VANISHING OF ALGEBRAIC BRAUER-MANIN OBSTRUCTIONS

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ABSTRACT. Let X be a homogeneous space of a quasi-trivial k-group G, with geometric stabilizer $\overline{H}$, over a number field k. We prove that under certain conditions on the character group of $\overline{H}$, certain algebraic Brauer-Manin obstructions to the Hasse principle and weak approximation vanish, because the abelian groups where they take values vanish. When $\overline{H}$ is connected or abelian, these algebraic Brauer-Manin obstructions to the Hasse principle and weak approximation are the only ones, so we prove the Hasse principle and weak approximation for X under certain conditions. As an application, we obtain new sufficient conditions for the Hasse principle and weak approximation for linear algebraic groups.

CONTENTS

1. INTRODUCTION: SANSUC'S RESULTS

We are inspired by Sansuc's paper [Sa]. In this section we state and discuss Sansuc's results on the Hasse principle and weak approximation for principal homogeneous spaces of connected linear algebraic groups admitting special coverings.

1.1. Let k be a field of characteristic 0 and $\bar{k}$ be a fixed algebraic closure of k. If X is an algebraic variety over k, we write $\overline{X} = X \times_k \bar{k}$.

A k-torus T is called quasi-trivial if its character group $\widehat{T} := \mathrm{Hom}(\overline{T}, \mathbb{G}_{m,\bar{k}})$ is a permutation $\mathrm{Gal}(\bar{k}/k)$-module, i.e. $\widehat{T}$ has a $\mathbb{Z}$-basis which the Galois group permutes.

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A *special covering* of a (connected) reductive k-group G is a short exact sequence

$$1 \to B \to G' \to G \to 1,$$

where G' is a product of a simply connected semisimple k-group and a quasi-trivial k-torus, and B is a finite central subgroup of G'.

Not all reductive groups admit special coverings. For example, a nonsplit one-dimensional k-torus does not admit such a covering.

A finite group Γ is called *metacyclic* if all its Sylow subgroups are cyclic. Any cyclic group is metacyclic. The group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is not metacyclic. A finite Galois extension L/k is called metacyclic if $\mathrm{Gal}(L/k)$ is a metacyclic group. Clearly any cyclic extension is metacyclic.

Let k be a field of characteristic 0 and M be a $\mathrm{Gal}(\overline{k}/k)$-module. We say that a field extension $K \subset \overline{k}$ of k *splits* M if $\mathrm{Gal}(\overline{k}/K)$ acts trivially on M. If T is a k-torus, then K splits T if and only if K splits $\widehat{T}$.

Now let k be a number field. We denote by $\mathscr{V}$ the set of all places v of k, and by $\mathscr{V}_\infty$ the set of infinite (archimedean) places. For $v \in \mathscr{V}$ we denote by k_v the completion of k at v.

1.2. Let G be a reductive k-group over a number field k admitting a special covering

$$1 \to B \to G' \to G \to 1.$$

Let $\widehat{B}$ denote the character group of B, i.e. $\widehat{B} = \mathrm{Hom}(B, \mathbb{G}_{m,\overline{k}})$. Let K be the smallest subfield of $\overline{k}$ splitting $\widehat{B}$ (i.e K is the subfield corresponding to the subgroup $\ker\left[\mathrm{Gal}(\overline{k}/k) \to \mathrm{Aut}(\widehat{B})\right]$). Let X be a right principal homogeneous space of G.

Sansuc [Sa] proved the following results:

Proposition 1.3 ([Sa], Cor. 5.2)**.** *Let k, G, B, K, and X be as in 1.2. If K/k is a metacyclic extension, then X satisfies the Hasse principle and weak approximation, i.e. if $X(k_v) \neq \emptyset$ for any place v of k, then $X(k) \neq \emptyset$ and, moreover, $X(k)$ is dense in $\prod_{v\in\mathscr{V}} X(k_v)$.*

Remark 1.4. In [Sa], Sansuc always assumes that G has no factors of type E_8. This assumption can be removed by Chernousov's result [Ch].

Remark 1.5. If the extension K/k is not metacyclic (e.g. $\mathrm{Gal}(K/k) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$), then the Hasse principle or weak approximation may fail for X, see [Se, §III.4.7] and [Sa, Examples 5.6, 5.7, and 5.8].

Proposition 1.6 ([Sa], Thm. 3.3 and Lemma 1.6)**.** *Let k, G, B, K, and X be as in 1.2. Assume that X has a k-point (hence we may identify X with G). Let $S \subset \mathscr{V}$ be a finite set formed by places of k with cyclic decomposition groups in K/k (for example, assume that K/k is unramified at all finite places in S). Then X has the weak approximation property in S, i.e. the set $X(k)$ is dense in $\prod_{v\in S} X(k_v)$.*

1.7. The results of Propositions 1.3 and 1.6 can be explained in terms of the Brauer group of X. Let X be a smooth k-variety. We write $\mathrm{Br}(X)$ for the cohomological Brauer group of X, i.e $\mathrm{Br}(X) = H^2_{\text{ét}}(X, \mathbb{G}_m)$. We set $\mathrm{Br}_1(X) = \ker[\mathrm{Br}(X) \to \mathrm{Br}(\overline{X})]$. We define the *algebraic Brauer group* $\mathrm{Br_a}(X)$ by $\mathrm{Br_a}(X) = \mathrm{coker}[\mathrm{Br}(k) \to \mathrm{Br}_1(X)]$.

When k is a number field and $S \subset \mathscr{V}$ is a finite set of places of k, we set

$$Б_S(X) = \ker\left[\mathrm{Br_a}(X) \to \prod_{v\notin S} \mathrm{Br_a}(X_{k_v})\right].$$

Set $Б_\omega(X) = \bigcup_S Б_S$. We set $Б(X) := Б_\emptyset(X)$ and $Б_{S,\emptyset}(X) = Б_S(X)/Б_\emptyset(X) = Б_S(X)/Б(X)$.

