal edge of T such that the polydifferential operator ω acts on the current assigned to this edge as the identity, and this edge can be chosen arbitrarily. So we may assume that E is not this edge. Moreover, employing $\xi_T(W)$ instead of $\kappa_T(W)$, we change the integral by a binomial coefficient, depending on the type of W, see (33). Thus we may assume that ω acts to the current assigned to this edge by $d^{\mathbb{C}}$. It remains to notice that $d^{\mathbb{C}}\omega = 0$.

We claim that if T_ω has internal edges, then the corresponding Hodge correlator type integral is zero. Indeed, the contribution to the integral of the tree T with the edge E decorated as above is a sum of rational multiples of

$$\xi_{T_\beta}(d^{\mathbb{C}}\beta \otimes ...) \cdot \eta_{T_\omega}(\omega \otimes ...), \quad \text{or} \quad \eta_{T_\beta}(d^{\mathbb{C}}\beta \otimes ...) \cdot \xi_{T_\omega}(\omega \otimes ...).$$

Since $d^{\mathbb{C}}\alpha_i = 0$ for every i, $\eta_{T_\beta}(d^{\mathbb{C}}\beta \otimes ...) = 0$. Indeed, if the tree T_β has internal edges, this follows from Lemma 5.2. Otherwise the integral is $\int_X d^{\mathbb{C}}\beta \wedge \alpha_1 \wedge \alpha_2 = \int_X d^{\mathbb{C}}(\beta \wedge \alpha_1 \wedge \alpha_2) = 0$. Since the tree T_ω has internal edges, $\eta_{T_\omega}(\omega \otimes ...) = 0$ by Lemma 5.2. The claim is proved.

So T_ω has no internal edges, and we get integral (80). The Lemma is proved.

We use a notation $\overset{(-)}{\partial}$ meaning "either ∂ or $\overline{\partial}$". Consider the following integrals

$$\mathrm{Sym}_{2m-2} \int_{X^{\mathrm{vert}(T^0)}} \overset{(-)}{\partial} g_1 \wedge \ldots \wedge \overset{(-)}{\partial} g_{2m-1} \wedge \alpha_1 \wedge \ldots \wedge \alpha_m. \tag{82}$$

$$\mathrm{Sym}_{2m-2} \int_{X^{\mathrm{vert}(T^0)}} g_1 \wedge \overset{(-)}{\partial} g_2 \wedge \ldots \wedge \overset{(-)}{\partial} g_{2m-2} \wedge \partial\overline{\partial} g_{2m-1} \wedge \alpha_1 \wedge \ldots \wedge \alpha_m. \tag{83}$$

Proposition 5.4 *If the tree T^0 is not empty and $d\alpha_i = 0$, then integrals (82) and (83) are zero.*

Lemma 5.5 *For any forms φ_i and a form ω such that $\partial\omega = \overline{\partial}\omega = 0$, one has*

$$\text{(i) } \mathrm{Sym}_2 \int_X \omega \wedge \overset{(-)}{\partial} \varphi_1 \wedge \overset{(-)}{\partial} \varphi_2 = 0, \qquad \text{(ii) } \mathrm{Sym}_3 \int_X \overset{(-)}{\partial} \varphi_1 \wedge \overset{(-)}{\partial} \varphi_2 \wedge \overset{(-)}{\partial} \varphi_3 = 0. \tag{84}$$

Proof. (i) There are three cases, depending on how many $\overline{\partial}$'s we have in the integrand. The two $\overline{\partial}$'s case is reduced by complex conjugation to the case without $\overline{\partial}$'s. In the latter case the integral equals $\int_X \partial(\omega \wedge \varphi_1 \wedge \partial\varphi_2) = 0$. Finally, in the one $\overline{\partial}$ case

$$(\partial\varphi_1 \wedge \overline{\partial}\varphi_2 + \overline{\partial}\varphi_1 \wedge \partial\varphi_2) \wedge \omega = \partial\left(\varphi_1 \wedge \overline{\partial}\varphi_2 \wedge \omega\right) + \overline{\partial}\left(\varphi_1 \wedge \partial\varphi_2 \wedge \omega\right).$$

These are complete derivatives. So integrating over X we get zero.

(ii) By using the complex conjugation we may assume that we have no or one $\overline{\partial}$. In the first case the form $\partial\varphi_1 \wedge \partial\varphi_2 \wedge \partial\varphi_3$ is a complete derivative. In the second the integrand is

$$(\partial\varphi_1 \wedge \overline{\partial}\varphi_2 + \overline{\partial}\varphi_1 \wedge \partial\varphi_2) \wedge \partial\varphi_3 + \partial\varphi_1 \wedge \partial\varphi_2 \wedge \overline{\partial}\varphi_3 =$$

$$\partial\left(\varphi_1 \wedge \overline{\partial}\varphi_2 \wedge \partial\varphi_3\right) + \overline{\partial}\left(\varphi_1 \wedge \partial\varphi_2 \wedge \partial\varphi_3\right) + \partial(\varphi_1 \wedge \partial\varphi_2 \wedge \overline{\partial}\varphi_3).$$

So it is also a complete derivative. So integrating over X we get zero. The lemma is proved.

Proof of Proposition 5.4. Applying $\overset{(-)}{\partial}$ to a current $g(x,y)$, we think about it as a sum of two operators, $\overset{(-)}{\partial}_x + \overset{(-)}{\partial}_y$, acting on x and y parts of $g(x,y)$.

We start from integral (82). Take an external edge E of the tree T^0. Let v be its external vertex (or one of them if T^0 has a single edge). Then by Lemma 5.5(i), or using $\int_X \alpha_1 \wedge \alpha_2 \wedge \overset{(-)}{\partial} \gamma = 0$, the integral over the copy of X assigned to v is zero for the component $\overset{(-)}{\partial}_x g(x,y)$ where x runs through the copy of X labeled by v. So we have only the component $\overset{(-)}{\partial}_y g(x,y)$. Using Lemma 5.5 we run a similar argument, until we find an internal vertex of T^0 such that the corresponding integral assigned to this vertex is of type (84), and thus vanishes by Lemma 5.5(ii).

Now let us consider (83). Let E be the edge contributing the Laplacian $\partial\overline{\partial} g_E$. Cutting this edge, we get two trees $T_E^{\pm}$. Denote by $v_\pm$ the vertex of the tree edge E entering the tree $T_E^{\pm}$. Assume that the edge F contributing just g_F is in the tree T_E^-. Then if the current g_E was hit by $\overset{(-)}{\partial}_{v_+}$, the integral related to T_E^+ is zero. This is proved just like we proved that (83) is zero: we start from the integral related to the vertex v_+. Otherwise the current g_E was hit by the Laplacian $\overline{\partial}_{v_-}\partial_{v_-}$. In this case the integral for the vertex v_+ is equivalent, by the Stokes formula, to the integral (ii) in (84). Proposition 5.4 is proved.

Lemma 5.6 *The Hodge correlator integral for a cyclic word $\mathcal{C}(\theta_0 \otimes \ldots \otimes \theta_m)$ where $d\theta_i = d^{\mathbb{C}}\theta_i = 0$, and $\theta_i = dd^{\mathbb{C}}\eta_i$ for at least two θ_i's, is zero.*

Proof. Suppose that $\theta_i = dd^{\mathbb{C}}\eta_i$ for $i = a, b$. Using the Stokes formula, we move $dd^{\mathbb{C}}$ from θ_a to the Green current assigned to an edge E of a tree T. Formula (38) for $dd^{\mathbb{C}}G_E$ gives us the usual three terms: the Casimir, the volume and the δ-terms. The last vanishes after taking the sum over all trees. The volume term always vanishes. So we are left with the Casimir term. Running the same argument for $dd^{\mathbb{C}}\eta_b$ we get zero since there are no edges F contributing G_F rather then $d^{\mathbb{C}}G_F$ to the operator ω left. The lemma is proved.

5.2 Proof of Theorem 4.1

Proof of (i) The degree is zero by Lemma 2.4. The second claim is equivalent to the shuffle relations for the Hodge correlators from Proposition 2.5. The part (i) is proved.

Proof of (ii). Recall the cyclic Casimir element $\mathcal{C}\mathrm{Id}_X$, see (54).

