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Nicolaas Govert de Bruijn, the enchanter of friable integers

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In memoriam: Nicolaas Govert ('Dick') de Bruijn (1918–2012)

Abstract

N.G. de Bruijn carried out fundamental work on integers having only small prime factors and the Dickman–de Bruijn function that arises on computing the density of those integers. In this he used his earlier work on linear functionals and differential–difference equations. We review his relevant work and also some later improvements by others.

Keywords: N.G. de Bruijn; Friable integers; Dickman–de Bruijn function

1. Introduction

The number theoretical work of Nicolaas Govert ('Dick') de Bruijn (1918–2012) comes into two highly distinct flavours: combinatorial and analytical. In his combinatorial number theoretical work de Bruijn even did some cowork with one of the all-time greats in this area: Paul Erdős (6 joint papers!). Here we will only discuss de Bruijn's work in analytic number theory. This was done mainly in two periods: 1948–1953 and 1962–1966. In the second period de Bruijn revisited his earlier subjects. Some of the later work is joint with Jacobus Hendricus ('Jack') van Lint (1932–2004) [34,99].

In *sieve theory* (see, e.g., [46,52] or for a very brief introduction Section 6 below) one is interested in estimating in terms of elementary functions the number of integers having a prescribed factorization in a prescribed sequence. Usually one is interested in the integers in the

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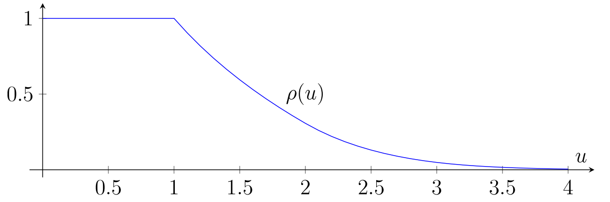

Fig. 1. The Dickman–de Bruijn function $\rho(u)$.

sequence that are primes and if one cannot handle those, integers that have only a few distinct prime factors. De Bruijn's work in analytic number theory belongs mainly to sieve theory, but he is not restricting the number of prime factors of the integers that remain after sieving. This is a milder form of sieving and here usually quite sharp estimates can be obtained. As a basic example one can take the friable number counting function $\Psi(x, y)$. Let $P(n)$ denote the largest prime divisor of an integer $n \geqslant 2$. Put $P(1) = 1$. A number n is said to be *y-friable*[1] if $P(n) \leqslant y$. We let $S(x, y)$ denote the set of integers $1 \leqslant n \leqslant x$ such that $P(n) \leqslant y$. The cardinality of $S(x, y)$ is denoted by $\Psi(x, y)$.

In 1930, Dickman[2] [38] proved that

$$\lim_{x\to\infty} \frac{\Psi(x, x^{1/u})}{x} = \rho(u), \tag{1}$$

where the *Dickman function* (today often also called *Dickman–de Bruijn function*) $\rho(u)$ is defined by

$$\rho(u) = \begin{cases} 1 & \text{for } 0 \leqslant u \leqslant 1; \\ \dfrac{1}{u}\displaystyle\int_0^1 \rho(u-t)\,\mathrm{d}t & \text{for } u > 1. \end{cases} \tag{2}$$

Note that $\rho(u) > 0$, for if not, then because of the continuity of $\rho(u)$ there is a smallest zero $u_0 > 1$ and on substituting u_0 in (2) we easily arrive at a contradiction. De Bruijn's contribution was to provide a very precise asymptotic estimate for $\rho(u)$ and in later works he provided much more precise estimates than hitherto known for $\Psi(x, y)$. As a consequence various authors call $\rho(u)$ the Dickman–de Bruijn function and some authors $\Psi(x, y)$ the Dickman–de Bruijn function. Both in depth and originality de Bruijn's work goes way beyond everything done on the subject until then. Only starting in the 1980s, when Hildebrand and Tenenbaum started to work in this area, various results of de Bruijn were greatly improved upon. Most introductions of $\Psi(x, y)$ related papers mention some de Bruijn papers. The techniques de Bruijn used in studying $\rho(u)$ are now standard in studying similar functions that arise in sieve theory; see, e.g. [46, Appendix B].

In the rest of this paper we first describe de Bruijn's work on $\rho(u)$ and then that on $\Psi(x, y)$. Before describing de Bruijn's work on $\rho(u)$ more in detail (Section 2.7), we will discuss the basic properties of $\rho(u)$ and very briefly the linear functional work of de Bruijn (Sections 2.5 and 2.6). In Section 4 de Bruijn's work on $\Psi(x, y)$ is described, after a discussion of preliminaries in Section 3. In Section 5 de Bruijn's work on $\omega(u)$, a function similar to $\rho(u)$ that arises on

[1] Some authors use y-smooth. Friable is an adjective meaning easily crumbled or broken.

[2] Karl Dickman (1861–1947) wrote his paper when he was 69 years old, after retiring from the Swedish insurance business world.

estimating the number of integers $n \leqslant x$ having no prime factors $\leqslant y$ is considered. In Section 6 it will be made clear that functions like $\omega(u)$ and $\rho(u)$ are ubiquitous in sieve theory and that de Bruijn's methods can also be applied to study this more general class of functions. In Section 7 arithmetic sums over friable integers are discussed and finally in Section 8 very short summaries of the relevant papers of de Bruijn are provided ordered by year of publication.

There is much more to friable integers than de Bruijn's work and its direct follow-up. The beautiful survey of Granville [51] makes very clear how many connections there are with other problems in (mainly) number theory, but also in probability, cycle structure of permutations (see Section 4.3), etc. Here we will not go into this rich tapestry.

It is assumed that the reader is familiar with the Landau–Bachmann O-notation (see wikipedia or any introductory text on analytic number theory). Instead of $\log\log x$ we sometimes write $\log_2 x$, instead of $(\log x)^A$, $\log^A x$.

This paper is the extended version of a paper [76] (in Dutch) written on request for an issue of *Nieuw Archief voor Wiskunde* dedicated to the mathematics and memory of de Bruijn. For a discussion of the work done on friable integers before de Bruijn, see Norton [78] and Moree [77].

2. Dick de Bruijn and the Dickman–de Bruijn function

2.1. *Elementary properties of $\rho(u)$*

From (2) we see that

$$\rho'(u) = -\frac{\rho(u-1)}{u}$$

for $u > 1$ and thus an alternative way of defining $\rho(u)$ is

$$\rho(u) = \begin{cases} 1 & \text{for } 0 \leqslant u \leqslant 1; \\ 1 - \displaystyle\int_1^u \frac{\rho(t-1)}{t}\,\mathrm{d}t & \text{for } u > 1. \end{cases} \tag{3}$$

It follows that $\rho(u) = 1 - \log u$ for $1 \leqslant u \leqslant 2$. For $2 \leqslant u \leqslant 3$, $\rho(u)$ can be expressed in terms of the dilogarithm. Put $\tau = (1+\sqrt{5})/2$. We have, see, e.g., Moree [74],

$$\rho(\tau^2) = 1 - 2\log\tau + (\log\tau)^2 - \frac{\pi^2}{60} \approx 0.1046\ldots$$

The author does not know any other explicit pair (u, v) satisfying $u > 2$ and $\rho(u) = v$.

Various papers discuss how to obtain *high accuracy numerical approximations* to $\rho(u)$; see, e.g., [69,100]. The interest of de Bruijn was in the *asymptotical behaviour* of $\rho(u)$ and similar functions. So that will be our main focus from now on.

Since $\rho'(u) = -\rho(u-1)/u$ for $u > 1$ and $\rho(u) > 0$ for $u > 0$, it follows that $\rho'(u) < 0$ and hence $\rho(u)$ is monotonically decreasing for $u > 1$. By (2) we find that $u\rho(u) \leqslant \rho(u-1)$, which on using induction leads to $\rho(u) \leqslant 1/[u]!$ for $u \geqslant 0$. It follows that $\rho(u)$ quickly tends to zero as u tends to infinity. De Bruijn was the first to obtain the following much more precise estimate for $\rho(u)$, for $u \geqslant 3$,

$$\rho(u) = \exp\left\{-u\left\{\log u + \log_2 u - 1 + \frac{\log_2 u - 1}{\log u} + O\left(\left(\frac{\log_2 u}{\log u}\right)^2\right)\right\}\right\}. \tag{4}$$

Note that (4) implies that, as $u \to \infty$,

$$\rho(u) = \frac{1}{u^{u+o(u)}}, \qquad \rho(u) = \left(\frac{\mathrm{e}+o(1)}{u\log u}\right)^u,$$

formulae that suffice for most purposes and are easier to remember.

De Bruijn obtained (4) as a corollary to his highly non-elementary Theorem 2, to be discussed below. In Section 2.8 an elementary proof of (4) is given.

