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INDUCED REPRESENTATIONS OF INFINITE-DIMENSIONAL GROUPS

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Abstract. The *induced representation* $\mathrm{Ind}_H^G S$ of a locally compact group G is the unitary representation of the group G associated with unitary representation $S : H \to U(V)$ of a subgroup H of the group G. Our aim is to *develop the concept of induced representations for infinite-dimensional groups*. The induced representations for infinite-dimensional groups in not unique, as in the case of a locally compact groups. It depends on two completions $\tilde{H}$ and $\tilde{G}$ of the subgroup H and the group G, on an extension $\tilde{S} : \tilde{H} \to U(V)$ of the representation $S : H \to U(V)$ and on a choice of the G-quasi-invariant measure μ on an appropriate completion $\tilde{X} = \tilde{H}\backslash\tilde{G}$ of the space $H\backslash G$. As the illustration we consider the "nilpotent" group $B_0^{\mathbb{Z}}$ of infinite in both directions upper triangular matrices and the induced representation corresponding to the so-called *generic orbits*.

Contents

1. Introduction

The *induced representations* were introduced and studied for a *finite groups* by F.G. Frobenius. Our aim is to develop the concept of *induced representations for infinite-dimensional groups*.

The content of the article is as follows. Section 2 is devoted to the notion of induced representations elaborated for a *locally compact groups* by G.W.Mackey [14, 15] and to the Kirillov *orbit methods* [4] for the nilpotent Lie groups $B(n,\mathbb{R})$.

In Section 3 we extend the notion of the induced representations for infinite-dimensional groups. We start the orbit method for infinite-dimensional "nilpotent" group $B_0^{\mathbb{Z}}$, construct the induced representations corresponding to the generic orbits and study its irreducibility.

In Section 4 we remind the Gauss decomposition of $n \times n$ matrices (Subsection 4.1), and Gauss decomposition of infinite order matrices (Subsection 4.2).

More precisely, we give the well-known definition of the induced representations for a locally compact groups in Subsection 2.1. In Subsection 2.2 we remind the Kirillov orbit method for finite-dimensional nilpotent group $G_n = B(n,\mathbb{R})$. The induced representations, corresponding to a generic orbits of the group G_n are discussed in Subsection 2.3. In the Subsection 2.4 we give a new proof of the irreducibility of the induced representations corresponding to a generic orbits in order to extend the proof of the irreducibility for infinite-dimensional "nilpotent" group $B_0^{\mathbb{Z}}$.

In Subsection 3.1 we remind the definition of the regular and quasiregular representations of infinite-dimensional groups. As in the case of a locally compact group these representations are the particular cases of the induced representations. This gives us the hint how to define the induced representations for infinite-dimensional groups. The definition is done in Subsection 3.2. The questions concerning the development of the orbit method for infinite-dimensional "nilpotent" group $B_0^{\mathbb{N}}$ and $B_0^{\mathbb{Z}}$ are discussed in Subsection 3.3.

The completions of the initial groups G are necessary to the definition of the induced representations for the initial infinite-dimensional group. The completions of the inductive limit $G = \varinjlim_n G_n$ of matrix groups G_n are studied in Subsection 3.4 and 3.5. We show that the *Hilbert-Lie groups* appear naturally in the representation theory of the infinite-dimensional matrix group. We define a family of the Hilbert-Lie group $\mathrm{GL}_2(a)$ (resp. $B_2(a)$), a Hilbert completions of the group $\mathrm{GL}_0(2\infty,\mathbb{R}) = \varinjlim_n \mathrm{GL}(2n-1,\mathbb{R})$ (resp. $B_0^{\mathbb{Z}} = \varinjlim_n B(2n-1,\mathbb{R})$). We show that *any continuous representation* of the group $\mathrm{GL}_0(2\infty,\mathbb{R})$ (resp. $B_0^{\mathbb{Z}}$) *is in fact continuous* in some stronger topology, namely *in a topology of a suitable Hilbert -Lie group* $\mathrm{GL}_2(a)$ (resp. $B_2(a)$) depending on the representation.

In Subsection 3.7 we construct the induced representations of the group $B_0^{\mathbb{Z}}$ corresponding to a generic orbits. The irreducibility of these representations is studied in Subsection 3.8. The very first steps to describe some part of the *dual* for the group $B_0^{\mathbb{N}}$ and $B_0^{\mathbb{Z}}$ are mentioned in Subsection 3.9

2. Induced representations, finite-dimensional case

2.1. **Induced representations.** The *induced representation* $\mathrm{Ind}_H^G S$ is the unitary representation of a group G associated with a unitary representation $S : H \to U(V)$ of a closed subgroup H of the group G. For details, see [7], Section 2.1. Suppose that $X = H \setminus G$ is a right $G-$space and that $s : X \to G$ is a Borel section of the projection

$p : G \to X = H\backslash G : g \mapsto Hg$. For Lie group, such a mapping s can be chosen to be smooth almost everywhere. Then every element $g \in G$ can be uniquely written in the form

$$g = hs(x),\ h \in H,\ x \in X, \tag{2.1}$$

and thus G (as a set) can be identified with $H \times X$. Under this identification, the Haar measure on G goes into a measure equivalent to the product of a quasi-invariant measure on X and a Haar measure on H. More precisely, if a quasi-invariant measure μ_s on X is appropriately chosen, then the following equalities are valid

$$d_r(g) = \frac{\Delta_G(h)}{\Delta_H(h)} d\mu_s(x) d_r(h), \tag{2.2}$$

$$\frac{d\mu_s(xg)}{d\mu_s(x)} = \frac{\Delta_H(h(x,g))}{\Delta_G(h(x,g))}, \tag{2.3}$$

where Δ_G is a modular function on the group G and $h(x,g) \in H$ is defined by the relation

$$s(x)g = h(x,g)s(xg). \tag{2.4}$$

Recall that a *modular function* on a group G is a homomorphism $G \ni t \mapsto \Delta_G(t) \in \mathbb{R}_+$ defined by the equality $h^{L_t} = \Delta_G(t)h$, where h is the right Haar measure on G, L is the left action of the group G on itself and $h^{L_t}(C) = h(tC)$.

Remark 2.1. *If the group G is unimodular, i.e $\Delta_G \equiv 1$, and it is possible to select a subgroup K that is complementary to H in the sense that almost every element of G can be uniquely written in the form*

$$g = hk,\ h \in H,\ k \in K, \tag{2.5}$$

then it is natural to identify $X = H\backslash G$ with K and to choose s as the embedding of K in G

$$s : K \mapsto G. \tag{2.6}$$

In such a case, the formula (2.2) assume the form

$$dg = \Delta_H(h)^{-1} d_r(h) d_r(k). \tag{2.7}$$

If both G and H are unimodular (or, more generally, if $\Delta_G(h)$ and $\Delta_H(h)$ coincide for $h \in H$), then there exist a G-invariant measure on $X{=}H\backslash G$. If it is possible to extend Δ_H to a multiplicative function on the group G, then there exist a *quasi-invariant measure* on X which is multiplied by the factor $\frac{\Delta_H(g)}{\Delta_G(g)}$ under translation by g.

Now we can define $\mathrm{Ind}_H^G S$ (see [7], section 2.3.). Let $S : H \to U(V)$ be a unitary representation of a subgroup H of the group G in a Hilbert space V and let μ be a measure on X satisfying condition (2.3). Let $\mathcal{H}$ denote the space of all vector-valued functions f on X with values in V such that

$$\|f\|^2 := \int_X \|f(x)\|_V^2 d\mu(x) < \infty.$$

Let us consider the representation T given by the formula

$$[T(g)f](x) = A(x,g)f(xg) = S(h)\left(\frac{d\mu_s(xg)}{d\mu_s(x)}\right)^{1/2} f(xg), \tag{2.8}$$

where

$$A(x,g)=\left[\frac{\Delta_H(h)}{\Delta_G(h)}\right]^{1/2}S(h), \tag{2.9}$$

and where the element $h=h(x,g)$ is defined by formula (2.4).

Definition 2.2. *The representation T is called the* unitary induced *representation and is denoted by* $\mathrm{Ind}_H^G S$.

Remark 2.3. *The* right (or the left) regular representation $\rho,\lambda : G\mapsto U(L^2(G,h))$ *of a locally compact group G is a particular case of the induced representation* $\mathrm{Ind}_H^G S$ *with $H=\{e\}$ and $S=Id$. The* guasiregular representation *is a particular case of the induced representation with some closed subgroup $H\subset G$ and $S=Id$.*

2.2. **Orbit method for finite-dimensional nilpotent group $B(n,\mathbb{R})$.** See Kirillov [6] and [7], Chapter 7, §2, p.129-130, for details. "Fix the group $G_n=B(n,\mathbb{R})$ of all upper triangular real matrices of order n with ones on the main diagonal. (The Kirillov notation for the group $B(n,\mathbb{R})$ is $N_+(n,\mathbb{R})$).

The basic result of the method of orbits, applied to nilpotent Lie groups, is the description of a one-to-one correspondence between two sets:

a) the set $\hat{G}$ of all equivalence classes of irreducible unitary representations of a connected and simply connected nilpotent Lie group G,

b) the set $\mathcal{O}(G)$ of all orbits of the group G in the space $\mathfrak{g}^*$ dual to the Lie algebra $\mathfrak{g}$ with respect to the coadjoint representation.

To construct this correspondence, we introduce the following definition. A subalgebra $\mathfrak{h}\subset\mathfrak{g}$ is *subordinate* to a functional $f\in\mathfrak{g}^*$ if

$$\langle f,[x,y]\rangle=0\quad for\ \ all\quad x,y\in\mathfrak{h},$$

i.e. if $\mathfrak{h}$ is an *isotropic subspace* with respect to the bilinear form defined by $B_f(x,y)=\langle f,[x,y]\rangle$ on $\mathfrak{g}$.

Lemma 2.4 (Lemma 7.7, [7]). *The following conditions are equivalent:*

(a) a subalgebra $\mathfrak{h}$ is subordinate to the functional f,

(b) the image of $\mathfrak{h}$ in the tangent space $T_f\Omega$ to the orbit Ω in the point f is an isotropic subspace,

(c) the map

$$x\mapsto\langle f,x\rangle$$

is a one-dimensional real representation of the Lie algebra $\mathfrak{h}$.

If the conditions of Lemma 2.4 are satisfied, we define the one-dimensional unitary representation $U_{f,H}$ of the group $H=\exp\mathfrak{h}$ by the formula

$$U_{f,H}(\exp\,x)=\exp\,2\pi i\langle f,x\rangle.$$

Theorem 2.5 (Theorem 7.2, [7]). *(a) Every irreducible unitary representation T of a connected and simply connected nilpotent Lie group G has the form*

$$T=\mathrm{Ind}_H^G U_{f,H},$$

where $H\subset G$ is a connected subgroup and $f\in\mathfrak{g}^*$;

(b) the representation $T_{f,H}=\mathrm{Ind}_H^G U_{f,H}$ is irreducible if and only if the Lie algebra $\mathfrak{h}$ of the group H is a subalgebra of $\mathfrak{g}$ subordinate to the functional f with maximal possible dimension;

(c) irreducible representations T_{f_1,H_1} and T_{f_2,H_2} are equivalent if and only if the functionals f_1 and f_2 belong to the same orbit of $\mathfrak{g}^*$."