Proposition 5.7 *One has*

$$\delta\mathrm{Cor}_{\mathcal{H}}\Big(\mathcal{C}\mathrm{Id}_X\Big) = \sum_T \int_{X^{\mathrm{vert}(T^0)}} d\kappa_T\Big(\ \mathcal{C}\mathrm{Id}_X\Big) = 0. \qquad (85)$$

Remark. Let $\gamma_0, \ldots, \gamma_m$ be harmonic representatives of cohomology classes in $H^{>0}(X)$ such that $\sum_i(\deg(\gamma_i) - 1) = 2\mathrm{dim}X - 3$. Then the current $\kappa_T(\gamma_1 \otimes \ldots \otimes \gamma_m)$ has degree one less then the maximal possible. So (85) just means that

$$\delta\mathbf{G} = \sum_W \frac{1}{\mathrm{Aut}(W)|} \sum_T \int_{X^{\mathrm{vert}(T^0)}} d\kappa_T(\gamma_0|h_{\gamma_0} \otimes \ldots \otimes \gamma_m|h_{\gamma_m}) = 0. \qquad (86)$$

The first sum is over a basis $W = \mathcal{C}(\gamma_0 \otimes \ldots \otimes \gamma_m)$ in the degree -1 subspace of $\mathcal{C}_{\mathbb{H}^*} \otimes \mathcal{H}$.

Proof of Proposition 5.7. Let us calculate the double sum before the integration. We use the same terminology and method as in Theorem 6.8 in [G1]. We employ formula (32) for dw, and use (38) to calculate $(2\pi i)^{-1}\overline{\partial}\partial G(x,y)$. There are three terms in the latter formula, called the δ-term, the volume term and the Casimir term. The δ-term vanishes just as in proof of Theorem 6.8. The volume term vanishes due to the absence of H^0-factors in the decoration and Lemma 2.4. By Lemma 5.3, the Casimir term for an edge E can be non-zero only if E is an external edge of T^0.

Let us assume that $E = E_0$ is an external edge of T^0, and F_0, F_1 are the two external edges incident to a vertex v of E_0. We assume that $\mathrm{sgn}(\mathbf{E}_0 \wedge \mathbf{F_0} \wedge \mathbf{F_1}) = 1$. Let T' be the tree obtained by removing the edges F_0, F_1, and $\mathrm{Or}_{T'}$ is an orientation of T'. Then

$$\mathrm{sgn}(\mathrm{Or}_{T'}) = \mathrm{sgn}(\mathrm{Or}_{T'} \wedge \mathbf{F}_0 \wedge \mathbf{F}_1). \qquad (87)$$

Let us calculate the contribution of the Casimir term assigned to the edge E_0. We assume that the external edges F_i are decorated by γ_i, and the internal edges are $E_0, ..., E_k$.

Lemma 5.8 *The contribution of Casimir term for the edge E_0, after summation over γ_0, γ_1, is*

$$\sum_{\gamma_i} \mathrm{Cor}_{\mathcal{H}}\Big(\delta(\alpha|h) \otimes \gamma_2|h_{\gamma_2} \otimes ... \otimes \gamma_m|h_{\gamma_m}\Big). \tag{88}$$

Proof. The element $\mathcal{C}(\overline{h}_{\gamma_0} \otimes \ldots \otimes \overline{h}_{\gamma_m})$ appears in (86) with the coefficient obtained by taking the sum over trees of the integrals of

$$d\omega\Big((G_{E_0} \wedge \mathbf{E}_0) \wedge \ldots \wedge (G_{E_k} \wedge \mathbf{E}_k)\Big) \wedge (\gamma_0 \wedge \mathbf{F}_0)|\overline{h}_{\gamma_0} \otimes (\gamma_1 \wedge \mathbf{F}_1)|\overline{h}_{\gamma_1} \otimes \ldots.$$

Observe that formula (32) just means that

$$d\omega\Big((\varphi_0 \wedge \mathbf{E}_0) \wedge \ldots \wedge (\varphi_k \wedge \mathbf{E}_k)\Big) =$$

$$-(-1)^{\deg\varphi_0}(\overline{\partial}\partial\varphi_0 \wedge \mathbf{E}_0) \wedge \omega\Big((\varphi_1 \wedge \mathbf{E}_1) \wedge \ldots \wedge (\varphi_k \wedge \mathbf{E}_k)\Big) + \ldots \tag{89}$$

Here ... means sum of the similar terms for φ_i, $i > 0$, plus the two summands (31).

The Green current has an even degree. So taking into account differential equation (39) for the Green current, we conclude that the contribution of the Casimir term for the edge E_0 equals

$$\sum_s (\alpha_s^\vee \wedge \alpha_s) \wedge \mathbf{E}_0 \bigwedge \omega\Big((G_{E_1} \wedge \mathbf{E}_1) \wedge \ldots \wedge (G_{E_k} \wedge \mathbf{E}_k)\Big) \bigwedge (\gamma_0 \wedge \mathbf{F}_0)|\overline{h}_{\gamma_0} \otimes (\gamma_1 \wedge \mathbf{F}_1)|\overline{h}_{\gamma_1} \otimes \ldots. \tag{90}$$

Notice that the $-$ signs in (89) and in front of the Casimir term in (39) canceled each other. Further, the form $\alpha_s^\vee \wedge \alpha_s$ is even, and $\omega(...)$ in (90) is odd. So moving $\alpha_s^\vee \wedge \alpha_s \wedge \mathbf{E}_0$ we get

$$-\omega\Big((G_{E_1} \wedge \mathbf{E}_1) \wedge \ldots \wedge (G_{E_k} \wedge \mathbf{E}_k)\Big) \bigwedge$$

$$(\gamma_0 \wedge \mathbf{F}_0)|\overline{h}_{\gamma_0} \otimes (\gamma_1 \wedge \mathbf{F}_1)|\overline{h}_{\gamma_1} \bigwedge \sum_s (\alpha_s^\vee \wedge \alpha_s) \wedge \mathbf{E}_0 \bigwedge (\gamma_2 \wedge \mathbf{F}_2)|\overline{h}_{\gamma_2} \otimes \ldots.$$

Lemma 5.9

$$\int_X \sum_{\gamma_0,\gamma_1} (\gamma_0 \wedge \mathbf{F}_0)|\overline{h}_{\gamma_0} \otimes (\gamma_1 \wedge \mathbf{F}_1)|\overline{h}_{\gamma_0} \bigwedge \sum_s (\alpha_s^\vee \wedge \alpha_s) \wedge \mathbf{E}_0 = -\delta(\alpha|h) \cdot \mathbf{E}_0 \wedge \mathbf{F}_0 \wedge \mathbf{F}_1.$$

Proof. Let $\mathrm{pr}_{\mathcal{H}}$ be the orthogonal projection onto the space of harmonic forms. Then

$$\mathrm{pr}_{\mathcal{H}}(\beta) = \sum_s (\beta, \alpha_s^\vee)\alpha_s. \tag{91}$$

Thus $\sum_s \int_X (\gamma_0 \wedge \gamma_1 \wedge \alpha_s^\vee) \cdot \alpha_s = \mathrm{pr}_{\mathcal{H}}(\gamma_0 \wedge \gamma_1)$. The Lemma follows from (73).

This proves Lemma 5.8. Notice that the two $-$ signs in the last two formulas canceled each other. Thus we proved Proposition 5.7 and hence the part (ii).

Proof of (iii). Recall that there are the following choices in the definition of the Green current:

1. A solution of equation (38) for the Green current G is unique up to
   (a) currents of type $\partial\alpha + \overline{\partial}\beta$.
   (b) harmonic forms of type $(n-1, n-1)$.

2. Differential equation (38) depends on the splitting of the Dolbeaut complex.

Let us investigate how these choices affect the resulting Hodge correlator **G**.

1a) The difference of the Hodge correlators of $\mathcal{C}(\alpha_1 \otimes \ldots \otimes \alpha_m)$ defined using $G(x,y)$ and $G(x,y) + \overset{(-)}{\partial}\gamma$ is a linear combination of integrals like (82) where g_i stands for G_E or γ.

If the tree T^0 is empty, the difference is zero by the very definition. If the tree T^0 is not empty, it is zero by Proposition 5.4.

1b) Let us add a closed form γ to a Green current. Using the Kunneth formula for $H^*(X \times X)$ and 1a) we may assume that it is decomposable: $\gamma = \omega_1 \times \omega_2$, $d\omega_i = 0$. Then its contribution is zero by Lemma 5.2, unless the tree T^0 has a single vertex. To handle the latter case we use Lemma 5.3. Thus if the tree T^0 has a single vertex, the contribution is shown on Fig 8. It equals $\mathcal{C}\left(\delta h_{\omega_1} \otimes \delta h_{\omega_2}\right)$. This is obviously a coboundary. The part 1. of (iii) is proved.