2.2. *The Laplace transform of $\rho(u)$*

In de Bruijn's work the function $(\mathrm{e}^x - 1)/x$ plays a prominent role. To understand its origin it is a very useful exercise to compute the Laplace transform of $\rho(u)$. This transform is defined by

$$\hat{\rho}(s) = \int_0^\infty \rho(u)\mathrm{e}^{-us}\,\mathrm{d}u.$$

Using that $0 < \rho(u) \leqslant 1/[u]!$ it is seen that this integral is absolutely convergent for all complex s, and thus defines an entire function of s. Using (2) we then find that

$$\begin{aligned}-\hat{\rho}'(s) &= \int_0^\infty u\rho(u)\mathrm{e}^{-us}\,\mathrm{d}u = \int_0^1 u\mathrm{e}^{-us}\,\mathrm{d}u + \int_1^\infty \left(\int_{u-1}^u \rho(t)dt\right)\mathrm{e}^{-us}\,\mathrm{d}u \\ &= \int_0^\infty \rho(t)\left(\int_t^{t+1}\mathrm{e}^{-us}\,\mathrm{d}u\right)\mathrm{d}t = \frac{1-\mathrm{e}^{-s}}{s}\hat{\rho}(s).\end{aligned}$$

It follows that $\hat{\rho}(s) = C\exp\{-\mathrm{Ein}(s)\}$, where C is some constant and

$$\mathrm{Ein}(s) := \int_0^s \frac{1-\mathrm{e}^{-t}}{t}\,\mathrm{d}t, \quad s \in \mathbb{C},$$

is the *complementary exponential integral*. The value of C can be deduced by comparing the behaviour of $\exp\{-\mathrm{Ein}(s)\}$ with that of $\hat{\rho}(s)$ as s tends to infinity along the positive real axis. On the one hand we have, by partial integration,

$$\int_0^1 \frac{\mathrm{e}^{-t}-1}{t}\,\mathrm{d}t + \int_1^\infty \frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t = \int_0^\infty \mathrm{e}^{-t}\log t\,\mathrm{d}t = \Gamma'(1) = -\gamma,$$

where γ denotes the Euler–Mascheroni constant and $\Gamma(z)$ the Gamma-function. Thus, with s tending to infinity along the positive real axis,

$$\begin{aligned}\mathrm{Ein}(s) &= \int_0^1 \frac{1-\mathrm{e}^{-t}}{t}\,\mathrm{d}t - \int_1^s \frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t + \log s \\ &= \int_0^1 \frac{1-\mathrm{e}^{-t}}{t}\,\mathrm{d}t - \int_1^\infty \frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t + \int_s^\infty \frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t + \log s \\ &= \gamma + \log s + O(\mathrm{e}^{-s}).\end{aligned}$$

It follows that $\hat{\rho}(s) \sim C\mathrm{e}^{-\gamma}/s$. On the other hand we have, by partial integration,

$$\int_0^\infty \rho'(u)\mathrm{e}^{-us}\,\mathrm{d}u = -1 + s\hat{\rho}(s)$$

and since the integral, in absolute value, is bounded by $1/s$, it follows that $\lim_{s\to\infty} s\hat{\rho}(s) = 1$ and hence $C = e^{\gamma}$. Thus we proved (see also [72, pp. 210–211] or [94, pp. 370–372]) the following result.

Lemma 1. *We have*

$$\hat{\rho}(s) = \exp\left\{\gamma - \int_0^s \frac{1 - e^{-t}}{t}\, dt\right\}. \tag{5}$$

As a consequence we obtain that for any $0 \leqslant \delta \leqslant 1$,

$$\hat{\rho}(0) = e^{\gamma} = \int_0^{\infty} \rho(t)\, dt = \delta + \sum_{n\geqslant 1}(n+\delta)\rho(n+\delta),$$

where the last equality follows on using (2). On applying the inverse Laplace transform to both sides in (5) we obtain, for any real σ_0,

$$\rho(u) = \frac{1}{2\pi i}\int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \exp\left\{\gamma + \int_0^s \frac{e^z - 1}{z} dz + us\right\} ds, \tag{6}$$

De Bruijn seems to have been the first to consider the Laplace transform of $\rho(u)$ (in [28]). Lemma 1 is due to him.

2.3. De Bruijn's ξ function

An important quantity in the asymptotic study of $\rho(u)$ is the function $\xi(u)$. For any given $u > 1$, $\xi(u)$ is defined as the unique positive solution of the transcendental equation

$$\frac{e^{\xi} - 1}{\xi} = u. \tag{7}$$

It is easy to show that $\xi(u)$ is indeed unique. Put $f(x) = (e^x - 1)/x$. Note that the Taylor series of $f(x)$ is

$$f(x) = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots$$

So $f(x)$ tends to 1 as x tends to $0+$ and is strictly increasing in x. This clearly shows that $f(x) = u$ has a unique solution $x > 0$ for each $u > 1$.

From (7) we infer, that as $u \to \infty$, we have

$$\xi = \log \xi + \log u + O\left(\frac{1}{u\xi}\right). \tag{8}$$

We want to find an expression for $\xi = \xi(u)$ for u large. Note that for u sufficiently large $1 < \xi < 2\log u$. Therefore $\log \xi = O(\log\log u)$. On feeding this into the right hand side of (8) we obtain that $\xi = \log u + O(\log\log u)$, which on feeding again into the right hand side of (8) gives

$$\xi = \log u + \log\log u + O\left(\frac{\log\log u}{\log u}\right). \tag{9}$$

Iterating often enough we can get an error term $O(\log^{-k} u)$ for any fixed $k > 0$.

By means of standard techniques of asymptotic analysis (see [33, 2.3]) it is possible to go beyond this process of iteration. Using these a convergent series for $\xi(u)$ can be given (see [62]).

We will also need some information on $\int_1^u \xi(t)\,\mathrm{d}t$. By integration of a precise enough estimate for $\xi(u)$ one obtains, cf. Moree [73, p. 34],

$$\int_1^u \xi(t)\,\mathrm{d}t = u\left\{\log u + \log_2 u - 1 + \frac{\log_2 u - 1}{\log u} + O\left(\left(\frac{\log_2 u}{\log u}\right)^2\right)\right\}. \tag{10}$$

As we will see the similarity between (10) and (4) is no coincidence (see Theorem 3).

We leave it as an exercise to the reader to show that

$$\lim_{u\to\infty} u\xi'(u) = 1,$$

and that the estimate (9) leads to the estimate

$$\int_1^u \xi(t)\,\mathrm{d}t = u\left\{\log u + \log_2 u - 1 + O\left(\frac{\log_2 u}{\log u}\right)\right\}.$$

2.4. *Ramanujan's hand...*

On June 1–5, 1987, a meeting was held at the University of Illinois to commemorate the centenary of Ramanujan's birth and to survey the many areas of mathematics (and of physics) that have been profoundly influenced by his work. One of the speakers, R. W. Gosper, remarked in his lecture, 'How can you love this man? He continually reaches his hand from his grave to snatch your theorems from you.' A few striking examples are given in an article by Berndt [11]. One example involves Dickman's result (1). Ramanujan [81, p. 337] made a claim (written at least ten years before Dickman's paper appeared!) that is equivalent to the following claim.

Claim 1. *Define, for $u \geqslant 0$,*

$$I_0(u) = 1, \qquad I_k(u) = \underset{\substack{t_1\geqslant 1,\ldots,t_k\geqslant 1\\ t_1+\cdots+t_k\leqslant u}}{\int\cdots\int\cdots\int} \frac{\mathrm{d}t_1\cdots\mathrm{d}t_k}{t_1\cdots t_k}, \quad k \geqslant 1.$$

Then, for $u \geqslant 0$, (1) holds with

$$\rho(u) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} I_k(u).$$

For more details, including a proof of Claim 1, we refer the reader to the book by Andrews and Berndt [5, Section 8.2]. The asymptotic behaviour of the $I_k(u)$ has been studied by Soundararajan [90]. Related formulae for $\rho(u)$ as an iterated integral can be, e.g., found in Buchstab [12], Chowla and Vijayaraghavan [19], Goncharov [49] (see also p. 22 of [8]) and Tenenbaum [96, (12) with $\lambda = 1/u$ and $t = 1$].

Ramanujan in his first letter to Hardy (January 16th, 1913) gave an asymptotic for $\Psi(x, 3)$. See Moree [77] for further details.

2.5. *Mahler's partition problem*

Mahler's partition problem is to find an asymptotic formula for the number $p_r(n)$ of representations of an integer n in the form $n = n_0 + n_1 r + n_2 r^2 + \cdots$, $r > 1$ a prescribed integer, in nonnegative integers $n_0, n_1, n_2, \ldots$, when n is large. By considering the functional equation

$$F'(y) = F(y/r), \tag{11}$$

K. Mahler [66,67] obtained an approximate formula for the number $p_r(n)$. De Bruijn [23] determined the asymptotic behaviour much more precisely. A generalization of de Bruijn's results was established by Pennington [80], with $1, r, r^2, \ldots$ replaced by a more general sequence $\lambda_1, \lambda_2, \ldots$ satisfying a certain growth condition.

By setting $y = r^x$ and $F(r^x) = G(x)$, Mahler's functional equation is transformed into a functional equation of the form

$$G'(x) = \mathrm{e}^{\alpha x+\beta} G(x-1), \tag{12}$$

with β real. Using saddle-point techniques de Bruijn [30] studied the asymptotic behaviour of real solutions of the above equation as x tends to infinity in case $\alpha > 0$ and β is a complex number.

2.6. The linear functional equations papers

De Bruijn's initial mathematical interest was in combinatorics. According to J. Korevaar, de Bruijn was really excited about Mahler's partition problem (de Bruijn attended lectures of Mahler on this problem). This problem leads one to consider linear functional equations and the author would not be surprised if this problem is the ultimate source of de Bruijn's strong interest in these types of equations. Also he had a Ph.D. student who wrote his Ph.D. Thesis in this area of research [9].