Sansuc [Sa, (6.2.3)], following Manin [M], defined a natural pairing

$$\prod_{v\in\mathscr{V}} X(k_v)\times \text{Б}_\omega(X)\to\mathbb{Q}/\mathbb{Z}$$

(see also [Sk, (5.2)]), which is continuous in the first argument and is additive in the second one. This pairing induces a continuous map

$$m\colon \prod_{v\in\mathscr{V}} X(k_v)\to \mathrm{Hom}(\text{Б}_\omega(X),\mathbb{Q}/\mathbb{Z}). \tag{1}$$

If $x_0\in X(k)\subset\prod_{v\in\mathscr{V}}X(k_v)$, then $m(x_0)=0$. If m is not identically 0, say, $m(x_\mathscr{V})\neq 0$ for some $x_\mathscr{V}=(x_v)_{v\in\mathscr{V}}\in\prod_{v\in\mathscr{V}}X(k_v)$, then $x_\mathscr{V}$ is not contained in the closure of $X(k)$, hence the Hasse principle or weak approximation fail for X (i.e. either $X(k)=\emptyset$, or $X(k)\neq\emptyset$ but $X(k)$ is not dense in $\prod_{v\in\mathscr{V}}X(k_v)$). We say that m is *the algebraic Brauer-Manin obstruction to the Hasse principle and weak approximation for X associated with* Б_ω.

Assume that X is a smooth k-variety having a k-point. Let $S\subset\mathscr{V}$ be a finite set of places of k. Inspired by [CTS] and [Sa], we defined in [B3, §1] a natural pairing

$$\prod_{v\in S} X(k_v)\times \text{Б}_{S,\emptyset}(X)\to\mathbb{Q}/\mathbb{Z}.$$

This pairing induces a continuous map

$$m_S\colon \prod_{v\in S} X(k_v)\to \mathrm{Hom}(\text{Б}_{S,\emptyset}(X),\mathbb{Q}/\mathbb{Z}). \tag{2}$$

If $x_0\in X(k)\subset\prod_{v\in S}X(k_v)$, then $m_S(x_0)=0$. If m_S is not identically 0, say $m_S(x_S)\neq 0$ for some $x_S\in\prod_{v\in S}X(k_v)$, then x_S is not contained in the closure of $X(k)$, hence weak approximation in S fails for X. We say that m_S is *the algebraic Brauer-Manin obstruction to weak approximation in S for X associated with* $\text{Б}_{S,\emptyset}(X)$.

Using Sansuc's methods and results, one can show that under the assumptions of Proposition 1.3 we have $\text{Б}_\omega=0$, hence $m=0$, see Proposition 1.8(ii) below. Moreover, the Hasse principle and weak approximation hold for X because there is no Brauer-Manin obstruction. Similarly, under the assumptions of Proposition 1.6 we have $\text{Б}_{S,\emptyset}=0$, hence $m_S=0$, see Proposition 1.8(i) below. Again, weak approximation in S holds for X because there is no Brauer-Manin obstruction. We provide some details.

Proposition 1.8. *Let k, G, B, K, and X be as in 1.2. Let $S\subset\mathscr{V}$ be a finite set of places of k.*

(i) If any place $v\in S$ has a cyclic decomposition group in K/k, then $\text{Б}_{S,\emptyset}(X)=0$.

(ii) If K/k is a metacyclic extension, then $\text{Б}_\omega(X)=0$.

Proof. We use the notation of §3.1 below. By [Sa, Lemma 6.8] $\mathrm{Br_a}(X)\cong\mathrm{Br_a}(G)$, hence $\text{Б}_\omega(X)\cong\text{Б}_\omega(G)$ and $\text{Б}_{S,\emptyset}(X)\cong\text{Б}_{S,\emptyset}(G)$. By [Sa, Prop. 9.8] $\text{Б}_\omega(G)\cong Ш^1_\omega(k,\widehat{B})$ and $\text{Б}(G)\cong Ш^1(k,\widehat{B})$. One proves similarly that $\text{Б}_S(G)\cong Ш^1_S(k,\widehat{B})$, hence $\text{Б}_{S,\emptyset}(G)\cong Ш^1_{S,\emptyset}(k,\widehat{B}):=Ш^1_S(k,\widehat{B})/Ш^1_\emptyset(k,\widehat{B})$. In case (i), since S is formed by places of k with cyclic decomposition groups in K/k, by [Sa, Lemma 1.1(iii)] (see also Lemma 3.2(iii) below) $Ш^1_{S,\emptyset}(k,\widehat{B})=0$, hence $\text{Б}_{S,\emptyset}(X)=0$. In case (ii), since K/k is metacyclic, by [Sa, Lemma 1.3] (see also Lemma 3.4 below) $Ш^1_\omega(k,\widehat{B})=0$, hence $\text{Б}_\omega(X)=0$. □

An alternative proof of Proposition 1.3. By Proposition 1.8(ii) the Brauer-Manin obstruction m of formula (1) is identically zero in our case, i.e there is no algebraic Brauer-Manin obstruction to the Hasse principle and weak approximation associated with Б_ω. Since by [Sa, Cor. 8.7 and Cor. 8.13] this obstruction is the only one (see Remark 1.4), we conclude that the Hasse principle and weak approximation hold for X. □

An alternative proof of Proposition 1.6. By Proposition 1.8(i) the Brauer-Manin obstruction m_S of formula (2) is identically zero in our case, i.e. there is no algebraic Brauer-Manin obstruction to weak approximation in S associated with $Б_{S,\emptyset}(X)$. Since by [Sa, Cor. 8.13] this obstruction is the only one, we conclude that weak approximation in S holds for X. □

2. Introduction (continued): our main results

In this section we state our generalizations of Sansuc's results in two cases: homogeneous spaces of quasi-trivial groups and principal homogeneous spaces of connected linear algebraic groups. Our main results are Theorems 2.1 and 2.7, generalizing Proposition 1.8 and proving that the groups $Б_{S,\emptyset}(X)$ and $Б_\omega(X)$ vanish under certain conditions on X.