Example 2.6. *Let us consider the Heisenberg group $G_3 = B(3,\mathbb{R})$, its Lie algebra $\mathfrak{g}$ and the dual space $\mathfrak{g}^*$. Fix the notations*

$$G = B(3,\mathbb{R}) = \left\{ \left(\begin{smallmatrix} 1 & x_{12} & x_{13} \\ 0 & 1 & x_{23} \\ 0 & 0 & 1 \end{smallmatrix}\right) \right\},$$

$$\mathfrak{g} = n_+(3,\mathbb{R}) = \left\{ \left(\begin{smallmatrix} 0 & x_{12} & x_{13} \\ 0 & 0 & x_{23} \\ 0 & 0 & 0 \end{smallmatrix}\right) \right\}, \ \mathfrak{g}^* = n_-(3,\mathbb{R}) = \left\{ \left(\begin{smallmatrix} 0 & 0 & 0 \\ y_{21} & 0 & 0 \\ y_{31} & y_{32} & 0 \end{smallmatrix}\right) \right\}.$$

The *adjoint action* $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ of the group G on its Lie algebra $\mathfrak{g}$ is:

$$\mathfrak{g} \ni x \mapsto \mathrm{Ad}_t(x) := txt^{-1} \in \mathfrak{g}, \quad t \in G, \tag{2.10}$$

the pairing between the $\mathfrak{g}$ and $\mathfrak{g}^*$:

$$\mathfrak{g}^* \times \mathfrak{g} \ni (y,x) \mapsto \langle y,x\rangle := tr(xy) = \sum_{1\leq k<n\leq 3} x_{kn}y_{nk} \in \mathbb{R}. \tag{2.11}$$

Since $tr(txt^{-1}y) = tr(xt^{-1}yt)$ the *coadjoint action* of G on the dual $\mathfrak{g}^*$ to $\mathfrak{g}$ is

$$\mathfrak{g}^* \ni y \mapsto \mathrm{Ad}^*_t(y) := (t^{-1}yt)_- \in \mathfrak{g}^*, \quad t \in G, \tag{2.12}$$

where $(z)_-$ means that we take lower triangular part of the matrix z.

To calculate $\mathrm{Ad}^*_t(y)$ explicitly for $n = 3$, we have

$$t^{-1}yt = \left(\begin{smallmatrix} 1 & t_{12} & t_{13} \\ 0 & 1 & t_{23} \\ 0 & 0 & 1 \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} 0 & 0 & 0 \\ y_{21} & 0 & 0 \\ y_{31} & y_{32} & 0 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & t_{12} & t_{13} \\ 0 & 1 & t_{23} \\ 0 & 0 & 1 \end{smallmatrix}\right)$$
$$= \left(\begin{smallmatrix} 1 & -t_{12} & -t_{13}+t_{12}t_{23} \\ 0 & 1 & -t_{23} \\ 0 & 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 0 & 0 & 0 \\ y_{21} & y_{21}t_{12} & y_{21}t_{13} \\ y_{31} & y_{31}t_{12}+y_{32} & y_{31}t_{13}+y_{32}t_{23} \end{smallmatrix}\right),$$

hence

$$\mathrm{Ad}^*_t(y) := (t^{-1}yt)_- = \left(\begin{smallmatrix} 0 & 0 & 0 \\ y_{21}-t_{23}y_{31} & 0 & 0 \\ y_{31} & y_{31}t_{12}+y_{32} & 0 \end{smallmatrix}\right).$$

We have two type of the orbits $\mathcal{O}$:

1) if $y_{31} = 0$, then $\left(\begin{smallmatrix} y_{21} & \\ 0 & y_{32} \end{smallmatrix}\right) \simeq (y_{21}, y_{32})$ for fixed y_{21}, y_{32} is 0-dimensional orbit;

2) if $y_{31} \neq 0$, then $\left(\begin{smallmatrix} \mathbb{R} & \\ y_{31} & \mathbb{R} \end{smallmatrix}\right)$ is 2-dimensional orbits.

In the case 1) fixe the point $f = (y_{21}, y_{32})$, the subordinate subalgebra $\mathfrak{h}$ coinside with all $\mathfrak{g}$, since $[\mathfrak{g},\mathfrak{g}] = \langle E_{13}\rangle := \{tE_{13} \mid t \in \mathbb{R}\}$. Corresponding one-dimensional representation of the algebra $\mathfrak{h} = \mathfrak{g}$ is

$$\mathfrak{g} \ni x \mapsto \langle f,x\rangle = tr(xf) = tr\left[\left(\begin{smallmatrix} 0 & x_{12} & x_{13} \\ 0 & 0 & x_{23} \\ 0 & 0 & 0 \end{smallmatrix}\right) \left(\begin{smallmatrix} 0 & 0 & 0 \\ y_{21} & 0 & 0 \\ 0 & y_{32} & 0 \end{smallmatrix}\right) \right] = x_{12}y_{21} + x_{23}y_{32} \in \mathbb{R}.$$

The corresponding representation of the group G is

$$G \ni \exp(x) \mapsto \exp(2\pi i\langle f,x\rangle) \in S^1. \tag{2.13}$$

So we have 1-dimensional representation

$$G_3 \ni \exp\left(\begin{smallmatrix} 0 & x_{12} & x_{13} \\ 0 & 0 & x_{23} \\ 0 & 0 & 0 \end{smallmatrix}\right) \mapsto \exp(2\pi i(x_{12}y_{21} + x_{23}y_{32})) \in S^1.$$

We note that

$$\exp(x) = \exp\left(\begin{smallmatrix} 0 & x_{12} & x_{13} \\ 0 & 0 & x_{23} \\ 0 & 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & x_{12} & x_{13}+\frac{1}{2}x_{12}x_{23} \\ 0 & 1 & x_{23} \\ 0 & 0 & 1 \end{smallmatrix}\right).$$

In the case 2) we have two subordinate subalgebras of the maximal dimension

$$\mathfrak{h}_1 = \left(\begin{smallmatrix} 0 & 0 & x_{13} \\ 0 & 0 & x_{23} \\ 0 & 0 & 0 \end{smallmatrix}\right), \quad \text{and} \quad \mathfrak{h}_2 = \left(\begin{smallmatrix} 0 & x_{12} & x_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right). \quad \text{Set} \quad f = \left(\begin{smallmatrix} 0 & 0 & 0 \\ y_{21} & 0 & 0 \\ y_{31} & y_{32} & 0 \end{smallmatrix}\right).$$

The corresponding one-dimensional representations of the subalgebras $\mathfrak{h}_i$, $i=1,2$ are

$$\mathfrak{h}_1 \ni x \mapsto \langle f, x\rangle = x_{13}y_{31} + x_{23}y_{32} \in \mathbb{R},$$

$$\mathfrak{h}_2 \ni x \mapsto \langle f, x\rangle = x_{12}y_{21} + x_{13}y_{31} \in \mathbb{R}.$$

The corresponding representations S of the subgroups H_1 and H_2 respectively are:

$$H_1 \ni \begin{pmatrix} 1 & 0 & x_{13} \\ 0 & 1 & x_{23} \\ 0 & 0 & 1 \end{pmatrix} = \exp(x) \mapsto \exp(2\pi i(x_{13}y_{31} + x_{23}y_{32})) \in S^1,$$

$$H_2 \ni \begin{pmatrix} 1 & x_{12} & x_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \exp(x) \mapsto \exp(2\pi i(x_{12}y_{21} + x_{13}y_{31})) \in S^1.$$

In the case H_1 we have the decomposition $G_3 = \mathbb{R}^2 \ltimes B(2,\mathbb{R}) \simeq H_1 \ltimes \mathbb{R}$, indeed we have

$$G_3 \ni \begin{pmatrix} 1 & x_{12} & x_{13} \\ 0 & 1 & x_{23} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & x_{13} \\ 0 & 1 & x_{23} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x_{12} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{R}^2 \ltimes B(2,\mathbb{R}),$$

hence the space $X = H_1\backslash G_3$ is isomorphic to $B(2,\mathbb{R}) \simeq \mathbb{R}$ and s can be choosing as the *embedding* $s : B(2,\mathbb{R}) \mapsto B(3,\mathbb{R})$.

$$B(2,\mathbb{R}) \ni \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} =: x \mapsto s(x) = \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in B(3,\mathbb{R}).$$

For general n we have

$$B(n+1,\mathbb{R}) = \mathbb{R}^n \ltimes B(n,\mathbb{R}). \tag{2.14}$$

To calculate the right action of G on X i.e. to find $h(x,t)$ such that

$$s(x)t = h(x,t)s(xt),$$

we have for $x \in B(2,\mathbb{R})$ and $t \in B(3,\mathbb{R})$

$$s(x)t = \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & t_{12} & t_{13} \\ 0 & 1 & t_{23} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+t_{12} & t_{13}+xt_{23} \\ 0 & 1 & t_{23} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & t_{13}+xt_{23} \\ 0 & 1 & t_{23} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x+t_{12} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= h(x,t)s(xt), \text{ hence } \ h(x,t) = \begin{pmatrix} 1 & 0 & t_{13}+xt_{23} \\ 0 & 1 & t_{23} \\ 0 & 0 & 1 \end{pmatrix}.$$

Finally, the induced unitary representation $\mathrm{Ind}_{H_1}^G S$ have the following form in the Hilbert space $L^2(\mathbb{R}, dx)$ (case H_1 and $f = y_{31}E_{31}$):

$$f(x) \mapsto S(h(x,t))f(xt) = \exp(2\pi i(t_{13} + t_{23}x)y_{31})f(x + t_{12}). \tag{2.15}$$

In the Kirillov [7] notations we have:

$$f(x) \mapsto \exp(2\pi i(c + bx)\lambda)f(x + a), \quad y_{31} = \lambda, \ \begin{pmatrix} 1 & t_{12} & t_{13} \\ 0 & 1 & t_{23} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}.$$

2.3. **The induced representations, corresponding to a generic orbits, finite-dimensional case.** We show following A. Kirillov [7] how the orbit method works for the nilpotent group $B(n,\mathbb{R})$ and small n.

For general $n \in \mathbb{N}$ the coadjoint action of the group G_n on $\mathfrak{g}$ is as follows

$$t = I + \sum_{1\leq k<m\leq n} t_{km}E_{km}, \ y = \sum_{1\leq m<k\leq n} y_{km}E_{km}, \ t^{-1} := I + \sum_{1\leq k<m\leq n} t_{km}^{-1}E_{km}$$

hence

$$(tyt^{-1})_{pq} = \sum_{m=1}^{q} (ty)_{pm}t_{mq}^{-1} = \sum_{m=1}^{q}\sum_{r=p}^{n} t_{pr}y_{rm}t_{mq}^{-1}, \ 1 \leq p, q \leq n,$$

and

$$\mathrm{Ad}_t^*(y) = (t^{-1}yt)_- = I + \sum_{1\leq q<p\leq n} (t^{-1}yt)_{pq}E_{pq}. \tag{2.16}$$

Example 2.7. Generic orbits *for the group* $G = B(n,\mathbb{R})$ (*see* [7], *Example 7.9*).

"The form of the action $\mathrm{Ad}^*_t(y) = (t^{-1}yt)_-$ implies, that Ad^*_t, $t \in G$ acts as follows: to a given column of $y \in \mathfrak{g}^*$, a linear combination of the previous columns is added and to a given row of y, a linear combination of the following rows is added. More generally, the minors Δ_k, $k = 1, 2, ..., [\frac{n}{2}]$, consisting of the last k rows and first k columns of y are invariant of the action. *It is possible to show that if all the numbers c_k are different from zeros, then the manifold given by the equation*

$$\Delta_k = c_k,\ 1 \le k \le \left[\frac{n}{2}\right] \tag{2.17}$$

is a G-orbit in $\mathfrak{g}^*$. Hence generic orbits have codimension equal to $[\frac{n}{2}]$ and dimension equal to $\frac{n(n-1)}{2} - [\frac{n}{2}]$. To obtain a representation for such an orbit, we can take a matrix y of the form

$$y = \left(\begin{smallmatrix} 0 & 0 \\ \Lambda & 0 \end{smallmatrix}\right),$$

where Λ is the matrix of order $[\frac{n}{2}]$ such that all nonzero elements are contained in the *anti-diagonal.* It is easy to find a subalgebra of dimension $[\frac{n}{2}] \times [\frac{n+1}{2}]$ subordinate to the functional y. It consist of all matrices of the form

$$\left(\begin{smallmatrix} 0 & A \\ 0 & 0 \end{smallmatrix}\right),$$

where A is an $[\frac{n}{2}] \times [\frac{n+1}{2}]$ or $[\frac{n+1}{2}] \times [\frac{n}{2}]$ matrix."