In [24], de Bruijn studies the equation $x^{-a} f(x) + f(x) - f(x-1) = 0$ and shows that depending on a the behaviour of $f(x)$ is quite distinct. He discusses an application to Mahler's partition problem (see Section 2.5). In a later paper [25], de Bruijn treats the more general equation $w(x)f(x) + f(x) - f(x-1) = 0$. It is shown that if $w(x)$ is not too small as x tends to infinity (a sufficient condition is $w(x) \geqslant 1/\log x$ as $x \to \infty$), then every solution tends to a constant $f(\infty)$, and an estimate is obtained for the difference $|f(x) - f(\infty)|$.

In [27], de Bruijn is concerned with equations of the type

$$f(x) = \int_0^1 K(x,t) f(x-t)\, \mathrm{d}t.$$

He is interested in imposing conditions on K which guarantee that every continuous solution $f(x)$ will be convergent, i.e., $\lim_{x\to\infty} f(x)$ exists and is finite (in such a case the kernel $K(x,t)$ is said to be 'stabilizing'). As an application he establishes the following result.

Theorem 1. *If F is continuous and satisfies*

$$xF(x) = \int_0^1 F(x-t)\, \mathrm{d}t \quad (x > 1), \tag{13}$$

then there exists a constant K such that $F(x) = \{K + O(x^{-1/2})\}\rho(x)$.

De Bruijn used Theorem 1 in order to prove Theorem 2.

2.7. De Bruijn's asymptotic formula for $\rho(u)$

In 1951, de Bruijn [28] proved the following theorem.

Theorem 2. *As $u \to \infty$ we have*

$$\rho(u) \sim \frac{1}{\sqrt{2\pi u}} \exp\left\{\gamma - u\xi(u) + \int_0^{\xi(u)} \frac{\mathrm{e}^s - 1}{s}\,\mathrm{d}s\right\}.$$

Note that the integral on the right hand side equals $-\mathrm{Ein}(-\xi)$. We remark that

$$u\xi + \mathrm{Ein}(-\xi) = \int_1^u \xi(t)\,\mathrm{d}t, \tag{14}$$

which can be seen on noting that

$$\frac{d}{du}(u\xi + \mathrm{Ein}(-\xi)) = \left(u - \frac{\mathrm{e}^{\xi} - 1}{\xi}\right)\xi'(u) + \xi(u) = \xi(u), \quad u > 1,$$

by the definition of ξ, and

$$\lim_{u\to 1+}(u\xi + \mathrm{Ein}(-\xi)) = 0,$$

since $\xi(u) \to 0$ as $u \to 1+$. Estimate (4) for $\rho(u)$ follows from Theorem 2 together with (14) and (10).

An outline of de Bruijn's remarkable proof of Theorem 2 follows.

(1) Note that

$$F_1(u) = \frac{1}{2\pi i}\int_W \exp\left\{-uz + \int_0^z \frac{\mathrm{e}^s - 1}{s}\,\mathrm{d}s\right\} dz$$

satisfies (13), where the contour W consists of $\{z : \Re z \geqslant 0,\ \Im z = -\pi\}$; followed by $\{z : \Re z = 0, -\pi \leqslant \Im z \leqslant +\pi\}$; followed by $\{z : \Re z \geqslant 0,\ \Im z = +\pi\}$.

(2) Use the saddle point method to estimate $F_1(u)$ for large u.

(3) Note that for any continuous G satisfying the adjoint equation

$$uG'(u-1) = G(u) - G(u-1), \quad u > 1, \tag{15}$$

and any continuous F satisfying $uF'(u-1) = -F(u-1),\ u > 1$, the function

$$\langle F, G\rangle = \int_{b-1}^{b} F(u)G(u)\,\mathrm{d}u - bF(b)G(b-1)$$

is a constant independent of $b > 0$.

(4) Note that

$$G_1(u) = \lim_{\delta\to 0}\left\{\int_{-\infty}^{\delta} + \int_{\delta}^{\infty}\right\} \exp\left\{uz - \int_0^z \frac{\mathrm{e}^s - 1}{s}\,\mathrm{d}s\right\}\frac{\mathrm{e}^z}{z} dz, \quad u > -1,$$

satisfies (15).

(5) Estimate $G_1(u)$, u large, by the saddle point method.

(6) Use the estimates of steps 2 and 5 to show that $\langle F_1, G_1\rangle = 1$.

(7) By considering appropriate limits, prove that $\langle \rho, G_1\rangle = \mathrm{e}^{\gamma}$.

(8) By Theorem 1 deduce that $\rho(u) \sim \mathrm{e}^{\gamma} F_1(u)$.

(9) Combine the latter result with the estimate for $F_1(u)$ obtained in step 2 in order to complete the proof. □

By applying the saddle-point method directly to (6) it is possible to give a rather shorter proof of Theorem 2; see [94, pp. 374–376].[3] Indeed, in this way a much sharper version of Theorem 2 can be obtained; see Smida [89].

In 1982, Canfield [14] gave a reproof of Theorem 2 using combinatorial analysis. He defines two step functions $p_N(u)$ and $q_N(u)$. The step size for each is N^{-1}; the value of p_N on $[jN^{-1},(j+1)N^{-1})$ is $A(N,j)$ for $j \geqslant 0$; the value of q_N on $((j-1)N^{-1}, jN^{-1}]$ is $B(N,j)$ for $j \geqslant 1$, where

$$\begin{aligned}&\frac{j}{N}A(N,j) = \frac{1}{N}(A(N,j-1)+\cdots+A(N,j-N))\\ &A(N,0)=A(N,1)=\cdots=A(N,N-1)=1;\end{aligned}$$

and

$$\begin{aligned}&\frac{j}{N}B(N,j) = \frac{1}{N}(B(N,j-1)+\cdots+B(N,j-N+1))\\ &B(N,1)=B(N,2)=\cdots=B(N,N-1)=1.\end{aligned}$$

The idea is that the step functions p_N and q_N are discrete models of (2). The generating series for $A(N,j)$ and $B(N,j)$ are not difficult to find:

$$\sum_{j=1}^{\infty} B(N+1,j)x^j = x\sum_{j=1}^{\infty} A(N,j)x^j = x\exp\left\{x+\frac{x^2}{2}+\cdots+\frac{x^N}{N}\right\}.$$

Canfield continues by showing that, for all N and u,

$$B(N,[uN]) \leqslant \rho(u) \leqslant A(N,[uN]).$$

Using the circle method he then shows that there exists an integer-valued function $\nu(u)$ such that

$$B(\nu(u),[u\nu(u)]) \sim A(\nu(u),[u\nu(u)]) \sim \frac{1}{\sqrt{2\pi u}}\exp(\gamma - u\xi - \mathrm{Ein}(-\xi)),$$

thus giving another proof of Theorem 2.

Alladi [1] generalized de Bruijn's result as follows (see also [94, p. 374]).

Theorem 3. *For $u \geqslant 1$ we have*

$$\rho(u) = \left(1+O\left(\frac{1}{u}\right)\right)\sqrt{\frac{\xi'(u)}{2\pi}}\exp\left(\gamma - \int_1^u \xi(t)\,\mathrm{d}t\right).$$

Since $\lim_{u\to\infty} u\xi'(u) = 1$ and because of the identity (14), Theorem 3 has Theorem 2 as a corollary. Alladi writes that 'this improvement is in fact obtained by iterating de Bruijn's method a second time, more carefully!' He starts by showing that in de Bruijn's Theorem 1 the error term can be substantially improved.

Hensley [56] gave another proof of Theorem 3. An arithmetic proof of Theorem 3 via the function $\Psi(x,y)$ is contained in Hildebrand and Tenenbaum [60]. Hildebrand [59] established an asymptotic estimate similar to the one given in Theorem 3 for every function in a one parameter family of differential–difference equations. Smida [89] and Xuan [102] gave (quite technical) further sharpenings of Theorem 3.

[3] The first to remark this was Tenenbaum in 1990 in the French original of his book [94].

De Bruijn in his proof of Theorem 2 apparently was the first to use an adjoint differential–difference equation. Its application to questions arising in sieve theory was developed in several papers by H. Iwaniec and others, in particular in Iwaniec [64].