In order to state our results we use the notion of a quasi-trivial group, introduced by Colliot-Thélène [CT, Definition 2.1], see also Definition 4.2 below.

Let X be a right homogeneous space of a quasi-trivial k-group G over a number field k. Let $\overline{H} \subset \overline{G}$ be the stabilizer of a $\bar{k}$-point $\bar{x} \in X(\bar{k})$ (we do not assume that $\overline{H}$ is connected or abelian). It is well known that the character group $\widehat{\overline{H}}$ of $\overline{H}$ has a canonical structure of a Galois module, see [B3, 4.1] or [BvH2, Rem. 5.7(1)], see also §8.1 below.

Theorem 2.1. *Let X be a right homogeneous space of a quasi-trivial k-group G over a number field k. Let $\overline{H} \subset \overline{G}$ be the stabilizer of a $\bar{k}$-point $\bar{x} \in X(\bar{k})$. Let $S \subset \mathscr{V}$ be a finite subset. Let K/k be the smallest Galois extension in $\bar{k}$ splitting the Galois module $\widehat{\overline{H}}$.*

(i) If any place $v \in S$ has a cyclic decomposition group in K/k, then $Б_{S,\emptyset}(X) = 0$.

(ii) If K/k is a metacyclic extension, then $Б_\omega(X) = 0$.

Theorem 2.1 will be proved in Section 5.

Corollary 2.2. *Let X be as in Theorem 2.1. Assume that X has a k-point, i.e. $X = H\backslash G$, where G is a quasi-trivial k-group and $H \subset G$ is a k-subgroup. Assume that H is connected or abelian. Let $S \subset \mathscr{V}$ be a finite set of places of k formed by places with cyclic decomposition groups in K/k (for example assume that K/k is unramified at all finite places in S). Then X has weak approximation in S.*

Proof. Since H is connected or abelian, the algebraic Brauer-Manin obstruction m_S associated with $Б_{S,\emptyset}$ is the only obstruction to weak approximation in S for X, see [B3, Thm. 2.3]. By Theorem 2.1(i) $m_S = 0$, hence X has weak approximation in S. □

Corollary 2.3. *Let X be as in Corollary 2.2, i.e. $X = H\backslash G$, where G is a quasi-trivial k-group and $H \subset G$ is a k-subgroup. Assume that H is connected or abelian. Then X has the real approximation property, i.e $X(k)$ is dense in $\prod_{v\in\mathscr{V}_\infty} X(k_v)$.*

Proof. For any $v \in \mathscr{V}_\infty$ the decomposition group of v in K/k is cyclic, and by Corollary 2.2 X has weak approximation in $\mathscr{V}_\infty$, i.e. real approximation. □

Question 2.4. Does there exist a homogeneous space $X = H\backslash G$, where G is a quasi-trivial k-group over a number field k, and $H \subset G$ is a nonconnected non-abelian k-subgroup, such that real approximation fails for X?

Corollary 2.5. *Let X be a homogeneous space having a k-rational point with connected stabilizer, of a connected linear algebraic group (not necessarily quasi-trivial) over a number field k; in other words, $X = H\backslash G$, where G is a connected k-group and $H \subset G$ is a connected k-subgroup. Then X has the real approximation property.*

Proof. By Lemma 4.3 below we can write $X = H'\backslash G'$, where G' is a quasi-trivial k-group and $H' \subset G'$ is a connected k-subgroup. Now by Corollary 2.3 X has real approximation. □

Corollary 2.6. *Let X be as in Theorem 2.1. Assume that $\overline{H}$ is connected or abelian. Assume that K/k is a metacyclic extension. Then X satisfies the Hasse principle and weak approximation.*

Proof. Since $\overline{H}$ is connected or abelian, the algebraic Brauer-Manin obstruction m associated with $Б_\omega$ is the only obstruction to the Hasse principle and weak approximation, see [B3, Thms. 2.2 and 2.3]. By Theorem 2.1(ii) we have $m = 0$, hence X satisfies the Hasse principle and weak approximation. □

Note that Propositions 1.3 and 1.6 (due to Sansuc) follow immediately from our Corollaries 2.6 and 2.2, respectively. Note also that the special case of Corollary 2.2 when G is simply connected and H is connected was earlier proved in [B1, Cor. 1.6] by a different method. The special case of Corollary 2.6 when $\overline{H}$ is connected and X has a k-point was proved in [B5, Thm. 4.2] by the method of [B1].

In order to state our results on principal homogeneous spaces of connected k-groups, we use the notion of the *algebraic fundamental group* $\pi_1(G)$ introduced in [B4, §1] (where we wrote $\pi_1(\overline{G})$ instead of $\pi_1(G)$), see also [CT, §6]. Note that $\pi_1(G)$ is a finitely generated Galois module.

Theorem 2.7. *Let G be a connected linear k-group over a number field k. Let X be a right principal homogeneous space (right torsor) of G over k. Let $S \subset \mathscr{V}$ be a finite set of places of k. Let K/k be the smallest Galois extension in $\bar{k}$ splitting the Galois module $\pi_1(G)$.*

(i) If any place $v \in S$ has a cyclic decomposition group in K/k, then $Б_{S,\emptyset}(X) = 0$.

(ii) If K/k is a metacyclic extension, then $Б_\omega(X) = 0$.

Theorem 2.7 will be proved in Section 6.

Proposition 2.8. *Let G be a connected linear algebraic group over a field k of characteristic 0. Let K/k be the smallest Galois extension in $\bar{k}$ splitting $\pi_1(G)$. Then there exists an exact sequence*

$$1 \to H \to G' \to G \to 1,$$

where G' is a quasi-trivial k-group and H is a central k-subgroup of multiplicative type, such that K splits both $\widehat{H}$ and $\widehat{G'}$.

Proposition 2.8 will be proved in Section 7. In Section 8 we shall give an alternative proof of Theorem 2.7 based on Proposition 2.8 and Theorem 2.1.