Example 2.8. *Let* $G = B(5,\mathbb{R})$, $\mathfrak{g} = n_+(5,\mathbb{R})$, $\mathfrak{g}^* = n_-(5,\mathbb{R})$. *We write the representations for generic orbit corresponding to the point* $y = y_{51}E_{51} + y_{42}E_{42} \in \mathfrak{g}^*$. *Set* $\mathfrak{h}_3 = \{t - I \mid t \in H_3\}$ *where*

$$G = \left\{\begin{pmatrix} 1 & x_{12} & x_{13} & x_{14} & x_{15} \\ 0 & 1 & x_{23} & x_{24} & x_{25} \\ 0 & 0 & 1 & x_{34} & x_{35} \\ 0 & 0 & 0 & 1 & x_{45} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right\},\quad H_3 = \left\{\begin{pmatrix} 1 & 0 & 0 & t_{14} & t_{15} \\ 0 & 1 & 0 & t_{24} & t_{25} \\ 0 & 0 & 1 & t_{34} & t_{35} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right\},\quad \mathfrak{g}^* = \left\{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ y_{21} & 0 & 0 & 0 & 0 \\ y_{31} & y_{32} & 0 & 0 & 0 \\ y_{41} & y_{42} & y_{43} & 0 & 0 \\ y_{51} & y_{52} & y_{53} & y_{54} & 0 \end{pmatrix}\right\}.$$

The corresponding representation S of the subgroup H_3 of the maximal dimension is:

$$H_3 \ni t \mapsto \exp(2\pi i\langle y,(t - I)\rangle) = \exp(2\pi i[t_{15}y_{51} + t_{24}y_{42}]) \in S^1.$$

For the group $B(5,\mathbb{R})$ holds the following decomposition

$$B(5,\mathbb{R}) = B_3B(3)B^{(3)} \text{ i.e. } x = x_3x(3)x^{(3)}, \tag{2.18}$$

where

$$B^{(3)} = \left\{\begin{pmatrix} 1 & x_{12} & x_{13} & 0 & 0 \\ 0 & 1 & x_{23} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right\},\quad B(3) = \left\{\begin{pmatrix} 1 & 0 & 0 & x_{14} & x_{15} \\ 0 & 1 & 0 & x_{24} & x_{25} \\ 0 & 0 & 1 & x_{34} & x_{35} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right\},\quad B_3 = \left\{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_{45} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right\}.$$

We *calculate $h(x,t)$ in the relation $s(x)t = h(x,t)s(xt)$,* but first we fix the section $s : X = H\backslash G \mapsto G$ of the projection $p : G \mapsto X$. To define the section $s : X \mapsto G$ we show that in addition to the decomposition (2.18) the following decomposition $B(5,\mathbb{R}) = B(3)B_3B^{(3)}$ also holds. Indeed, to find $h \in H_3 = B(3)$ such that $x = hx_3x^{(3)}$, we get $x_3x(3)x^{(3)} = hx_3x^{(3)}$, hence

$$h = x_3x(3)x_3^{-1} = \begin{pmatrix} 1 & 0 & 0 & x_{14} & x_{15} \\ 0 & 1 & 0 & x_{24} & x_{25} \\ 0 & 0 & 1 & x_{34} & x_{35} \\ 0 & 0 & 0 & 1 & x_{45} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -x_{45} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & x_{14} & x_{15}-x_{14}x_{45} \\ 0 & 1 & 0 & x_{24} & x_{25}-x_{24}x_{45} \\ 0 & 0 & 1 & x_{34} & x_{35}-x_{34}x_{45} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in B(3).$$

We have two different decompositions

$$B_3B(3)B^{(3)} \ni x_3x(3)x^{(3)} = hx_3x^{(3)} \in B(3)B_3B^{(3)}, \quad \text{with} \quad h = x_3x(3)x_3^{-1}.$$

Remark 2.9. *For an arbitrary n, $m\in\mathbb{N}$, $1<m<n$, we have for the group $G_n=B(n,\mathbb{R})$ two decompositions:*
(2.19)
$$G_n{=}B_mB(m)B^{(m)}\ni x_mx(m)x^{(m)}=hx_mx^{(m)}\in B(m)B_mB^{(m)},\ h{=}x_mx(m)x_m^{-1},$$
where
$$B_m=\{I+\sum_{m<k<r\leq n}x_{kr}E_{kr}\},\ B(m)=\{I+\sum_{1\leq k\leq m<r\leq n}x_{kr}E_{kr}\},\ B^{(m)}=\{I+\sum_{1\leq k<r\leq m}x_{kr}E_{kr}\}.$$

Since $X=B(m)\backslash G_n$ is isomorphic to $B_mB^{(m)}$ by decomposition (2.19), the section s can be choosing, by Remark 2.1, as the embedding
$$B_mB^{(m)}\ni x_mx^{(m)}\mapsto s(x_mx^{(m)})=x_mx^{(m)}\in B_mB(m)B^{(m)}.$$
Since $s(x)t=h(x,t)s(xt)$, we have $h(x,t)=s(x)t(s(xt))^{-1}$. It remains to calculate $s(x)t$ and $s(xt)$.

Remark 2.10. *We have*
$$h(x,t)-I=\left\{\begin{array}{cc}0, & \textit{for}\quad t\in B_mB^{(m)}\\ x^{(m)}(t-I)x_m^{-1}, & \textit{for}\quad t\in B(m)\end{array}\right. .$$

Indeed, let $t=t_mt^{(m)}\in B_mB^{(m)}$ then $s(x)t=x_mx^{(m)}t_mt^{(m)}=x_mt_mx^{(m)}t^{(m)}$. We get also $xt=x_mx^{(m)}t_mt^{(m)}=x_mt_mx^{(m)}t^{(m)}$, so $s(xt)=x_mt_mx^{(m)}t^{(m)}$, hence $s(x)t=s(xt)$ and we get $h(x,t){=}e$. For $t:=t(m)\in B(m)$ and $x=x_mx^{(m)}\in B_mB(m)$ we get
$$s(x)t=x_mx^{(m)}t=x_mx^{(m)}t(x^{(m)})^{-1}x^{(m)}=x_m\tilde{x}(m)x^{(m)}=hx_mx^{(m)}=h(x,t)s(xt),$$
where $\tilde{x}(m)=x^{(m)}t(x^{(m)})^{-1}$. Then we get by (2.19)
$$h(x,t)=h=x_m\tilde{x}(m)x_m^{-1}=x_mx^{(m)}t(x^{(m)})^{-1}x_m^{-1}=x_mx^{(m)}t(x_mx^{(m)})^{-1}, \tag{2.20}$$
$$h(x,t)=\begin{pmatrix}x^{(m)}&0\\0&x_m\end{pmatrix}\begin{pmatrix}1&t-I\\0&1\end{pmatrix}\begin{pmatrix}(x^{(m)})^{-1}&0\\0&x_m^{-1}\end{pmatrix}=\begin{pmatrix}1&x^{(m)}(t-I)x_m^{-1}\\0&1\end{pmatrix}=\begin{pmatrix}1&H(x,t)\\0&1\end{pmatrix}, \tag{2.21}$$
where
$$H(x,t):=x^{(m)}(t-I)x_m^{-1}. \tag{2.22}$$
Denote by $E_{kr}(t):=I{+}tE_{kr},\ t\in\mathbb{R}$ the one-parameter subgroups of the groups $B(n,\mathbb{R})$. We would like *to find the generators $A_{kn}=\frac{d}{dt}T_{I+tE_{kn}}|_{t=0}$ of the induced representation* T_t (2.28).

Set for $G_n=B_mB(m)B^{(m)}$ and $1\leq k\leq m<r\leq n$
(2.23)
$$S_{kr}(t_{kr}):=\langle y,(h(x,E_{kr}(t_{kr}))-I)\rangle,\ \text{then}\ A_{kr}=\frac{d}{dt}\exp(2\pi iS_{kr}(t))|_{t=0}=2\pi iS_{kr}(1).$$
Let us denote by $\mathbb{S}$ the following matrix:
$$\mathbb{S}=(S_{kr})_{1\leq k\leq m<r\leq n},\quad\text{where}\quad S_{kr}=S_{kr}(1),\quad\text{then}\quad \mathbb{S}=(2\pi i)^{-1}(A_{kr})_{k,r}. \tag{2.24}$$

Lemma 2.11. *Let $B=(b_{kr})_{k,r=1}^n\in\mathrm{Mat}(n,\mathbb{C})$. Define the matrix $C=(c_{kr})_{k,r=1}^n\in\mathrm{Mat}(n,\mathbb{C})$ by*
$$c_{kr}=\mathrm{tr}(E_{kr}B),\quad 1\leq k,r\leq n,\quad \textit{then we have}\quad C=B^T, \tag{2.25}$$
where E_{kr} are matrix units and B^T means transposed matrix to the matrix B. The equality $C=B^T$ holds also in the case when B is an arbitrary $m\times n$ rectangular matrix. The statement is true also for matrices $B\in\mathrm{Mat}(\infty,\mathbb{C})$.

Proof. Indeed, we have $\mathrm{tr}(E_{kr}B) = b_{rk}$. □

We calculate now the matrix $\mathbb{S}(t) = (S_{kr}(t_{kr}))_{k,r}$ and the matrix $\mathbb{S} = (S_{kr}(1))_{k,r}$ using Lemma 2.11. Using (2.22) we have

$$\langle y, h(x,t) - I\rangle = \mathrm{tr}\,(H(x,t)y) = \mathrm{tr}\left(x^{(m)}t_0x_m^{-1}y\right) = \mathrm{tr}\left(t_0x_m^{-1}yx^{(m)}\right) = \mathrm{tr}\,(t_0B(x,y))\,,$$

where $t_0 = t - I$ and

$$B(x,y) = x_m^{-1}yx^{(m)} \cong \begin{pmatrix} 1 & 0 \\ 0 & x_m^{-1}\end{pmatrix}\begin{pmatrix} 0 & 0 \\ y & 0\end{pmatrix}\begin{pmatrix} x^{(m)} & 0 \\ 0 & 1\end{pmatrix} = \begin{pmatrix} 0 & 0 \\ x_m^{-1}yx^{(m)} & 0\end{pmatrix}. \tag{2.26}$$

By definition we have

$$S_{kr}(t_{kr}) = \langle y, (h(x, E_{kr}(t_{kr})) - I)\rangle = \mathrm{tr}(t_{kr}E_{kr}B(x,y)),$$

hence by Lemma 2.11 and (2.26) we conclude that

(2.27)

$$\mathbb{S} = (S_{kr}(1))_{kr} = (\mathrm{tr}\,(E_{kr}B(x,t)))_{k,r} = B^T(x,y) = (x^{(m)})^Ty^T(x_m^{-1})^T = \begin{pmatrix} 0 & (x^{(m)})^Ty^T(x_m^{-1})^T \\ 0 & 0\end{pmatrix}.$$