2.8. *An elementary proof of de Bruijn's estimate (4)*

Evertse et al. [42, pp. 109–110] showed that for $u \geqslant 1$

$$\exp\left(-\int_2^{u+1} \xi(t)\,\mathrm{d}t\right) \leqslant \rho(u) \leqslant \exp\left(-\int_1^{u} \xi(t)\,\mathrm{d}t\right). \tag{16}$$

The short and elementary proof of (16) only uses that $\rho'(u)/\rho(u)$ is non-decreasing for $u > 1$. There is also an elementary argument for that; see, e.g., [73, p. 35]. Using the elementary estimate (10) we infer from (16) that

$$\rho(u) = \exp\left(-\int_1^{u} \xi(t)\,\mathrm{d}t + O(\log u)\right).$$

On inserting (10) in this, the proof of (4) is completed. □

3. Preliminaries on $\Psi(x, y)$

3.1. *Counting primes*

In order to study $\Psi(x, y)$ it is essential to have some understanding of the distribution of prime numbers. As usual we let $\pi(x) = \sum_{p\leqslant x} 1$ denote the number of primes $p \leqslant x$. We let $\mathrm{li}(x) = \int_2^x \mathrm{d}t/\log t$ denote the *logarithmic integral*. Hadamard and independently de la Vallée Poussin in 1896 established the celebrated prime number theorem, stating that

$$\pi(x) \sim \frac{x}{\log x}, \quad x \to \infty. \tag{17}$$

It follows from the proof of Hadamard and de la Vallée Poussin that

$$\pi(x) = \mathrm{li}(x) + O(E(x)), \tag{18}$$

with

$$E(x) = x\mathrm{e}^{-c_1\sqrt{\log x}},$$

and c_1 some positive constant. The currently best error term, due to I.M. Vinogradov and Korobov in 1958, is

$$E(x) = x \exp\{-c_2 \log^{3/5} x(\log_2 x)^{-1/5}\}, \tag{19}$$

with c_2 some positive constant (see [17, Ch. 3]). Important to keep in mind is that for a large class of functions

$$\sum_{p\leqslant x} f(p) \approx \int_2^x \frac{f(t)}{\log t}\,\mathrm{d}t. \tag{20}$$

Indeed, writing $\pi(x) = \mathrm{li}(x) + E_1(x)$ we obtain, using the Stieltjes integral,

$$\sum_{p\leqslant x} f(p) = \int_2^x f(t)d\pi(t) = \int_2^x \frac{f(t)}{\log t}\,\mathrm{d}t + \int_2^x f(t)\,\mathrm{d}E_1(t),$$

making (20) more precise. E.g., taking $f(x) = 1/x$ and using that $E_1(x) = O(x \log^{-2} x)$, we obtain the classical estimate

$$\sum_{p \leqslant x} \frac{1}{p} = \log\log x + c_3 + O\left(\frac{1}{\log x}\right), \tag{21}$$

with c_3 a constant.

3.2. The Buchstab functional equation for $\Psi(x, y)$

Historically the first functional equation to be derived for the $\Psi(x, y)$ function was the *Buchstab functional equation* [12] (some authors call this the Buchstab–de Bruijn equation),

$$\Psi(x, y) = \Psi(x, z) - \sum_{y < p \leqslant z} \Psi\left(\frac{x}{p}, p\right), \tag{22}$$

where $1 \leqslant y < z \leqslant x$. In the special case when $y = 1$ and $z > 1$ we obtain

$$\Psi(x, z) = 1 + \sum_{p \leqslant z} \Psi\left(\frac{x}{p}, p\right).$$

In order to derive the Buchstab functional equation we start by observing that the difference $\Psi(x, z) - \Psi(x, y)$ equals precisely the cardinality of the set of natural numbers $\leqslant x$ having greatest prime factor in $(y, z]$. Let p be some prime number. Notice that the cardinality of the set of natural numbers $\leqslant x$ having greatest prime factor p equals $\Psi(x/p, p)$. Thus, the Buchstab functional equation results. It can be used to determine $\Psi(x, y)$ successively in the regions $y \geqslant x, \sqrt{x} < y \leqslant x, x^{1/3} \leqslant y < \sqrt{x}$. In terms of $u := \log x/\log y$ this corresponds with $u \leqslant 1,\ 1 < u \leqslant 2, 2 < u \leqslant 3 \ldots$. Note that u is very close to the quotient of the number of digits of x and the number of digits of y.

For $u \leqslant 1$ we obviously have $\Psi(x, y) = [x]$ and so in particular $\Psi(x/p, p) = [x/p]$ for $\sqrt{x} \leqslant p \leqslant x$. On taking $z = x$ in the Buchstab functional equation it then follows, for $1 \leqslant u \leqslant 2$, that

$$\Psi(x, y) = [x] - \sum_{y < p \leqslant x} \left[\frac{x}{p}\right]. \tag{23}$$

Proceeding in this way one gets an exact expression for $\Psi(x, y)$ in increasingly large u regions. Suppose we have computed $\Psi(x, y)$ for $u \leqslant h$. For a term $\Psi(x/p, p)$ in the Buchstab functional equation the logarithm of the first argument divided by the logarithm of the second argument equals $(\log x/\log p) - 1$ and this is $\le u - 1$. So once we can compute $\Psi(x, y)$ for $u \leqslant h$ we can compute $\Psi(x, y)$ for $u \leqslant h + 1$. With each step of this process the expression for $\Psi(x, y)$ becomes more complicated. The idea now is to invoke analytic number theory to 'smoothen' the sums, e.g., for $1 \leqslant u \leqslant 2$ we obtain from (23) and (17) the estimate

$$\Psi(x, y) = x - \sum_{y < p \leqslant x} \frac{x}{p} + O(\pi(x)) = x - \sum_{y < p \leqslant x} \frac{x}{p} + O\left(\frac{x}{\log x}\right). \tag{24}$$

This together with (21) yields, for all $x \geqslant 2$ and $1 < u \leqslant 2$,

$$\Psi(x, x^{1/u}) = x(1 - \log u)\left(1 + O\left(\frac{1}{\log x}\right)\right).$$

We infer that, for $0 < u \leq 2$,

$$\lim_{x\to\infty} \frac{\Psi(x, x^{1/u})}{x} = \rho(u). \tag{25}$$

(For $u \leq 1$ this result is trivial.) Heuristically it can be now readily justified that this result also should hold for *every* $u > 2$. We proceed by induction and assume that (25) is satisfied for every $v \leqslant h$, with $h \geq 2$ an integer and then heuristically establish it for $u \leqslant h + 1$. In the Buchstab functional equation we put $y = x^{1/u}$ and $z = x^{1/v}$ with $v \leqslant h$ and $u \leqslant h + 1$ arbitrary real numbers satisfying $v < u$. We then infer that the right hand side of (22) is

$$\begin{aligned}
&\approx x\rho(v) - \sum_{x^{1/u} < n \leqslant x^{1/v}} \frac{x}{p}\rho\left(\frac{\log x}{\log p} - 1\right)\\
&\approx x\rho(v) - \int_{x^{1/u}}^{x^{1/v}} \frac{x}{t}\rho\left(\frac{\log x}{\log t} - 1\right)\frac{\mathrm{d}t}{\log t}\\
&= x\rho(v) - x\int_v^u \frac{\rho(w-1)}{w}dw,\\
&= x\rho(u),
\end{aligned}$$

where in the second step we used the approximation principle (20) and in the third step the transformation $t = x^{1/w}$ was made.

3.3. The Hildebrand functional equation for $\Psi(x, y)$

The *Hildebrand functional equation* [57] reads

$$\Psi(x, y)\log x = \int_1^x \frac{\Psi(t, y)}{t}\,\mathrm{d}t + \sum_{\substack{p^m \leqslant x\\ p \leqslant y}} \Psi\left(\frac{x}{p^m}, y\right)\log p.$$

The advantage over the Buchstab functional equation is that one of the parameters is held fixed and that the coefficients are non-negative. It was the use of this functional equation that led to serious improvements of several of de Bruijn's results. For most purposes it is enough to work with the approximate functional equation

$$\Psi(x, y)\log x = \sum_{p \leqslant y} \Psi\left(\frac{x}{p}, y\right)\log p + O(E(x, y)),$$

where one can take $E(x, y) = x$. For all $y \geqslant \log^{2+\epsilon} x$ one can take $E(x, y) = \Psi(x, y)$. A heuristic proof of $\Psi(x, y) \sim x\rho(u)$ based on the Hildebrand functional equation can also be given; see [61, pp. 416–417].

3.4. Rankin's method

In his work in the 1930s on the gaps between consecutive primes Rankin [82] introduced a simple idea to estimate $\Psi(x, y)$ which turns out to be remarkably effective and can be used in similar situations. The starting point is the observation that for any $\sigma > 0$

$$\Psi(x, y) \leqslant \sum_{n \in S(x,y)} \left(\frac{x}{n}\right)^{\sigma} \leqslant x^{\sigma} \sum_{P(n) \leqslant y} \frac{1}{n^{\sigma}} = x^{\sigma}\zeta(\sigma, y), \tag{26}$$

where

$$\zeta(s, y) = \prod_{p \leqslant y} (1 - p^{-s})^{-1},$$

is the partial Euler product up to y for the Riemann zeta function $\zeta(s)$. Recall that, for $\Re s > 1$,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - p^{-s}}.$$

Note that the logarithmic derivative of $x^s \zeta(s, y)$ with respect to s equals

$$\log x - \sum_{p \leqslant y} \frac{\log p}{p^s - 1}.$$

It is minimized for $\sigma = \alpha(x, y)$, where $\alpha(x, y)$ is the (unique) solution of

$$\sum_{p \leqslant y} \frac{\log p}{p^{\alpha(x,y)} - 1} = \log x.$$

Using a sufficiently sharp form of the prime number theorem one can derive, see [60], the estimate

$$\alpha(x, y) = \frac{\log(1 + y/\log x)}{\log y} \left(1 + O\left(\frac{\log\log(1 + y)}{\log y}\right)\right).$$

The upper bound $x^\alpha \zeta(\alpha, y)$ for $\Psi(x, y)$ is only too large asymptotically by a factor

$$\alpha\sqrt{2\pi\phi_2(\alpha, y)}, \tag{27}$$

where $\phi_k(s, y)$ is defined as the kth partial derivative of $\log \zeta(s, y)$ with respect to s. The factor (27) turns out to be relatively small. This follows from one of the deepest results in this area, due to Hildebrand and Tenenbaum [60], namely that uniformly in the range $x \geqslant y \geqslant 2$ one has

$$\Psi(x, y) = \frac{x^\alpha \zeta(\alpha, y)}{\alpha\sqrt{2\pi\phi_2(\alpha, y)}} \left(1 + O\left(\frac{1}{u}\right) + O\left(\frac{\log y}{y}\right)\right).$$

Of course, in (26) one is free to make any choice of σ. E.g., the choice $\sigma = 1 - 1/(2\log y)$ leads to

$$\zeta(\sigma, y) \ll \exp\left\{\sum_{p \leqslant y} \frac{1}{p^\sigma}\right\} \leqslant \exp\left\{\sum_{p \leqslant y} \frac{1}{p} + O\left((1 - \sigma)\sum_{p \leqslant y} \frac{\log p}{p}\right)\right\} \ll \log y,$$

which gives rise to $\Psi(x, y) \ll x\mathrm{e}^{-u/2} \log y$. The factor $\log y$ can be easily removed; see [94, p. 359].

