VECTOR BUNDLES AND MONADS ON ABELIAN THREEFOLDS

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Abstract. Using the Serre construction, we give examples of stable rank 2 vector bundles on principally polarized abelian threefolds (X, Θ) with Picard number 1. The Chern classes (c_1, c_2) of these examples realize roughly one half of the classes that are a priori allowed by the Bogomolov inequality and Riemann-Roch (the latter gives a certain divisibility condition).

In the case of even c_1, we study deformations of these vector bundles $\mathscr{E}$, using a second description in terms of monads, similar to the ones studied by Barth–Hulek on projective space. By an explicit analysis of the hyperext spectral sequence associated to the monad, we show that the space of first order infinitesimal deformations of $\mathscr{E}$ equals the space of first order infinitesimal deformations of the monad. This leads to the formula

$$\dim \operatorname{Ext}^1(\mathscr{E}, \mathscr{E}) = \tfrac{1}{3}\Delta(\mathscr{E}) \cdot \Theta + 5$$

(we emphasize that its validity is only proved for special bundles $\mathscr{E}$ coming from the Serre construction), where Δ denotes the discriminant $4c_2 - c_1^2$.

Finally we show that, in the first nontrivial example of the above construction (where $c_1 = 0$ and $c_2 = \Theta^2$), the infinitesimal identification between deformations of $\mathscr{E}$ and of the monad can be extended to a Zariski local identification: this leads to an explicit description of a Zariski open neighbourhood of $\mathscr{E}$ in its moduli space $M(0, \Theta^2)$. This neighbourhood is a ruled, nonsingular variety of dimension 13, birational to a $\mathbb{P}^1$-bundle over a finite quotient of $X^2 \times_X X^2 \times_X X^2$, where X^2 is considered as a variety over X via the group law.

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1. Introduction

The geometry of moduli spaces for stable vector bundles on Calabi-Yau (by which we just mean having trivial canonical bundle) threefolds is largely unknown, but of high interest, for instance due to their relevance for string theory, and the computation of Donaldson-Thomas invariants. In lower dimension, vector bundles on Calabi-Yau curves (i.e. elliptic) were classified by Atiyah, and are parametrized by the same curve. Moduli spaces for stable bundles, or coherent sheaves, on Calabi-Yau surfaces (i.e. K3 or abelian) are holomorphic symplectic varieties (Mukai [13], generalizing Beauville [2], generalizing Fujiki [4]). This is a very rare geometric structure, at least on complete varieties. Our leitfaden is the question whether equally interesting geometries exist in higher dimension.

We study here examples of rank 2 vector bundles on *abelian* threefolds, partly for the intrinsic interest, and partly in the hope that the abelian case may shed light on the case of general Calabi-Yau threefolds, but be more accessible. Our central tool, besides the Serre construction, is monads: these are usually put to work on rational varieties, and it may be slightly surprising that they can be useful also in our context. On the other hand, we do not know whether the bundles we construct, and their moduli, show typical or exceptional behaviour.

1.1. **Notation.** We work over an algebraically closed field k of characteristic zero. Our terminology regarding (semi-) stable sheaves and their moduli follows Simpson [15]; in particular, stability for a coherent $\mathscr{O}_X$-module on a polarized projective variety (X, H) is defined using the normalized Hilbert polynomial with respect to the polarization H. Stable sheaves admit a coarse moduli space M, with a compactification $\overline{M}$ parametrizing S-equivalence classes of semistable sheaves. The sheaves we construct in this text will in fact have the stronger property of μ-stability, in the sense of Mumford and Takemoto, which is measured by the slope, i.e. the ratio of degree with respect to H to the rank, and which implies stability. Conversely, semistability implies μ-semistability.

The words line bundles and vector bundles are used as synonyms of invertible and locally free sheaves. In particular, an inclusion of vector bundles means an inclusion as sheaves, i.e. the quotient need not be locally free. We take Chern classes to live in the Chow ring modulo numerical equivalence.

Let (X, Θ) be a principally polarized abelian variety. If $x \in X$ is a point, we write $T_x\colon X \to X$ for the translation map, and define Θ_x as $T_x(\Theta) = \Theta + x$. We identify X with its dual $\operatorname{Pic}^0(X)$ by associating with x the line bundle $\mathscr{P}_x = \mathscr{O}_X(\Theta - \Theta_x)$. The normalized Poincaré line bundle on $X \times X$ is denoted $\mathscr{P}$; its restriction to $X \times \{x\}$ is $\mathscr{P}_x$.

2. The Serre construction

In this section we apply the standard Serre construction to produce rank 2 vector bundles on principally polarized abelian threefolds, including examples with small c_2. This is the content of Theorem 2.3. These examples (in the case of even c_1) will be our objects of study for the rest of this paper.

2.1. **The bundles/curves correspondence.** Let $\mathscr{E}$ be rank 2 vector bundle on a projective variety X, and let $s \in \Gamma(X, \mathscr{E})$ be a section. If the vanishing locus $V(s)$ has codimension 2, then: (1) it is a locally complete intersection, and (2) its canonical bundle is $(\omega_X \otimes \bigwedge^2 \mathscr{E})\big|_{V(s)}$. The Serre construction says (under a cohomological condition on $\bigwedge^2 \mathscr{E}$) that any codimension two subscheme $Y \subset X$ with these two properties is of the form $V(s)$. More precisely:

Theorem 2.1. *Let X be a projective variety with a line bundle $\mathscr{L}$ satisfying $H^p(X, \mathscr{L}^{-1}) = 0$ for $p = 1, 2$. Let $Y \subset X$ be a codimension two locally complete intersection subscheme with canonical bundle isomorphic to $(\omega_X \otimes \mathscr{L})|_Y$. Then there is a canonical isomorphism*

$$\operatorname{Hom}((\omega_X \otimes \mathscr{L})|_Y, \omega_Y) \cong \operatorname{Ext}^1(\mathscr{I}_Y \otimes \mathscr{L}, \mathscr{O}_X)$$

which is functorial in Y with respect to inclusions, and such that isomorphisms on the left correspond to locally free extensions on the right.

For the proof we refer to Hartshorne [7, Thm. 1.1 and Rem. 1.1.1], who attributes "all essential ideas" to Serre [14].

It follows that, whenever we choose an isomorphism $(\omega_X \otimes \mathscr{L})|_Y \cong \omega_Y$, the theorem gives an extension

$$0 \longrightarrow \mathscr{O}_X \xrightarrow{\ s\ } \mathscr{E} \longrightarrow \mathscr{I}_Y \otimes \mathscr{L} \longrightarrow 0 \tag{1}$$

with $\mathscr{E}$ locally free, and hence $Y = V(s)$ as required.

Definition 2.2. We say that $\mathscr{E}$ and Y *corresponds* if there is a short exact sequence (1).

Note that, if Y has several connected components, there may be several non-isomorphic bundles $\mathscr{E}$ corresponding to Y. See Proposition 2.8.

2.2. **Construction of bundles.** For the rest of this paper, we fix a principally polarized abelian threefold (X, Θ). We assume that its Picard number is one, although this assumption is not essential in later sections. Thus every divisor is numerically equivalent to an integral multiple of Θ. Moreover (see e.g. Debarre [3]), an application of the endomorphism construction of Morikawa [11] and Matsusaka [10] shows that every 1-cycle is numerically equivalent to an integral multiple of $\Theta^2/2$.

So fix classes $c_1 = m\Theta$ and $c_2 = n\Theta^2/2$, where m and n are integers. If these are the Chern classes of a rank two vector bundle $\mathscr{E}$, then, by Riemann-Roch

$$\chi(\mathscr{E}) = \tfrac{1}{6}(c_1^3 - 3c_1c_2) = m^3 - \tfrac{3}{2}nm,$$

so either m or n is even. Moreover, if $\mathscr{E}$ is μ-semistable, then Bogomolov's inequality reads $m^2 \le 2n$.

Theorem 2.3. *Let (X, Θ) be a principally polarized abelian threefold of Picard number 1, and let $c_1 = m\Theta$ and $c_2 = n\Theta^2/2$, with m and n integers. Assume*

(1) *the strict Bogomolov inequality holds, i.e. $m^2 < 2n$, and*
(2) *n is even and mn is divisible by 4.*

Then there exist μ-stable rank 2 vector bundles with Chern classes c_1 and c_2.

Remark 2.4. For each $c_1 \in \mathrm{NS}(X)$, the theorem realizes every second c_2 that is allowed by (strict) Bogomolov and Riemann-Roch. The other half seems much more subtle. In fact, we do not know any example of a rank 2 vector bundle, stable or not, that violates condition (2). The situation in which equality occurs in the Bogomolov inequality will be analysed in Proposition 2.6.

Before proving the theorem, we rephrase μ-stability for $\mathscr{E}$ as a condition on the corresponding curve Y. The argument is similar to that of Hartshorne [7, Prop. 3.1] in the case of $\mathbb{P}^3$.

Lemma 2.5. *Let (X, Θ) be as in the theorem, and $\mathscr{E}$ be a rank 2 vector bundle corresponding to a curve $Y \subset X$. Let $c_1(\mathscr{E}) = m\Theta$. Then the following are equivalent.*

(1) *$\mathscr{E}$ is μ-stable.*
(2) *$m > 0$ and Y is not contained in any translate of any divisor in the linear system $|k\Theta|$, where k is the round down of $m/2$.*

Proof. Since $\mathscr{E}$ has a section, it is clear that $m > 0$ is necessary for its μ-stability. Write $\lfloor m/2 \rfloor$ and $\lceil m/2 \rceil$ for the round down and round up of $m/2$. The bundle $\mathscr{E}$ fails μ-stability if and only if it contains a line bundle $\mathscr{P}_x(l\Theta) \subset \mathscr{E}$ with $l \ge m/2$. Since $\mathscr{P}_x(l\Theta)$ has global sections for l positive, it suffices to test with $l = \lceil m/2 \rceil$. Thus $\mathscr{E}$ is μ-stable if and only if

$$H^0(X, \mathscr{E}(-\lceil m/2 \rceil\Theta) \otimes \mathscr{P}_x)) = 0 \quad \text{for all } x \in X. \tag{2}$$

Now twist the short exact sequence (1) with $-\lceil m/2 \rceil\Theta$ and take cohomology. Since $H^i(X, \mathscr{O}_X(-\lceil m/2 \rceil\Theta))) = 0$ for $i = 0, 1$, and the determinant of $\mathscr{E}$ has the form $\mathscr{P}_a(m\Theta)$ for some $a \in X$, we find that the vanishing (2) is equivalent to the vanishing of $H^0(X, \mathscr{I}_Y(\lfloor m/2 \rfloor\Theta) \otimes \mathscr{P}_x)$ for all $x \in X$. Since Θ is ample, this is equivalent to

$$H^0(X, \mathscr{I}_Y \otimes T_x^*\mathscr{O}_X(\lfloor m/2 \rfloor\Theta)) = 0 \quad \text{for all } x \in X$$

which is condition (2). □

Proof of Theorem 2.3. Since μ-stability, and the conditions (1) and (2) in the statement of the theorem, are preserved under tensor product with line bundles, it suffices to prove the theorem for $m = 2$ and $m = 3$.