So the induced representation $\mathrm{Ind}_H^G(S) : G \to U(L^2(X,\mu))$ corresponding to the point $y \in \mathfrak{g}^*$ has the following form

$$(T_tf)(x) = S(h(x,t))\left(\frac{d\mu(xt)}{d\mu(x)}\right)^{1/2} f(xt),\;\; f \in L^2(X,\mu),\;\; x \in X = H\backslash G,\; t \in G, \tag{2.28}$$

where

$$S(h(x,t)) = \exp(2\pi i\langle y, (h(x,t) - I)\rangle) = \exp\Big(2\pi i\mathrm{tr}\,((t-I)B(x,y))\Big). \tag{2.29}$$

We *calculate* $B(x,y)$ *and* $\mathbb{S}$ *for different groups* G_n. For G_5 we get by (2.26):

$$G_5 = \left\{\begin{pmatrix} 1 & x_{12} & x_{13} & x_{14} & x_{15} \\ 0 & 1 & x_{23} & x_{24} & x_{25} \\ 0 & 0 & 1 & x_{34} & x_{35} \\ 0 & 0 & 0 & 1 & x_{45} \\ 0 & 0 & 0 & 0 & 1\end{pmatrix}\right\},\;\; y = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & y_{42} & 0 & 0 & 0 \\ y_{51} & 0 & 0 & 0 & 0\end{pmatrix},\;\; x^{(3)} = \begin{pmatrix} 1 & x_{12} & x_{13} \\ 0 & 1 & x_{23} \\ 0 & 0 & 1\end{pmatrix},\;\; x_3 = \begin{pmatrix} 1 & x_{45} \\ 0 & 1\end{pmatrix},$$

$$B(x,y) = \begin{pmatrix} 1 & x_{45}^{-1} \\ 0 & 1\end{pmatrix}\begin{pmatrix} 0 & y_{42} & 0 \\ y_{51} & 0 & 0\end{pmatrix}\begin{pmatrix} 1 & x_{12} & x_{13} \\ 0 & 1 & x_{23} \\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix} x_{45}^{-1}y_{51} & y_{42}+x_{45}^{-1}y_{51}x_{12} & y_{42}x_{23}+x_{45}^{-1}y_{51}x_{13} \\ y_{51} & y_{51}x_{12} & y_{51}x_{13}\end{pmatrix},$$

hence by (2.27) we have

$$\mathbb{S} := B(x,y)^T = \begin{pmatrix} 1 & 0 & 0 \\ x_{12} & 1 & 0 \\ x_{13} & x_{23} & 1\end{pmatrix}\begin{pmatrix} 0 & y_{51} \\ y_{42} & 0 \\ 0 & 0\end{pmatrix}\begin{pmatrix} 1 & x_{45}^{-1} \\ 0 & 1\end{pmatrix} = \begin{pmatrix} x_{45}^{-1}y_{51} & y_{51} \\ y_{42}+x_{45}^{-1}y_{51}x_{12} & y_{51}x_{12} \\ y_{42}x_{23}+x_{45}^{-1}y_{51}x_{13} & y_{51}x_{13}\end{pmatrix}. \tag{2.30}$$

Remark 2.12. *For the matrix* $x = I + \sum_{1\leq k<n\leq m} x_{kn}E_{kn} \in B(m,\mathbb{R})$ *we denote by* x_{kn}^{-1} *the matrix elements of the matrix* x^{-1}*, i.e.* $x^{-1} =: I + \sum_{1\leq k<n\leq m} x_{kn}^{-1}E_{kn} \in B(m,\mathbb{R})$*. The explicit expressions for* x_{kn}^{-1} *are as follows (see* [8]*, formula (4.4))* $x_{kk+1}^{-1} = -x_{kk+1}$*,*

$$x_{kn}^{-1} = -x_{kn} + \sum_{r=1}^{n-k-1}(-1)^{r-1}\sum_{k<i_1<i_2<\ldots<i_r<n} x_{ki_1}x_{i_1i_2}\ldots x_{i_rn},\;\; k < n-1. \tag{2.31}$$

The generators $A_{kn} = \frac{d}{dt}T_{I+tE_{kn}}|_{t=0}$ of the one-parameter subgroups $E_{kn}(t) := I + tE_{kn}$, $t \in \mathbb{R}$ generated by *the representation* T_t (2.28) are as follows (see (2.24) and (2.30)):

$$A_{12} = D_{12}, \quad A_{13} = D_{13}, \quad A_{23} = x_{12}D_{13} + D_{23}, \quad A_{45} = D_{45}, \tag{2.32}$$

$$\mathbb{S} = \frac{1}{2\pi i}\begin{pmatrix} A_{14} & A_{15} \\ A_{24} & A_{25} \\ A_{34} & A_{35}\end{pmatrix} = \begin{pmatrix} x_{45}^{-1}y_{51} & y_{51} \\ y_{42}+x_{45}^{-1}y_{51}x_{12} & y_{51}x_{12} \\ y_{42}x_{23}+x_{45}^{-1}y_{51}x_{13} & y_{51}x_{13}\end{pmatrix}, \tag{2.33}$$

where $D_{kn}=\frac{\partial}{\partial x_{kn}}$. For example, to obtain the expression $A_{23}=x_{12}D_{13}+D_{23}$ we note that

$$B(3,\mathbb{R})\ni x(I+tE_{23})=\begin{pmatrix}1&x_{12}&x_{13}\\0&1&x_{23}\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&t\\0&0&1\end{pmatrix}=\begin{pmatrix}1&x_{12}&x_{13}+tx_{12}\\0&1&x_{23}+t\\0&0&1\end{pmatrix}.$$

Here we denote by $D_{kn}=D_{kn}(h)$ the operator of the partial derivative corresponding to the shift $x\mapsto x+tE_{kn}$ on the group $B_m\times B^{(m)}\ni x=(x_{kn})_{k,n}$ and the Haar measure h:

$$(D_{kn}(h)f)(x)=\frac{d}{dt}\left(\frac{dh(x+tE_{kn})}{dh(x)}\right)^{1/2}f(x+tE_{kn})\mid_{t=0},\quad D_{kn}(h):=\frac{\partial}{\partial x_{kn}}. \tag{2.34}$$

Example 2.13. *Let* $G=B(4,\mathbb{R})=\left\{\begin{pmatrix}1&x_{23}&x_{24}&x_{25}\\0&1&x_{34}&x_{35}\\0&0&1&x_{45}\\0&0&0&1\end{pmatrix}\right\}$. *The representations for generic orbit corresponding to the point* $y=y_{43}E_{43}+y_{52}E_{52}\in\mathfrak{g}^*$.

We calculate $\mathbb{S}$ in two different ways. First using (2.26) we get

$$B(x,y)=x_m^{-1}yx^{(m)}=\begin{pmatrix}1&x_{45}^{-1}\\0&1\end{pmatrix}\begin{pmatrix}0&y_{43}\\y_{52}&0\end{pmatrix}\begin{pmatrix}1&x_{23}\\0&1\end{pmatrix}=\begin{pmatrix}x_{45}^{-1}y_{52}&y_{43}+x_{45}^{-1}y_{52}x_{23}\\y_{52}&x_{23}y_{52}\end{pmatrix},$$

$$\frac{1}{2\pi i}\begin{pmatrix}A_{24}&A_{25}\\A_{34}&A_{35}\end{pmatrix}=\mathbb{S}=B^T(x,y)=\begin{pmatrix}1&0\\x_{23}&1\end{pmatrix}\begin{pmatrix}0&y_{52}\\y_{43}&0\end{pmatrix}\begin{pmatrix}1&0\\x_{45}^{-1}&1\end{pmatrix}=\begin{pmatrix}x_{45}^{-1}y_{52}&y_{52}\\y_{43}+x_{45}^{-1}y_{52}x_{23}&y_{52}x_{23}\end{pmatrix},$$

$$A_{23}=D_{23},\quad A_{45}=D_{45}.$$

From the other hand, by (2.21) we get $h(x,t)=\begin{pmatrix}1&H(x,t)\\0&1\end{pmatrix}$, where

$$H(x,t)=x^{(3)}(t-I)x_3^{-1}=\begin{pmatrix}1&x_{23}\\0&1\end{pmatrix}\begin{pmatrix}t_{24}&t_{25}\\t_{34}&t_{35}\end{pmatrix}\begin{pmatrix}1&x_{45}^{-1}\\0&1\end{pmatrix}=\begin{pmatrix}t_{24}+x_{23}t_{34}&(t_{24}+x_{23}t_{34})x_{45}^{-1}+t_{25}+x_{23}t_{35}\\t_{34}&t_{34}x_{45}^{-1}+t_{25}+t_{35}\end{pmatrix}. \tag{2.35}$$

Therefore,

$$\langle y,(h(x,t)-I)\rangle=h(x,t)_{34}y_{43}+h(x,t)_{25}y_{52}=t_{34}y_{43}+[(t_{24}+x_{23}t_{34})x_{45}^{-1}+t_{25}+x_{23}t_{35}]y_{52},$$

hence

$$\mathbb{S}_2(t):=\begin{pmatrix}S_{24}(t_{24})&S_{25}(t_{25})\\S_{34}(t_{34})&S_{35}(t_{35})\end{pmatrix}=\begin{pmatrix}t_{24}x_{45}^{-1}y_{52}&t_{25}y_{52}\\t_{34}y_{43}+x_{23}t_{34}x_{45}^{-1}y_{52}&x_{23}t_{35}y_{52}\end{pmatrix},$$

$$\mathbb{S}_2:=\mathbb{S}_2(1)=\begin{pmatrix}S_{24}&S_{25}\\S_{34}&S_{35}\end{pmatrix}=\begin{pmatrix}x_{45}^{-1}y_{52}&y_{52}\\y_{43}+x_{45}^{-1}y_{52}x_{23}&y_{52}x_{23}\end{pmatrix}=\begin{pmatrix}1&0\\x_{23}&1\end{pmatrix}\begin{pmatrix}0&y_{52}\\y_{43}&0\end{pmatrix}\begin{pmatrix}1&0\\x_{45}^{-1}&1\end{pmatrix}. \tag{2.36}$$

Example 2.14. *Let* $G=B(6,\mathbb{R})$, $\mathfrak{g}=n_+(6,\mathbb{R})$, $\mathfrak{g}^*=n_-(6,\mathbb{R})$. *We write the representations for generic orbit corresponding to the point* $y=y_{43}E_{43}+y_{52}E_{52}+y_{61}E_{61}\in\mathfrak{g}^*$. *Set*

$$G_6=\left\{\begin{pmatrix}1&x_{12}&x_{13}&x_{14}&x_{15}&x_{16}\\0&1&x_{23}&x_{24}&x_{25}&x_{26}\\0&0&1&x_{34}&x_{35}&x_{36}\\0&0&0&1&x_{45}&x_{46}\\0&0&0&0&1&x_{56}\\0&0&0&0&0&1\end{pmatrix}\right\},\quad H_3=\left\{\begin{pmatrix}1&0&0&t_{14}&t_{15}&t_{16}\\0&1&0&t_{24}&t_{25}&t_{26}\\0&0&1&t_{34}&t_{35}&t_{36}\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix}\right\},\quad y=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&y_{43}&0&0&0\\0&y_{52}&0&0&0&0\\y_{61}&0&0&0&0&0\end{pmatrix},$$

$\mathfrak{h}_3=\{t-I\mid t\in H_3\}$. The corresponding representations S of the subgroup H_3 is:

$$H_3\ni\exp(t-I)=t\mapsto\exp(2\pi i\langle y,(t-I)\rangle)=\exp(2\pi i[t_{34}y_{43}+t_{25}y_{52}+t_{16}y_{61}])\in S^1.$$