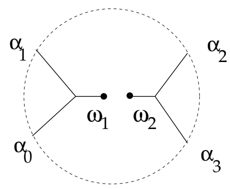


Figure 8: The contribution of $\gamma = \omega_1 \times \omega_2$.

2. We want to compare two splittings s and $\widetilde{s}$ of the Dolbeaut complex. We assume that they differ only by choice of another representative $\widetilde{\omega}^\vee$ of a (p,q)-harmonic form $\omega^\vee$, which is dual to a harmonic form ω. By the $\overline{\partial}\partial$-lemma there exists a $(p-1,q-1)$-form β such that $\overline{\partial}\partial\beta = \widetilde{\omega}^\vee - \omega^\vee$. Let $\widetilde{G} = \widetilde{G}(x,y)$ be a Green current for the splitting $\widetilde{s}$. One can choose it as

$$\widetilde{G}(x,y) = G(x,y) + p_1^*\omega \wedge p_2^*\beta, \tag{92}$$

so the second term is decomposable. Let $\mathrm{Cor}_{\mathcal{H}}^{\widetilde{G}}$ be the Hodge correlator map for the Green current $\widetilde{G}$. Let us calculate the difference $\widetilde{\mathbf{G}} - \mathbf{G}$ between the Hodge correlators for the Green currents $\widetilde{G}(x,y)$ and $G(x,y)$.

The integrand of the Hodge correlator is obtained by applying the polydifferential operator ω to the (pull backs of) Green currents assigned to the edges of T^0, and multiplying the result by the closed forms assigned to the external edges of T. The difference $\widetilde{\mathbf{G}} - \mathbf{G}$ is a sum of such integrals, where one or more of the Green currents $G(x,y)$ are replaced by $p_1^*\omega \wedge p_2^*\beta$. Let us pick an edge E of T where $p_1^*\omega \wedge p_2^*\beta$ was employed. By Lemma 5.3 the contribution of such a pair (E,T) to $\widetilde{G} = \widetilde{G}(x,y)$ is zero unless the condition (79) holds.

Consider expressions of the following types (s_ω was introduced in the end of Section 5.1):

$$(i)\ \delta(d^{\mathbb{C}}\beta | s_\omega), \qquad (ii)\ dd^{\mathbb{C}}\beta | s_\omega, \qquad (iii)\ \alpha_i | h_{\alpha_i}. \tag{93}$$

We make cyclic words out of them. Since ω, s_ω, β have the same parity of the grading, these expressions are plain cyclic invariant. Let us introduce a notation

$$\Delta(\gamma|h) := d\gamma|h + (-1)^{|\gamma|}\gamma|\delta h. \qquad \text{So} \quad \Delta^2 = 0.$$

It agrees with (72) since there we have $d\gamma = 0$.

Denote by $\mathrm{Cor}^*_{\mathcal{H}}$ the Hodge correlator map defined by using the polydifferential operator ξ instead of ω, see (33). The relationship between $\mathrm{Cor}^*_{\mathcal{H}}$ and $\mathrm{Cor}_{\mathcal{H}}$ is the following:

$$\mathrm{Cor}^*_{\mathcal{H}} = \mu \circ \mathrm{Cor}_{\mathcal{H}} \circ \mu^{-1}.$$

Here μ is the operator multiplying a homogeneous element of bidegree (p,q) by $\binom{p+q-2}{p-1}$. Set

$$\mathbf{A}^* := \mathbf{A}'^* + \mathbf{A}''^* := \sum_Z \mathrm{Cor}^{*,G}_{\mathcal{H}}(\mathcal{C}(Z)) \ + \ \mathrm{Cor}^{*,G}_{\mathcal{H}}(\alpha|h \otimes \alpha|h \otimes \omega)\mathrm{Cor}^{*,G}_{\mathcal{H}}(\beta \otimes \alpha|h \otimes \alpha|h).$$

Here Z runs through a basis in the space of all cyclic products of expressions $\Delta(d^{\mathbb{C}}\beta|s_w)$ and $\alpha_s|h_{\alpha_s}$. The second term in $\mathbf{A}^*$ corresponds to the diagram on Fig. 8 where $\omega_1 = \omega$ and $\omega_2 = \beta$.

Lemma 5.10 *One has the following, where n is the number of factors* Id_X*:*

$$\mathbf{A}'^* = \sum_{n\geq 1} \frac{1}{n}\mathrm{Cor}^{*,G}_{\mathcal{H}}\Big(\Delta(d^{\mathbb{C}}\beta|s_w) \otimes \mathrm{Id}_X \otimes \ldots \otimes \Delta(d^{\mathbb{C}}\beta|s_w) \otimes \mathrm{Id}_X\Big). \tag{94}$$

Remark. The coefficient n is the order of the automorphism group of the summand.

Example. Choose a basis $\{A_i\}$ in the tensor algebra of the vector space with the basis $\{\alpha_k|h_{\alpha_k}\}$. The beginning of $\mathbf{A}'^*$ looks as follows:

$$\sum_i \mathrm{Cor}^{*,G}_{\mathcal{H}}\Big(\Delta(d^{\mathbb{C}}\beta|s_w) \otimes A_i\Big) + \frac{1}{2}\sum_{i\neq j} \mathrm{Cor}^{*,G}_{\mathcal{H}}\Big(\Delta(d^{\mathbb{C}}\beta|s_w) \otimes A_i \otimes \Delta(d^{\mathbb{C}}\beta|s_w) \otimes A_j\Big) +$$

$$\frac{1}{2}\sum_i \mathrm{Cor}^{*,G}_{\mathcal{H}}\Big(\Delta(d^{\mathbb{C}}\beta|s_w) \otimes A_i \otimes \Delta(d^{\mathbb{C}}\beta|s_w) \otimes A_i\Big) + \ldots$$

Here the coefficient $1/2$ in front of the second term appears since the cyclic word for the pair (A_i, A_j) is the same as for the one (A_j, A_i). The coefficient $1/2$ in front of the next term appears for a different reason: the corresponding cyclic word has an automorphism group of order two.

Proof of Lemma 5.10. Given a cyclic word

$$Z = \mathcal{C}\Big(\Delta(d^{\mathbb{C}}\beta|s_w) \otimes A_1 \otimes \ldots \otimes \Delta(d^{\mathbb{C}}\beta|s_w) \otimes A_n\Big),$$

write $n = ab$ where $\mathbb{Z}/a\mathbb{Z}$ is the automorphism group of Z – so $A_{i+a} = A_i$. Then Z enters with the coefficient $1/a \cdot 1/b = 1/n$. Indeed, the coefficient $1/a$ appears since $a = |\mathrm{Aut}(Z)|$, and the coefficient $1/b$ is needed since the cyclic shift which moves A_k to A_{k+i}, where $0 \leq i \leq b-1$, produces a cyclic word which equals to Z. The lemma is proved.

Proposition 5.11 *One has* $\widetilde{\mathbf{G}}^* - \mathbf{G}^* = \mathbf{A}^*$.

Proof. 1) Denote by $\mathcal{C}\widetilde{\mathrm{Id}}_X$ the cyclic Casimir element written by using the forms $\widetilde{\alpha}_s$ instead of α_s. Since $\widetilde{\omega}^\vee - \omega^\vee = dd^{\mathbb{C}}\beta$, we have

$$\mathrm{Cor}^{*,\widetilde{G}}_{\mathcal{H}}\Big(\mathcal{C}\widetilde{\mathrm{Id}}_X - \mathcal{C}\mathrm{Id}_X\Big) = \sum_X \mathrm{Cor}^{*,\widetilde{G}}_{\mathcal{H}}(\mathcal{C}(X)). \tag{95}$$

The sum on the right of (95) is over a basis X in the cyclic tensor algebra of the vector space with a basis $dd^{\mathbb{C}}\beta|s_\omega$, $\{\alpha_s|h_{\alpha_s}\}$. The equality is proved by an argument similar to the one in the proof of Lemma 5.10.