When $m = 2$, the theorem claims that there are μ-stable rank 2 bundles with $c_1 = 2\Theta$ and $c_2 = N\Theta^2$ for all integers $N \geq 2$. For this, choose N generic points $a_i \in X$ and let

$$Y = \bigcup_{i=1}^{N} Y_i, \quad Y_i = \Theta_{a_i} \cap \Theta_{-a_i}.$$

We want to apply the Serre construction to this curve.

First we claim that the Y_i's are pairwise disjoint, for a_i chosen generically. In fact, for $i \neq j$ write

$$Y_i \cap Y_j = \underbrace{(\Theta_{a_i} \cap \Theta_{a_j})}_{V} \cap \underbrace{(\Theta_{-a_i} \cap \Theta_{-a_j})}_{W},$$

where V and W have codimension 2. By an easy moving lemma for abelian varieties [9, Lemma 5.4.1], a general translate $V + x$ intersects W properly, hence empty. Thus (replacing x by a "square root" $x/2$) also $V + x$ and $W - x$ are disjoint. So Y_i and Y_j will be disjoint after a small perturbation $a_i \mapsto a_i + x$, $a_j \mapsto a_j + x$.

The normal bundle of each $Y_i \subset X$ is $\mathscr{O}_{Y_i}(\Theta_{a_i}) \oplus \mathscr{O}_{Y_i}(\Theta_{-a_i})$, hence the canonical bundle ω_{Y_i} is $\mathscr{O}_{Y_i}(\Theta_{a_i} + \Theta_{-a_i})$. The theorem of the square shows that $\Theta_{a_i} + \Theta_{-a_i}$ is linearly equivalent to 2Θ. Since the Y_i's are disjoint, we conclude that Y is a locally complete intersection with canonical bundle $\mathscr{O}_Y(2\Theta)$. The Serre construction produces a bundle $\mathscr{E}$ with determinant $\mathscr{O}_X(2\Theta)$ and second Chern class $[Y] = \sum_i [Y_i] = N\Theta^2$.

Next we show μ-stability. We claim that the only theta-translates containing Y_i are Θ_{a_i} and Θ_{-a_i}. This is a standard result: the intersection of two theta-translates are never contained in a third one. In fact, consider the Koszul complex:

$$0 \to \mathscr{O}_X(-\Theta_{a_i} - \Theta_{-a_i}) \to \mathscr{O}_X(-\Theta_{a_i}) \oplus \mathscr{O}_X(-\Theta_{-a_i}) \to \mathscr{I}_{Y_i} \to 0.$$

Twist with an arbitrary theta-translate Θ_x and apply cohomology to obtain an isomorphism

$$H^0(X, \mathscr{O}_X(\Theta_x - \Theta_{a_i})) \oplus H^0(X, \mathscr{O}_X(\Theta_x - \Theta_{-a_i})) \cong H^0(X, \mathscr{I}_{Y_i}(\Theta_x)).$$

Thus Θ_x contains Y_i if and only if $x = \pm a_i$ as claimed. It follows that, for $N \geq 2$, no theta-translate contains Y, and so $\mathscr{E}$ is μ-stable by Lemma 2.5.

In the case $m = 3$, we take

$$Y = \bigcup_{i=1}^{N} Y_i, \quad Y_i = D_i \cap \Theta_{-2a_i}$$

for N generic points $a_i \in X$ and generic divisors $D_i \in |2\Theta_{a_i}|$. A similar argument to the one above shows that the Serre construction produces a μ-stable rank 2 vector bundle with determinant $\mathscr{O}_X(3\Theta)$ and second Chern class $2N\Theta^2$, for each $N \geq 2$. □

Recall that a vector bundle $\mathscr{E}$ is *semihomogeneous* if, for every $x \in X$, there exists a line bundle $\mathscr{L} \in \mathrm{Pic}^0(X)$ such that $T_x^*(\mathscr{E})$ is isomorphic to $\mathscr{E} \otimes \mathscr{L}$ (in short, homogeneous means translation invariant, and semihomogeneous means translation invariant up to twist). Semihomogeneous bundles are well understood thanks to work of Mukai [12].

Proposition 2.6. *Let (X, Θ) be as in the theorem, and let $c_1 = m\Theta$ and $c_2 = n\Theta^2/2$ satisfy $m^2 = 2n$, i.e. equality occurs in the Bogomolov inequality. Then $\mathscr{E}$ is a non simple, semihomogeneous vector bundle. In particular, it is semistable, but not stable.*

More precisely, there are a line bundle $\mathscr{L}_0$ and points $x, y \in X$ such that $\mathscr{E}_0 = \mathscr{E} \otimes \mathscr{L}_0^{-1}$ is an extension (necessarily split if $x \neq y$)

$$0 \to \mathscr{P}_x \to \mathscr{E}_0 \to \mathscr{P}_y \to 0.$$

Proof. Semihomogenous bundles of rank r are numerically characterized (Yang [16]) by the property that the Chern roots may be taken to be c_1/r. This means that the Chern character takes the form $ch = r\exp(c_1/r)$, or, equivalently, the total Chern class is $c = (1 + c_1/r)^r$. If $r = 2$, this is equivalent to $c_1^2 = 4c_2$. Thus $\mathscr{E}$ is semihomogeneous.

Now we use several results by Mukai [12] on semihomogeneous bundles. *Simple* semihomogeneous vector bundles are classified, up to twist by homogeneous line bundles, by the element $\delta = c_1/r$ in $\mathrm{NS}(X) \otimes \mathbb{Q}$. But m is even, since $m^2 = 2n$, so there exist line bundles with class $c_1/2$, which rules out the possibility that $\mathscr{E}$ is simple. Moreover, any semihomogeneous bundle is Gieseker-semistable, and it is simple if and only if it is Gieseker-stable. This proves the first part.

For the last part, we use Mukai's Harder-Narasimhan filtration for semihomogeneous bundles, which in particular says that any semihomogeneous vector bundle with $\delta = c_1/r$ has a filtration whose factors are simple semihomogeneous bundles with the same invariant δ. Choosing $\mathscr{L}_0$ in the (integral) class $c_1/2$, we ensure that $\mathscr{E}_0$ has $\delta = 0$. Since $\mathscr{E}_0$ is semihomogeneous, but not simple, its Harder-Narasimhan factors are necessarily line bundles with $c_1 = 0$. □

2.3. **A note on the curves $\Theta_a \cap \Theta_{-a}$.** For later use, we make an observation regarding the curve obtained by intersecting two general theta-translates, which was used as input for the Serre construction above (in the even c_1 case).

First note that there is a Zariski open subset $U \subset X$ such that $\Theta \cap \Theta_x$ is a nonsingular irreducible curve for all $x \in U$. This is standard: since Θ is a nonsingular surface, generic smoothness shows that $\Theta \cap \Theta_x$ is nonsingular, but possibly disconnected, for generic x (see Hartshorne [6, III 10.8]). On the other hand, $\Theta \cap \Theta_x$ is an ample divisor on Θ, hence it is connected (see Hartshorne [6, III 7.9]).

Lemma 2.7. *Let a and b be two points in X and define $Y_a = \Theta_a \cap \Theta_{-a}$ and $Y_b = \Theta_b \cap \Theta_{-b}$. Then, for a and b generic, no divisor in $|2\Theta|$ contains both Y_a and Y_b.*

Proof. Begin by imposing the conditions on a and b that Y_a and Y_b are disjoint irreducible curves, and also that the two curves $\Theta_a \cap \Theta_{\pm b}$ are irreducible. Assume there is a divisor $D \in |2\Theta|$ containing both Y_a and Y_b. We will prove the lemma by producing a curve C such that $C \cap \Theta_b = C \cap \Theta_{-b}$, and then deduce from this that b is not generic.

First we observe that D meets $\Theta_a \cap \Theta_b$ properly. As the latter is irreducible, it suffices to verify that it is not contained in D. In fact, one checks (determine $H^0(\mathscr{I}_{\Theta_a \cap \Theta_b}(2\Theta))$ using the Koszul resolution) that the linear subsystem of $|2\Theta|$, consisting of divisors containing $\Theta_a \cap \Theta_b$, is the pencil spanned by $\Theta_a + \Theta_{-a}$ and $\Theta_b + \Theta_{-b}$. The only element of this pencil containing Y_a is $\Theta_a + \Theta_{-a}$, and the only element containing Y_b is $\Theta_b + \Theta_{-b}$, so no element contains both.

In particular, D and Θ_a intersects properly, so $D \cap \Theta_a$ is a curve containing Y_a. Since $D \cap \Theta_a$ has cohomology class $2\Theta^2$, and Y_a has class Θ^2, there is another effective 1-cycle C of class Θ^2 such that

$$D \cap \Theta_a = Y_a + C$$

as 1-cycles. We saw above that $D \cap \Theta_a$ meets Θ_b properly, so we consider the 0-cycle

$$D \cap \Theta_a \cap \Theta_b = Y_a \cap \Theta_b + C \cap \Theta_b.$$

The left hand side contains $Y_b \cap \Theta_a$. Since Y_a and Y_b are disjoint, this means that $C \cap \Theta_b$ contains $Y_b \cap \Theta_a$, i.e. their difference is an effective cycle. But these are 0-cycles of the same degree, so they are equal. None of the arguments given distinguish between b and $-b$, so we find that also $C \cap \Theta_{-b}$ equals $Y_b \cap \Theta_a$. Thus we have established

$$C \cap \Theta_b = C \cap \Theta_{-b}.$$

To conclude, we apply the endomorphism construction of Morikawa [11] and Matsusaka [10], which we briefly recall. The endomorphism $\alpha = \alpha(C, \Theta)$ associated to C and Θ is defined by

$$\alpha(x) = \sum (C \cdot \Theta_x) - \sum (C \cdot \Theta)$$

where each term means the sum, using the group law, of the points in the intersection cycle appearing. This is well defined as a point in X, although the intersection cycle is only defined up to rational equivalence. The constant term is included to force $\alpha(0) = 0$, i.e. to make α a group homomorphism. We have just established that C intersects Θ_b and Θ_{-b} properly, and the two intersections are equal already as cycles. In particular $\alpha(b) = \alpha(-b)$, so all we need to know to prove the lemma is that α is not constant, so that $\alpha(2b) \neq 0$ defines a nonempty Zariski open subset. But in fact, a theorem of Matsusaka [10] tells us that α is multiplication by 2 (the intersection number $C\cdot\Theta = 3!$ divided by $\dim X = 3$), so the condition required is just that $4b \neq 0$, i.e. b is not a 4-torsion point. □

As an immediate consequence of the Lemma, we find that if $\mathscr{E}$ corresponds to a curve with at least two components of the form $\Theta_{a_i}\cap\Theta_{-a_i}$, for sufficiently general points a_i, then the short exact sequence

$$0 \longrightarrow \mathscr{O}_X \xrightarrow{s} \mathscr{E} \longrightarrow \mathscr{I}_Y(2\Theta) \longrightarrow 0,$$

shows that $H^0(X,\mathscr{E})$ is spanned by s.