For the group $B(6,\mathbb{R})$ holds the following decomposition (see Remark 2.9)

$$B(6,\mathbb{R})=B_3B(3)B^{(3)}\quad\text{i.e.}\quad x=x_3x(3)x^{(3)}, \tag{2.37}$$

where

$$x^{(3)}=\begin{pmatrix}1&x_{12}&x_{13}&0&0&0\\0&1&x_{23}&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix},\quad x(3)=\begin{pmatrix}1&0&0&x_{14}&x_{15}&x_{16}\\0&1&0&x_{24}&x_{25}&x_{26}\\0&0&1&x_{34}&x_{35}&x_{36}\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix},\quad x_3=\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&x_{45}&x_{46}\\0&0&0&0&1&x_{56}\\0&0&0&0&0&1\end{pmatrix}.$$

We get by (2.26) and (2.27)

$$B(x,y)=\begin{pmatrix}1&x_{45}^{-1}&x_{46}^{-1}\\0&1&x_{56}^{-1}\\0&0&1\end{pmatrix}\begin{pmatrix}0&0&y_{43}\\0&y_{52}&0\\y_{61}&0&0\end{pmatrix}\begin{pmatrix}1&x_{12}&x_{13}\\0&1&x_{23}\\0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}x_{46}^{-1}y_{61}&x_{45}^{-1}y_{52}+x_{46}^{-1}y_{61}x_{12}&y_{43}+x_{45}^{-1}y_{52}x_{23}+x_{46}^{-1}y_{61}x_{13}\\x_{56}^{-1}y_{61}&y_{52}+x_{56}^{-1}y_{61}x_{12}&y_{52}x_{23}+x_{56}^{-1}y_{61}x_{13}\\y_{61}&y_{61}x_{12}&y_{61}x_{13}\end{pmatrix},$$

hence

$$\mathbb{S}=B^T(x,y)=\begin{pmatrix}1&0&0\\x_{12}&1&0\\x_{13}&x_{23}&1\end{pmatrix}\begin{pmatrix}0&0&y_{61}\\0&y_{52}&0\\y_{43}&0&0\end{pmatrix}\begin{pmatrix}1&0&0\\x_{45}^{-1}&1&0\\x_{46}^{-1}&x_{56}^{-1}&1\end{pmatrix}$$

$$=\begin{pmatrix}x_{46}^{-1}y_{61}&x_{56}^{-1}y_{61}&y_{61}\\x_{45}^{-1}y_{52}+x_{46}^{-1}y_{61}x_{12}&y_{52}+x_{56}^{-1}y_{61}x_{12}&y_{61}x_{12}\\y_{43}+x_{45}^{-1}y_{52}x_{23}+x_{46}^{-1}y_{61}x_{13}&y_{52}x_{23}+x_{56}^{-1}y_{61}x_{13}&y_{61}x_{13}\end{pmatrix}.$$

Using again (2.24), (2.28) and Remark 2.10 we get the following expressions for the generators $A_{kn}=\frac{d}{dt}T_{I+tE_{kn}}|_{t=0}$ of one-parameter subgroups $I+tE_{kn},\ t\in\mathbb{R}$:

$$A_{12}=D_{12},\ A_{13}=D_{13},\ A_{23}=x_{12}D_{13}+D_{23}, \tag{2.38}$$

$$A_{45}=D_{45},\ A_{46}=D_{46},\ A_{56}=x_{45}D_{46}+D_{56}, \tag{2.39}$$

$$\mathbb{S}=\frac{1}{2\pi i}\begin{pmatrix}A_{14}&A_{15}&A_{16}\\A_{24}&A_{25}&A_{26}\\A_{34}&A_{35}&A_{36}\end{pmatrix}=\begin{pmatrix}x_{46}^{-1}y_{61}&x_{56}^{-1}y_{61}&y_{61}\\x_{45}^{-1}y_{52}+x_{46}^{-1}y_{61}x_{12}&y_{52}+x_{56}^{-1}y_{61}x_{12}&y_{61}x_{12}\\y_{43}+x_{45}^{-1}y_{52}x_{23}+x_{46}^{-1}y_{61}x_{13}&y_{52}x_{23}+x_{56}^{-1}y_{61}x_{13}&y_{61}x_{13}\end{pmatrix}. \tag{2.40}$$

We *recall the expressions* for $B(x,y)$ and hence for $\mathbb{S}=B(x,y)^T$ *for small* n. For $n=4$ we have

$$B(x,y)=x_m^{-1}yx^{(m)}=\begin{pmatrix}1&x_{45}^{-1}\\0&1\end{pmatrix}\begin{pmatrix}0&y_{43}\\y_{52}&0\end{pmatrix}\begin{pmatrix}1&x_{23}\\0&1\end{pmatrix}=\begin{pmatrix}x_{45}^{-1}y_{52}&y_{43}+x_{45}^{-1}y_{52}x_{23}\\y_{52}&y_{52}x_{23}\end{pmatrix},$$

$$\mathbb{S}=\begin{pmatrix}1&0\\x_{23}&1\end{pmatrix}\begin{pmatrix}0&y_{52}\\y_{43}&0\end{pmatrix}\begin{pmatrix}1&0\\x_{45}^{-1}&1\end{pmatrix}=\begin{pmatrix}x_{45}^{-1}y_{52}&y_{52}\\y_{43}+x_{45}^{-1}y_{52}x_{23}&y_{52}x_{23}\end{pmatrix}.$$

For $G_2^3\simeq B(6,\mathbb{R})$ (see (2.41) for the notation G_n^m) holds:

$$B(x,y)=\begin{pmatrix}1&x_{45}^{-1}&x_{46}^{-1}\\0&1&x_{56}^{-1}\\0&0&1\end{pmatrix}\begin{pmatrix}0&0&y_{43}\\0&y_{52}&0\\y_{61}&0&0\end{pmatrix}\begin{pmatrix}1&x_{12}&x_{13}\\0&1&x_{23}\\0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}x_{46}^{-1}y_{61}&x_{45}^{-1}y_{52}+x_{46}^{-1}y_{61}x_{12}&y_{43}+x_{45}^{-1}y_{52}x_{23}+x_{46}^{-1}y_{61}x_{13}\\x_{56}^{-1}y_{61}&y_{52}+x_{56}^{-1}y_{61}x_{12}&y_{52}x_{23}+x_{56}^{-1}y_{61}x_{13}\\y_{61}&y_{61}x_{12}&y_{61}x_{13}\end{pmatrix}$$

hence

$$\mathbb{S}=\begin{pmatrix}x_{46}^{-1}y_{61}&x_{56}^{-1}y_{61}&y_{61}\\x_{45}^{-1}y_{52}+x_{46}^{-1}y_{61}x_{12}&y_{52}+x_{56}^{-1}y_{61}x_{12}&y_{61}x_{12}\\y_{43}+x_{45}^{-1}y_{52}x_{23}+x_{46}^{-1}y_{61}x_{13}&y_{52}x_{23}+x_{56}^{-1}y_{61}x_{13}&y_{61}x_{13}\end{pmatrix}$$

$$=\begin{pmatrix}1&0&0\\x_{12}&1&0\\x_{13}&x_{23}&1\end{pmatrix}\begin{pmatrix}0&0&y_{61}\\0&y_{52}&0\\y_{43}&0&0\end{pmatrix}\begin{pmatrix}1&0&0\\x_{45}^{-1}&1&0\\x_{46}^{-1}&x_{56}^{-1}&1\end{pmatrix}.$$

For $G_3^3\simeq B(8,\mathbb{R})$ holds:

$$\begin{pmatrix}1&x_{01}&x_{02}&x_{03}&t_{04}&t_{05}&t_{06}&t_{07}\\0&1&x_{12}&x_{13}&t_{14}&t_{15}&t_{16}&t_{17}\\0&0&1&x_{23}&t_{24}&t_{25}&t_{26}&t_{27}\\0&0&0&1&t_{34}&t_{35}&t_{36}&t_{37}\\0&0&0&0&1&x_{45}^{-1}&x_{46}^{-1}&x_{47}^{-1}\\0&0&0&0&0&1&x_{56}^{-1}&x_{57}^{-1}\\0&0&0&0&0&0&1&x_{67}^{-1}\\0&0&0&0&0&0&0&1\end{pmatrix},\quad y=\begin{pmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&y_{43}&0&0&0&0\\0&0&y_{52}&0&0&0&0&0\\0&y_{61}&0&0&0&0&0&0\\y_{70}&0&0&0&0&0&0&0\end{pmatrix}.$$

As before we have

$$B(x,y)=\begin{pmatrix}1&x_{45}^{-1}&x_{46}^{-1}&x_{47}^{-1}\\0&1&x_{56}^{-1}&x_{57}^{-1}\\0&0&1&x_{67}^{-1}\\0&0&0&1\end{pmatrix}\begin{pmatrix}0&0&0&y_{43}\\0&0&y_{52}&0\\0&y_{61}&0&0\\y_{70}&0&0&0\end{pmatrix}\begin{pmatrix}1&x_{01}&x_{02}&x_{03}\\0&1&x_{12}&x_{13}\\0&0&1&x_{23}\\0&0&0&1\end{pmatrix},$$

$$\mathbb{S}=(x^{(m)})^Ty^T(x_m^{-1})^T=\begin{pmatrix}1&0&0&0\\x_{01}&1&0&0\\x_{02}&x_{12}&1&0\\x_{03}&x_{13}&x_{23}&1\end{pmatrix}\begin{pmatrix}0&0&0&y_{70}\\0&0&y_{61}&0\\0&y_{52}&&0\\y_{43}&0&0&0\end{pmatrix}\begin{pmatrix}1&0&0&0\\x_{45}^{-1}&1&0&0\\x_{46}^{-1}&x_{56}^{-1}&1&0\\x_{47}^{-1}&x_{57}^{-1}&x_{67}^{-1}&1\end{pmatrix}.$$

2.4. New proof of the irreducibility of the induced representations corresponding to a generic orbits.

Remark 2.15. *By Kirillov's Theorem 2.5 the induced representation* $T_{f,H}=\mathrm{Ind}_H^G U_{f,H}$ *is irreducible if and only if the Lie algebra* $\mathfrak{h}$ *of the group* H *is a subalgebra of* $\mathfrak{g}$ *subordinate to the functional* f *with maximal possible dimension.*

The condition of "maximal possible dimension" is difficult to extend for the infinite-dimensional case. That is why in this section we give another proof of the irreducibility of the induced representation of a nilpotent group $B(n,\mathbb{R})$ *that will be extended in Section 3.8 for the infinite-dimensional analog* $B_0^{\mathbb{Z}}$ *of the group* $B(n,\mathbb{R})$.