As a further example of working with Rankin's method, let us try to estimate $\Psi(x, \log^A x)$ for $A > 1$. Letting $\sigma = 1 - 1/A$, we get

$$\log \zeta(\sigma, y) \ll \sum_{p \leqslant y} p^{-\sigma} = \sum_{p \leqslant y} \frac{p^{1/A}}{p} \ll \frac{y^{1/A}}{\log y} \ll \frac{\log x}{\log\log x}.$$

We conclude that

$$\Psi(x, \log^A x) \leqslant x^{1-1/A+O(1/\log\log x)}. \tag{28}$$

We will see in Section 4.4 that this upper bound is actually sharp.

3.5. A binomial lower bound for $\Psi(x, y)$

Let $2 = p_1 < p_2 < p_3 < \cdots$ denote the consecutive primes. Let p_k be the largest prime $p \leqslant y$ (thus $k = \pi(y)$). Evidently n is in $S(x, y)$ if and only if we can write n in the form $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, where the e_k are non-negative integers and $n \leqslant x$, that is,

$$e_1 \log p_1 + e_2 \log p_2 + \cdots + e_k \log p_k \leqslant \log x. \tag{29}$$

Since each $\log p_i \leqslant \log y$, any solution to $e_1 + e_2 + \cdots + e_k \leqslant \left[\frac{\log x}{\log y}\right] = [u]$ gives a solution to (29) with $x = y^u$. It follows that

$$\Psi(x, x^{1/u}) \geqslant \sum_{\substack{e_1+\cdots+e_k \leqslant [u] \\ e_i \geqslant 0}} 1 = \binom{[u] + \pi(y)}{\pi(y)}. \tag{30}$$

3.6. The lattice point method

If y is fixed or tends to infinity very slowly in comparison with x, say $y = o(\log x)$, the behaviour of $\Psi(x, y)$ is, not surprisingly, heavily influenced by the irregularities in the distribution of the primes. Approximating $\Psi(x, y)$ by smooth functions in this (x, y)-region is therefore bound to be not very successful. However, $\Psi(x, y)$ can be quite well approximated by the volume of a certain tetrahedron. Set $k = \pi(y)$. Notice that $\Psi(x, y)$ equals the number of $(e_1, \ldots, e_k)$ satisfying (29) and $e_i \geqslant 0$. Thus $\Psi(x, y)$ equals the number of lattice points in a $\pi(y)$-dimensional tetrahedron with sides of length $\log x / \log 2, \ldots, \log x / \log p_k$ having volume

$$\frac{1}{\pi(y)!} \prod_{p \leqslant y} \left(\frac{\log x}{\log p}\right).$$

Ennola [39] proved that for $2 \leqslant y \leqslant \sqrt{\log x}$ one has

$$\Psi(x, y) = \frac{1}{\pi(y)!} \prod_{p \leqslant y} \left(\frac{\log x}{\log p}\right) \left(1 + O\left(\frac{y^2}{\log x \log y}\right)\right).$$

There are sharper results in this y-region due to Ennola [39] (quoted in Norton [78, pp. 24–26]) and Specht [91]. In these results Bernoulli numbers make their appearance.

4. De Bruijn's work on $\Psi(x, y)$

4.1. De Bruijn's Λ function

Let us define (as many authors do) $\rho(u) = 0$ for $u < 0$. In [29] de Bruijn introduced the function $\Lambda(x, y)$. He defines it as follows:

$$\Lambda(x, y) = x \int_0^\infty \rho\left(\frac{\log x - \log t}{\log y}\right) d\left(\frac{[t]}{t}\right). \tag{31}$$

Since $\rho(u) = 0$ for $u < 0$ and $\rho(u)$ is continuous everywhere apart from the jump at $u = 0$, the integral is well-defined, unless the jump of ρ coincides with a jump of $[t]/t$. This is the case if x is a positive integer n; this turns out to be a minor nuisance and can be dealt with by defining $\Lambda(n, y) = \Lambda(n + 0, y)$. The function $\Lambda(x, y)$ satisfies a functional equation which is similar to the Buchstab functional equation (22). It arises from (22) by writing the sum appearing in it formally as a Stieltjes integral $\int \Psi(x/t,t)d\pi(t)$ and replacing $\pi(t)$ by $\mathrm{li}(t)$. One can check that indeed

$$\Lambda(x,y)=\Lambda(x,z)-\int_y^z \Lambda\left(\frac{x}{t},t\right)\frac{\mathrm{d}t}{\log t}. \tag{32}$$

Further, for $0<x\leqslant y$ we have

$$\Lambda(x,y)=x\int_0^x d\left(\frac{[t]}{t}\right)=x\frac{[x]}{x}=[x]=\Psi(x,y).$$

Thus $\Lambda(x,y)$ satisfies the same initial condition as $\Psi(x,y)$ and obeys a functional equation which is the smoothened version of Buchstab's functional equation (22).

De Bruijn showed that $\Lambda(x,y)$ closely approximates $\Psi(x,y)$ in a certain (x,y)-region (this region was substantially extended in 1989 by Saias [84]). Further he shows that both $\Lambda(x,y)$ and $\Psi(x,y)$ satisfy an expansion of the form

$$x\sum_{r=0}^{n} a_r\frac{\rho^{(r)}(u)}{\log^r y}+E_n(x),$$

with n an arbitrary integer, $u>n+1$, $E_n(x)$ a term he describes rather explicitly and $a_0+a_1z+a_2z^2+\cdots$ the Taylor series around $z=0$ of $z\zeta(1+z)/(1+z)$. The term $E_n(x)$ is only an error term if u is not close to an integer from above. For a deeper understanding of this phenomenon, see Theorem 1.1 and Lemma 3.1 in Hanrot et al. [54].

4.2. De Bruijn's uniform version of Dickman's result

In [29] de Bruijn gives estimates for the differences

$$|\Psi(x,y)-\Lambda(x,y)| \quad \text{and} \quad |\Lambda(x,y)-x\rho(u)|.$$

These estimates involve the error term $E(x)$ in the prime number theorem; cf. (18). These bounds, when combined with the sharpest known form of the prime number theorem (having error term (19)), yield the following uniform estimate of Dickman's limit result (1).

Theorem 4. *The estimate*

$$\Psi(x,y)=x\rho(u)\left\{1+O\left(\frac{\log(u+1)}{\log y}\right)\right\}, \tag{33}$$

holds for $1\leqslant u\leqslant \log^{3/5-\epsilon} y$, *that is*, $y>\exp(\log^{5/8+\epsilon} x)$.

Hildebrand [58] improved this substantially to the range

$$1\leqslant u\leqslant \exp(\log^{3/5-\epsilon} y), \quad \text{that is, } y>\exp((\log\log x)^{5/3+\epsilon}).$$

There are limitations as to how far this range can be extended. Hildebrand [57] showed that the estimate (33) holds uniformly for

$$1\leqslant u\leqslant y^{1/2-\epsilon}, \quad \text{that is, } y\geqslant \log^{2+\epsilon} x,$$

if and only if the Riemann hypothesis is true.

One might wonder in which region $\Psi(x, y) \gg x\rho(u)$ is valid, as did Hensley [55]. Canfield et al. [15] proved that there is a constant C such that $\Psi(x, y) \geqslant x\rho(u)\exp\{C(\log_2 u/\log u)^2\}$ uniformly for $x \geq 1$ and $u \geq 3$. It seems that $\Psi(x, y) \geqslant x\rho(u)$ for a very large (x, y)-region.