Proposition 2.8. *Fix $N \geq 2$ general points $a_i \in X$, and let Y be the union of the curves $Y_i = \Theta_{a_i}\cap\Theta_{-a_i}$. Then the vector bundles $\mathscr{E}$ corresponding to the union Y of $Y_i = \Theta_{a_i}\cap\Theta_{-a_i}$ form an $(N-1)$-dimensional family, parametrized by $\mathbb{G}_m^{N-1}$.*

Proof. The Serre construction gives a one-one correspondence between isomorphisms $\omega_Y \cong \mathscr{O}_Y(2\Theta)$ modulo scale, and isomorphism classes of vector bundles $\mathscr{E}$ which admit a section vanishing at Y: the choice of a section can be left out, since we just observed that it is unique modulo scale. But isomorphisms $\omega_Y \cong \mathscr{O}_Y(2\Theta)$ constitute a homogeneous $\mathbb{G}_m^N$-space, as Y has N connected components. Dividing by scale, we are left with $\mathbb{G}_m^N/\mathbb{G}_m \cong \mathbb{G}_m^{N-1}$. □

3. Monads

It turns out that the vector bundles constructed in Theorem 2.3 admit more deformations than are visible in the Serre construction, i.e. more deformations than those obtained by varying the curve Y and the isomorphism $\omega_Y \cong \mathscr{O}_X(2\Theta)$. In this section we rephrase the construction in terms of certain monads (Proposition 3.2). This new viewpoint is then used in the remaining sections to analyse first order deformations.

Definition 3.1 (Barth–Hulek [1]). A *monad* is a composable pair of maps of vector bundles

$$\mathscr{A} \xrightarrow{\phi} \mathscr{B} \xrightarrow{\psi} \mathscr{C}$$

such that $\psi\circ\phi$ is zero, ψ is surjective and ϕ is an embedding of vector bundles (i.e. injective as a homomorphism of sheaves, and with locally free cokernel).

Thus $\mathscr{E} = \operatorname{Ker}(\psi)/\operatorname{Im}(\phi)$ is a vector bundle, and we say that the monad is a *monad for $\mathscr{E}$*.

We will also use chain complex notation $(M^\bullet, d)$ for monads, so that $M^{-1} = \mathscr{A}$, $M^0 = \mathscr{B}$, $M^1 = \mathscr{C}$ and M^i is zero otherwise, and the differential d consists of two nonzero components $d^{-1} = \phi$ and $d^0 = \psi$. Thus $M^\bullet$ is exact except in degree zero, where its cohomology is $\mathscr{E} = H^0(M^\bullet)$.

3.1. **Decomposable monads.** Consider rank 2 vector bundles $\mathscr{E}$ with trivial determinant $\bigwedge^2\mathscr{E} \cong \mathscr{O}_X$ on the principally polarized abelian threefold (X, Θ). From the construction in Theorem 2.3, we have a series of such vector bundles, such that $\mathscr{E}(\Theta)$ corresponds to a curve $Y = \bigcup_i Y_i$, where $Y_i = \Theta_{a_i} \cap \Theta_{-a_i}$. (The assumption that X has Picard number 1 is not needed here; this was only needed to establish μ-stability of $\mathscr{E}$, which is not relevant in this section.)

We now show that, corresponding to the decomposition of Y into its connected components Y_i, there is a way of building up $\mathscr{E}$ from the Koszul complexes[1]

(3)

$$\xi_i\colon \quad 0 \longrightarrow \mathscr{O}_X(-\Theta) \xrightarrow{\begin{pmatrix} \vartheta_i^+ \\ \vartheta_i^- \end{pmatrix}} \mathscr{P}_{a_i} \oplus \mathscr{P}_{-a_i} \xrightarrow{\begin{pmatrix} \vartheta_i^- & -\vartheta_i^+ \end{pmatrix}} \mathscr{I}_{Y_i}(\Theta) \longrightarrow 0$$

where $\vartheta_i^\pm$ are nonzero global sections of $\mathscr{O}_X(\Theta_{\pm a_i})$. This can be conveniently phrased in terms of a monad.

Proposition 3.2. *Let $a_1, \dots, a_N \in X$ be generically chosen points and $Y_i = \Theta_{a_i} \cap \Theta_{-a_i}$. Then $\mathscr{E}(\Theta)$ corresponds to $Y = \bigcup_{i=1}^N Y_i$ if and only if $\mathscr{E}$ is isomorphic to the cohomology of a monad*

$$(N-1)\mathscr{O}_X(-\Theta) \xrightarrow{\phi} \bigoplus_{i=1}^N (\mathscr{P}_{a_i} \oplus \mathscr{P}_{-a_i}) \xrightarrow{\psi} (N-1)\mathscr{O}_X(\Theta)$$

where, if we decompose ϕ and ψ into pairs

$$\phi^\pm\colon (N-1)\mathscr{O}_X(-\Theta) \to \bigoplus_{i=1}^N \mathscr{P}_{\pm a_i}$$
$$\psi^\pm\colon \bigoplus_{i=1}^N \mathscr{P}_{\pm a_i} \to (N-1)\mathscr{O}_X(\Theta)$$

then we have

$$\phi^\pm = \begin{pmatrix} \vartheta_1^\pm & & & \\ & \vartheta_2^\pm & & \\ & & \ddots & \\ & & & \vartheta_{N-1}^\pm \\ \vartheta_N^\pm & \vartheta_N^\pm & \dots & \vartheta_N^\pm \end{pmatrix}, \qquad \psi^\pm = \pm(\phi^\mp)^\vee$$

for nonzero sections $\vartheta_i^\pm \in \Gamma(X, \mathscr{O}_X(\Theta_{\pm a_i}))$.

Proof. One immediately verifies that homomorphisms ϕ and ψ of this form do define a monad.

The statement that $\mathscr{E}(\Theta)$ and Y correspond means that $\mathscr{E}$ is an extension

$$\xi\colon \quad 0 \longrightarrow \mathscr{O}_X(-\Theta) \longrightarrow \mathscr{E} \longrightarrow \mathscr{I}_Y(\Theta) \longrightarrow 0.$$

Giving such an extension is, by Theorem 2.1, equivalent to giving an isomorphism $\mathscr{O}_Y(2\Theta) \cong \omega_Y$. The obvious decomposition

$$\operatorname{Hom}(\mathscr{O}_Y(2\Theta), \omega_Y) \cong \bigoplus_{i=1}^N \operatorname{Hom}(\mathscr{O}_{Y_i}(2\Theta), \omega_{Y_i})$$

[1]Here and elsewhere, whenever $f\colon \mathscr{F}_1 \to \mathscr{F}_2$ is a homomorphism of sheaves, we use the same symbol to denote any twist $f\colon \mathscr{F}_1(D) \to \mathscr{F}_2(D)$.

gives, when applying Theorem 2.1 also to each Y_i, a corresponding decomposition

$$\operatorname{Ext}^1(\mathscr{I}_Y(\Theta), \mathscr{O}_X(-\Theta)) \cong \bigoplus_{i=1}^N \operatorname{Ext}^1(\mathscr{I}_{Y_i}(\Theta), \mathscr{O}_X(-\Theta)), \tag{4}$$

which sends ξ to an N-tuple of extensions ξ_i. Each $\operatorname{Hom}(\mathscr{O}_{Y_i}(2\Theta), \omega_{Y_i})$ is one dimensional, since Y_i is connected, so $\operatorname{Ext}^1(\mathscr{I}_{Y_i}(\Theta), \mathscr{O}_X(-\Theta))$ is one dimensional, too. This shows that each ξ_i is of the form (3).

From the functoriality in Theorem 2.1, it follows that the inclusion of each direct summand in (4) is the natural map, induced by the inclusion $\mathscr{I}_Y \subset \mathscr{I}_{Y_i}$. Thus ξ is obtained from the ξ_i's by pulling them back over this inclusion of ideals, and adding the results in $\operatorname{Ext}^1(\mathscr{I}_Y(\Theta), \mathscr{O}_X(-\Theta))$. By definition of (Baer) addition in Ext-groups, this means that there is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N\mathscr{O}_X(-\Theta) & \longrightarrow & \bigoplus_{i=1}^N(\mathscr{P}_{a_i} \oplus \mathscr{P}_{-a_i}) & \longrightarrow & \bigoplus_{i=1}^N \mathscr{I}_{Y_i}(\Theta) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\beta} & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathscr{O}_X(-\Theta) & \longrightarrow & \mathscr{F} & \longrightarrow & \bigoplus_{i=1}^N \mathscr{I}_{Y_i}(\Theta) & \longrightarrow & 0 \\
 & & \| & & \cup\!\uparrow & & \cup\!\uparrow{\scriptstyle\alpha} & & \\
0 & \longrightarrow & \mathscr{O}_X(-\Theta) & \longrightarrow & \mathscr{E} & \longrightarrow & \mathscr{I}_Y(\Theta) & \longrightarrow & 0
\end{array}$$

where the top row is $\bigoplus_i \xi_i$, the bottom row is ξ, the top left square is pushout over the N-fold addition β, the bottom right square is pullback along the inclusion α, and $\mathscr{F}$ is just an intermediate sheaf (in fact a vector bundle) that we do not care about. This diagram presents $\mathscr{E}$ as the middle cohomology of a complex

$$\operatorname{Ker}(\beta) \xrightarrow{\phi} \bigoplus_{i=1}^N(\mathscr{P}_{a_i} \oplus \mathscr{P}_{-a_i}) \xrightarrow{\psi} \operatorname{Coker}(\alpha).$$

Now identify $\operatorname{Ker}(\beta)$ with $(N-1)\mathscr{O}_X(-\Theta)$ by means of the monomorphism

$$(N-1)\mathscr{O}_X \to N\mathscr{O}_X, \quad (f_1, \ldots, f_{N-1}) \mapsto (f_1, \ldots, f_{N-1}, -\textstyle\sum_i f_i)$$

and similarly identify $\operatorname{Coker}(\alpha)$ with $(N-1)\mathscr{O}_X(\Theta)$ by means of the epimorphism

$$N\mathscr{O}_X \to (N-1)\mathscr{O}_X, \quad (f_1, \ldots, f_N) \mapsto (f_1 - f_N, \ldots, f_{N-1} - f_N)$$

(the latter is surjective even when restricted to $\bigoplus_i \mathscr{I}_{Y_i}$ because the Y_i's are pairwise disjoint). Via these identifications, the homomorphisms ϕ and ψ are represented by the matrices as claimed, except that $\vartheta_N^{\pm}$ appears with opposite sign. Change its sign, and we are done. $\square$

Definition 3.3. A monad is *decomposable* if it is isomorphic, as a complex, to a monad of the form appearing in Proposition 3.2.