2) Let us calculate

$$\sum_X \Big(\mathrm{Cor}^{*,\widetilde{G}}_{\mathcal{H}}(\mathcal{C}(X)) - \mathrm{Cor}^{*,G}_{\mathcal{H}}(\mathcal{C}(X)) \Big). \tag{96}$$

By Lemma 5.3 we can get non-zero contributions only from an external edges E of T^0 such that

there is a vertex of E shared by two external edges, F_- and F_+, and these edges are decorated by α's, not by $dd^{\mathbb{C}}\beta$ – we denote them by α_- and α_+, see Fig 9.

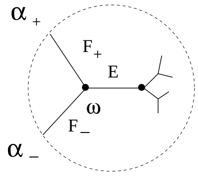


Figure 9:

If the tree T^0 has a single vertex, we get the second term in $\mathbf{A}^*$.

Lemma 5.12 *Assume that the tree T^0 has more then one vertex. Then the contribution to (96) of an external edge E of the tree T^0 as above is described by the following procedure:*

- *Cut the edge E and throw away the small tree T''_E,*
- *Decorate the new external edge E' by $\delta(d^{\mathbb{C}}\beta|s_\omega)$.*

The same recipe is applied to every edge E where G_E is replaced by $\widetilde{G}_E$. Using the same argument as in the proof of Lemma 5.10, we get

$$(96) = \sum_Y \mathrm{Cor}^{*,\widetilde{G}}_{\mathcal{H}}(\mathcal{C}(Y)) \tag{97}$$

where Y is a basis in the cyclic tensor algebra of the vector space spanned by $\delta(d^{\mathbb{C}}\beta|s_\omega)$ and $\{\alpha_s|h_{\alpha_s}\}$. Proposition 5.11 would follow from this by using the Lemma 5.10 argument again.

Proof of Lemma 5.12. It is similar to the proof of Proposition 5.7. The key point is this:[6]

$$\int \xi(\widetilde{G}_{E_0} \wedge \ldots \wedge \widetilde{G}_{E_k}) \wedge \ldots = \int \xi(\widetilde{G}_{E_0} \wedge \ldots \wedge \widetilde{G}_{E_{k-1}}) \wedge d^{\mathbb{C}}\widetilde{G}_{E_k} \wedge \ldots. \tag{98}$$

We arrange the edges so that $E = E_k$, one has $\mathrm{sgn}(\mathbf{F}_- \wedge \mathbf{F}_+ \wedge \mathbf{E}) = 1$, and $F_\pm$ is decorated by $\alpha_\pm$. Lemma 5.12 boils down to the computation of

$$(d^{\mathbb{C}}(\widetilde{G}_E - G_E) \wedge \mathbf{E}) \bigwedge (\alpha_- \wedge \mathbf{F}_-)|\overline{h}_{\alpha_-} \otimes (\alpha_+ \wedge \mathbf{F}_+)|\overline{h}_{\alpha_+}$$

Using (74), we prove Lemma 5.12. Proposition 5.11 is proved.

[6] This is the reason to use the version Cor^* of the Hodge correlator, see Remark at the end of this Subsection.

Set

$$\mathbf{B} := \sum_{n\geq 1} \frac{1}{n}\mathrm{Cor}^G_{\mathcal{H}}\Big(d^{\mathbb{C}}\beta|s_w \otimes \mathrm{Id}_X \otimes \Delta(d^{\mathbb{C}}\beta|s_w) \otimes \mathrm{Id}_X \otimes \ldots \otimes \Delta(d^{\mathbb{C}}\beta|s_w) \otimes \mathrm{Id}_X\Big). \tag{99}$$

Here there are $(n-1)$ factors $\Delta(d^{\mathbb{C}}\beta|s_w) \otimes \mathrm{Id}_X$. We denote by $\mathbf{A}'$ the expression similar to $\mathbf{A}'^*$ defined using the Hodge correlator Cor instead of Cor^*.

Proposition 5.13 *One has* $\mathbf{A}' = \delta\mathbf{B}$.

Proof. Applying the differential δ to $d^{\mathbb{C}}\beta|s_w$ in $\mathbf{B}$ we get a part of the sum $\mathbf{A}'$:

$$\mathbf{A}'_1 := \sum_{n>0} \mathrm{Cor}_{\mathcal{H}}\Big(d^{\mathbb{C}}\beta|\delta s_\omega \otimes \mathrm{Id}_X \otimes \Delta(d^{\mathbb{C}}\beta|s_w) \otimes \mathrm{Id}_X \otimes \ldots\Big).$$

Lemma 5.14 *Applying δ to every factor h_{α_i} in* $\mathbf{B}$ *except s_ω, and taking the sum we get*

$$\mathbf{A}'_2 := \sum_{n>0} \mathrm{Cor}_{\mathcal{H}}\Big(dd^{\mathbb{C}}\beta|s_\omega \otimes \mathrm{Id}_X \otimes \Delta(d^{\mathbb{C}}\beta|s_w) \otimes \mathrm{Id}_X \otimes \ldots\Big).$$

Proof. The claim we calculate the following sum, similar to the one defining $\mathbf{B}$:

$$0 = \sum_{n>0} \int \sum_T d\kappa_T\Big(d^{\mathbb{C}}\beta|s_\omega \otimes \mathrm{Id}_X \otimes \Delta(d^{\mathbb{C}}\beta|s_w) \otimes \mathrm{Id}_X \otimes \ldots\Big).$$

Let us calculate $d\kappa_T(d^{\mathbb{C}}\beta|s_\omega \otimes \mathrm{Id}_X \otimes \ldots)$. Applying d to $d^{\mathbb{C}}\beta|s_\omega$ we get $dd^{\mathbb{C}}\beta|s_\omega$. The other decorating forms are $d^{\mathbb{C}}$-closed. Thus differentiating the Green currents in $\kappa_T(d^{\mathbb{C}}\beta|s_\omega \otimes \mathrm{Id}_X \otimes \ldots)$, we get, similarly to Lemma 5.8, an expression containing the orthogonal projections onto the harmonic forms of products of three kinds: $d^{\mathbb{C}}\beta \wedge \alpha_*$, $dd^{\mathbb{C}}\beta \wedge \alpha_*$, or $\alpha_i \wedge \alpha_{i+1}$. The first two of them are zero since we integrate complete differentials. For example, using $d^{\mathbb{C}}\alpha_j = 0$,

$$\mathrm{pr}_{\mathcal{H}}(d^{\mathbb{C}}\beta \wedge \alpha_*) = \int_X d^{\mathbb{C}}\beta \wedge \alpha_* \wedge \alpha_k = \int_X d^{\mathbb{C}}(\beta \wedge \alpha_* \wedge \alpha_k) = 0.$$

The sum of remaining integrals coincides with the negative (because $d^{\mathbb{C}}\beta|s_\omega$ is odd) of the expression obtained by applying the differential δ to every factor h_{α_i} in $\alpha|h_{\alpha_i}$ except s_ω, and taking the sum. Lemma 5.14, hence Proposition 5.13, and thus the part 2) of (iii) are proved.

Clearly $\mathbf{A}' = \mathbf{A}'_1 + \mathbf{A}'_2$. Furthermore, $\mathbf{A}' = \delta\mathbf{B}$ implies $\mathbf{A}'^* = \delta\mathbf{B}^*$ for the element $\mathbf{B}^*$ defined using Cor^*. Thus Theorem 4.1 is proved.

Remark. To write down a formula for the coboundary, we employ two forms of the Hodge correlator: the one $\mathrm{Cor}_{\mathcal{H}}$ using the polydifferential operator ω, and the one $\mathrm{Cor}^*_{\mathcal{H}}$ using ξ. Dealing with $dd^{\mathbb{C}}G_E$ it is convenient to use ω, while handling the effect of replacing $\widetilde{G}_E$ by G_E it is handy to employ ξ, because of formula (98).

5.3 Functoriality: Proof of Theorem 1.4

We work in the category of compact complex manifolds. Let $f : X \to Y$ be a map of manifolds. There are two standard functors: the pull back $f^* : H^*(Y) \to H^*(X)$, and the dual map $f_* : H_*(X) \to H_*(Y)$. We define a map $f^! : H_*(Y) \to H_*(X)$ by using the Poincare duality:

$$f^!(c) \cap \mathcal{H}_X := f^*(c \cap \mathcal{H}_Y). \tag{100}$$

Therefore one has

$$\langle c_0 \cap \mathcal{H}_Y \cap f_*h\rangle_Y = \langle f^*(c_0 \cap \mathcal{H}_X) \cap h\rangle_Y \stackrel{(100)}{=} \langle f^! c_0 \cap \mathcal{H}_X \cap h\rangle_X. \tag{101}$$

Let $f_! : H^*(X) \to H^*(Y)$ be the dual map. There is the projection formula

$$f_!(\alpha \cup f^*\gamma) = f_!\alpha \cup \gamma. \tag{102}$$

We reserve the notation f^* for the inverse image of smooth forms, and $f_! = f_*$ for the direct image of distributions – the adjoint map. They are well defined in the category of manifolds.