4.3. The average largest prime factor of n

Recall that $P(n)$ denotes the largest prime factor of n. De Bruijn [29] applies his results on $\Lambda(x, y)$ to show that

$$\sum_{n\leqslant x}\log P(n) = \lambda x\log x + O(x), \quad \text{with } \lambda = \int_0^\infty \frac{\rho(u)}{(1+u)^2}\,\mathrm{d}u. \tag{34}$$

By partial integration it follows that

$$\lim_{x\to\infty}\frac{1}{x}\sum_{2\leqslant n\leqslant x}\frac{\log P(n)}{\log n} = \lambda,$$

thus proving a heuristic claim by Dickman [38]. The interpretation of the result is that for an average integer with m digits, its greatest prime factor has about λm digits. De Bruijn's estimate can be further sharpened to

$$\sum_{n\leqslant x}\log P(n) = \lambda x\log x - \lambda(1-\gamma)x + O(x\exp(-\log^{3/8-\epsilon}x));$$

see [94, Exercise III.5.3]. The constant λ is now known as the *Golomb–Dickman constant*. Mitchell [71] computed that $\lambda = 0.62432998854\ldots$. Golomb et al. [48] have defined a constant μ which is related to the limiting properties of random permutations. Let L_n be the expected length of the longest cycle of a random permutation of n letters. Define $\mu_n = L_n/n$. (Thus $\mu_1 = 1, \mu_2 = 3/4, \mu_3 = 13/18, \mu_4 = 67/96$.) It can be shown that the numbers μ_n are monotonically decreasing with n and hence a limit μ exists. It turns out that $\lambda = \mu$. Thus on average the longest cycle of a permutation of n elements has length λn. Much earlier Goncharov announced in [49] and proved in [50] that the ratio of permutations in S_n having longest cycle $\leqslant n/u$, with $u \geq 1$ and fixed, tends to $\rho(u)$ as n tends to infinity. Goncharov, unaware of the occurrence of $\rho(u)$ in number theory, also expressed $\rho(u)$ as an iterated integral; see Arratia et al. [8, p. 22]. Shepp and Lloyd [88] derived a second formula for λ:

$$\lambda = \int_0^\infty \exp\left\{-x - \int_x^\infty \frac{\mathrm{e}^{-y}}{y}\,\mathrm{d}y\right\}\mathrm{d}x.$$

It is striking that $\rho(u)$ turns up both in the setting of greatest cycle length and greatest prime factor. It also shows up in the setting of greatest irreducible polynomial factor of a polynomial over a finite field; see, e.g., Car [16]. For a unified treatment, see Arratia et al. [7] (or their book [8]).

Meanwhile there are many results (see, e.g., [41,86]) involving $P(n)$, one of the more interesting, see [41], being

$$\sum_{n\leqslant x}\frac{1}{P(n)} = x\int_2^x \rho\left(\frac{\log x}{\log t}\right)\frac{\mathrm{d}t}{t^2}\left(1 + O\left(\left(\frac{\log_2 x}{\log x}\right)^{1/2}\right)\right).$$

Later Scourfield [86] improved on the error term. Sums of the form $\sum_{n\leqslant x} f(P(n))$, where f is some arithmetic function can be evaluated using the identity

$$\sum_{n\leqslant x} f(P(n)) = \sum_{p\leqslant x} f(p)\,\Psi\left(\frac{x}{p}, p\right),$$

together with sufficiently sharp estimates for $\Psi(x, y)$.

A quite general result is due to Tenenbaum and Wu [97], who gave an asymptotic expansion for $\sum_{n\leqslant x} f(n)\{\log P(n)\}^r$, where f is a fairly general non-negative arithmetical function (which need not be multiplicative) and r is any positive real number.

4.4. *De Bruijn's 1966 estimate for* $\log \Psi(x, y)$

We now turn our attention to de Bruijn's 1966 paper [32]. In it he is concerned with estimating $\log \Psi(x, y)$. Set

$$Z := \frac{\log x}{\log y}\log\left(1 + \frac{y}{\log x}\right) + \frac{y}{\log y}\log\left(1 + \frac{\log x}{y}\right). \tag{35}$$

Theorem 5. *We have, uniformly for $x \geqslant y \geqslant 2$,*

$$\log \Psi(x, y) = Z\left\{1 + O\left(\frac{1}{\log y} + \frac{1}{u} + \frac{1}{\log\log 2x}\right)\right\}.$$

Actually, in the error term u^{-1} can be left out; see Tenenbaum [94, pp. 359–362] for a proof. Let us apply Theorem 5 with $y = \log^A x$ and $A > 1$ fixed. One then finds that

$$\Psi(x, \log^A x) = x^{1-1/A+O(1/\log\log x)},$$

thus improving on (28). Already the results of de Bruijn in [29] imply as a special case that $\Psi(x, \log^A x) = x^{1-1/A+o(1)}$. Following de Bruijn's argument very closely Chowla and Briggs [18] gave a reproof of the latter estimate that they believed to be considerably simpler.

The idea of the proof of Theorem 5 is very simple (but takes several pages to work out). The upper bound is deduced from Rankin's upper bound (26) with an optimal choice of the parameter σ. The lower bound is based on the inequality (30) together with a lower bound for the right hand side of (30) obtained by Stirling's formula.

Theorem 5 clearly shows that there is a change of behaviour of $\Psi(x, y)$ in the (x, y)-region where $y \approx \log x$. If $y/\log x \to \infty$, then the first term in (35) dominates, whereas Z is asymptotic to the second term in (35) when $y = o(\log x)$. The change in behaviour is due to the fact that if y is small compared to $\log x$, then many prime factors of a 'typical' integer in $S(x, y)$ occur to high powers, whereas for larger values of y most prime factors occur only to the first power. P. Erdős [40] already had shown that, if $x \to \infty$, we have

$$\log \Psi(x, \log x) \sim (\log 4)\frac{\log x}{\log\log x}.$$

5. De Bruijn and the sieve of Eratosthenes

Define $\Phi(x, y)$ as the number of integers $n \leqslant x$ having no prime factors $\leqslant y$. In the sieve of Eratosthenes all the integers $1 \leqslant n \leqslant x$, which are multiples of the primes p with $2 \leqslant p < \sqrt{x}$ are removed. What remains after this process is the number 1 and all the prime numbers p in the range $\sqrt{x} \leqslant p \leqslant x$. It follows that

$$\Phi\left(x, \sqrt{x}\right) = 1 + \pi(x) - \pi\left(\sqrt{x}\right). \tag{36}$$

Buchstab [12] showed that

$$\Phi(x,y) \sim x e^{\gamma} \omega(u) \prod_{p<y} \left(1 - \frac{1}{p}\right), \tag{37}$$

where $\omega(u) = 1/u$ for $1 \leqslant u \leqslant 2$ and, for all $u \geqslant 2$,

$$u\omega(u) = 1 + \int_1^{u-1} \omega(t)\, dt.$$

By Mertens' formula (see, e.g., [94, Theorem 11]) we have

$$\prod_{p<y} \left(1 - \frac{1}{p}\right)^{-1} \sim \frac{e^{-\gamma}}{\log y},$$

and thus we can reformulate (37) as

$$\Phi(x,y) \sim u\omega(u) \frac{x}{\log x}.$$

It follows, e.g., that $\Phi\left(x, \sqrt{x}\right) \sim x/\log x$, an asymptotic one also finds from (17) and (36).

De Bruijn [25, Example 1] showed that $\lim_{u\to\infty} \omega(u)$ exists, and, if we denote it by A, then

$$\omega(u) = A + O\left(\frac{1}{[u]!}\right). \tag{38}$$

In a later paper de Bruijn [26] showed that $A = e^{-\gamma}$. De Bruijn stated that he thought that probably

$$|\omega(u) - e^{-\gamma}| = O(\exp(-u \log u - u \log_2 u)). \tag{39}$$

This was proved subsequently by L.K. Hua [63] by an 'advanced calculus argument' and by Buchstab [13] using an arithmetic argument. The formula (39) reminds one of the asymptotic estimate (4) and indeed it turns out to be natural to write $\omega(u) - e^{-\gamma} = \rho(u)h(u)$ in order to give a sharp estimate for the difference; see Tenenbaum [95, Lemme 4]. A further connection between $\omega(u)$ and $\rho(u)$ is that their Laplace transforms $\hat{\rho}(s)$ and $\hat{\omega}(s)$ are related by

$$1 + \hat{\omega}(s) = \frac{1}{s\hat{\rho}(s)}, \quad s \neq 0.$$

De Bruijn [26] writes

$$\Phi(x,y) = x\,\Psi_1(x,y) \prod_{p<y} \left(1 - \frac{1}{p}\right)$$

and shows that

$$\left| \Psi_1(x,y) - e^{\gamma} \log y \int_1^u y^{t-u} \omega(t)\, dt \right| < c_4 \exp\left(-c_5 \sqrt{\log y}\right), \quad u \geqslant 1,\ y \geqslant 2,$$

for appropriate constants c_4 and c_5.

A surprising and important application of the properties of $\omega(u)$ has been made by Maier. Under the assumption of the Riemann hypothesis Selberg [87] showed in 1943 that given $\epsilon > 0$

there exists $x_0(\epsilon)$ such that

$$1-\epsilon < \frac{\pi(x+\log^C x)-\pi(x)}{\log^{C-1} x} < 1+\epsilon$$

for almost all $x \geq x_0(\epsilon)$, in the sense of Lebesgue measure, provided that $C > 2$. Maier [68] proved that the above ratio does not tend to 1 as x tends to infinity. Put $W(u) = \mathrm{e}^{\gamma}\omega(u) - 1$. Put

$$M_+(v) = \max_{u \geqslant v} W(u), \qquad M_-(v) = \min_{u \geqslant v} W(u).$$

Maier showed that for any fixed $C > 1$,

$$\limsup_{x\to\infty} \frac{\pi(x+\log^C x)-\pi(x)}{\log^{C-1} x} \geqslant 1 + M_+(C);$$
$$\liminf_{x\to\infty} \frac{\pi(x+\log^C x)-\pi(x)}{\log^{C-1} x} \leqslant 1 + M_-(C).$$

Using a method due to de Bruijn involving the adjoint equation of $\omega(u)$, Maier showed that $W(u)$ changes sign in every interval of length one. Hence, for all $C \geqslant 1$, $M_+(C) > 0$ and $M_-(C) < 0$. Asymptotically one has [45, p. 40]

$$M_+(v) = e^{-v\xi(v)+O(v)}, \qquad M_-(v) = -e^{-v\xi(v)+O(v)}.$$

The paper of Maier led to a lot of follow-up work on the lack of equi-distribution of the primes. In this work methods from the study of friable integers and differential–difference equations play an important role; see e.g. [44,45].