With this terminology, a rank 2 vector bundle $\mathscr{E}$ can be resolved by a decomposable monad if and only if $\mathscr{E}(\Theta)$ corresponds to a disjoint union $Y = \bigcup_i Y_i$, where $Y_i = \Theta_{a_i} \cap \Theta_{-a_i}$, via the Serre construction.

Remark 3.4. The symmetry seen in the decomposable monads is no accident, but reflects the self duality of $\mathscr{E}$ corresponding to the natural pairing $\wedge$ on $\mathscr{E}$ with values in $\bigwedge^2(\mathscr{E}) \cong \mathscr{O}_X$. See Barth–Hulek [1].

4. Digression on the hyperext spectral sequence

Our basic aim is to understand first order deformations of the bundles $\mathscr{E}$ appearing as the cohomology of a decomposable monad. The strategy is to analyse $\operatorname{Ext}^1(\mathscr{E}, \mathscr{E})$ using the first hyperext spectral sequence associated to the monad. This is in principle straight forward, but requires some honest calculation. As preparation, we collect in this section a few standard constructions in homological algebra, for ease of reference. We fix an abelian category $\mathcal{A}$ with enough injectives and infinite direct sums, and denote by $K(\mathcal{A})$ the homotopy category of complexes and by $D(\mathcal{A})$ the derived category.

4.1. **The spectral sequence.** Let $(M^\bullet, d_M)$ and $(N^\bullet, d_N)$ denote complexes in $\mathcal{A}$, and assume that $N^\bullet$ is bounded from below. The *first hyperext spectral sequence* is a spectral sequence

$$E_1^{pq} = \bigoplus_i \operatorname{Ext}^q(M^i, N^{i+p}) \Rightarrow \operatorname{Ext}^{p+q}(M^\bullet, N^\bullet).$$

Briefly, take a double injective resolution $N^\bullet \to I^{\bullet\bullet}$ with $I^{\bullet\bullet}$ concentrated in the upper half plane (for instance a Cartan-Eilenberg resolution), and form the double complex $\operatorname{Hom}^{\bullet\bullet}(M^\bullet, I^{\bullet\bullet})$. The required spectral sequence is the first spectral sequence associated to this double complex.

4.2. **The edge map.** Along the axis $q = 0$, the first sheet of the spectral sequence in Section 4.1 has the usual hom-complex $\operatorname{Hom}^\bullet(M^\bullet, N^\bullet)$. Its cohomology is

$$E_2^{p,0} = \operatorname{Hom}_{K(\mathcal{A})}(M^\bullet, N^\bullet[p])$$

where the right hand side denotes homotopy classes of morphisms of complexes. Since all differentials emanating from $E_r^{p,0}$ for $r \geq 2$ vanish, there are canonical *edge maps*

$$E_2^{p,0} \twoheadrightarrow E_\infty^{p,0} \subset \operatorname{Ext}^p(M^\bullet, N^\bullet)$$

Viewing the right hand side as the group $\operatorname{Hom}_{D(\mathcal{A})}(M^\bullet, N^\bullet[p])$ of morphisms in the derived category, it is reasonable to expect, and not hard to verify, that the edge map is in fact the canonical map

$$\operatorname{Hom}_{K(\mathcal{A})}(M^\bullet, N^\bullet[p]) \to \operatorname{Hom}_{D(\mathcal{A})}(M^\bullet, N^\bullet[p]).$$

Thus the image of $E_2^{p,0}$ in the limit object $\operatorname{Ext}^p(M^\bullet, N^\bullet)$, consists of those p-extensions that can be realized by actual morphisms $M^\bullet \to N^\bullet[p]$ between complexes, without inverting quasi-isomorphisms.

4.3. **Differentials at E_2.** For $q = 1$, it is convenient to view elements of $\mathrm{Ext}^1(M^i, N^{i+p})$ as extensions, in the sense of short exact sequences, and this viewpoint leads to the following interpretation of the differentials d_2^{p1} at the E_2-level:

Lemma 4.1. *Let $\xi \in E_1^{p1}$ be given as a collection of extensions*

$$\xi_i \colon 0 \to N^{i+p} \to X^i \to M^i \to 0.$$

(1) *We have $d_1^{p1}(\xi) = 0$ if and only if there are maps f^i such that the diagram*

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & N^{i+p-1} & \xrightarrow{d_N} & N^{i+p} & \xrightarrow{d_N} & N^{i+p+1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & X^{i-1} & \xrightarrow{f^{i-1}} & X^i & \xrightarrow{f^i} & X^{i+1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & M^{i-1} & \xrightarrow{d_M} & M^i & \xrightarrow{d_M} & M^{i+1} & \longrightarrow & \cdots
\end{array}$$

commutes.

(2) *If we have such a collection of maps (f^i), then ξ represents an element of E_2^{p1}, and the differential*

$$d_2^{p1} \colon E_2^{p1} \to E_2^{p+2,0} = \mathrm{Hom}_{K(\mathcal{A})}(M^\bullet, N^\bullet[p+2])$$

sends ξ to the morphism having components $M^{i-1} \to N^{i+p+1}$ induced by $f^i \circ f^{i-1}$. In particular $d_2^{p1}(\xi) = 0$ if and only if there exists a collection (f^i) making the middle row in the diagram in (1) a complex.

Proof. This is straight forward, although tedious, to verify directly from the construction of the spectral sequence. □

4.4. **Serre duality.** Let X be a scheme of pure dimension d over a field, with a dualizing sheaf ω_X such that Grothendieck-Serre duality holds. Let $M^\bullet$ be a bounded below complex of coherent $\mathscr{O}_X$-modules. We obtain two spectral sequences from (5): one abutting to $\mathrm{Ext}^n(\mathscr{O}_X, M^\bullet) = H^n(X, M^\bullet)$, which we denote by E, and one abutting to $\mathrm{Ext}^n(M^\bullet, \omega_X)$, which we denote by $\hat{E}$. Then E is nothing but the first hypercohomology spectral sequence, and the E_1-levels of E and $\hat{E}$ are Grothendieck-Serre dual. We need to know that the duality extends to all sheets.

Lemma 4.2. *The two spectral sequences E and $\hat{E}$ are dual in the following sense:*

(1) *There are canonical dualities between the vector spaces E_r^{pq} and $\hat{E}_r^{-p,d-q}$ for all p, q, r, extending the Grothendieck-Serre duality between $H^q(X, M^p)$ and $\mathrm{Ext}^{n-q}(M^p, \omega)$ for $r = 1$.*

(2) *The differentials*

$$d_r^{pq} \colon E_r^{pq} \to E_r^{p+r,q-r+1}$$
$$\hat{d}_r^{-p-r,d-q+r-1} \colon \hat{E}_r^{-p-r,d-q+r-1} \to \hat{E}_r^{-p,d-q}$$

are dual maps.

(The statement can be extended to give a full duality between the two spectral sequences, including the filtrations on the abutments and all maps involved. The above is sufficient for our needs.)

Proof. This seems to be well known. We include a sketch, following Herrera–Liebermann [8] (they work in a context where the complexes have differentials that are differential operators of degree one; this demands more care than in our situation). Firstly, for any three complexes $L^\bullet$, $M^\bullet$, $N^\bullet$, the Yoneda pairing

$$\operatorname{Ext}^i(L^\bullet, M^\bullet) \times \operatorname{Ext}^j(M^\bullet, N^\bullet) \to \operatorname{Ext}^{i+j}(L^\bullet, N^\bullet)$$

can be defined on hyperext groups by resolving $M^\bullet$ and $N^\bullet$ by injective double complexes, and taking the double hom complex. On this "resolved" level, the Yoneda pairing is given by composition, and there is an induced pairing of hyperext spectral sequences in the appropriate sense, which specializes to the usual Yoneda pairing between ext groups of the individual objects L^l, M^m, N^n at the E_1-level. Specialize to the situation $L^\bullet = \mathscr{O}_X$ and $N^\bullet = \omega_X$ to obtain a morphism of spectral sequences from E to the dual of $\hat{E}$, in the above sense. At the E_1-level this is the Grothendieck-Serre duality map, hence an isomorphism, which is enough to conclude that it is an isomorphism of spectral sequences [5, Section 11.1.2]. □

Remark 4.3. If $M^\bullet$ and $N^\bullet$ denote two complexes of vector bundles, then we may apply the Lemma to the complex $(M^\bullet)^\vee \otimes N^\bullet$ to obtain a duality between the two hyperext spectral sequences abutting to $\operatorname{Ext}^n(M^\bullet, N^\bullet)$ and $\operatorname{Ext}^n(N^\bullet, M^\bullet \otimes \omega_X)$, respectively.

5. Deformations of decomposable monads

We now apply the homological algebra from the previous section to analyse first order deformations of vector bundles $\mathscr{E}$ which can be resolved by a decomposable monad. Firstly, we find that deformations obtained by varying the isomorphism $\omega_Y \cong \mathscr{O}_Y(2\Theta)$ in the Serre construction coincides with the deformations obtained by varying the differential in the monad, while keeping the objects fixed. Secondly, and this is the nontrivial part, we find that all first order deformations of $\mathscr{E}$ can be obtained by also deforming the objects in the monad, and there are more of these deformations than those obtained by varying Y in the Serre construction. Since the objects in the monad are sums of line bundles, their first order deformations are easy to understand, so we are able to compute the dimension of $\operatorname{Ext}^1(\mathscr{E}, \mathscr{E})$, in Theorem 5.7.

$$
\begin{array}{ccccccccc}
E_1^{-2,3} & \longrightarrow & E_1^{-1,3} & \longrightarrow & E_1^{0,3} & & & & \\
 & & & & E_1^{0,2} & & & & \\
 & & & & E_1^{0,1} & & & & \\
 & & & & E_1^{0,0} & \longrightarrow & E_1^{1,0} & \longrightarrow & E_1^{2,0}
\end{array}
$$

FIGURE 1. The first sheet in the spectral sequence for $\operatorname{Ext}^i(\mathscr{E},\mathscr{E})$

5.1. **Calculations in the spectral sequence.** Let $\mathscr{E}$ be the rank 2 vector bundle given as the cohomology of a decomposable monad

$$M^\bullet\colon\ \mathscr{A} \xrightarrow{\phi} \mathscr{B} \xrightarrow{\psi} \mathscr{C}$$

given explicitly in Proposition 3.2. In particular, $\mathscr{C} = \mathscr{A}^\vee$, and $\mathscr{B}$ is self dual. If we fix the self duality $\iota\colon \mathscr{B} \to \mathscr{B}^\vee$, given by the direct sum of the skew symmetric

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : \mathscr{P}_{a_i} \oplus \mathscr{P}_{-a_i} \to \mathscr{P}_{-a_i} \oplus \mathscr{P}_{a_i},$$

then $\psi = \phi^\vee \circ \iota$. More generally, for any map $f\colon \mathscr{A} \to \mathscr{B}$, we define its transpose $f^t\colon \mathscr{B} \to \mathscr{C}$ by

$$f^t = f^\vee \circ \iota.$$

Thus ψ is the transpose of ϕ.[2]

The spectral sequence from Section 4.1 gives

$$E_1^{pq} = \bigoplus_i \operatorname{Ext}^q(M^i, M^{i+p}) \Rightarrow \operatorname{Ext}^{p+q}(\mathscr{E},\mathscr{E}). \tag{5}$$

Using that, for any $x \in X$, line bundles of the form $\mathscr{P}_x(m\Theta)$ have sheaf cohomology concentrated in degree 0 when $m > 0$ and in top degree when $m < 0$, we see that the nonzero terms in the first sheet have the shape depicted in Figure 1. It follows that all differentials at level E_r vanish for $r = 3$ and $r > 4$. Also, the duality of Section 4.4, applied to $M^\bullet \otimes (M^\bullet)^\vee$, shows that each term E_r^{pq} is dual to $E_r^{-p,3-q}$, and similarly for the differentials. In this section we analyse the E_2-sheet, and get as a consequence that the spectral sequence in fact degenerates at the E_3-level.