Corollary 2.9. *Let k, G, and K be as in Theorem 2.7. Let $S \subset \mathscr{V}$ be a finite set of places of k formed by places with cyclic decomposition groups in K/k (for example assume that K/k is unramified at all finite places in S). Then G has weak approximation in S.*

Proof. Under our assumptions the algebraic Brauer-Manin obstruction m_S associated with $Б_{S,\emptyset}(G)$ is the only obstruction to weak approximation in S for G, see [Sa, Cor. 8.13]. By Theorem 2.7(i) $m_S = 0$, hence G has weak approximation in S. Alternatively, the corollary follows from Proposition 2.8 and Corollary 2.2. □

Corollary 2.10. *Let k, G, X, and K be as in Theorem 2.7. Assume that K/k is a metacyclic extension. Then X satisfies the Hasse principle and weak approximation.*

Proof. Under our assumptions the algebraic Brauer-Manin obstruction m associated with $Б_\omega$ is the only obstruction to the Hasse principle and weak approximation for X, see [Sa, Cor. 8.7 and Cor. 8.13] (see Remark 1.4). By Theorem 2.7(ii) we have $m = 0$, hence X satisfies the Hasse principle and weak approximation. Alternatively, the corollary follows from Proposition 2.8 and Corollary 2.6 (because K splits $\widehat{\overline{H}}$, where $\overline{H}$ is the stabilizer of $\bar{x}$ in $\overline{G'}$, see §8.2 below). □

Remark 2.11. Sansuc proved in [Sa, Cor. 3.5(iii)] that any connected k-group over a number field k has the real approximation property. This follows from our Corollary 2.9 (because infinite places have cyclic decomposition groups in $\mathrm{Gal}(K/k)$) and from our Corollary 2.5 (because we may write $G = \{1\}\backslash G$, and $\{1\}$ is a connected k-subgroup).

Note that Sansuc proved the following result:

Proposition 2.12 (Sansuc)**.** *Let G be a connected linear k-group over a number field k. Let $S \subset \mathscr{V}$ be a finite subset. Assume that G splits over a finite Galois extension K/k.*

(i) If any place $v \in S$ has a cyclic decomposition group in K/k, then G has weak approximation in S (cf. [Sa, Cor. 3.5(ii)]*).*

(ii) If K/k is a metacyclic extension, then $Б_\omega(X) = 0$ (cf. [Sa, Prop. 9.8]*), hence any principal homogeneous space X of G over k satisfies the Hasse principle and weak approximation.*

Proposition 2.12 follows from our Theorem 2.7: if a finite Galois extension K/k splits G, then it splits $\pi_1(G)$. The following example shows that Theorem 2.7 is indeed stronger than Proposition 2.12.

Example 2.13. Let k be a number field, and let K_1 and K_2 be two different quadratic extensions of k in $\bar{k}$. Let K be the composite of K_1 and K_2, then K/k is a Galois extension with non-metacyclic Galois group $\mathrm{Gal}(K/k) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Set $G_1 = \mathrm{SU}_{2,K_1}$, it is a k-group. Set

$$G_2 = R^1_{K_2/k}\mathbb{G}_m := \ker[N_{K_2/k}\colon R_{K_2/k}\mathbb{G}_{m,K_2} \to \mathbb{G}_{m,k}],$$

where $N_{K_2/k}$ is the norm map. Set $\mu = \{(-1,-1),(1,1)\} \subset G_1 \times G_2$, and set $G = (G_1 \times G_2)/\mu$.

Clearly G does not admit a special covering. Let L/k be any finite Galois extension in $\bar{k}$ splitting G. Then L splits both G_1 and G_2, hence $L \supset K$, and therefore L/k is not metacyclic. We see that we cannot prove the Hasse principle and weak approximation for a principal homogeneous space X of G using Proposition 1.3 or Proposition 2.12.

However, the quadratic extension K_2/k splits $\pi_1(G)$. Indeed, consider the composed homomorphism $G_1 \hookrightarrow G_1 \times G_2 \to G$, it is injective. We obtain a short exact sequence of connected reductive k-groups

$$1 \to G_1 \to G \to G_2/\mu_2 \to 1,$$

where $\mu_2 = \{1,-1\} \subset G_2$. Since $\pi_1(G_1) = 0$, we see that $\pi_1(G) \cong \pi_1(G_2/\mu_2)$, cf. [B4, Lemma 1.5] or [CT, Prop. 6.8]. The quadratic extension K_2/k splits the one-dimensional torus G_2/μ_2, hence K_2/k splits $\pi_1(G_2/\mu_2)$ and $\pi_1(G)$.

Now by Theorem 2.7(ii) $Б_\omega(X) = 0$, and by Corollary 2.10 X satisfies the Hasse principle and weak approximation.

The plan of the rest of this paper is as follows. In Sections 3 we give preliminaries on Galois cohomology of finitely generated Galois modules. In Section 4 we give preliminaries on quasi-trivial groups. In Sections 5 and 6 we prove Theorems 2.1 and 2.7, respectively. In Section 7 we prove Proposition 2.8, and in Section 8 we use this proposition in order to give an alternative proof of Theorem 2.7. Our proofs are based on the results of Section 3 and of our papers [BvH1] and [BvH2].

3. Preliminaries on Galois cohomology of finitely generated Galois modules

3.1. In this section k denotes a number field, and B is a discrete $\mathrm{Gal}(\bar{k}/k)$-module which is finitely generated as an abelian group (we say just "a finitely generated Galois module"). By S we always denote a finite subset of $\mathscr{V}$. We write

$$Ш^i_S(k,B) = \ker\left[H^i(k,B) \to \prod_{v\notin S} H^i(k_v,B)\right].$$

We have $Ш^i_\emptyset(k,B) = Ш^i(k,B)$. We set $Ш^i_{S,\emptyset}(k,B) := Ш^i_S(k,B)/ Ш^i_\emptyset(k,B)$ and $Ш^i_\omega(k,B) = \bigcup_S Ш^i_S(k,B)$.