6. How special are ρ and ω?

The reader might wonder how special $\rho(u)$ and $\omega(u)$ are. The answer is that many similar functions occur in number theory, especially in sieve theory (the sieve of Eratosthenes being the easiest example of a sieve). Let us give some examples.

(1) Let Q be a set of primes, and denote by $\Psi(x, y, Q)$ the number of positive integers not exceeding x that have no prime factors from Q exceeding y. Suppose that the number of primes $\pi(x, Q)$ of primes $p \leqslant x$ in Q satisfies

$$\pi(x, Q) = \delta \mathrm{li}(x) + O\left(\frac{x}{\log^B x}\right),$$

with $0 < \delta < 1$. Then Goldston and McCurley [47] showed that

$$\Psi(x, y, Q) = x\tau_\delta(u)\left(1 + O\left(\frac{1}{\log y}\right)\right),$$

uniformly for $u \geqslant 1$ and $y \geqslant 1.5$. Here $\tau_\delta(u)$ is a function similar to $\rho(u)$, it is the unique solution of $\tau_\delta(u) = 1$ for $0 \leqslant u \leqslant 1$ and

$$u\tau_\delta'(u) = -\delta\tau_\delta(u-1) \quad \text{for } u > 1.$$

This family of differential–difference equations had been earlier investigated by Beenakker [9], a Ph.D. student of de Bruijn.

(2) Let $k \geqslant 1$ be an integer. Let $P_k(n)$ denote the kth largest prime factor of n. Let $\Psi_k(x, y)$ denote the number of integers $n \leqslant x$ such that $P_k(n) \leqslant y$. Knuth and Trabb Pardo [65] showed that

$$\lim_{x\to\infty} \frac{\Psi_k(x, x^{1/u})}{x} = \rho_{1,k}(u),$$

where $\rho_{1,k}(u)$ is a function similar to the Dickman–de Bruijn function.

(3) In sieving we consider a finite sequence A of integers and a set P of primes. Let X denote the number of integers in A. The goal is to count the number of elements $S(A, P, z)$ that remain after sifting A by all the primes in P less than z, that is we are interested in estimating

$$S(A, P, z) := |\{a \in A : (a, P(z)) = 1\}|,$$

where $P(z) := \prod_{p\in P,\ p<z} p$. This formulation is much too general if one wants to obtain non-trivial results and so we impose regularity conditions for $d|P(z)$ on the subsequence

$$A_d := \{a \in A : a \equiv 0 (\mathrm{mod}\ d)\}.$$

We assume that we have approximations of the form

$$|A_d| = \frac{\omega(d)}{d} X + r_d, \tag{40}$$

where $\omega(p)$ is a multiplicative function satisfying $0 \le \omega(p) < p$ such that r_d is a remainder that is small. For $p \notin P$ we put $\omega(p) = 0$. We define

$$V(z) := \prod_{p<z} \left(1 - \frac{\omega(p)}{p}\right).$$

Intuitively $V(z)$ can be regarded as the probability that an element of A is not divisible by any prime $p < z$ with p in P. We further assume that there is a positive constant c_6 such that

$$0 \le \frac{\omega(p)}{p} \le 1 - \frac{1}{c_6} \tag{41}$$

for every p in P and that there are constants $\kappa > 0$ and $c_7 \ge 1$ such that

$$\left| \sum_{w_1\le p<w_2} \frac{\omega(p)}{p} \log p - \kappa \log\left(\frac{w_2}{w_1}\right) \right| \le c_7, \quad \text{for all } w_2 \ge w_1 \ge 2. \tag{42}$$

The constant κ is called the *dimension of the sieve* and is roughly the average of $\omega(p)$ over the primes as

$$\sum_{p\le w_2} \frac{\log p}{p} \sim \log w_2$$

as w_2 tends to infinity. Under the assumptions (40)–(42) one obtains the following inequalities for $S(A, P, z)$:

$$S(A, P, z) \le X V(z) \left\{ F_\kappa\left(\frac{\log y}{\log z}\right) + o \right\} + R_y, \tag{43}$$

and

$$S(A,P,z) \geq XV(z)\left\{f_\kappa\left(\frac{\log y}{\log z}\right)+o\right\}+R_y, \tag{44}$$

respectively, where $F_\kappa(u)$ and $f_\kappa(u)$ are *sieve auxiliary functions* depending on the dimension κ, y is a free parameter, o is an error term and R_y is an error term formed of the r_d's of (40); cf. the inequality (45). In applications the free parameter y is chosen in such a way that the remainder term R_y is small in comparison with the main term containing X.

The Buchstab functional equation (22) has a counterpart in sieve theory:

$$S(A,P,z)=S(A,P,z_1)-\sum_{z_1\leq p<z,\,p\in P} S(A_p,P,p),\quad z\geq z_1\geq 2.$$

From this we can obtain a lower bound for $S(A,P,z)$ on assuming that a lower bound for $S(A,P,z_1)$ is already known, and that we have an upper bound for the terms in the sum. For z_1 small enough an estimate for $S(A,P,z)$ follows from the so called fundamental lemma of sieve theory which states that under the regularity conditions (40)–(42) supplemented with the condition that $|r_d|\leq\omega(d)$ for $d|P(z)$ and d square free, we have for $z_1\leq X$,

$$S(A,P,z_1)=XV(z_1)\{1+O(\mathrm{e}^{-u/2})\},$$

where $u=\log X/\log z_1$ [52, Theorem 2.5].

As a demonstration of the above let us consider the Selberg upper bound sieve [52,83]. Let $\kappa\geq 0, u=\log y/\log z>0$ and $\nu(d)$ the number of distinct prime divisors of d. Under the conditions (40)–(42), we have

$$S(A,P,z)\leq\frac{XV(z)}{\sigma_\kappa(u)}\left\{1+O\left(\frac{(\log_2 y)^{2\kappa+1}}{\log y}\right)\right\}+\sum_{d<y,d|P(z)}3^{\nu(d)}|r_d|, \tag{45}$$

where $\sigma_\kappa(u)$ is the Ankeny–Onishi–Selberg function. The implied constant depends only on κ, c_6 and c_7. One has $\sigma_\kappa(u)=\rho_\kappa(u)u^{\kappa-1}$, where $\rho_\kappa(u)=1$ for $0\leq u\leq 1$ and

$$\rho_\kappa'(u)=-u^{-\kappa}(1-u)^{\kappa-1}\rho_\kappa(u-1),\quad u>1.$$

Note that $\sigma_1(u)=\rho(u)$. It seems that $\sigma_\kappa(u)$ was introduced independently by Ankeny and Onishi [6] and de Bruijn and van Lint [37].

The function $\sigma_\kappa(u)$ satisfies the differential–difference equation

$$uf'(u)+af(u)+bf(u-1)=0, \tag{46}$$

with $a=1-\kappa$ and $b=\kappa$. The Buchstab sifting function $\omega(u)$ satisfies (46) with $a=1$ and $b=-1$. Solutions to Eq. (46) appear to have been first studied by Iwaniec [64] in connection with his work on Rosser's sieve. A more systematic investigation was carried out by Wheeler [101]. Both Iwaniec and Wheeler make intensive use of the so called adjoint equation

$$ug'(u)+(1-a)g(u)-bg(u+1)=0, \tag{47}$$

which, in some sense, is easier to deal with than (46). Whereas the functions satisfying (46) can exhibit a rather erratic behaviour, it turns out that there exists a solution of (47) which is analytic in the right half plane. Since solutions to Eqs. (46) and (47) are connected to a simple integral relation (the right hand side of (48), which turns out to be a constant), one can derive asymptotic information for a given solution to (46) by studying the asymptotic behaviour of a suitable solution to (47). Notice that this is precisely the approach followed by de Bruijn in proving Theorem 2! It has been established by Hildebrand and Tenenbaum [62] that also in this general case a much more direct approach is possible (cf. the paragraph in Section 2.7 following the proof of de Bruijn).

The problem of describing the general solution to (46) amounts to describing solutions $f(u) = f(u;\varphi)$ satisfying (46) for $u > u_0$ and $f(u) = \phi(u)$ for $u_0 - 1 \leqslant u \leqslant u_0$, where u_0 is any positive real number and $\phi(u)$ is any given continuous function on $[u_0 - 1, u_0]$. Given two functions f and g defined on $[u_0 - 1, u_0]$ and $[u_0, u_0 + 1]$, respectively, we set

$$\langle f, g\rangle = u_0 f(u_0)g(u_0) - b\int_{u_0-1}^{u_0} f(u)g(u+1)\,\mathrm{d}u. \tag{48}$$

If f and g are solutions of (46), respectively (47), then this 'scalar product' is independent of u_0.