Let us consider a sequence of a Lie groups G_n^m and its Lie algebras $\mathfrak{g}_n^m,\ m\in\mathbb{Z},\ n\in\mathbb{N}$ defined as follows

$$G_n^m=\{I+\sum_{m-n\leq k<n\leq m+n+1}x_{kn}E_{kn}\},\quad \mathfrak{g}_n^m=\{\sum_{m-n\leq k<n\leq m+n+1}x_{kn}E_{kn}\}. \tag{2.41}$$

We note that for any $m\in\mathbb{N}$ holds $B_0^{\mathbb{Z}}=\varinjlim_n G_n^m$. We have the decomposition (see (2.9))

$$G_n^m=B_{m,n}B(m,n)B^{(m,n)},$$

where

$$B_{m,n}=\{I+\sum_{(k,r)\in\Delta_{m,n}}x_{kr}E_{kr}\},\ B(m,n)=\{I+\sum_{(k,r)\in\Delta(m,n)}x_{kr}E_{kr}\},$$
$$B^{(m,n)}=\{I+\sum_{(k,r)\in\Delta^{(m,n)}}x_{kr}E_{kr}\},$$

and

$$\Delta(m,n)=\{(k,r)\in\mathbb{Z}^2\mid m-n\leq k\leq m<r\leq m+n+1\},$$
$$\Delta_{m,n}=\{(k,r)\in\mathbb{Z}^2\mid m+1\leq k<r\leq m+n+1\},$$
$$\Delta^{(m,n)}=\{(k,r)\in\mathbb{Z}^2\mid m-n\leq k<r\leq m\}.$$

The corresponding elements of the group G_n^m are as follows

$$\begin{pmatrix}1&x_{m-n,m-n+1}&\dots&x_{m-n,m-1}&x_{m-n,m}&t_{m-n,m+1}&t_{m-n,m+2}&\dots&t_{m-n,m+n+1}\\0&1&\dots&x_{m-n+1,m-1}&x_{m-n+1,m}&t_{m-n+1,m+1}&t_{m-n+1,m+2}&\dots&t_{m-n+1,m+n+1}\\&&\dots&&&&&\dots&\\0&0&\dots&1&x_{m-1,m}&t_{m-1,m+1}&t_{m-1,m+2}&\dots&t_{m-1,m+n+1}\\0&0&\dots&0&1&t_{m,m+1}&t_{m,m+2}&\dots&t_{m,m+n+1}\\0&0&\dots&0&0&1&x_{m+1,m+2}&\dots&x_{m+1,m+n+1}\\0&0&\dots&0&0&0&1&\dots&x_{m+2,m+n+1}\\&&\dots&&&&&\dots&\\0&0&\dots&0&0&0&0&\dots&x_{m+n,m+n+1}\\0&0&\dots&0&0&0&0&\dots&1\end{pmatrix}.$$

The induced representation of the group G_n^m is defined in the space $L^2(X,d\mu)$ by the following formula

(2.42)
$$(T_t^{m,y_n}f)(x)=S(h(x,t))\left(\frac{d\mu(xt)}{d\mu(x)}\right)^{1/2}f(xt),\ \ f\in L^2(X,\mu),\ \ x\in X=H\backslash G,\ t\in G$$

where $X=B(m,n)\backslash G_n^m\cong B_{m,n}\times B^{(m,n)}$ (see (2.4)),

$$d\mu(x_m,x^{(m)})=dx_m\otimes dx^{(m)}=\otimes_{(k,n)\in\Delta_{m,n}}dx_{kn}\otimes\otimes_{(k,n)\in\Delta^{(m,n)}}dx_{kn} \tag{2.43}$$

be the Haar measure on the group $B_{m,n}\times B^{(m,n)}$. Denote by $\mathcal{H}^{m,n}=L^2(B_{m,n}\times B^{(m,n)},dx_m\otimes dx^{(m)})$.

Theorem 2.16. *The induced representation T^{m,y_n} of the group G_n^m defined by formula (2.42), corresponding to generic orbit $\mathcal{O}_{y_n}$, generated by the point $y_n\in(\mathfrak{g}_n^m)^*$, $y_n=\sum_{r=0}^{n-1}y_{m+r+1,m-r}E_{m+r+1,m-r}$ is irreducible. Moreover the generators of one-parameter groups $A_{kr}=\frac{d}{dt}T_{I+tE_{kr}}^{m,y_n}\mid_{t=0}$ are as follows*

$$A_{kr}=\sum_{s=m-n}^{k-1}x_{ks}D_{rs}+D_{kr},\ \ (k,r)\in\Delta^{(m,n)},\quad A_{kr}=\sum_{s=m+1}^{k-1}x_{ks}D_{rs}+D_{kr},\ \ (k,r)\in\Delta_{m,n},$$

$$(2\pi i)^{-1}\big(A_{kr}\big)_{(k,r)\in\Delta(m,n)}=\mathbb{S}_n^{(m)}=(S_{kr})_{(k,r)\in\Delta(m,n)}=\big(x_m^{-1}yx^{(m)}\big)^T.$$

The irreducibility of the induced representation of the group G_n^m is based on the following lemma.

Lemma 2.17. *Two von Neumann algebra $\mathfrak{A}^S$ and $\mathfrak{A}^x$ in the space $\mathcal{H}^{m,n}$ generated respectively by the sets of unitary operators $U_{kr}(t)$ and $V_{kr}(t)$ coincides, where*

$$(U_{kr}(t)f)(x)=\exp(2\pi iS_{kr}(t))f(x),\quad (V_{kr}(t)f)(x):=\exp(2\pi itx_{kr})f(x), \tag{2.44}$$
$$\mathfrak{A}^S=\big(U_{kr}(t)=T_{I+tE_{kr}}^{m,y_n}=\exp(2\pi iS_{kr}(t))\mid t\in\mathbb{R},\ (k,r)\in\Delta(m,n)\big)'',$$
$$\mathfrak{A}^x=\big(V_{kr}(t):=\exp(2\pi itx_{kr})\mid t\in\mathbb{R},\ (k,r)\in\Delta_{m,n}\bigcup\Delta^{(m,n)}\big)''.$$

Proof. Using the decomposition (see (2.26) and (2.27))

$$\mathbb{S}_n^{(m)}=(x_m^{-1}yx^{(m)})^T=(x^{(m)})^Ty^T(x_m^{-1})^T \tag{2.45}$$

we conclude that $\mathfrak{A}^S\subseteq\mathfrak{A}^x$. Indeed, we get $V_{kr}(t):=\exp(2\pi itx_{kr})\in\mathfrak{A}^x$ hence the operators x_{kr} of multiplication by the independent variable $f(x)\mapsto x_{kr}f(x)$ in the space $\mathcal{H}^{m,n}$ are affiliated with the von Neumann algebra $\mathfrak{A}^x$ i.e. $x_{kr}\ \eta\ \mathfrak{A}^x$ for $(k,r)\in\Delta_{m,n}\bigcup\Delta^{(m,n)}$.

Definition 2.18. *Recall* (*c.f. e.g.* [3]) *that a non necessarily bounded self-adjoint operator A in a Hilbert space H is said to be affiliated with a von Neumann algebra M of operators in this Hilbert space H, if* $\exp(itA)\in M$ *for all* $t\in\mathbb{R}$. *One then writes $A\ \eta\ M$.*

By (2.31) the matrix elements x_{kr}^{-1} of the matrix $x_m^{-1}\in B_{m,n}$ are also affiliated $x_{kr}^{-1}\ \eta\ \mathfrak{A}^x$. Using (2.45) we conclude that the matrix elements $S_{kr},\in\Delta(m,n)$ of the matrix $\mathbb{S}_n^{(m)}$ are affiliated: $S_{kr}\ \eta\ \mathfrak{A}^x,\ (k,r)\in\Delta(m,n)$, so $\mathfrak{A}^S\subseteq\mathfrak{A}^x$.

To prove that $\mathfrak{A}^S\supseteq\mathfrak{A}^x$ we find the expressions of the matrix element of the matrix $x^{(m)}\in B^{(m,n)}$ and $x_m^{-1}\in B_{m,n}$ in terms of the matrix elements of the matrix $\mathbb{S}_n^{(m)}=(S_{kr})_{(k,r)\in\Delta(m,n)}$. To do that we connect the above decomposition $\mathbb{S}_n^{(m)}=$

$(x^{(m)})^Ty^T(x_m^{-1})^T$ and the Gaussian decomposition $C=LDU$ (see Theorem 4.1). Let us denote by J the $n\times n$ anti-diagonal matrix $J=\sum_{r=0}^{n-1}E_{m-r,m+r+1}$ Using $J^2=I$ and (2.27) we get

$$\mathbb{S}J=B^T(x,y)J=(x^{(m)})^Ty^T(x_m^{-1})^TJ=(x^{(m)})^T(y^TJ)(J(x_m^{-1})^TJ). \tag{2.46}$$

The latter decomposition (2.46) is in fact the Gauss decomposition of the matrix $\mathbb{S}J$ i.e. we get

$$\mathbb{S}J=LDU,\quad\text{where}\quad L=(x^{(m)})^T,\quad D=y^TJ,\quad U=J(x_m^{-1})^TJ.$$

Using the Theorem 4.1 we can find the matrix elements of the matrix $x^{(m)}\in B^{(m,n)}$ and $x_m^{-1}\in B_{m,n}$ in terms of the matrix elements of the matrix $\mathbb{S}_n^{(m)}$, hence we can also find the matrix elements of the matrix $x_m\in B_{m,n}$. This finish the proof of the lemma. □

We give below the expressions for $\mathbb{S}_nJ$. For $m=3$ and $n=1$ i.e. for G_1^3 we have (remind that $J^2=I$)

$$\mathbb{S}_2=\begin{pmatrix}1&0\\x_{23}&1\end{pmatrix}\begin{pmatrix}0&y_{52}\\y_{43}&0\end{pmatrix}\begin{pmatrix}1&0\\x_{45}^{-1}&1\end{pmatrix}=\begin{pmatrix}1&0\\x_{23}&1\end{pmatrix}\begin{pmatrix}y_{52}&0\\0&y_{43}\end{pmatrix}\begin{pmatrix}x_{45}^{-1}&1\\1&0\end{pmatrix},$$

$$\mathbb{S}_2J=\begin{pmatrix}1&0\\x_{23}&1\end{pmatrix}\begin{pmatrix}y_{52}&0\\0&y_{43}\end{pmatrix}\begin{pmatrix}1&x_{45}^{-1}\\0&1\end{pmatrix}.$$

For G_2^3 we get

$$\mathbb{S}_3=\begin{pmatrix}1&0&0\\x_{12}&1&0\\x_{13}&x_{23}&1\end{pmatrix}\begin{pmatrix}y_{61}&0&0\\0&y_{52}&0\\0&0&y_{43}\end{pmatrix}\begin{pmatrix}x_{46}^{-1}&x_{56}^{-1}&1\\x_{45}^{-1}&1&0\\1&0&0\end{pmatrix},$$

$$\mathbb{S}_3J=\begin{pmatrix}1&0&0\\x_{12}&1&0\\x_{13}&x_{23}&1\end{pmatrix}\begin{pmatrix}y_{61}&0&0\\0&y_{52}&0\\0&0&y_{43}\end{pmatrix}\begin{pmatrix}1&x_{56}^{-1}&x_{46}^{-1}\\0&1&x_{45}^{-1}\\0&0&1\end{pmatrix}.$$

For G_3^3 we have

$$\mathbb{S}_4=\begin{pmatrix}1&0&0&0\\x_{01}&1&0&0\\x_{02}&x_{12}&1&0\\x_{03}&x_{13}&x_{23}&1\end{pmatrix}\begin{pmatrix}y_{70}&0&0&0\\0&y_{61}&0&0\\0&0&y_{52}&0\\0&0&0&y_{43}\end{pmatrix}\begin{pmatrix}x_{47}^{-1}&x_{57}^{-1}&x_{67}^{-1}&1\\x_{46}^{-1}&x_{56}^{-1}&1&0\\x_{45}^{-1}&1&0&0\\1&0&0&0\end{pmatrix},$$

$$\mathbb{S}_4J=\begin{pmatrix}1&0&0&0\\x_{01}&1&0&0\\x_{02}&x_{12}&1&0\\x_{03}&x_{13}&x_{23}&1\end{pmatrix}\begin{pmatrix}y_{70}&0&0&0\\0&y_{61}&0&0\\0&0&y_{52}&0\\0&0&0&y_{43}\end{pmatrix}\begin{pmatrix}1&x_{67}^{-1}&x_{57}^{-1}&x_{47}^{-1}\\0&1&x_{56}^{-1}&x_{46}^{-1}\\0&0&1&x_{45}^{-1}\\0&0&0&1\end{pmatrix}. \tag{2.47}$$