We denote by $f^!$ the inverse image of distributions. It is not always defined: a sufficient condition is that the conormal bundle to the graph of f is transversal to the wave front of the distribution.

A map $f : X \to Y$ induces a map $f_* : \mathbb{H}_*(X) \to \mathbb{H}_*(Y)$, and hence a push forward $f_*\mathbf{H}_{X,x}$.

Lemma 5.15 *The induced map* $f_* : \mathcal{CL}ie_{\mathbb{H}_*(X)} \longrightarrow \mathcal{CL}ie_{\mathbb{H}_*(Y)}$ *is a map of complexes.*

Proof. Given a finite dimensional vector space V, denote by $\mathrm{Id}_V \in V^* \otimes V$ the Casimir element corresponding to the identity map. Given a linear map $f : V \to W$, let $f^* : W^* \to V^*$ be the dual map. Set

$$F_* = (f, \mathrm{Id}) : V^* \otimes V \longrightarrow V^* \otimes W, \quad F^* = (\mathrm{Id}, f^*) : W^* \otimes W \longrightarrow V^* \otimes W.$$

Then one has

$$F_*\mathrm{Id}_V = F^*\mathrm{Id}_W. \tag{103}$$

Let δ_X (respectively δ_Y) be the differential in $\mathcal{CL}ie_{\mathbb{H}_*(X)}$ (respectively $\mathcal{CL}ie_{\mathbb{H}_*(Y)}$). We denote by $\{\alpha = \alpha_{(s)}\}$ a basis in $s_X\mathbb{H}^*(X)$, by $\{a = a_{(s)}\}$ the dual basis in $\mathbb{H}_*(X)$, and by $\{\gamma\}$ and $\{c\}$ a basis and the dual basis in $s_Y\mathbb{H}^*(Y)$ and $\mathbb{H}_*(Y)$.

Conventions. 1. We assume that if both α and a enter to a formula, we have the corresponding Casimir element $\alpha|a := \sum_s \alpha_{(s)} \otimes a_{(s)}$ there. Similarly with γ and c.

2. We often ignore signs in the Hodge correlator integrals since they were carefully explained in the Section 5.2.

Given a generator $h \in \mathbb{H}_*(X)$ of the Lie algebra $\mathcal{L}ie_{\mathbb{H}_*(X)}$, one has

$$f_*\delta_X(h) = \sum_{\mathcal{C}(\alpha_0\otimes\alpha_1\otimes\alpha_2)} \int_X (\alpha_0 \wedge \alpha_1 \wedge \alpha_2)\langle a_2 \cap \mathcal{H}_X \cap h\rangle_X \cdot f_*a_0 \otimes f_*a_1.$$

$$\delta_Y(f_*h) = \sum_{\mathcal{C}(\gamma_0\otimes\gamma_1\otimes\gamma_2)} \int_Y (\gamma_0 \wedge \gamma_1 \wedge \gamma_2)\langle c_2 \cap \mathcal{H}_Y \cap f_*h\rangle_Y \cdot c_0 \otimes c_1.$$

Here and below the sum is over bases $\mathcal{C}(\alpha_0 \otimes \alpha_1 \otimes \alpha_2)$, $\mathcal{C}(\gamma_0 \otimes \gamma_1 \otimes \gamma_2)$. One has

$$\delta_Y(f_*h) \overset{(101)}{=} \sum \int_Y (\gamma_0 \wedge \gamma_1 \wedge \gamma_2)\langle f^! c_2 \cap \mathcal{H}_X \cap h\rangle_X \cdot c_0 \otimes c_1.$$

Applying the Casimir elements identity $\gamma_2 | f^! c_2 = f_! \alpha_2 | a_2$, see (103), we write this as

$$\sum \int_Y (\gamma_0 \wedge \gamma_1 \wedge f_!\alpha_2)\langle a_2 \cap \mathcal{H}_X \cap h\rangle_X \cdot c_0 \otimes c_1 \overset{(102)}{=}$$

$$\sum \int_X (f^*(\gamma_0 \wedge \gamma_1) \wedge \alpha_2)\langle a_2 \cap \mathcal{H}_X \cap h\rangle_X \cdot c_0 \otimes c_1 \overset{(103)}{=}$$

$$\sum \int_X (\alpha_0 \wedge \alpha_1 \wedge \alpha_2)\langle a_2 \cap \mathcal{H}_X \cap h\rangle_X \cdot f_*a_0 \otimes f_*a_1 = f_*\delta_X(h).$$

The lemma is proved.

Let us choose direct sum decompositions provided by the map $f^* : H^*(Y) \longrightarrow H^*(X)$:

$$H^*(Y) = \mathrm{Ker} f^* \oplus \widetilde{\mathrm{CoIm}} f^*, \qquad H^*(X) = \mathrm{Im} f^* \oplus \widetilde{\mathrm{CoKer}} f^*. \qquad f^* : \widetilde{\mathrm{CoIm}} f^* \xrightarrow{\sim} \mathrm{Im} f^*.$$

Using Theorem 4.1 we reduce the functoriality proof to the case when the splittings s_X and s_Y of the Dolbeaut complexes of X and Y are compatible in the following sense:

$$f^*\Big(s_Y \widetilde{\mathrm{CoIm}} f^*\Big) = s_X \mathrm{Im} f^*.$$

We will assume this below.

Denote by $\{\gamma''_{(j)}\}$ (respectively $\{\gamma'_{(i)}\}$) a basis in $s_Y\mathrm{Ker} f^*$ (respectively $s_Y\mathrm{CoIm} f^*$), and by $\{c''_{(j)}\}$ (respectively $\{c'_{(i)}\}$) the dual bases in the homology. In particular, $\{c'_{(i)}\}$ is a basis in $f_*\mathbb{H}_*(X)$. The basis $\{\gamma'_{(i)}\}$ gives rise to a basis $\{\alpha'_{(i)} := f^*\gamma'_{(i)}\}$ in $s_X\mathrm{Im} f^*$.

The induced by f maps

$$f_1 : X \times X \longrightarrow Y \times X, \qquad f_2 : Y \times X \longrightarrow Y \times Y,$$

provide the following two currents on $Y \times X$:

(i) the push forward $f_{1*}G_X(y_1, x_2)$ of the Green current $G_X(x_1, x_2)$;

(ii) the restriction $f_2^! G_Y(y_1, x_2)$ of the Green current $G_Y(y_1, y_2)$.

To check that $f_2^! G_Y$ is defined, decompose f into a composition of a closed embedding and a projection $X \times \mathbb{P}^n \to X$.

Denote by p_Y and p_X the projections from $Y \times X$ onto Y and X. The following Lemma is another incarnation of identity (103).