Lemmas 3.2 and 3.4 below are straightforward generalizations of [Sa, Lemmas 1.1 and 1.3] (Sansuc assumes that B is a *finite* Galois module).

Lemma 3.2. *Let K/k be a a finite Galois extension with Galois group $\mathfrak{g}$ and S be a finite set of places of k.*

(i) If B is a constant $\mathrm{Gal}(\overline{k}/k)$*-module (i.e.* $\mathrm{Gal}(\overline{k}/k)$ *acts trivially), then* $Ш^1_S(K/k,B) = 0$.

(ii) If the extension K/k trivializes B, there is a reduction

$$Ш^1_S(k,B) = Ш^1_S(K/k,B).$$

(iii) If S' is a finite subset of $\mathscr{V}$ formed of places with cyclic decomposition groups in K/k, then

$$Ш^1_{S\cup S'}(K/k,B) = Ш^1_S(K/k,B).$$

□

Consider the homomorphism $\mathrm{Gal}(\overline{k}/k) \to \mathrm{Aut}(B)$. The image $\mathfrak{g}$ of this homomorphism is finite. Let K denote the subfield in $\overline{k}$ corresponding to the kernel of this homomorphism, then K/k is a finite Galois extension with Galois group $\mathfrak{g}$. We say that K/k be the smallest Galois extension in $\overline{k}$ splitting B.

Corollary 3.3. *Let K/k be the smallest Galois extension in $\overline{k}$ splitting B, and let $S \subset \mathscr{V}$ be a finite set of places of k. If any place $v \in S$ has a cyclic decomposition group in* $\mathrm{Gal}(K/k)$*, then* $Ш^1_{S,\emptyset}(k,B) = 0$.

Idea of proof. We have $Ш^1_S(k,B) = Ш^1_S(K/k,B) = Ш^1_\emptyset(K/k,B) = Ш^1_\emptyset(k,B)$. □

Recall that the exponent of a finite group is the least common multiple of the orders of its elements. A finite group is metacyclic if and only if its exponent is equal to its order.

Lemma 3.4. *Let K/k be the smallest Galois extension in $\overline{k}$ splitting B, and let n and e be the order and the exponent of* $\mathfrak{g} = \mathrm{Gal}(K/k)$*, respectively. Then multiplication by n/e equals* 0 *in* $Ш^1_\omega(k,B)$. *In particular, if K/k is a metacyclic extension, then*

$$Ш^1_\omega(k,B) = 0.$$

□

4. Preliminaries on quasi-trivial groups

4.1. Let k be a field of characteristic 0, $\overline{k}$ a fixed algebraic closure of k. Let G be a connected linear k-group. We set $\overline{G} = G \times_k \overline{k}$. We use the following notation:
G^{u} is the unipotent radical of G;
$G^{\mathrm{red}} = G/G^{\mathrm{u}}$ (it is reductive);
G^{ss} is the derived group of G^{red} (it is semisimple);
G^{sc} is the universal cover of G^{ss} (it is simply connected);
$G^{\mathrm{tor}} = G^{\mathrm{red}}/G^{\mathrm{ss}}$ (it is a torus).

Definition 4.2 ([CT], Prop. 2.2)**.** A connected linear k-group G over a field k of characteristic 0 is called *quasi-trivial,* if G^{tor} is a quasi-trivial torus and G^{ss} is simply connected.

Lemma 4.3 (well known)**.** *Let k be a field of characteristic 0 and let X be a right homogeneous space of a connected linear k-group G. Let $\overline{H} \subset \overline{G}$ be the stabilizer of a point $\overline{x} \in X(\overline{k})$; we assume that $\overline{H}$ is connected. Then the variety X is a homogeneous space of some quasi-trivial k-group G' such that the stabilizer $\overline{H'} \subset \overline{G'}$ of $\overline{x}$ in $\overline{G'}$ is connected.*

Proof. The lemma follows from [CT, Prop.-Def. 3.1], cf. [B3, Proof of Lemma 5.2]. □

5. Homogeneous spaces of quasi-trivial groups

In this section we prove Theorem 2.1.

5.1. Let k be a field of characteristic 0, and let $A \to B$ be a morphism of $\mathrm{Gal}(\bar{k}/k)$-modules. We write $\mathbb{H}^i(k, A \to B)$ for the Galois hypercohomology of the complex $A \to B$, where A is in degree 0 and B is in degree 1. When k is a number field, we define $\text{Ш}^i_S(k, A \to B)$, $\text{Ш}^i_{S,\emptyset}(k, A \to B)$, and $\text{Ш}^i_\omega(k, A \to B)$ as in §3.

The following lemma must be well known (see [B3, Proof of Lemma 4.4] and [BvH1, Proof of Cor. 2.15] for similar results) but we do not know a reference where it was stated, so we state and prove it here.

Lemma 5.2. *Let k be a number field and $P \to L$ a complex of* $\mathrm{Gal}(\bar{k}/k)$*-modules in degrees 0 and 1, where P is a permutation* $\mathrm{Gal}(\bar{k}/k)$*-module. Then for any finite set S of places of k we have a canonical isomorphism* $\text{Ш}^1_S(k, L) \xrightarrow{\sim} \text{Ш}^2_S(k, P \to L)$.

Proof. We have an exact sequence

$$0 = H^1(k, P) \to H^1(k, L) \to \mathbb{H}^2(k, P \to L) \to H^2(k, P),$$

and similar exact sequences for Galois cohomology over k_v for $v \notin S$. We obtain a commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 \longrightarrow & H^1(k, L) & \longrightarrow & \mathbb{H}^2(k, P \to L) & \longrightarrow & H^2(k, P) \\
& \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & \prod_{v \notin S} H^1(k_v, L) & \longrightarrow & \prod_{v \notin S} \mathbb{H}^2(k_v, P \to L) & \longrightarrow & \prod_{v \notin S} H^2(k_v, P).
\end{array}$$

Since P is a permutation module, we have $\text{Ш}^2_S(k, P) = 0$, cf. [Sa, (1.9.1)]. An easy diagram chase shows that the homomorphism $\text{Ш}^1_S(k, L) \to \text{Ш}^2_S(k, P \to L)$ induced by this diagram is an isomorphism. □

Proposition 5.3. *Let X be a homogeneous space of a quasi-trivial k-group G over a number field k. Let $\overline{H} \subset \overline{G}$ be the stabilizer of a $\bar{k}$-point $\bar{x} \in X(\bar{k})$. Let $S \subset \mathcal{V}$ be a finite set of places of k. Then there is a canonical isomorphism* $\text{Б}_S(X) \cong \text{Ш}^1_S(k, \widehat{\overline{H}})$.