Proof. of the Theorem 2.16. The irreducibility follows from the Kirillov results (see Remark 2.15). To give another proof of the irreducibility of the induced representation consider the restriction $T^{m,y_n}\mid_{B(m,n)}$ of this representation to the commutative subgroup $B(m,n)$ of the group G_n^m. Note that

$$\mathfrak{A}^x=\big(\exp(2\pi itx_{kr})\mid t\in\mathbb{R},\ (k,r)\in\Delta_{m,n}\bigcup\Delta^{(m,n)}\big)''=L^\infty(B_{m,n}\times B^{(m,n)},dx_m\otimes dx^{(m)}).$$

By Lemma 2.17 the von Neumann algebra $\mathfrak{A}^S$ generated by this restriction coincides with $L^\infty(B_{m,n}\times B^{(m,n)},dx_m\otimes dx^{(m)})$. Let now a bounded operator A in a Hilbert space $\mathcal{H}^{m,n}$ commute with the representation T^{m,y_n}. Then A commute by the above arguments with $L^\infty(B_{m,n}\times B^{(m,n)},dx_m\otimes dx^{(m)})$, therefore the operator A itself is an operator of multiplication by some essentially bounded function $a\in L^\infty$ i.e. $(Af)(x)=a(x)f(x)$ for $f\in\mathcal{H}^{m,n}$. Since A commute with the representation T^{m,y_n} i.e. $[A,T_t^{m,y_n}]=0$ for all $t\in B_{m,n}\times B^{(m,n)}$ we conclude that

$$a(x)=a(xt)\ (\mathrm{mod}\ dx_m\otimes dx^{(m)})\quad\text{for}\quad\text{all}\quad t\in B_{m,n}\times B^{(m,n)}.$$

Since the measure $dh = dx_m \otimes dx^{(m)}$ is the Haar measure on $G = B_{m,n} \times B^{(m,n)}$, this measure is G-right ergodic. We conclude that $a(x) = const \ (\mathrm{mod}\, dx_m \otimes dx^{(m)})$. □

3. Induced representations, infinite-dimensional case

3.1. **Regular and quasiregular representations of infinite-dimensional groups.** To define the induced representation we explain first how to define the regular representation of *infinite-dimensional group* G. Since the initial group in not locally compact there is neither Haar (invariant) measure on G (Weil, [18]), nor a G-quasi-invariant measure (Xia Dao-Xing, [19]). We can try to find some bigger topological group $\widetilde{G}$ and the G-quasi-invariant measure μ on $\widetilde{G}$ such that G is the dense subgroup in $\widetilde{G}$. In this case we define the *right or left regular representation* of the group G in the space $L^2(\tilde{G},\mu)$ if $\mu^{R_t} \sim \mu$ (resp. $\mu^{L_t} \sim \mu$) for all $t \in G$ as follows:

$$(T_t^{R,\mu} f)(x) = (d\mu(xt)/d\mu(x))^{1/2} f(xt),\ \ f \in L^2(\tilde{G},\mu),\ \ t \in G, \tag{3.1}$$

$$(T_t^{L,\mu} f)(x) = (d\mu(t^{-1}x)/d\mu(x))^{1/2} f(t^{-1}x),\ \ f \in L^2(\tilde{G},\mu),\ \ t \in G. \tag{3.2}$$

Conjecture 3.1 (Ismagilov, 1985)**.** *The right regular representation $T^{R,\mu} : G \to U(L^2(\tilde{G},\mu))$ is irreducible if and only if*
1) $\mu^{L_t} \perp \mu\ \forall t \in G\backslash\{e\}$,
2) the measure μ is G-ergodic.

Analogously we can define the *quasiregular representation.* Namely, if H is a closed subgroup of the group G, then on the space $X = \widetilde{H\backslash G} = \tilde{H}\backslash\tilde{G}$ the right action of the group G is well defined, where $\tilde{G}$ (resp. $\tilde{H}$) is some completion of the group G (resp. H). If we have some G-right-quasi-invariant measure μ on X one may define the "quasiregular representation" of the group G in the space $L^2(X,\mu)$ as in a locally compact case:

$$(\pi_t^{R,\mu,X} f)(x) = (d\mu(xt)/d\mu(x))^{1/2} f(xt), \quad t \in G.$$

The regular and quasiregular representations for general infinite-dimensional groups were introduced and investigated in e.g. [1, 9, 10, 11, 13].

3.2. **Induced representations for infinite-dimensional groups.** The induced representation $\mathrm{Ind}_H^G S$ of a locally-compact group is the unitary representation of the group G associated with a unitary representation S of a subgroup H of the group G (see Section 2).

As it was mentioned in section 2.2 (see [4, 7]) all unitary irreducible representations up to equivalence $\hat{G}_n$ of the nilpotent group $G_n = B(n,\mathbb{R})$, are obtained as induced representations $\mathrm{Ind}_H^{G_n} U_{f,H}$ associated with a points $f \in \mathfrak{g}_n^*$ and the corresponding *subordinate* subgroup $H \subset G_n$. The induced representation $\mathrm{Ind}_H^{G_n} U_{f,H}$ is defined canonically in the Hilbert space $L^2(H\backslash G_n,\mu)$.

A. Kirillov [7], Chapter I, §4, p.10 says: *"The method of induced representations is not directly applicable to infinite-dimensional groups (or more precisely to a pair $G \supset H$) with an infinite-dimensional factor $H\backslash G$)".*

Our aim is to *develop the concept of induced representations for infinite-dimensional groups.* Let we have the infinite-dimensional group G and a unitary representation $S : H \to U(V)$ in a Hilbert space V of a subgroup H of the group G such that the factor space $H\backslash G$ is infinite-dimensional.

In general, it is difficult to construct G-quasi-invariant measure on an infinite-dimensional homogeneous space $H\backslash G$. As is the case of the regular and quasiregular representations of infinite-dimensional groups G (see Subsection 3.1) it is reasonable to construct some G-quasi-invariant measure on a *suitable completion* $\widetilde{H\backslash G} = \tilde{H}\backslash\tilde{G}$ of the initial space $H\backslash G$ *in a certain topology*, where $\tilde{H}$ (resp. $\tilde{G}$) is some completion of the group H (resp. G). To go further we should be able to *extend the representation* $S : H \to U(V)$ of the group H to the representation $\tilde{S} : \tilde{H} \to U(V)$ of the completion $\tilde{H}$ of the group H.

Finally, the induced representation of the group G associated with a unitary representation S of a subgroup H will depend on two completions $\tilde{H}$ and $\tilde{G}$ of the subgroup H and the group G, on an extension $\tilde{S} : \tilde{H} \to U(V)$ of the representation $S : H \to U(V)$ and on a choice of the G-quasi-invariant measure μ on an appropriate completion $\tilde{X} = \tilde{H}\backslash\tilde{G}$ of the space $H\backslash G$.

Hence the procedure of induction will not be unique but nevertheless well-defined (if a G-quasi-invariant measure on $\widetilde{H\backslash G}$ exists). So the uniquely defined induced representation $\mathrm{Ind}_H^G S$ in the Hilbert space $L^2(H\backslash G, V, \mu)$ (in the case of a locally-compact group G) should be replaced by the family of induced representations $\mathrm{Ind}_{\tilde{H},H}^{\tilde{G},G,\mu}(\tilde{S}, S)$ in the Hilbert spaces $L^2(\tilde{H}\backslash\tilde{G}, V, \mu)$ depending on different completions $\tilde{G}$ of the group G, completions $\tilde{H}$ of the group H and different G-quasi-invariant measures μ on $\tilde{H}\backslash\tilde{G}$.

Example 3.2 ([9, 11])**.** Regular representations $T^{R,\mu}$ *of the infinite-dimensional group* G *in the space* $L^2(\tilde{G}, \mu)$*, associated with the completion* $\tilde{G}$ *of the group* G *and a* G*-right -quasi-invariant measure* μ *on* $\tilde{G}$*, is a particular case of the induced representation (see Remark 2.3)*

$$T^{R,\mu} = \mathrm{Ind}_e^{\tilde{G},G,\mu}(Id),$$

generated by the trivial representation $S = Id$ *of the trivial subgroup* $H = \{e\}$ *(as in the case of a locally compact groups).*

Example 3.3 ([1, 13])**.** Quasi-regular representations $\pi^{R,\mu,X}$ *of the infinite-dimensional group* G *in the space* $L^2(X, \mu)$ *where* $X = \tilde{H}\backslash\tilde{G}$ *and* H *is some subgroup of the group* G *is a particular case of the induced representation (see Remark 2.3)*

$$\pi^{R,\mu,X} = \mathrm{Ind}_{\tilde{H},H}^{\tilde{G},G,\mu}(Id)$$

generated by the trivial representation $S = Id$ *of the* completion $\tilde{H}$ *in the group* $\tilde{G}$ *of the subgroup* H *in the group* G*.*

Let G be an infinite-dimensional group and $S : H \to U(V)$ be a unitary representation in a Hilbert space V of the subgroup $H \subset G$, such that the space $H\backslash G$ is infinite-dimensional. We give the following definition.

Definition 3.4. *The induced representation*

$$\mathrm{Ind}_{\tilde{H},H}^{\tilde{G},G,\mu}(\tilde{S}, S),$$

generated by the unitary representations $S : H \to U(V)$ *of the subgroup* H *in the group* G *is defined (similarly to (3.2) and (3.3)) as follows:*

1) we should first find some completion $\tilde{H}$ *of the group* H *such that*

$$\tilde{S} : \tilde{H} \to U(V)$$

is the continuous unitary representation of the group $\tilde{H}$*, such that* $\tilde{S}|_H = S$*,* *2) take any G-right-quasi-invariant measure μ on the an appropriate completion $\tilde{X}=\tilde{H}\backslash\tilde{G}$ of the space $X=H\backslash G$, on which the group G acts from the right, where $\tilde{H}$ (resp. $\tilde{G}$) is a suitable completion of the group H (resp. G),*

3) in the space $L^2(\tilde{X},V,\mu)$ of all vector-valued functions f on $\tilde{X}$ with values in V such that

$$\|f\|^2:=\int_{\tilde{X}}\|f(x)\|_V^2d\mu(x)<\infty,$$

define the representation of the group G by the following formula

$$(T_tf)(x)=S(\tilde{h}(x,t))\left(\frac{d\mu(xt)}{d\mu(x)}\right)^{1/2}f(xt),\quad x\in\tilde{X},\ t\in G, \tag{3.3}$$

where $\tilde{h}$ is defined by

$$\tilde{s}(x)t=\tilde{h}(x,t)\tilde{s}(xt).$$

The section $s:H\to G$ of the projection $p:G\to H$ should be extended to the appropriate section $\tilde{s}:\tilde{H}\to\tilde{G}$ of the extended projection $\tilde{p}:\tilde{G}\to\tilde{H}$.