[2]One can show that, in the affine space of all homomorphisms $f\colon \mathscr{A} \to \mathscr{B}$, the locally closed subset U defined by (1) f is an embedding of vector bundles, and (2) the composition $f^t \circ f$ is zero, has an irreducible connected component corresponding to decomposable monads. It seems plausible that this component is all of U.

5.1.1. *The objects* E_2^{pq}. By duality, it suffices to consider the lower half of Figure 1. The only nonzero differentials in this area, at the E_1-level, are in the lower row $q = 0$. We observed in Section 4.2 that the cohomology groups of this row are the groups of morphisms $M^\bullet \to M^\bullet[p]$ modulo homotopy.

Lemma 5.1. *The dimensions of* $E_2^{p,0}$ *for* $p = 0, 1, 2$ *are* 1, $N-1$ *and* $6(N-1)^2 - N + 2$, *respectively.*

Proof. The vector spaces in question are the cohomologies of the complex

$$0 \longrightarrow E_1^{0,0} \xrightarrow{d_1^{0,0}} E_1^{1,0} \xrightarrow{d_1^{1,0}} E_1^{2,0} \longrightarrow 0,$$

where

$$\begin{aligned}
\dim E_1^{0,0} &= \dim\big(\operatorname{Hom}(\mathscr{A},\mathscr{A}) \oplus \operatorname{Hom}(\mathscr{B},\mathscr{B}) \oplus \operatorname{Hom}(\mathscr{C},\mathscr{C})\big)\\
&= 2(N-1)^2 + 2N\\
\dim E_1^{1,0} &= \dim\big(\operatorname{Hom}(\mathscr{A},\mathscr{B}) \oplus \operatorname{Hom}(\mathscr{B},\mathscr{C})\big)\\
&= 4N(N-1)\\
\dim E_1^{2,0} &= \dim \operatorname{Hom}(\mathscr{A},\mathscr{C})\\
&= 8(N-1)^2
\end{aligned} \tag{6}$$

(using that the space of global sections of $\mathscr{O}_X(\Theta_{\pm a_i})$ has dimension 1, and the space of global sections of $\mathscr{O}_X(2\Theta)$ has dimension 8). Thus it suffices to compute the dimensions of the kernels of the two differentials $d_1^{0,0}$ and $d_1^{1,0}$, i.e. the vector spaces of morphisms of degree 0 and 1 from the monad to itself.

One checks immediately that any morphism $M^\bullet \to M^\bullet$ (of degree 0) is multiplication with a scalar, so

$$\dim E_2^{0,0} = 1. \tag{7}$$

Next we compute the dimension of the space of morphisms $M^\bullet \to M^\bullet[1]$. Since $\mathscr{C} = \mathscr{A}^\vee$, such a morphism is given by an element of

$$\operatorname{Hom}(\mathscr{A},\mathscr{B}) \oplus \operatorname{Hom}(\mathscr{B},\mathscr{A}^\vee),$$

which we may write as $(\mu, -\nu^t)$, where both μ and ν are homomorphisms $\mathscr{A} \to \mathscr{B}$. The sign on $-\nu^t$ is inserted to compensate for the sign on the differential in the shifted complex $M^\bullet[1]$; thus $(\mu, -\nu^t)$ defines a morphism $M^\bullet \to M^\bullet[1]$ if and only if $\nu^t \circ \phi = \phi^t \circ \mu$.

As in Proposition 3.2, we decompose these homomorphisms into pairs $\mu^\pm$ and $\nu^\pm$, and then

$$\begin{aligned}
\nu^t \circ \phi &= (\nu^-)^\vee \circ \phi^+ - (\nu)^{+\vee} \circ \phi^-\\
\phi^t \circ \mu &= (\phi^-)^\vee \circ \mu^+ - (\phi^+)^\vee \circ \mu^-.
\end{aligned} \tag{8}$$

Choosing generators $\vartheta_i^\pm \in \Gamma(X, \mathscr{O}_X(\Theta_{\pm a_i}))$, we may represent μ by a matrix with entries $\mu_{ij}^\pm \vartheta_i^\pm$, where $\mu_{ij}^\pm$ are scalars. Similarly for ν. Then the two compositions (8) are given by $(N-1)\times(N-1)$ scalar matrices with entries

$$\begin{aligned}(\nu^t\circ\phi)_{ij} &= (\mu^+_{ij}-\mu^-_{ij})\vartheta^+_i\vartheta^-_i + (\mu^+_{Nj}-\mu^-_{Nj})\vartheta^+_N\vartheta^-_N\\ (\phi^t\circ\mu)_{ij} &= (\nu^+_{ji}-\nu^-_{ji})\vartheta^+_j\vartheta^-_j + (\nu^+_{Ni}-\nu^-_{Ni})\vartheta^+_N\vartheta^-_N.\end{aligned}\tag{9}$$

Recall that the Kummer map $X\to|2\Theta|$ sends $a_i\in X$ to the divisor $\Theta_{a_i}+\Theta_{-a_i}$. This implies that, for sufficiently general points a_i, and $i\neq j$, the three elements $\vartheta^+_i\vartheta^-_i$, $\vartheta^+_j\vartheta^-_j$ and $\vartheta^+_N\vartheta^-_N$ are linearly independent in $\Gamma(X,\mathscr{O}_X(2\Theta))$. It follows easily that the two expressions in (9) coincide for all i and j if and only if there are equalities of scalar $(N-1)\times(N-1)$ matrices

$$(\mu^+_{ij})-(\mu^-_{ij})=(\nu^+_{ij})-(\nu^-_{ij})=\begin{pmatrix}c_1 & & & \\ & c_2 & & \\ & & \ddots & \\ & & & c_{N-1}\\ c_N & c_N & \cdots & c_N\end{pmatrix},$$

where $c_1,\dots,c_N$ are arbitrary scalars. Thus the vector space of morphisms $M^\bullet\to M^\bullet[1]$ has a basis corresponding to the (μ^+_{ij}), (ν^+_{ij}) and (c_i), hence has dimension $2N(N-1)+N$. The expressions for $\dim E_2^{p,0}$ follow from this, together with (6) and (7). □

5.1.2. *The differentials d_2^{pq}.* The only nonzero differentials at the E_2-level are $d_2^{0,1}$ and its dual $d_2^{-2,3}$. So it suffices to analyse $d_2^{0,1}$. This is, by Lemma 4.1, an obstruction map for equipping first order infinitesimal deformations of the objects M^i with differentials, and will henceforth be denoted ob.

The domain

$$E_2^{0,1}=\bigoplus\nolimits_i \operatorname{Ext}^1(M^i,M^i)\tag{10}$$

of $\mathrm{ob}=d_2^{0,1}$ is canonically isomorphic to a direct sum of a large number of copies of $H^1(X,\mathscr{O}_X)$. More precisely, for each i and j from 1 to $N-1$, apply the bifunctor $\operatorname{Ext}^1(-,-)$ to the i'th projection $\mathscr{A}\to\mathscr{O}_X(-\Theta)$ in the first argument and the j'th inclusion $\mathscr{O}_X(-\Theta)\to\mathscr{A}$ in the second argument. This defines an inclusion

$$f_{ij}\colon H^1(X,\mathscr{O}_X)\cong\operatorname{Ext}^1(\mathscr{O}_X(-\Theta),\mathscr{O}_X(-\Theta))\hookrightarrow\operatorname{Ext}^1(\mathscr{A},\mathscr{A})$$

and clearly the direct sum of all the f_{ij}'s is an isomorphism. Similarly, for all i and j from 1 to $N-1$, we define inclusions

$$h_{ij}\colon H^1(X,\mathscr{O}_X)\cong\operatorname{Ext}^1(\mathscr{O}_X(\Theta),\mathscr{O}_X(\Theta))\hookrightarrow\operatorname{Ext}^1(\mathscr{C},\mathscr{C})$$

whose direct sum is an isomorphism. Finally, for all i from 1 to N, and each sign $\pm$, define inclusions

$$g_i^\pm\colon H^1(X,\mathscr{O}_X)\cong\operatorname{Ext}^1(\mathscr{P}_{\pm a_i},\mathscr{P}_{\pm a_i})\hookrightarrow\operatorname{Ext}^1(\mathscr{B},\mathscr{B})$$

induced by projection to and inclusion of the summand $\mathscr{P}_{\pm a_i}$ of $\mathscr{B}$. Note that also the direct sum of the $g_i^{\pm}$'s is an isomorphism, since $\mathrm{Ext}^1(\mathscr{P}_x, \mathscr{P}_y) = H^1(X, \mathscr{P}_{y-x})$ vanishes unless $x = y$.

The obstruction map ob takes values in homotopy classes of morphisms $M^\bullet \to M^\bullet[2]$ of complexes, modulo homotopy. Such a morphism is given by a single homomorphism from $\mathscr{A}$ to $\mathscr{C}$, which can be presented as an $(N-1)\times(N-1)$ matrix with entries in $\Gamma(X, \mathscr{O}_X(2\Theta))$. We now give such a matrix representative for the homotopy class $\mathrm{ob}(\xi)$, for any element ξ in each summand $H^1(X, \mathscr{O}_X)$ of $E_2^{0,1}$.