Lemma 5.16 *$f_{1*}G_X - f_2^! G_Y$ is a sum of decomposable forms on $Y \times X$:*

$$f_{1*}G_X - f_2^! G_Y \;=\; \sum_j p_Y^* \gamma''^{\vee}_{(j)} \wedge p_X^* \varphi_{(j)} - \sum_k p_Y^* \psi_{(k)} \wedge p_X^* \alpha''_{(k)} + \text{a closed form}. \qquad (104)$$

Remark. There are three terms in this formula. The first two are of the form

$$\text{(a harmonic form)} \times \text{(an arbitrary form)} + \text{(an arbitrary form)} \times \text{(a harmonic form)}. \quad (105)$$

Proof. Let Δ_f be the graph of the map $f : X \to Y$, and Har_D is a harmonic representative of the cohomology class of a cycle $D \subset X \times Y$. The currents (i), (ii) satisfy differential equations

$$(4\pi i)^{-1} dd^{\mathbb{C}} G' = \delta_{\Delta_f} - f_{1*}\mathrm{Har}_{\Delta_X}, \qquad (4\pi i)^{-1} dd^{\mathbb{C}} G'' = \delta_{\Delta_f} - f_2^{*}\mathrm{Har}_{\Delta_Y}.$$

One has $\mathrm{Ker} f_! = (\mathrm{Im} f^*)^{\perp}$, where the orthogonal is for the form on the cohomology. Recall the dual basis $\{\alpha^{\vee}_{(k)}\}$. So $\{\alpha''^{\vee}_{(k)}\}$ is a basis in $\mathrm{Ker} f_!$. By the $dd^{\mathbb{C}}$-lemma there exists a form $\psi_{(k)}$ such that

$$f_*\alpha''^{\vee}_{(k)} = (4\pi i)^{-1} dd^{\mathbb{C}} \psi_{(k)}. \quad (106)$$

Since the cohomology class $f^*[\gamma''_{(j)}] = 0$, by the $dd^{\mathbb{C}}$-lemma there exists a form $\varphi_{(j)}$ such that

$$f^*\gamma''_{(j)} = (4\pi i)^{-1} dd^{\mathbb{C}} \varphi_{(j)}. \quad (107)$$

Thus

$$f_2^{!}\mathrm{Har}_{\Delta_Y} - f_{1*}\mathrm{Har}_{\Delta_X} = (4\pi i)^{-1}\Big(\sum_j p_Y^*\gamma''^{\vee}_{(j)} \wedge p_X^* dd^{\mathbb{C}}\varphi_{(j)} - \sum_k p_Y^* dd^{\mathbb{C}}\psi_{(k)} \wedge p_X^*\alpha''_{(k)}\Big).$$

The lemma follows by the $dd^{\mathbb{C}}$-lemma.

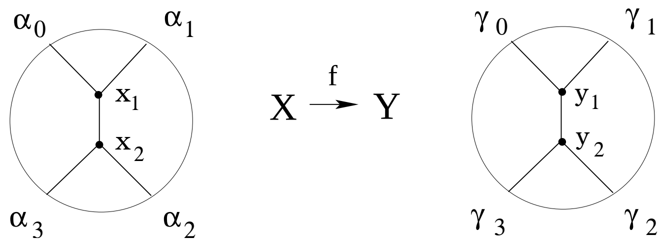


Figure 10: Proof of functoriality: the running example.

Let us denote by T the tensor algebra of the vector space spanned by

$$\gamma_{(s)}|c_{(s)} \quad \text{and} \quad \delta(f_* d^{\mathbb{C}}\varphi_{(j)}|c''_{(j)}).$$

We denote by T_k (respectively $\mathrm{T}_{\geq k}$) its component of the tensor degree k (respectively $\geq k$).
We denote by T^0 the subalgebra of T spanned by $\gamma_{(s)}|c_{(s)}$. Recall (53) the element Id_Y.

Theorem 5.17 *Let* $\{A\}$ *be a basis in* T_2*, and* $\{B\}$ *a basis in* $\mathrm{T}_{\geq 3}$*. Then*

$$f_*\delta_X(h) - \delta_X(f_*h) =$$

$$\delta \sum_m \mathrm{Cor}_{\mathcal{H},Y}\Big(\mathrm{Id}_Y \otimes d^{\mathbb{C}}\varphi_i|c''_i \otimes \mathrm{Id}_Y \otimes \delta(d^{\mathbb{C}}\varphi_i|c''_i) \otimes \ldots \otimes \mathrm{Id}_Y \otimes \delta(d^{\mathbb{C}}\varphi_i|c''_i) \otimes \gamma_m\Big)\langle f^! c_m \cap \mathcal{H}_X \cap h\rangle \quad (108)$$

$$+\,\delta \sum_{k;\ A,B} \mathrm{Cor}_{\mathcal{H},Y}\Big(A \otimes \psi_{(k)} + B \otimes d^{\mathbb{C}}\psi_{(k)}\Big)\langle a''^{\vee}_{(k)} \cap \mathcal{H}_X \cap h\rangle. \quad (109)$$

Proof. We call the external edge of a tree T decorated by the form α_m *the special leg*, and show it by a thick edge on pictures. Its vertices are called *special vertices*. Every edge of T is oriented "towards the special edge". Given a vertex v of T, there is a unique edge $\overrightarrow{E_v}$ incident to v and closest to the special edge.

One has

$$f_*\delta_X(h) = \sum_m \mathrm{Cycle}_{m+1} \sum_W \frac{1}{|\mathrm{Aut}W|} \mathrm{Cor}_{\mathcal{H},X}(\alpha_0|a_0 \otimes \ldots \otimes \alpha_{m-1}|a_{m-1} \otimes \alpha_m)\langle a_m \cap \mathcal{H}_X \cap h\rangle. \quad (110)$$

We are going to transform the sum of integrals (110) over products of copies of X into a similar sum of integrals over products of copies of Y, getting $\delta_X(f_*h) + (109) + (108)$.

We use the tree on Fig. 10 decorated by a cyclic word $\mathcal{C}(\alpha_0 \otimes \ldots \otimes \alpha_3)$ as a running example.

Step 1 – replacing $\alpha_i|a_i$ by $f^*\gamma'_i|c'_i$ for non-special external vertices of T. Taking the sum over a basis $\mathcal{C}(\alpha_0 \otimes \ldots \otimes \alpha_3)$, we get

$$\sum \int_{X\times X} \Big(\alpha_0 \wedge \alpha_1 \wedge G_X(x_1, x_2) \wedge \alpha_2 \wedge \alpha_3\Big)\langle a_3 \cap \mathcal{H}_X \cap h\rangle \cdot f_*a_0 \otimes f_*a_1 \otimes f_*a_2 =$$

$$\sum \int_{X\times X} \Big(f^*\gamma'_0 \wedge f^*\gamma'_1 \wedge G_X(x_1, x_2) \wedge f^*\gamma'_2 \wedge \alpha_3\Big)\langle a_3 \cap \mathcal{H}_X \cap h\rangle \cdot c'_0 \otimes c'_1 \otimes c'_2.$$

The first sum is over a'_i only, since a''_i is in $\mathrm{Ker}f_*$. Since $\alpha'_i = f^*\gamma'_i$, we get the second sum.

The general case is completely similar.

Step 2 – replacing the Casimir factor $f^*\gamma'_i|c'_i$ by the one $f^*\gamma_i|c_i$. By (107)

$$f^*\gamma_i|c_i = f^*\gamma'_i|c'_i + (4\pi i)^{-1}dd^{\mathbb{C}}\varphi_i|c''_i, \qquad dd^{\mathbb{C}}\varphi_i|c''_i := \sum_s dd^{\mathbb{C}}\varphi_{i,(s)} \otimes c''_{i,(s)}.$$

So we have to subtract Casimir factors $(4\pi i)^{-1}dd^{\mathbb{C}}\varphi_i|c''_i$. If we have at least two such factors in the integral, we get zero after the integration thanks to Lemma 5.6.

Step 3 – Using the projection formula. Since we can have at most one factor like $dd^{\mathbb{C}}\varphi_i|c''_i$, we can take any non-special external vertex of the tree T^0, and use the projection formula for the integral related to this vertex. In our running example such a vertex is x_1, and we get

$$\stackrel{1}{\sim} \sum \int_{Y\times X} \Big(\gamma_0 \wedge \gamma_1 \wedge f_{1*}G_X(y_1, x_2) \wedge f^*\gamma_2 \wedge \alpha_3\Big)\langle a_3 \cap \mathcal{H}_X \cap h\rangle \cdot c_0 \otimes c_1 \otimes c_2. \quad (111)$$

Here we may have $-(4\pi i)^{-1}dd^{\mathbb{C}}f_*\varphi_i|c''_i$ instead of $\gamma_i|c_i$, or $-(4\pi i)^{-1}dd^{\mathbb{C}}\varphi_i|c''_i$ instead of $f^*\gamma_i|c_i$. Hence put $\stackrel{1}{\sim}$ instead of $=$ in front of the formula. Below we transform formula (111), keeping in mind the terms with φ_i's.

We perform this procedure for all non-special external vertices of T^0.

Step 4 – replacing $f_{1*}G_X$ by $f_2^!G_Y$. In our running example, using Lemma 5.16, we replace $f_{1*}G_X(y_1, x_2)$ by $f_2^!G_Y(y_1, x_2)$ plus the terms in (104). Observe the following:

(a) The closed form does not change the Hodge correlator class – see Step 7.