Proof. By [BvH2, Thm. 7.2] we have a canonical, functorial in k isomorphism

$$\mathrm{Br_a}(X) \xrightarrow{\sim} \mathbb{H}^2(k, \widehat{G} \to \widehat{\overline{H}}),$$

whence we obtain a canonical isomorphism

$$\text{Б}_S(X) \cong \text{Ш}^2_S(k, \widehat{G} \to \widehat{\overline{H}}).$$

Since $\widehat{G}$ is a permutation module, by Lemma 5.2 we have a canonical isomorphism

$$\text{Ш}^1_S(k, \widehat{\overline{H}}) \xrightarrow{\sim} \text{Ш}^2_S(k, \widehat{G} \to \widehat{\overline{H}}).$$

Thus we obtain a canonical isomorphism $\text{Б}_S(X) \cong \text{Ш}^1_S(k, \widehat{\overline{H}})$. □

5.4. *Proof of Theorem 2.1.* By Proposition 5.3 we have a canonical isomorphism $\text{Б}_S(X) \cong \text{Ш}^1_S(k, \widehat{\overline{H}})$, hence we obtain a canonical isomorphism $\text{Б}_{S,\emptyset}(X) \cong \text{Ш}^1_{S,\emptyset}(k, \widehat{\overline{H}})$ and a canonical isomorphism $\text{Б}_\omega(X) \cong \text{Ш}^1_\omega(k, \widehat{\overline{H}})$. In case (i) by Corollary 3.3 we have $\text{Ш}^1_{S,\emptyset}(k, \widehat{\overline{H}}) = 0$, hence $\text{Б}_{S,\emptyset}(X) = 0$. In case (ii) by Lemma 3.4 we have $\text{Ш}^1_\omega(k, \widehat{\overline{H}}) = 0$, hence $\text{Б}_\omega(X) = 0$. □

6. Principal homogeneous spaces of connected groups

In this section we prove Theorem 2.7.

6.1. Let M be a $\mathrm{Gal}(\bar{k}/k)$-module, finitely generated over $\mathbb{Z}$. Choose a $\mathbb{Z}$-free resolution

$$0 \to L \to P \to M \to 0, \tag{3}$$

where L and P are finitely generated $\mathbb{Z}$-free Galois modules. We write

$$\mathbb{H}^i(k, M^D) := \mathbb{H}^i(k, P^\vee \to L^\vee),$$

where $P^\vee := \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Z})$ is in degree 0 and $L^\vee := \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$ is in degree 1. We regard $M^D := (P^\vee \to L^\vee)$ as a dual complex to M. Since the isomorphism class of M^D in the derived category does not depend on the choice of the resolution (3), the hypercohomology $\mathbb{H}^i(k, M^D)$ also does not depend on the resolution.

Lemma 6.2. *Let M be as in 6.1. Let K/k be the smallest Galois extension in $\bar{k}$ splitting M. Let $S \subset \mathcal{V}$ be finite set of places of k.*

(i) If any place $v \in S$ has a cyclic decomposition group in $\mathrm{Gal}(K/k)$, *then* $Ш^2_{S,\emptyset}(k, M^D) = 0$.

(ii) If K/k is a metacyclic extension, then $Ш^2_\omega(k, M^D) = 0$.

Proof. Set $\mathfrak{g} = \mathrm{Gal}(K/k)$, then M is a $\mathfrak{g}$-module. We can choose a resolution (3) such that P is a permutation $\mathfrak{g}$-module and L is a $\mathbb{Z}$-free $\mathfrak{g}$-module. Then $P^\vee$ is a permutation module as well, and by Lemma 5.2 we have a canonical isomorphism

$$Ш^1_S(k, L^\vee) \xrightarrow{\sim} Ш^2_S(k, P^\vee \to L^\vee) = Ш^2_S(k, M^D),$$

whence we obtain canonical isomorphisms

$$Ш^1_{S,\emptyset}(k, L^\vee) \xrightarrow{\sim} Ш^2_{S,\emptyset}(k, M^D),$$
$$Ш^1_\omega(k, L^\vee) \xrightarrow{\sim} Ш^2_\omega(k, M^D).$$

Since K splits $L^\vee$, in case (i) by Corollary 3.3 we have $Ш^1_{S,\emptyset}(k, L^\vee) = 0$, hence $Ш^2_{S,\emptyset}(k, M^D) = 0$. In case (ii) by Lemma 3.4 we have $Ш^1_\omega(k, L^\vee) = 0$, hence $Ш^2_\omega(k, M^D) = 0$. ☐

6.3. *Proof of Theorem 2.7.* By [Sa, Lemma 6.8] there is a canonical isomorphism $\mathrm{Br_a}(X) \xrightarrow{\sim} \mathrm{Br_a}(G)$. By [BvH1, Cor. 7] there is a canonical isomorphism $\mathrm{Br_a}(G) \xrightarrow{\sim} \mathbb{H}^2(k, \pi_1(G)^D)$. Hence $Б_S(X) \cong Ш^2_S(k, \pi(G)^D)$, whence $Б_{S,\emptyset}(X) \cong Ш^2_{S,\emptyset}(k, \pi_1(G)^D)$ and $Б_\omega(X) \cong Ш^2_\omega(k, \pi_1(G)^D)$. In case (i) by Lemma 6.2(i) we have $Ш^2_{S,\emptyset}(k, \pi_1(G)^D) = 0$, hence $Б_{S,\emptyset}(X) = 0$. In case (ii) by Lemma 6.2(ii) we have $Ш^2_\omega(k, \pi_1(G)^D) = 0$, hence $Б_\omega(X) = 0$. ☐

7. Connected groups as homogeneous spaces of quasi-trivial groups

In this section we prove Proposition 2.8.