Lemma 5.2. *For every i, the boundary map of the long exact cohomology sequence associated to the Koszul complex*

$$0 \longrightarrow \mathscr{O}_X \xrightarrow{\begin{pmatrix}\vartheta_i^+\\ \vartheta_i^-\end{pmatrix}} \mathscr{O}_X(\Theta_{a_i}) \oplus \mathscr{O}_X(\Theta_{-a_i}) \xrightarrow{\begin{pmatrix}\vartheta_i^- & -\vartheta_i^+\end{pmatrix}} \mathscr{I}_{Y_i}(2\Theta) \longrightarrow 0$$

induces an isomorphism

$$H^0(X, \mathscr{I}_{Y_i}(2\Theta))/\langle \vartheta_i^+\vartheta_i^-\rangle \cong H^1(X, \mathscr{O}_X)$$

where $\langle \vartheta_i^+\vartheta_i^-\rangle$ denotes the one dimensional vector space spanned by the section $\vartheta_i^+\vartheta_i^-$.

Proof. Since $H^1(X, \mathscr{O}_X(\Theta_{\pm a_i})) = 0$, there is an induced right exact sequence

$$\begin{matrix} H^0(X, \mathscr{O}_X(\Theta_{a_i})) \\ \oplus \\ H^0(X, \mathscr{O}_X(\Theta_{-a_i})) \end{matrix} \longrightarrow H^0(X, \mathscr{I}_{Y_i}(2\Theta)) \longrightarrow H^1(X, \mathscr{O}_X) \longrightarrow 0.$$

Each summand $H^0(X, \mathscr{O}_X(\Theta_{\pm a_i}))$ is spanned by $\vartheta_i^{\pm}$, which is sent to $\mp\vartheta_i^+\vartheta_i^-$ in $H^0(X, \mathscr{I}_{Y_i}(2\Theta))$. ☐

Proposition 5.3. *Let $\xi \in H^1(X, \mathscr{O}_X)$. The obstruction map* ob *does the following on each summand in its domain:*

1. *Lift ξ to sections u and v of $\mathscr{I}_{Y_i}(2\Theta)$ and $\mathscr{I}_{Y_N}(2\Theta)$, respectively, using the lemma. Then $\mathrm{ob}(f_{ij}(\xi))$ is represented by the $(N-1)\times(N-1)$ matrix having j'th column (the transpose of)*

$$\begin{matrix}(v & \cdots & v & u+v & v & \cdots & v)\\ & & & \uparrow & & & \\ & & & \textit{(entry } i\textit{)} & & & \end{matrix}$$

and zeros everywhere else.

2. *Lift ξ to a section u of $\mathscr{I}_{Y_i}(2\Theta)$. If $i \neq N$, then $\mathrm{ob}(g_i^{\pm}(\xi))$ is represented by the $(N-1)\times(N-1)$ matrix having u at entry (i,i), and zeros everywhere else. The remaining case $\mathrm{ob}(g_N^{\pm}(\xi))$ is represented by the $(N-1)\times(N-1)$ matrix having all entries equal to u.*

(3) *Lift ξ to sections u and v of $\mathscr{I}_{Y_j}(2\Theta)$ and $\mathscr{I}_{Y_N}(2\Theta)$, respectively. Then $\mathrm{ob}(h_{ij}(\xi))$ is represented by the $(N-1)\times(N-1)$ matrix having i'th row*

$$\begin{pmatrix} v & \cdots & v & \underset{\substack{\uparrow \\ \text{(entry } j)}}{u+v} & v & \cdots & v \end{pmatrix}$$

and zeros everywhere else.

Proof. Notation: In the commutative diagrams that follow, we will use dotted arrows roughly to indicate maps that are not given to us, but need to be filled in by some construction.

Part 1: We view ξ as an extension in $\mathrm{Ext}^1(\mathscr{O}_X(-\Theta), \mathscr{O}_X(-\Theta))$. Writing out the description of ob from Lemma 4.1 in this situation, one arrives at the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{O}_X(-\Theta) & \longrightarrow & \mathscr{F} & \longrightarrow & \mathscr{O}_X(-\Theta) & \longrightarrow & 0 \\ & & \big\downarrow & \searrow^{\phi_i} & \big\downarrow{\scriptstyle \hat\phi_i} & & \big\downarrow & & \\ & & \mathscr{A} & \xrightarrow{\ \phi\ } & \mathscr{B} & \xrightarrow{\ \psi\ } & \mathscr{C} & & \end{array} \tag{11}$$

constructed as follows: The top row is the extension ξ and the bottom row is the monad. The leftmost vertical map is inclusion of the i'th summand, so in terms of matrices, ϕ_i is the i'th column of ϕ. We are required to extend ϕ_i to a map $\hat\phi_i$ making the left part of the diagram commute: one way of doing this is detailed below. The induced vertical map on the right, precomposed with projection $\mathscr{A} \to \mathscr{O}_X(-\Theta)$ on the j'th summand, is a representative for $\mathrm{ob}(f_{ij}(\xi))$.

The assumption that u is a lifting of ξ, means that there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{O}_X(-\Theta) & \xrightarrow{\begin{pmatrix}\vartheta_i^+ \\ \vartheta_i^-\end{pmatrix}} & \mathscr{P}_{a_i} \oplus \mathscr{P}_{-a_i} & \xrightarrow{\begin{pmatrix}\vartheta_i^- & -\vartheta_i^+\end{pmatrix}} & \mathscr{I}_{Y_i}(\Theta) & \longrightarrow & 0 \\ & & \Big\| & & \big\uparrow{\scriptstyle \tilde u} & & \big\uparrow{\scriptstyle u} & & \\ 0 & \longrightarrow & \mathscr{O}_X(-\Theta) & \longrightarrow & \mathscr{F} & \longrightarrow & \mathscr{O}_X(-\Theta) & \longrightarrow & 0 \end{array}$$

in which the rightmost square is a pullback. Similarly, the section v fits in the pullback diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{O}_X(-\Theta) & \xrightarrow{\begin{pmatrix}\vartheta_N^+ \\ \vartheta_N^-\end{pmatrix}} & \mathscr{P}_{a_N} \oplus \mathscr{P}_{-a_N} & \xrightarrow{\begin{pmatrix}\vartheta_N^- & -\vartheta_N^+\end{pmatrix}} & \mathscr{I}_{Y_N}(\Theta) & \longrightarrow & 0 \\ & & \Big\| & & \big\uparrow{\scriptstyle \tilde v} & & \big\uparrow{\scriptstyle v} & & \\ 0 & \longrightarrow & \mathscr{O}_X(-\Theta) & \longrightarrow & \mathscr{F} & \longrightarrow & \mathscr{O}_X(-\Theta) & \longrightarrow & 0 \end{array}$$

Now define $\hat\phi_i$ to be

$$(\tilde u, \tilde v) \colon \mathscr{F} \longrightarrow (\mathscr{P}_{a_i} \oplus \mathscr{P}_{-a_i}) \oplus (\mathscr{P}_{a_N} \oplus \mathscr{P}_{-a_N})$$

followed by the appropriate inclusion to $\mathscr{B}$. One verifies immediately that $\hat{\phi}_i$ extends ϕ_i in (11), and that the induced map in the rightmost part of that diagram is given by the vector as claimed in part 1.

Part 2: We view ξ as an extension in $\mathrm{Ext}^1(\mathscr{P}_{\pm a_i}, \mathscr{P}_{\pm a_i})$. In this situation, the description of ob from Lemma 4.1 boils down to a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{P}_{\pm a_i} & \longrightarrow & \mathscr{G} & \longrightarrow & \mathscr{P}_{\pm a_i} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi_i^{\pm}} & \swarrow & & \nwarrow & \uparrow{\scriptstyle \phi_i^{\pm}} & & \\ & & \mathscr{C} & & & & \mathscr{A} & & \end{array}$$

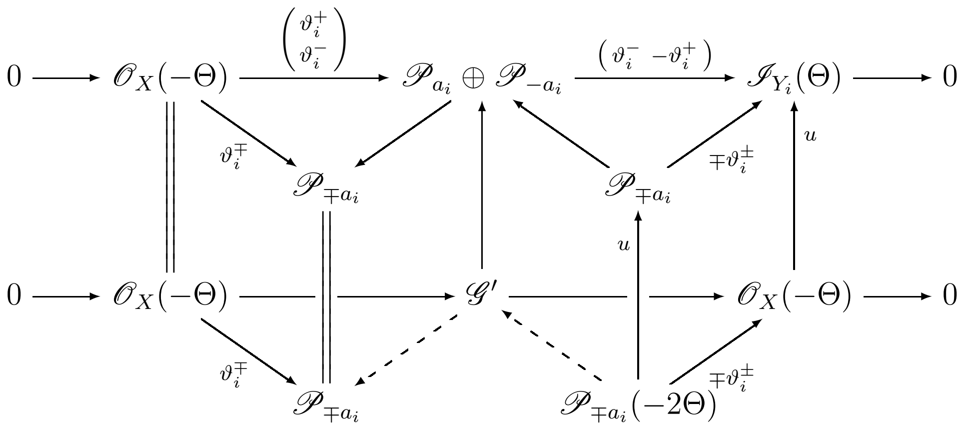
constructed as follows: The top row is ξ and the two vertical maps $\phi_i^{\pm}$ and $\psi_i^{\pm}$ denote the i'th row of $\phi^{\pm}$ and the i'th column of $\psi^{\pm}$. The task is to lift $\phi_i^{\pm}$ to the rightmost dotted arrow, and to extend $\psi_i^{\pm}$ to the leftmost dotted arrow. The composition of the two dotted arrows is then a representative for $\mathrm{ob}(g_i^{\pm}(\xi)))$.

First consider the problem of lifting $\mp\vartheta_i^{\pm}$ and extending $\vartheta_i^{\mp}$ to maps s and t as in the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{P}_{\pm a_i} & \longrightarrow & \mathscr{G} & \longrightarrow & \mathscr{P}_{\pm a_i} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \vartheta_i^{\mp}} & \swarrow{\scriptstyle t} & & \nwarrow{\scriptstyle s} & \uparrow{\scriptstyle \mp\vartheta_i^{\pm}} & & \\ & & \mathscr{O}_X(-\Theta) & & & & \mathscr{O}_X(\Theta) & & \end{array} \tag{12}$$

Suppose such a diagram is given. If $i < N$, then $\mp(0, \ldots, 0, s, 0, \ldots, 0)$ would lift $\phi_i^{\pm}$ and $\pm(0, \ldots, 0, t, 0, \ldots, 0)$ would extend $\psi_i^{\pm}$. Their composition is the matrix having $t \circ s$ in entry (i, i), and zeros elsewhere. If $i = N$, then similarly $\mp(s, \ldots, s)$ and $\pm(t, \ldots, t)$ would be the required lift and extension. Their composition is the matrix having all entries equal to $t \circ s$. Thus part 2 of the proposition will be established once we have constructed such maps s and t having composition $t \circ s = u$.