(b)The term $\sum_k p_Y^*\psi_{(k)} \wedge p_X^*\alpha''_{(k)}$ gives zero after the integration, since

$$\int_X \alpha''_{(k)} \wedge f^*\gamma'_2 \wedge \alpha_3 = \int_Y f_*(\alpha''_{(k)}) \wedge \gamma'_2 \wedge f_*(\alpha_3) = 0.$$

(c) We may get a non-zero contribution of the term $\sum_j p_Y^*\gamma''^{\vee}_{(j)} \wedge p_X^*\varphi_{(j)}$.
So we get

$$\overset{2}{\sim} \sum \int_{Y\times X} \Big(\gamma_0 \wedge \gamma_1 \wedge f_2^! G_Y(y_1, x_2) \wedge f^*\gamma_2 \wedge \alpha_3\Big) \langle a_3 \cap \mathcal{H}_X \cap h\rangle \cdot c_0 \otimes c_1 \otimes c_2. \qquad (112)$$

Here $\overset{2}{\sim}\sum$ means that we keep in mind the new terms with $\varphi_{(j)}$ coming from (104) – see (c).

In general we perform this procedure at the edge $\overrightarrow{E_v}$ for every non-special external vertex v of T^0. The term in (105) with a harmonic form assigned to the other vertex then v gives zero contribution. In the running example this is the case (b) above. Otherwise this follows from Lemma 5.3. The closed form (case (a)) also contributes zero – see Step 7.

Step 5 – using the projection formula. The current $f_2^! G_Y(y_1, x_2)$ is given by the form $f_2^* G_Y(y_1, x_2)$ at the generic point. We can use projection formula (102). In the running example, using it for the vertex x_2 we get

$$(112) = \sum \int_{Y\times Y} \Big(\gamma_0 \wedge \gamma_1 \wedge G_Y(y_1, y_2) \wedge \gamma_2 \wedge f_{2*}\alpha_3\Big) \langle a_3 \cap \mathcal{H}_X \cap h\rangle \cdot c_0 \otimes c_1 \otimes c_2. \qquad (113)$$

Here one of the forms γ_i can be replaced by the correction term $-(4\pi i)^{-1} f_* dd^{\mathbb{C}}\varphi_i$ from Step 2.
In general we do this for every external edge E of T^0.

Step 6 – Going down the tree. We move down the tree towards the special edge:
(a) By using the projection formula argument at a vertex v (like in Step 5), and

(b) By replacing $f_{1*}G_X$ by $f_2^! G_Y$ at the corresponding edge $\overrightarrow{E_v}$ (like in Step 4). See Fig. 11, where we use the projection formula for the copy of X assigned to the vertex v_3 as follows:

$$\int_{X_3} \omega\Big(f_2^! G_Y(y_1, x_3) \wedge f_2^! G_Y(y_2, x_3) \wedge G(x_3, x_4)\Big) = \int_{Y_3} \omega\Big(G_Y(y_1, y_3) \wedge G_Y(y_2, y_3) \wedge f_{1*}G_X(y_3, x_4)\Big).$$

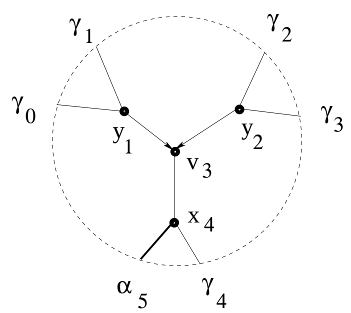


Figure 11: Using the projection formula at the vertex v_3.

Notice that if $\overrightarrow{E_v}$ is not an external edge of T^0, the latter procedure does not introduce extra terms. This follows easily from Lemmas 5.2 and 5.3.

Step 7 – independence of the closed form in (104). Adding to $f_2^! G_Y(y_1, x_2)$ a closed form does not change the Hodge correlator class. Indeed, one easily sees by using the projection formula that it results to adding a closed form to a Green current G_Y. By Theorem 4.1 this does not change the Hodge correlator class.

Step 8. The previous steps end up with an expression

$$\sim \sum \mathrm{Cor}_{\mathcal{H},Y}(\gamma_0|c_0 \otimes \ldots \otimes \gamma_{m-1}|c_{m-1} \otimes f_*\alpha_m)\langle a_m \cap \mathcal{H}_X \cap h\rangle. \tag{114}$$

Similarly to the proof of Lemma 5.15, we write it as

$$\overset{3}{\sim} \sum \mathrm{Cor}_{\mathcal{H},Y}(\gamma_0|c_0 \otimes \ldots \otimes \gamma_{m-1}|c_{m-1} \otimes \gamma_m)\langle f^! c_m \cap \mathcal{H}_X \cap h\rangle. \tag{115}$$

Here $\overset{3}{\sim}$ means equality modulo the extra terms provided by $f_*\alpha''^{\vee}_{(k)} \otimes a''^{\vee}_{(k)}$ and (106).
By (101) we have $\langle f^! c_m \cap \mathcal{H}_X \cap h\rangle_X = \langle c_m \cap \mathcal{H}_X \cap f_* h\rangle_Y$. Thus the sum in (115) equals $\delta_Y f_*(h)$.

It remains to take care of the extra terms, and show that their sum is a coboundary. We can get them from the following sources, marked $\overset{i}{\sim}$ in the proof:

(1) At most one term $-(4\pi i)^{-1} dd^{\mathbb{C}} f_*\varphi_i|c_i''$, see Step 2, for an external non-special edge of T.
(2) Extra terms $\sum_j \gamma''^{\vee}_{(j)} \wedge d^{\mathbb{C}} f_*\varphi_{(j)}$ from Step 4(c) for non-special external vertices of T^0.
(3) The extra term for the special external vertex of T^0 at Step (8), provided by $f_*\alpha''^{\vee}_{(k)} \otimes a''^{\vee}_{(k)}$.

Step 9 – Calculating the extra terms. By Lemma 5.6, terms with more then one factor $f_* dd^{\mathbb{C}}\eta_i$ are zero. Thus in the case (3) there are no extra terms (1).

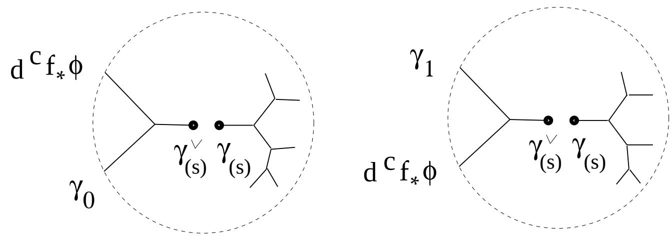


Figure 12: The sum of these contributions is zero.

(1). Moving $-(4\pi i)^{-1} d$ (or $-(4\pi i)^{-1} dd^{\mathbb{C}}$ if the tree T has a single internal edge) from $-(4\pi i)^{-1} dd^{\mathbb{C}} f_*\varphi_i|c_i''$ to the operator ω applied to the Green currents of the internal edges of the tree T, and using the formula for $d\omega$ (respectively $dd^{\mathbb{C}} G$), we conclude that, by Lemma 5.3, only the Casimir terms for the external edges of T^0 can give non-zero contribution. Denote by T''_E the small tree obtained by cutting the external edge E of T^0. Then there are two cases:

(i) None of the edges of the tree T''_E is decorated by $f_* d^{\mathbb{C}}\varphi$ (respectively $f_*\varphi$). Then by Lemma 5.8 the corresponding contribution is $\delta(\gamma_i|c_i'')$.

(ii) One of the edges of the tree T''_E is decorated by $f_* d^{\mathbb{C}}\varphi$ (respectively $f_*\varphi$), see Fig. 12. Then the contribution is zero. Indeed, there are two possible decorations, see Fig. 12. Their contribution is the simplest shuffle relation:

$$\mathrm{Cor}_{\mathcal{H}}(\gamma_1|c_1 \otimes f_* d^{\mathbb{C}}|c'' \otimes \gamma_{(s)}^{\vee}) + \mathrm{Cor}_{\mathcal{H}}(f_* d^{\mathbb{C}}|c'' \otimes \gamma_1|c_1 \otimes \gamma_{(s)}^{\vee}) = 0.$$

(2) The contribution of a term (2) is $\delta(d^{\mathbb{C}} f_* \varphi_i|c_i'')$. This is checked using Lemma 5.8. Taking into account the case (1)(i), we arrive at the term (108).