7.1. *Proof of Proposition 2.8.* We may and shall assume that G is reductive, cf. [CT, proof of Prop.-Def. 3.1]. Consider the largest quotient torus G^{tor} of G. Set $M = \pi_1(G)$, then $\mathbb{X}_*(G^{\mathrm{tor}}) = M/M_{\mathrm{tors}}$, where M_{tors} denotes the torsion subgroup of M. Since K splits M, we see that K splits $\mathbb{X}_*(G^{\mathrm{tor}})$.

We follow the construction in [CT, proof of Prop.-Def. 3.1]. Let Z^0 denote the radical (the identity component of the center) of our reductive group G. Since Z^0 is isogenous to G^{tor}, we see that K splits $\mathbb{X}_*(Z^0)$. Set $\mathfrak{g} = \mathrm{Gal}(K/k)$, then $\mathbb{X}_*(Z^0)$ is a $\mathfrak{g}$-module. Choose a surjective homomorphism of $\mathfrak{g}$-modules $P \twoheadrightarrow \mathbb{X}_*(Z^0)$, where P is a finitely generated permutation $\mathfrak{g}$-module. We regard P as a $\mathrm{Gal}(\bar{k}/k)$-module, then K splits P. Let Q be the quasi-trivial k-torus with $\mathbb{X}_*(Q) = P$. We have a surjective homomorphism $\theta\colon Q \twoheadrightarrow Z^0$.

Consider the canonical homomorphism

$$\rho\colon G^{\mathrm{sc}} \twoheadrightarrow G^{\mathrm{ss}} \hookrightarrow G.$$

Set $G' = G^{\mathrm{sc}} \times_k Q$, then G' is a quasi-trivial group and K splits $\widehat{G'} = P^\vee$. We define a surjective homomorphism

$$\alpha\colon G' \to G, \quad \alpha(g,q) = \rho(g)\theta(q), \text{ where } g \in G^{\mathrm{sc}},\ q \in Q.$$

Set $H = \ker \alpha$, then H is a central k-subgroup of G'. We have an exact sequence

$$1 \to H \to G' \to G \to 1. \tag{4}$$

Set $M' := \pi_1(G') = P$, then K splits M and M'. By Lemma 7.2 below we have a canonical isomorphism of Galois modules $\widehat{H} \cong \mathrm{Ext}^0_{\mathbb{Z}}(M' \to M,\ \mathbb{Z})$ (where M' is in degree 0). It follows that K splits $\widehat{H}$, which proves the proposition. □

Lemma 7.2. *Assume we have a short exact sequence*

$$1 \to H \to G' \xrightarrow{\varphi} G \to 1,$$

where G and G' are connected reductive k-groups over a field k of characteristic 0, and $H \subset G'$ is a central k-subgroup. Set $M := \pi_1(G)$, $M' := \pi_1(G')$. Then there is a canonical isomorphism of Galois modules

$$\mathbb{X}^*(H) \cong \mathrm{Ext}^0_{\mathbb{Z}}(M' \to M,\ \mathbb{Z}),$$

where in the complex $M' \to M$ the Galois module M' is in degree 0 and M is in degree 1 and we write $\mathbb{X}^*(H) := \widehat{H}$.

Proof (C. Demarche). Consider the induced homomorphsm $\varphi^{\mathrm{ss}}\colon G'^{\mathrm{ss}} \to G^{\mathrm{ss}}$, it is surjective and its kernel is a central k-subgroup in G'^{ss}, hence finite. Consider the induced homomorphism $\varphi^{\mathrm{sc}}\colon G'^{\mathrm{sc}} \to G^{\mathrm{sc}}$, it is surjective and has finite kernel. Since G^{sc} is simply connected, we conclude that φ^{sc} is an isomorphism.

Choose compatible maximal tori $T_G \subset G$, $T_{G'} \subset G'$, $T_{G^{\mathrm{sc}}} \subset G^{\mathrm{sc}}$ and $T_{G'^{\mathrm{sc}}} \subset G'^{\mathrm{sc}}$. It follows from the definition of M and M' that we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{X}_*(T_{G'^{\mathrm{sc}}}) & \longrightarrow & \mathbb{X}_*(T_{G'}) & \longrightarrow & M' & \longrightarrow & 0 \\ & & \downarrow{\cong} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{X}_*(T_{G^{\mathrm{sc}}}) & \longrightarrow & \mathbb{X}_*(T_G) & \longrightarrow & M & \longrightarrow & 0. \end{array} \tag{5}$$

Since the homomorphism $\varphi_*\colon G'^{\mathrm{sc}} \to G^{\mathrm{sc}}$ is an isomorphism, the left-hand vertical arrow $\mathbb{X}_*(T_{G'^{\mathrm{sc}}}) \to \mathbb{X}_*(T_{G^{\mathrm{sc}}})$ in diagram (5) is an isomorphism, and the five lemma shows that the morphism of complexes of Galois modules

$$(\mathbb{X}_*(T_{G'}) \to \mathbb{X}_*(T_G)) \longrightarrow (M' \to M)$$

given by this diagram is a quasi-isomorphism.