Now use that ξ is the pullback of the Koszul complex for Y_i along u. This enables us to construct the commutative diagram



as follows: The top row is the Koszul complex. In the top part, the unlabelled diagonal arrows are the canonical inclusion of and projection to the summand $\mathscr{P}_{\mp a_i}$. In particular their composition is the identity map. Pull back along u to get the short exact sequence in the lower part of the diagram. Thus this sequence coincides with ξ, twisted by $\mathscr{P}_{\mp a_i}(-\Theta)$. There are now uniquely determined dotted arrows making the diagram commute, and their composition is u. Twisting back by $\mathscr{P}_{\pm a_i}(\Theta)$, the lower part of the diagram is thus the required diagram (12). This ends the proof of part 2.

Part 3 is essentially dual to part 1, and is left out. □

By Lemma 2.7, we have

$$H^0(\mathscr{I}_{Y_i}(2\Theta)) \oplus H^0(\mathscr{I}_{Y_j}(2\Theta)) = H^0(\mathscr{O}_X(2\Theta)) \tag{13}$$

for all $i \neq j$ (the Lemma gives an inclusion of the left hand side into the right hand side, and by Riemann-Roch, the two sides have the same dimension). This decomposition of sections of $\mathscr{O}_X(2\Theta)$, together with the explicit description of ob in the proposition, enables us to conclude:

Corollary 5.4. *The obstruction map* ob *is surjective.*

Proof. We show that any $(N-1) \times (N-1)$ matrix of sections of $\mathscr{O}_X(2\Theta)$ represents an element in the image of ob. Let i and j be arbitrary indices between 1 and $N-1$.

Step 1: Let v be a section of $\mathscr{I}_{Y_N}(2\Theta)$. Let $\xi \in H^1(X, \mathscr{O}_X)$ be the image of v under the boundary map in Lemma 5.2, and lift ξ to another section u of $\mathscr{I}_{Y_i}(2\Theta)$. By parts 1 and 2 of the Proposition, $\mathrm{ob}(f_{ii}(\xi) - g_i^{\pm}(\xi))$ is represented by the matrix having zeros except for in column i, where all elements equal v. Similarly $\mathrm{ob}(h_{ii}(\xi) - g_i^{\pm}(\xi))$ is represented by a matrix having all entries of row i equal to v, and zeros elsewhere.

Step 2: Let u be a section of $\mathscr{I}_{Y_i}(2\Theta)$. By part 1 of the Proposition and the previous step, we can find a matrix representing an element in the image of ob, with u as entry (i, j) and zeros elsewhere. Similarly, for any section u of $\mathscr{I}_{Y_j}(2\Theta)$, combining part 3 of the proposition with the previous step, we obtain a matrix having u as entry (i, j) and zeros elsewhere.

Step 3: If $i \neq j$, the previous step and (13) enables us to construct a matrix with arbitrary entries outside the diagonal. Combining this with step 1, we can construct a matrix having any given section of $\mathscr{I}_{Y_N}(2\Theta)$ at entry (i, i), and zeros elsewhere. By step 2 we can also construct a matrix having any given section of $\mathscr{I}_{Y_i}(2\Theta)$ at (i, i), and zeros elsewhere. By (13) with $j = N$, this enables us to hit arbitrary elements along the diagonal, too. □

Corollary 5.5. *The spectral sequence* (5) *degenerates at* E_3.

Proof. The previous corollary implies $E_3^{2,0} = 0$. By duality also $E_3^{-2,3} = 0$. It follows from the shape of the first sheet, Figure 1, that all differentials vanish at the E_3-level and beyond. □

5.2. **First order deformations.** From the calculations in the previous section, we can understand infinitesimal deformations of the vector bundle $\mathscr{E}$ in terms of its monad. Let $k[\epsilon]$ be the ring of dual numbers. By a first order deformation of $M^\bullet$, we mean a monad over $X \otimes_k k[\epsilon]$, with $M^\bullet$ as fibre over $\epsilon = 0$, modulo isomorphism.

Theorem 5.6. *Let $M^\bullet$ be a decomposable monad with cohomology $\mathscr{E}$. The vector spaces of first order infinitesimal deformations of $M^\bullet$ and of $\mathscr{E}$ are isomorphic via the natural map, sending a first order deformation of $M^\bullet$ to its cohomology.*

Proof. Since the hyperext spectral sequence associated to the monad degenerates at E_3, and the only E_3^{pq} terms with $p+q=1$ are $E_3^{0,1}$ and $E_3^{1,0}$, there is a short exact sequence

$$0 \longrightarrow E_3^{1,0} \longrightarrow \operatorname{Ext}^1(\mathscr{E},\mathscr{E}) \longrightarrow E_3^{0,1} \longrightarrow 0. \tag{14}$$

Let $D(M^\bullet)$ be the vector space of first order deformations of $M^\bullet$. Thus the claim is that the natural map $D(M^\bullet) \to \operatorname{Ext}^1(\mathscr{E},\mathscr{E})$ is an isomorphism. It suffices to show that $D(M^\bullet) \to E_3^{0,1}$ is surjective, and that its kernel maps isomorphically to $E_3^{1,0}$.

Now $E_3^{0,1}$ is the kernel of the obstruction map $\mathrm{ob} = d_2^{0,1}$. By Lemma 4.1, this is the space of those first order deformations of the objects in $M^\bullet$, that allow the differential d_M to extend (non uniquely) to the deformed objects. Via this identification, $D(M^\bullet) \to E_3^{0,1}$ is the natural forgetful map, so it is surjective. Moreover, its kernel is the space of first order deformations of the differential in $M^\bullet$, keeping the objects fixed. It remains to see that this space gets identified with $E_3^{1,0}$.

By the shape of the spectral sequence (Figure 1) we have $E_3^{1,0} = E_2^{1,0}$, and, by Lemma 5.1, this is

$$E_2^{1,0} = \operatorname{Hom}_{K(X)}(M^\bullet, M^\bullet[1]).$$

The inclusion of $E_2^{1,0}$ into $\operatorname{Ext}^1(\mathscr{E},\mathscr{E})$ is the edge map discussed in Section 4.2, i.e. the canonical map

$$\operatorname{Hom}_{K(X)}(M^\bullet, M^\bullet[1]) \to \operatorname{Hom}_{D(X)}(M^\bullet, M^\bullet[1]). \tag{15}$$

This can be factored as follows: a morphism of complexes in an arbitrary abelian category $f\colon X^\bullet \to Y^\bullet[1]$ gives rise to a short exact sequence of complexes

$$0 \longrightarrow Y^\bullet \xrightarrow{\ \beta\ } Z^\bullet \xrightarrow{\ \alpha\ } X^\bullet \longrightarrow 0 \tag{16}$$

where $Z^\bullet = C(f[-1])$ is the mapping cone of $f[-1]$, which has objects $X^i \oplus Y^i$ in degree i and differential $(x,y) \mapsto (dx, f(x)+dy)$. The maps α and β are the canonical ones. Moreover, the usual Yoneda construction of elements in Ext^1 from short exact sequences (of objects) can be extended to complexes, by associating to any short sequence (16) the roof

$$X^\bullet \xleftarrow{\text{qism}} C(\beta) \longrightarrow Y^\bullet[1]$$

where $C(\beta)$ is the mapping cone, with objects $Y^{i+1} \oplus Z^i$ in degree i and differential $(y, z) \mapsto (-dy, \beta(y) + dz)$. The leftmost map is given by projection, and is a quasi-isomorphism, whereas the rightmost map is projection followed by α. This roof defines a morphism $X^\bullet \to Y^\bullet[1]$ in the derived category. Moreover, the diagram obtained from the roof by adding the negative of the map $f\colon X^\bullet \to Y^\bullet[1]$ we started with, is commutative up to homotopy, so the roof and $-f$ defines the same map in the derived category.

Thus we have factored the edge map (15) via short exact sequences, by sending $f\colon M^\bullet \to M^\bullet[1]$ to the short exact sequence

$$0 \longrightarrow M^\bullet \longrightarrow C(f[-1]) \longrightarrow M^\bullet \longrightarrow 0.$$

The associated element in $\operatorname{Hom}_{D(X)}(M^\bullet, M^\bullet[1])$ corresponds, up to sign, to the Yoneda class in $\operatorname{Ext}^1(\mathscr{E}, \mathscr{E})$ obtained by taking the H^0 cohomology of each complex in this short exact sequence. To phrase this in terms of first order deformations, we rewrite the cone $C(f[-1])$ as the complex $M^\bullet \otimes_k k[\epsilon]$ equipped with the differential $d_M \otimes 1 + f \otimes \epsilon$. The corresponding deformation of $\mathscr{E}$ is the H^0 cohomology of this complex. But this is the required result, since any differential on $M^\bullet \otimes_k k[\epsilon]$ that specializes to d_M for $\epsilon = 0$ has the form $d_M \otimes 1 + f \otimes \epsilon$, for some f satisfying

$$(d_M \otimes 1 + f \otimes \epsilon)^2 = 0.$$

Since $d_M^2 = 0$ and $\epsilon^2 = 0$, this says that $f \circ d_M + d_M \circ f = 0$, i.e. f defines a morphism $M^\bullet \to M^\bullet[1]$. This gives the required identification between $E_2^{1,0}$ and deformations of the differential. □

Next, we give the dimension formula for $\operatorname{Ext}^1(\mathscr{E}, \mathscr{E})$, which we phrase in a twist invariant way.

Theorem 5.7. *Let $\mathscr{E}$ be a rank 2 vector bundle obtained as the cohomology of a decomposable monad, or the twist of such a bundle by a line bundle. Then*

$$\dim \operatorname{Ext}^1(\mathscr{E}, \mathscr{E}) = \tfrac{1}{3}\Delta(\mathscr{E}) \cdot \Theta + 5$$

where Δ denotes the discriminant $4c_2 - c_1^2$.

Proof. Both sides of the equation are invariant under twist, so it suffices to verify the formula when $\mathscr{E}$ is the cohomology of a decomposable monad. Consider again the short exact sequence (14).

The space $E_3^{0,1}$ is the kernel of the map $\mathrm{ob} = d_2^{0,1}$ studied in Section 5.1.2. Its domain (10) has dimension

$$(2(N-1)^2 + 2N)\dim H^1(X, \mathscr{O}_X) = 6(N-1)^2 + 6N$$

and its codomain $E_2^{2,0}$ has dimension $6(N-1)^2 - N + 2$, by Lemma 5.1. Since ob is surjective, the dimension of its kernel $E_3^{0,1}$ is thus $7N-2$. Moreover, the dimension of $E_3^{1,0} = E_2^{1,0}$ is $N-1$ by the same Lemma, so $\mathrm{Ext}^1(\mathscr{E}, \mathscr{E})$ has dimension $8N-3$.