(3) Let us handle first the running example. The extra term is

$$\sum_k \sum \int_{Y\times Y} \Big(\gamma_0 \wedge \gamma_1 \wedge G_Y(y_1, y_2) \wedge \gamma_2 \wedge (4\pi i)^{-1} dd^{\mathbb{C}} \psi_{(k)}\Big) \langle a''^{\vee}_{(k)} \cap \mathcal{H}_X \cap h\rangle \cdot c_0 \otimes c_1 \otimes c_2. \quad (116)$$

Moving $(4\pi i)^{-1} dd^{\mathbb{C}}$ to $G_Y(y_1, y_2)$, and using the defining equation for $(4\pi i)^{-1} dd^{\mathbb{C}} G_Y$ we observe the following: The δ-term vanishes after the summation over the (two) trivalent trees. The volume term vanishes. The Casimir term $\sum_s \gamma_{(s)}^{\vee} \otimes \gamma_{(s)}$ leads to an equality

$$(116) = \sum_{s,k} \sum \int_Y \Big(\gamma_0 \wedge \gamma_1 \wedge \gamma_{(s)}^{\vee}\Big) \cdot \int_Y \Big(\gamma_{(s)} \wedge \gamma_2 \wedge \psi_{(k)}\Big) \langle a''^{\vee}_{(k)} \cap \mathcal{H}_X \cap h\rangle \cdot c_0 \otimes c_1 \otimes c_2 =$$

$$\sum_{s,k} \sum_{\gamma_2|c_2} \mathrm{Cor}_{\mathcal{H}}\Big((-1)^{|\gamma_{(s)}|} \gamma_{(s)}|\delta h_{\gamma_{(s)}} \otimes \gamma_2|c_2 \otimes \psi_{(k)}\Big) \langle a''^{\vee}_{(k)} \cap \mathcal{H}_X \cap h\rangle.$$

It is illustrated by the diagram on the left on Fig 13. Adding a similar contribution of the right

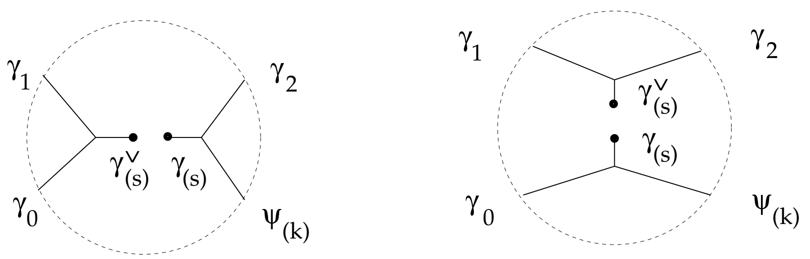


Figure 13: Correlator integrals for the extra term (3) in the running example.

diagram on Fig 13 we conclude that the total contribution is

$$\delta\Big(\sum_{s,k} \sum \mathrm{Cor}_{\mathcal{H}}\Big(\gamma_0|c_0 \otimes \gamma_1|c_1 \otimes \psi_{(k)}\Big) \langle a''^{\vee}_{(k)} \cap \mathcal{H}_X \cap h\rangle\Big).$$

In general, if the tree T^0 has more then one edge, the extra term (3) for integral (114) is

$$\delta\Big(\sum_{s,k} \sum \mathrm{Cor}_{\mathcal{H}}\Big(\gamma_0|c_0 \otimes \ldots \otimes \gamma_{m-1}|c_{m-1} \otimes d^{\mathbb{C}} \psi_{(k)}\Big) \langle a''^{\vee}_{(k)} \cap \mathcal{H}_X \cap h\rangle\Big).$$

The difference with the case when the tree T^0 has one edge is that we have now $d^{\mathbb{C}}\psi_{(k)}$ instead of $\psi_{(k)}$. The proof is similar: we move the differential $(4\pi i)^{-1} d$ from $d^{\mathbb{C}}\psi_{(k)}$ to the operator ω applied to the Green currents assigned to the internal edges of the tree T, and use the formula for $d\omega$. The Casimir term at certain internal edge E of the tree T is the only one which may contribute. If the edge E is an internal edge of the tree T^0 its contribution is zero by Lemma 5.3. Otherwise the contribution is just as in Lemma 5.8.

Finally, for certain external edges E of T the factor $\gamma_i|c_i$ attached to E may be replaced by the one of type (2). Then the Casimir term for the external edge of T^0 incident to E provides zero contribution in the above argument. Thus we arrive at the term (109).

Theorem 5.17 is proved.

References

[AKSZ] Alexandrov M., Kontsevich M., Schwarz A., Zaboronsky O. *The Geometry of the Master Equation and Topological Quantum Field Theory.* Internat. J. Modern Phys. A 12 (1997), no. 7, 1405–1429.

[B] Beilinson A.A.: *Height pairings between algebraic cycles.* K-theory, arithmetic and geometry (Moscow, 1984–1986), 1–25, Lecture Notes in Math., 1289, Springer, Berlin, 1987.

[Ch] Chen, Kuo Tsai: *Iterated path integrals.* Bull. Amer. Math. Soc. 83 (1977), no. 5, 831–879.

[D] Deligne P.: *Hodge theory II.* Publ. IHES No. 40 (1971), 5–57;

[D2] Deligne P.: *Structures des Hodges mixte réeles.* Proc. Symp. in Pure Math, vol. 55, (1994), part 1, 509-516.

[DG] Deligne P., Goncharov, A.B. *Groupes fondamentaux motiviques de Tate mixte.* Ann. Sci. Ecole Norm. Sup. (4) 38 (2005), no. 1, 1–56. math.NT/0302267.

[DGMS] Deligne P., Griffith Ph., Morgan J., Sullivan D.: Rational homotopy theory of algebraic varieties. Invent. Math. 29 (1975), no. 3, 245–274.

[Dr] Drinfeld V.G.: *On quasitriangular quasi-Hopf algebras and a group closely connected with* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Leningrad Math. J. 2 (1991), no. 4, 829–860.

[G1] Goncharov A.B.: *Hodge correlators*, arXiv:0803.0297.

[G2] Goncharov A.B.: *Multiple ζ-values, Galois groups and geometry of modular varieties*, European Congress of Mathematics, Vol. I (Barcelona, 2000), 361–392, Progr. Math., 201, Birkhauser, Basel. arXiv:math/0005069.

[G3] Goncharov, A.B. *Galois symmetries of fundamental groupoids and noncommutative geometry.* Duke Math. J. 128 (2005), no. 2, 209–284, ArXive math.AG/0208144,

[G4] Goncharov A.B.: *Mixed elliptic motives.* Galois representations in arithmetic algebraic geometry (Durham, 1996), 147–221, London Math. Soc. Lecture Note Ser., 254, Cambridge Univ. Press, Cambridge, 1998.

[H] Hain R.: *The de Rham homotopy theory of complex algebraic varieties I*, Journal of K-Theory 1 (1987), 271–324.

[H2] Hain R.: *Iterated integrals and homotopy periods.* Mem. Amer. Math. Soc. 47 (1984), no. 291.

[HZ] Hain R, Zucker S.: *Unipotent variations of mixed Hodge structure*. Invent. Math. 88 (1987), no. 1, 83–124.

[Ka] Kapranov M.: *Real mixed Hodge structures.* Preprint 2008, arXiv:0802.0215.

[K] Kontsevich M.: *Formal (non)commutative symplectic geometry.* The Gelfand mathematical seminars, Birkhauser, 1993, p. 173-187.

[KS] Kontsevich M., Soibelman Ya.: *Homological mirror symmetry and torus fibrations.* Symplectic geometry and mirror symmetry (Seoul, 2000), 203–263, World Sci. Publ., River Edge, NJ, 2001. math/0011041.

[L] Levin A. *Variations of* $\mathbb{R}$*-Hodge-Tate structures*. Preprint MPI-2001.

[M] Morgan J.: *The algebraic topology of smooth algebraic varieties.* Inst. Hautes tudes Sci. Publ. Math. No. 48 (1978), 137–204.

[Q] Quillen D.: *Rational homotopy theory.* Ann. of Math. (2) 90 1969 205–295.

[S] Sullivan, D.: *Infinitesimal computations in topology.* Inst. Hautes Etudes Sci. Publ. Math. No. 47 (1977), 269–331 (1978).

[Wit] Witten E.: *Perturbative quantum field theory.* In: Quantum fields and strings: a course for mathematicians. Vol. 1, 2. Edited by Pierre Deligne and others. American Mathematical Society, Providence, RI; 1999. Vol. 1.