The short exact sequence of complexes

$$0 \to (0 \to \mathbb{X}_*(T_G)) \to (\mathbb{X}_*(T_{G'}) \to \mathbb{X}_*(T_G)) \to (\mathbb{X}_*(T_{G'}) \to 0) \to 0$$

induces an exact sequence of Ext-groups

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{X}_*(T_G),\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{X}_*(T_{G'}),\mathbb{Z}) \to \mathrm{Ext}^0_{\mathbb{Z}}(\mathbb{X}_*(T_{G'}) \to \mathbb{X}_*(T_G),\ \mathbb{Z}) \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{X}_*(T_G),\mathbb{Z}) = 0.$$

Since the complex $\mathbb{X}_*(T_{G'}) \to \mathbb{X}_*(T_G)$ is quasi-isomorphic to $M' \to M$, we obtain an exact sequence

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{X}_*(T_G),\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{X}_*(T_{G'}),\mathbb{Z}) \to \mathrm{Ext}^0_{\mathbb{Z}}(M' \to M,\ \mathbb{Z}) \to 0,$$

which we can write as

$$\mathbb{X}^*(T_G) \to \mathbb{X}^*(T_{G'}) \to \mathrm{Ext}^0_{\mathbb{Z}}(M' \to M,\ \mathbb{Z}) \to 0, \tag{6}$$

where $\mathbb{X}^*(T_G) = \widehat{T_G}$ and $\mathbb{X}^*(T_{G'}) = \widehat{T_{G'}}$. On the other hand, the exact sequence of k-groups of multiplicative type

$$1 \to H \to T_{G'} \to T_G \to 1$$

gives an exact sequence

$$0 \to \mathbb{X}^*(T_G) \to \mathbb{X}^*(T_{G'}) \to \mathbb{X}^*(H) \to 0. \tag{7}$$

Comparing exact sequences (6) and (7), we obtain a canonical isomorphism of Galois modules

$$\mathbb{X}^*(H) \cong \mathrm{Ext}^0_{\mathbb{Z}}(M' \to M,\ \mathbb{Z}).$$ □

Remark 7.3. Constructing and arguing as in the proof of [CT, Prop.-Def. 3.1], we can construct an exact sequence (4) with G' a quasi-trivial k-group and H a flasque k-torus (and not just some k-group of multiplicative type) such that the smallest Galois extension K/k in $\overline{k}$ splitting $\pi_1(G)$ splits both tori $G'^{\,\mathrm{tor}}$ and H. This strengthens Remark 3.1.1 of [CT].

8. Principal homogeneous spaces of connected groups again

In this section we give an alternative proof of Theorem 2.7 based on Proposition 2.8 and Theorem 2.1.

8.1. Let X be a right homogeneous space of a quasi-trivial k-group G over a number field k. Let $\overline{H} \subset \overline{G}$ be the stabilizer of a $\overline{k}$-point $\overline{x} \in X(\overline{k})$. We describe the action of $\mathrm{Gal}(\overline{k}/k)$ on $\widehat{\overline{H}}$ defined by the homogeneous space X.

Let $h \in \overline{H}(\overline{k})$, then $\overline{x}.h = \overline{x}$. Let $\sigma \in \mathrm{Gal}(\overline{k}/k)$, then ${}^\sigma x.{}^\sigma h = {}^\sigma x$. For any $\sigma \in \mathrm{Gal}(\overline{k}/k)$ we choose $g_\sigma \in G(\overline{k})$ such that ${}^\sigma x = x.g_\sigma$ and the function $\sigma \mapsto g_\sigma$ is locally constant, then

$$g_\sigma \cdot {}^\sigma h \cdot g_\sigma^{-1} \in H(\overline{k}).$$

The map $h \mapsto g_\sigma \cdot {}^\sigma h \cdot g_\sigma^{-1}$ comes from some σ-semialgebraic automorphism (see [B2, §1.1] for a definition) ν_σ of $\overline{H}$, which induces an automorphism $\widehat{\nu_\sigma}$ of $\widehat{\overline{H}}$ (namely, $\widehat{\nu_\sigma}(\chi)(h) = \chi(\nu_\sigma^{-1}(h))$ for $\chi \in \widehat{\overline{H}}$ and $h \in \overline{H}(\overline{k})$). If we choose another element $g'_\sigma \in G(\overline{k})$ such that ${}^\sigma x = x.g'_\sigma$, then $g'_\sigma = h' g_\sigma$ for some $h' \in \overline{H}(\overline{k})$. Then we obtain $\nu'_\sigma = \mathrm{Inn}(h') \circ \nu_\sigma$, where $\mathrm{Inn}(h')$ is the inner automorphism of $\overline{H}$ defined by h'. We have $\widehat{\nu'_\sigma} = \widehat{\nu_\sigma}$, because $\mathrm{Inn}(h')$ acts trivially on $\widehat{\overline{H}}$. The well-defined map $\sigma \mapsto \widehat{\nu_\sigma}$ is a homomorphism defining an action of $\mathrm{Gal}(\overline{k}/k)$ on $\widehat{\overline{H}}$.

8.2. *Alternative proof of Theorem 2.7.* We deduce Theorem 2.7 from Proposition 2.8 and Theorem 2.1. Since K splits $\pi_1(G)$, by Proposition 2.8 we can write $G = G'/H$, where G' is a quasi-trivial k-group and H is a central k-subgroup of multiplicative type in G' such that K splits $\widehat{H}$.

The group G' acts on X via G. Let $\overline{x} \in X(\overline{k})$, then the stabilizer of $\overline{x}$ in $\overline{G'}$ is $\overline{H} := H_{\overline{k}}$. Consider the action $\sigma \mapsto \widehat{\nu_\sigma}$ of $\mathrm{Gal}(\overline{k}/k)$ on $\widehat{\overline{H}}$ defined in 8.1. Write ${}^\sigma x = x.g'_\sigma$, where $g'_\sigma \in G'(\overline{k})$, then for $h \in H(\overline{k})$ we have

$$\nu_\sigma(h) = g'_\sigma \cdot {}^\sigma h \cdot (g'_\sigma)^{-1} = {}^\sigma h,$$

because H is central in G'. It follows that the action of $\mathrm{Gal}(\overline{k}/k)$ on $\widehat{\overline{H}}$ defined by the homogeneous space X coincides with the action on $\widehat{H}$ defined by the k-structure of H.

Now, since K splits $\widehat{H}$, we see that K splits $\widehat{\overline{H}}$, and Theorem 2.7 follows from Theorem 2.1. □

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