In 1993, Hildebrand and Tenenbaum [62] showed that there is a solution $F(u; a, b)$ and that there are solutions $F_n(u; a, b)$ for every integer n that form a basis of the solution space, that is any solution can be expressed in the form

$$f(u) = \alpha F(u; a, b) + \sum_{n=-\infty}^{\infty} \alpha_n F_n(u; a, b),$$

with suitable coefficients α and α_n depending on the initial function ϕ. De Bruijn [30] by a different approach had earlier found a similar result for solutions of the differential–difference equation (12).

Special cases of (46) had been studied before 1993 by Alladi [4], Beenakker [9], Hensley [56], Hildebrand [59] and Wheeler [101] amongst others.

7. Arithmetic sums over friable integers

In [37] de Bruijn and van Lint consider sums of the form

$$\sum_{n\in S(x,y)} f(n),$$

with f a non-negative multiplicative function. Many of the usual techniques can be applied here, e.g., the analogues of the Buchstab and Hildebrand functional equation hold; see [75].

For an enlightening historical introduction and an extensive bibliography on the subject, see Tenenbaum and Wu [96]. For the state of the art of this area, see Tenenbaum and Wu [97].

Let $\Psi_m(x, y)$ denote the number of y-friables that are $\leqslant x$ and coprime to m. It turns out that evaluating the ratio $\Psi_m(x/d, y)/\Psi(x, y)$ for $1 \leqslant d \leqslant x,\ m \geqslant 1,\ x \geqslant y \geqslant 2$, is a crucial step for estimating arithmetic sums over friable integers; see de la Bretèche and Tenenbaum [20]. Alladi [2,3] noticed a duality which shows that $\Psi(x, y)$ is related to the sum of the Möbius function over the uncancelled elements in the sieve of Eratosthenes, and the sum of the Möbius function over the smooth numbers is related to $\Phi(x, y)$.

An interesting related theme is that of the *friable Turán–Kubilius inequality*. The Turán–Kubilius inequality

$$\sum_{n\leqslant x} |f(n) - A_f(x)|^2 \ll x B_f(x),$$

holds uniformly for strongly additive functions f, where $A_f(x)$, $B_f(x)$ stand for certain sums over primes $p \leqslant x$, depending on f. If

$$B_f(x) = o(A_f(x)),$$

it can be shown that f has *normal order* $A_f(x)$, that is, for any $\epsilon > 0$, we have

$$|f(n) - A_f(n)| \leqslant \epsilon |A_f(n)|$$

on a set of integers n of density 1. One can try to obtain an inequality analogous to the Turán–Kubilius inequality for the corresponding sum over the elements of the set $S(x, y)$. The first result in this direction is due to Alladi [1], who found that he needed a stronger version (Theorem 3) of de Bruijn's Theorem 2 for this purpose. For further developments the reader is referred to works of Tenenbaum and his collaborators [20–22,53,70].

8. De Bruijn's analytic number theory work in a nutshell

–[23], 1948–

Mahler's partition problem is to find an asymptotic formula for $p_r(n)$, the number of partitions of n into powers of a fixed integer $r > 1$. De Bruijn gives an asymptotic for $\log p_r(n)$ improving on Mahler's result [66] (cf. Section 2.5).

–[24], 1949–

This paper is mostly in pure analysis. At the end of the paper there is a brief discussion of connections with Mahler's partition problem and a promise to come back to this in a future paper [30].

–[25], 1950–

De Bruijn shows that $\omega(u) = A + O(1/[u]!)$ for some constant A as an application of more general results.

–[26], 1950–

He gives a sharp estimate for $\Phi(x, y)$. Further he gives the first proof that the Buchstab sifting function $\omega(u)$ tends to $e^{-\gamma}$ as u tends to infinity and states that probably the estimate (39) holds. This estimate was later proved by 'the father of modern Chinese mathematics' (Loo-Keng Hua) [63].

–[27], 1950–

He proves Theorem 1.

–[28], 1951–

The main result he proves here is Theorem 2, further Lemma 1 and the inverse Laplace transform formula (6). As a corollary of his main theorem he obtains (4). Further, he introduces the function $\xi(u)$.

–[29], 1951–

He introduces $\Lambda(x, y)$, shows that it closely approximates both $\Psi(x, y)$ and $x\rho(u)$. These estimates in combination with the error estimate (19) in the prime number theorem lead to a uniform version of Dickman's result (1), namely Theorem 4. Further, he uses his $\Lambda(x, y)$ function to prove the estimate (34).

–[30], 1953–

He studies, using saddle-point techniques, the asymptotic behaviour in x of real solutions of the differential–difference equation

$$F'(x) = \mathrm{e}^{\alpha x+\beta} F(x-1), \quad \text{with } \alpha > 0, \beta \in \mathbb{C}.$$

This is related to Mahler's partition problem (cf. Section 2.5), in which case β is real.

–[31], 1962/1963–

If n and x are positive integers, let $f(n, x)$ denote the number of integers $1 \le m \le x$ such that all prime factors of m divide n. Let $\gamma(k) = \prod_{p|k} p$ denote the squarefree kernel of k. Erdős in a letter to de Bruijn conjectured that $F(x) := \sum_{n\leqslant x} f(n, x)$ satisfies $\log(F(x)/x) = O(\log^{1/2+\epsilon} x)$. Put $G(x) = \sum_{n\leqslant x} f(n, n)$. De Bruijn, using a Tauberian theorem by Hardy and Ramanujan, proves that

$$\log\left(\frac{F(x)}{x}\right) \sim \log\left(\frac{G(x)}{x}\right) \sim \log\left(\sum_{n\leqslant x}\frac{1}{\gamma(n)}\right) \sim \sqrt{\frac{8\log x}{\log_2 x}},$$

thus proving a stronger form of Erdős conjecture. (Later W. Schwarz [85] gave an asymptotic for $\sum_{n\leqslant x}\gamma(n)^{-1}$ itself.)

–[32], 1966–

De Bruijn establishes Theorem 5, using Rankin's method, a binomial lower bound for $\Psi(x, y)$ and Stirling's formula.

–[36], 1963–

De Bruijn and van Lint show that $F(x) \sim G(x)$ (in the notation of [31]), thus answering a question raised by Erdős.

–[37], 1964–

De Bruijn and van Lint consider sums of the form $\sum_{n\in S(x,y)} f(n)$, with f a non-negative multiplicative function (cf. Section 7). They introduced a generalization of the Dickman–de Bruijn function (see Section 6) that around the same time showed up in the analysis of Ankeny and Onishi [6] of the Selberg upper bound sieve.

8.1. De Bruijn's writing style

When de Bruijn introduces a new result or method, he also describes to what extent this improves on earlier work. Interesting special cases of his result he also likes to consider. However, he also discusses limitations and might also describe an alternative method. Thus he looks at a topic from various angles and gives an honest evaluation of merits and drawbacks of his work. In this context, I was struck by the fact that in a lecture he gave at age 90 [35] he tried to give an honest evaluation of the positive and negative sides of his own character! In terms of giving details of proofs he is a bit on the brief side (as was usual in mathematical research papers written in 1945–1955). Often he promises to come back to some aspect in a future paper (and keeps his promise!).

De Bruijn's book [33] because of its clarity and pleasant style found many adepts. Rather than being theoretical the book proceeds through discussing interesting examples, several of which are from combinatorics, e.g. asymptotics for the Bell numbers.[4] De Bruijn's book was the

[4] De Bruijn's book predates the time when the Bell numbers had this name.

starting point and inspiration for several later works specializing on the use of analytic methods in combinatorics and algorithms, e.g., [10,43,79].[5]

9. Further reading

Firstly place the reader is advised to consult the book [33] and the papers of de Bruijn! As a first introduction to friable numbers I highly recommend Granville's 2008 survey [51]. It has a special emphasis on friable numbers and their role in algorithms in computational number theory. Mathematically more demanding is the 1993 survey by Hildebrand and Tenenbaum [61]. The literature up to 1971 is discussed in extenso in Norton's monograph [78]. For a discussion of the work prior to 1950 (when de Bruijn entered the scene), see Moree [77]. Chapter III.5 in Tenenbaum's book [94] deals with $\rho(u)$ and approximations to $\Psi(x, y)$ by the saddle point method, Chapter III.6 with the dual problem of counting integers $n \leqslant x$ having no prime factors $\leqslant y$. For an introductory account of the saddle point method, see, e.g., Tenenbaum [93].

Acknowledgements

The surveys by Granville [51] and Hildebrand and Tenenbaum [61] were very helpful to me. In Sections 3.2, 3.6 and 6 I copied close to verbatim material (not published in a journal) from my Ph.D. Thesis [73]. I copied close to verbatim the outline of de Bruijn's proof of Theorem 2 in Section 2.7 from Canfield [14]. The title of this paper is inspired by the title of a biography of Ada Lovelace [98]. I thank J. Sorenson and A. Weisse for their kind assistance in creating Fig. 1. A. Vershik pointed out to me the work of W. Goncharov [49,50] and E. Bach kindly sent me [49]. K. Alladi, A.D. Barbour, B. Berndt, D. Bradley, E.R. Canfield, L. Holst, A. Ivic, J. Korevaar, K.K. Norton, J. Sorenson and P. Tegelaar are heartily thanked for their comments. My special thanks are due to G. Tenenbaum and R. Tijdeman for their very extensive comments and helpful correspondence.

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