On the other hand, we know from the Serre construction that $\mathscr{E}(\Theta)$ has Chern classes $c_1 = 2\Theta$ and $c_2 = N\Theta^2$, and thus discriminant $(4N-4)\Theta^2$. The formula follows. □

Remark 5.8. The space of first order deformations obtained by varying the isomorphism $\omega_Y \cong \mathscr{O}_Y(2\Theta)$, coincides with the space of first order deformations of the differential in $M^\bullet$. In fact, it is trivial that the former is contained in the latter, and these spaces have the same dimension $N-1$, using Proposition 2.8.

Remark 5.9. The short exact sequence (14), and its interpretation given in the proof of Theorem 5.6, is not intrinsic to $\mathscr{E}$, but results from our choice of representing $\mathscr{E}$ by a decomposable monad. However, deformation of the differential in $M^\bullet$, or equivalently, variation of the isomorphism $\mathscr{O}_Y(2\Theta) \cong \omega_Y$ in the Serre construction, defines a rational $(N-1)$-dimensional subvariety through $\mathscr{E}$ in its moduli space, whose tangent space is $E_3^{0,1}$. It seems plausible that this $(N-1)$-dimensional rational variety can be intrinsically characterized as the unique (maximal) rational variety through $\mathscr{E}$.

6. Birational description of $M(0, \Theta^2)$

As before, let (X, Θ) be a principally polarized abelian threefold with Picard number 1. We write $M(c_1, c_2)$ for the coarse moduli space of stable rank 2 vector bundles on X with the indicated Chern classes.

The main point in the preceding section is that all first order infinitesimal deformations of the vector bundles constructed in Section 4.4, in the case of even c_1, can be realized as first order infinitesimal deformations of a monad. In this section we show that in the first nontrivial example, corresponding to $N = 2$, this statement holds not only infinitesimally, but Zariski locally: by deforming the monad, we realize a Zariski open neighbourhood of the vector bundle in its moduli space. In terms of the Serre construction, this is the case corresponding to curves $Y_1 \cup Y_2$ with two components $Y_i = \Theta_{a_i} \cap \Theta_{-a_i}$.

Theorem 6.1. *Let $\mathscr{E}$ be the rank 2 cohomology vector bundle of a decomposable monad, as in Proposition 3.2 for $N = 2$. Then, Zariski locally around $\mathscr{E}$, the moduli space $M(0, \Theta^2)$ is a uniruled, nonsingular variety of dimension 13.*

More precisely, there is a Zariski open neighbourhood around $\mathscr{E}$ which is isomorphic to a nonsingular Zariski open subset of a $\mathbb{P}^1$-bundle over a finite quotient of $X^2 \times_X X^2 \times_X X^2$, where X^2 is considered as a scheme over X via the group law.

Proof. We write down a parameter space for the family of monads

$$0 \longrightarrow \mathscr{P}_{b'}(-\Theta) \xrightarrow{\phi} \bigoplus_{i=1}^2(\mathscr{P}_{a_i} \oplus \mathscr{P}_{a'_i}) \xrightarrow{\psi} \mathscr{P}_b(\Theta) \longrightarrow 0 \tag{17}$$

where a_i, a'_i, b, b' are sufficiently general points in X satisfying

$$a_1 + a'_1 = a_2 + a'_2 = b + b', \tag{18}$$

and

$$\phi = (\vartheta_1, \vartheta'_1, \vartheta_2, \vartheta'_2), \quad \psi = (\vartheta'_1, -\vartheta_1, \vartheta'_2, -\vartheta_2)$$

and where the ϑ's are required to be nonzero, but otherwise arbitrary.

Viewing X^2 as a variety over X via the group law, the fibred product $X^2 \times_X X^2 \times_X X^2$ is the subvariety of X^6 defined by (18). Let

$$T \subset X^2 \times_X X^2 \times_X X^2$$

be the open subset consisting of sixtuples $(a_1, a'_1; a_2, a'_2; b, b')$ where the leading four entries are all distinct. Later we may have to shrink T further. With the help of the Poincaré line bundle on $X \times X$ it is clear that, on $T \times X$, there exist vector bundles $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ whose fibres over a sixtuple in T are the three objects in (17). The sixtuples $(a_1, a'_1; a_2, a_2; b, b')$ in T corresponding to the same three objects constitute an orbit for the action of

$$G = (\mathbb{Z}/(2) \oplus \mathbb{Z}/(2)) \rtimes \mathbb{Z}/(2) \tag{19}$$

on T, where the action of the first semidirect factor is given by the transpositions $a_1 \leftrightarrow a'_1$ and $a_2 \leftrightarrow a'_2$, and the last factor acts by $(a_1, a'_1) \leftrightarrow (a_2, a'_2)$. Thus T/G is a parameter space for the objects in (17).

Next we parametrize the maps ϕ and ψ, which are given by four nonzero sections

$$\begin{aligned} &\vartheta_1 \in \Gamma(X, \mathscr{P}_{a_1-b'}(\Theta)) \quad \vartheta'_1 \in \Gamma(X, \mathscr{P}_{a'_1-b'}(\Theta)) \\ &\vartheta_2 \in \Gamma(X, \mathscr{P}_{a_2-b'}(\Theta)) \quad \vartheta'_2 \in \Gamma(X, \mathscr{P}_{a'_2-b'}(\Theta)) \end{aligned} \tag{20}$$

There exist line bundles L_1, L'_1, L_2, L'_2 on T, whose fibres over a sixtuple $(a_1, a'_1; a_2, a'_2; b, b')$ are these (one dimensional) spaces of global sections. Thus, writing

$$F = \bigoplus_{i=1}^2(L_i \oplus L'_i) \xrightarrow{p} T,$$

a point of F, whose four entries are all nonzero, corresponds to a monad (17). More precisely, writing p_X for the product $p \times \mathrm{id}_X \colon F \times X \to T \times X$, there exists a monad

$$p_X^*\mathscr{A} \xrightarrow{\Phi} p_X^*\mathscr{B} \xrightarrow{\Psi} p_X^*\mathscr{C}$$

on $F \times X$, whose restriction to the point in F given by (20) is (17). Let $F' \subset F$ be the open subset consisting of quadruples with only nonzero entries. The cohomology of the "universal" monad above is a family of vector bundles over F', giving rise to a morphism of schemes

$$\phi\colon F' \to M(0, \Theta^2). \tag{21}$$

To make this morphism an embedding, we will divide by the group G to get rid of the ambiguity in the parametrization of the objects by T, and further divide by another group Γ to take care of distinct maps ϕ, ψ which give isomorphic monads.

For a fixed base point in T, and hence fixed objects in (17), the tuples $(\vartheta_1, \vartheta_1', \vartheta_2, \vartheta_2')$ which define isomorphic monads constitute orbits under the following group action on F: view $\mathbb{G}_m^2$ as a variety over $\mathbb{G}_m$ via the multiplication map, and let

$$\Gamma = \mathbb{G}_m^2 \times_{\mathbb{G}_m} \mathbb{G}_m^2.$$

Its closed points are tuples $(\lambda_1, \lambda_1'; \lambda_2, \lambda_2')$ satisfying $\lambda_1\lambda_1' = \lambda_2\lambda_2'$. The action on the fibres of F is given on closed points by

$$(\vartheta_1, \vartheta_1', \vartheta_2, \vartheta_2') \mapsto (\lambda_1\vartheta_1, \lambda_1'\vartheta_1', \lambda_2\vartheta_2, \lambda_2'\vartheta_2').$$

There is a short exact sequence of group varieties

$$1 \longrightarrow \mathbb{G}_m^2 \longrightarrow \Gamma \longrightarrow \mathbb{G}_m \longrightarrow 1$$

where the inclusion sends (λ_1, λ_1') to $(\lambda_1, \lambda_1^{-1}, \lambda_2, \lambda_2^{-1})$ and the projection sends $(\lambda_1, \lambda_1', \lambda_2, \lambda_2')$ to $\lambda_i\lambda_i'$. Correspondingly, we determine F/Γ in two steps. Firstly, the categorical quotient by the $\mathbb{G}_m^2$-action is

$$F/\mathbb{G}_m^2 \cong \bigoplus_{i=1}^2 (L_i \otimes L_i')$$

and the quotient map $F \to F/\mathbb{G}_m^2$ corresponds to multiplication in the fibres (locally on T, this is the product of two copies of the quotient $\operatorname{Spec} R[x,y] \to \operatorname{Spec} R[xy]$ for the $\mathbb{G}_m$-action $(x,y) \to (\lambda x, \lambda^{-1}x)$ on $\mathbb{A}_R^2$ over an arbitrary ring R). The induced action of $\Gamma/\mathbb{G}_m^2 \cong \mathbb{G}_m$ on the rank two vector bundle $F/\mathbb{G}_m^2$ is multiplication in the fibres, so the quotient $P = \{F \setminus 0\}/\Gamma$ is

$$P = \mathbb{P}(\bigoplus_{i=1}^2 (L_i \otimes L_i')^\vee),$$

which is a $\mathbb{P}^1$-bundle over T. The image $P' \subset P$ of $F' \subset F$ is an open subset; in fact it is the complement of the two natural sections corresponding to the subbundles $L_i \otimes L_i'$ of $F/\mathbb{G}_m^2$. The restricted quotient map

$$F' \to F'/\Gamma = P'$$

is a geometric quotient; in particular its fibres are orbits in F'. It is clear that the morphism (21) is invariant with respect to the Γ-action on F', so there is an induced morphism

$$\overline{\phi}\colon P' \to M(0, \Theta^2).$$

Moreover, the (free) action (19) of G on T has a canonical lift to P, and P' is G-invariant. Again $\overline{\phi}$ is invariant under this action, so we obtain the $\mathbb{P}^1$-bundle P/G over T/G, together with an open subset P'/G and an induced morphism

$$\overline{\overline{\phi}}\colon P'/G \to M(0, \Theta^2).$$

By construction, the domain P'/G parametrizes isomorphism classes of monads of the form (17). Given two such monads $M_1^\bullet$ and $M_2^\bullet$, with cohomology $\mathscr{E}_1$ and $\mathscr{E}_2$, the first hyperext spectral sequence gives an isomorphism $\operatorname{Hom}(M_1^\bullet, M_2^\bullet) \xrightarrow{\sim} \operatorname{Hom}(\mathscr{E}_1, \mathscr{E}_2)$. Here, the domain is the $E_2^{0,0}$-term in the spectral sequence, which is the group of morphisms of complexes (there are no homotopies, since $E_1^{-1,0}$ vanishes). It follows that $M_1^\bullet$ and $M_2^\bullet$ are isomorphic as complexes if and only if $\mathscr{E}_1$ and $\mathscr{E}_2$ are isomorphic vector bundles. In other words $\overline{\overline{\phi}}$ is injective on closed points. Shrinking T if necessary, we may apply Theorem 5.6 to see that $\overline{\overline{\phi}}$ is étale at points where $b = b' = 0$. By shrinking its domain further if necessary, we can assume that it is étale everywhere. An étale and injective morphism is an open embedding, so we are done. □

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