The comparison of the induced representation for locally compact group and the above definition for infinite-dimensional groups may be given in the following table:

1	G	G loc.comp.	$\dim G=\infty$
2	H	$H\subset G$	$H\subset G$
3	S	$S:H\to U(V)$	$S:H\to U(V)\Rightarrow\tilde{S}:\tilde{H}\to U(V)$
4	X	$X=H\backslash G$	$\tilde{X}=\widetilde{H\backslash G}=\tilde{H}\backslash\tilde{G}$
5	$\mathcal{H}$	$L^2(X=H\backslash G,V,\mu)$	$L^2(\tilde{X}=\tilde{H}\backslash\tilde{G},V,\mu)$
6	Ind	Ind_H^GS	$\mathrm{Ind}_{\tilde{H},H}^{\tilde{G},G,\mu}(\tilde{S},S)$
7	T_t	$(T_tf)(x)=S(h(x,t))(\frac{d\mu(xt)}{d\mu(x)})^{1/2}f(xt)$	$(T_tf)(x)=\tilde{S}(\tilde{h}(x,t))(\frac{d\mu(xt)}{d\mu(x)})^{1/2}f(xt)$
8	p	$p:G\to X$	$\tilde{p}:\tilde{G}\to\tilde{X}$
9	s	$s:X\to G$	$s:H\backslash G\to G\Rightarrow\tilde{s}:\widetilde{H\backslash G}\to\tilde{G}$
10	$h(x,t)$	$s(x)t=h(x,t)s(xt)$	$\tilde{s}(x)t=\tilde{h}(x,t)\tilde{s}(xt)$

3.3. **How to develop the orbit method for infinite-dimensional "nilpotent" group $B_0^{\mathbb{N}}$ and $B_0^{\mathbb{Z}}$?** *We would like to develop the orbit method for infinite-dimensional "nilpotent" group $G=\varinjlim_n G_n$* with $G_n=B(n,\mathbb{R})$. The corresponding Lie algebra $\mathfrak{g}$ is the *inductive limit* $\mathfrak{g}=\varinjlim_n\mathfrak{b}_n$ of upper triangular matrices, so as the linear space it is isomorphic to the space $\mathbb{R}_0^\infty$ of finite sequences $(x_k)_{k\in\mathbb{N}}$ hence the dual space $\mathfrak{g}^*$ is isomorphic to the space $\mathbb{R}^\infty$ of all sequences $(x_k)_{k\in\mathbb{N}}$, but *the latter space $\mathbb{R}^\infty$ is too large to manage with it, for example to equip with a Hilbert structure or to describe all orbits.* To make it less it is reasonable to increase the initial group G or to make completion $\tilde{G}$ of this group in some stronger topology.

To develop the orbit method for groups $B_0^{\mathbb{N}}$ and $B_0^{\mathbb{Z}}$ we should answer some questions:

(1) How to define the *appropriate completion* $\tilde{G}$ of the group G, corresponding Lie algebras $\mathfrak{g}$ (resp. $\tilde{\mathfrak{g}}$) and corresponding dual spaces $\mathfrak{g}^*$ (resp.$\tilde{\mathfrak{g}}^*$)?

(2) Which *pairing* should we use between $\mathfrak{g}$ and $\mathfrak{g}^*$?

(3) Let the dual space $\mathfrak{g}^*$, some element $f\in\mathfrak{g}^*$ and corresponding algebra $\mathfrak{h}$, subordinate to the element f, are chosen. How to *define the corresponding induced representation* $\mathrm{Ind}_H^G U_{f,H}$ and *study its irreducibility*?

(4) Shall we get all irreducible representations of the corresponding groups, using induced representations?

(5) Find the criteria of irreducibility and equivalence of induced representations.

The problem of *completion* of the inductive limit group $G=\varinjlim_n G_n$, where G_n are finite-dimensional classical groups were studied by A. Kirillov ([5], 1972) for the group $U(\infty)=\varinjlim_n U(n)$ and G. Olshanskiĭ([16], 1990) for inductive limit of classical groups. They described all unitary irreducible representations of the corresponding groups $G=\varinjlim_n G_n$, *continuous* in stronger topology, namely *in the strong operator topology.* The description of the dual $\hat{G}$ of the initial group $G=\varinjlim_n G_n$ is much more complicated.

In [8] (see details in section 3.4) we have constructed for the group $\mathrm{GL}_0(2\infty,\mathbb{R})$ $=\varinjlim_n \mathrm{GL}(2n-1,\mathbb{R})$ a family of the Hilbert-Lie groups $\mathrm{GL}_2(a)$, $a\in\mathfrak{A}$ such that
a) $\mathrm{GL}_0(2\infty,\mathbb{R})\subset\mathrm{GL}_2(a)$ and $\mathrm{GL}_0(2\infty,\mathbb{R})$ is dense in $\mathrm{GL}_2(a)$ for all $a\in\mathfrak{A}$,
b) $\mathrm{GL}_0(2\infty,\mathbb{R})=\cap_{a\in\mathfrak{A}}\mathrm{GL}_2(a)$,
c) *any continuous representation* of the group $\mathrm{GL}_0(2\infty,\mathbb{R})$ *is in fact continuous* in some stronger topology, namely *in a topology of a suitable Hilbert -Lie group* $\mathrm{GL}_2(a)$.

(1) Therefore, as we show in Sections 3.5, 3.4 it is sufficient to consider a *Hilbert-Lie completions* $B_2(a)$ of the initial group $B_0^{\mathbb{Z}}$.

(2) In this case the *pairing* between the corresponding Hilbert-Lie algebra $\mathfrak{b}_2(a)$ and its dual $\mathfrak{b}_2(a)^*$ is correctly defined by the trace (as in the finite-dimensional case).

(3.1) We define in Section 3.7 *the induced representations* of the group $B_0^{\mathbb{Z}}$ corresponding to a special orbits, *generic orbits*, using schema given in Section 3.2. We consider only the simplest example of $G-$quasi-invariant measures on $\tilde{X}=\tilde{H}\setminus\tilde{G}$, namely the infinite product of one-dimensional Gaussian measures.

(3.2) How to construct the *induced representation corresponding to an arbitrary orbit*?

Conjecture 3.5. *Two induced representations* $\mathrm{Ind}_{H_1}^{\tilde{G},\mu_1}U_{f_1,H_1}$ *and* $\mathrm{Ind}_{H_2}^{\tilde{G},\mu_2}U_{f_2,H_2}$ *are equivalent if and only if the corresponding measures* μ_1 *and* μ_2 *are equivalent and the functionals* f_1 *and* f_2 *belong to the same orbit of* $(\tilde{\mathfrak{g}})^*$.

3.4. **Hilbert-Lie groups** $\mathrm{GL}_2(a)$**.** We show that the *Hilbert-Lie groups* appear naturally in the representation theory of infinite-dimensional matrix group. The remarkable fact is that for the inductive limit $G=\varinjlim_n G_n$ of matrix groups $G_n\subset\mathrm{GL}(2n-1,\mathbb{R})$ it is sufficient to consider only the *Hilbert completions* of the initial group G and of the spaces $H\backslash G$.

Let us consider the group $\mathrm{GL}_0(2\infty,\mathbb{R})=\varinjlim_n\mathrm{GL}(2n-1,\mathbb{R})$ with respect to the symmetric embedding $i_n^s: G_n\mapsto G_{n+1}$, $G_n\ni x\mapsto x+E_{-n,-n}+E_{nn}\in G_{n+1}$, where $G_n=\mathrm{GL}(2n-1,\mathbb{R})$. We consider here only the real matrices.

The *Hilbert-Lie group* $\mathrm{GL}_2(a)$ we define (see [8]) by its *Hilbert-Lie algebra* $\mathfrak{g}l_2(a)$ with composition $[x,y]=xy-yx$

$$\mathfrak{g}l_2(a)=\{x=\sum_{k,n\in\mathbb{Z}}x_{kn}E_{kn}\mid \|x\|^2_{\mathfrak{g}l_2(a)}=\sum_{k,n\in\mathbb{Z}}\mid x_{kn}\mid^2 a_{kn}<\infty\},\ a\in\mathfrak{A}_{\mathrm{GL}},$$

$$\mathrm{GL}_2(a)=\{I+x\ \mid\ (I+x)^{-1}=1+y\quad x,y\in\mathfrak{g}l_2(a)\}.$$

To be more precise, let us consider an analogue $\sigma_2(a)$ of the algebra of the Hilbert-Schmidt operators $\sigma_2(H)$ in a Hilbert space H:

$$\sigma_2(a)=\{x=\sum_{k,n\in\mathbb{Z}}x_{kn}E_{kn}\mid \|x\|^2_{\sigma_2(a)}=\sum_{k,n\in\mathbb{Z}}\mid x_{kn}\mid^2 a_{kn}<\infty\}.$$

Lemma 3.6 ([8]). *The Hilbert space $\sigma_2(a)$ is an (associative) Hilbert algebra (i.e. $\|xy\|\leq C\|x\|\|y\|$, $x,y\in\sigma_2(a)$) if and only if the weight $a=(a_{kn})_{(k,n)\in\mathbb{Z}^2}$ belongs to the set $\mathfrak{A}_{\rm GL}$ defined as follows:*

$$\mathfrak{A}_{\rm GL}=\{a=(a_{kn})_{(k,n)\in\mathbb{Z}^2}\mid 0<a_{kn}\leq Ca_{km}a_{mn},\quad k,n,m\in\mathbb{Z},\ C>0\}. \tag{3.4}$$

We define the Hilbert-Lie algebra $\mathfrak{g}l_2(a)$ as the Hilbert space $\sigma_2(a)$ with an operation $[x,y]=xy-yx$.

Corollary 3.7. *The Hilbert space $\mathfrak{g}l_2(a)$ is a Hilbert-Lie algebra if and only if the weight $a=(a_{kn})_{(k,n)\in\mathbb{Z}^2}$ belongs to the set $\mathfrak{A}_{\rm GL}$.*

We remark also [8] that $\mathrm{GL}_0(2\infty,\mathbb{R})=\cap_{a\in\mathfrak{A}_{\rm GL}}\mathrm{GL}_2(a)$.

Theorem 3.8 (Theorem 6.1 [8]). *Every continuous unitary representation U of the group $\mathrm{GL}_0(2\infty,\mathbb{R})$ in a Hilbert space H can be extended by continuity to a unitary representation $U_2(a):\mathrm{GL}_2(a)\to U(H)$ of some Hilbert-Lie group $\mathrm{GL}_2(a)$ depending on the representation.*

3.5. **Hilbert-Lie groups $B_2(a)$.** Let us consider the following Hilbert-Lie group $B_2(a):=B_2^{\mathbb{Z}}(a)$

$$B_2(a)=\{I+x\ \mid\ x\in\mathfrak{b}_2(a)\}, \tag{3.5}$$

where the corresponding Hilbert-Lie algebra $\mathfrak{b}_2(a):=\mathfrak{b}_2^{\mathbb{Z}}(a)$ is defined as

$$\mathfrak{b}_2(a)=\{x=\sum_{(k,n)\in\mathbb{Z}^2,k<n}x_{kn}E_{kn}\mid \|x\|^2_{\mathfrak{b}_2(a)}=\sum_{(k,n)\in\mathbb{Z}^2,k<n}\mid x_{kn}\mid^2 a_{kn}<\infty\}. \tag{3.6}$$

Lemma 3.9 ([8]). *The Hilbert space $\mathfrak{b}_2(a)$ (with an operation $(x,y)\mapsto xy$) is a Banach algebra if and only if the weight $a=(a_{kn})_{(k,n)\in\mathbb{Z}^2,k<n}$ satisfies the conditions*

$$a=(a_{kn})_{k<n},\ a_{kn}\leq Ca_{km}a_{mn},\ k<m<n,\ k,m,n\in\mathbb{Z}. \tag{3.7}$$

Denote by $\mathfrak{A}$ the set of all weight a satisfying the mentioned condition.

3.6. **Orbit method for infinite-dimensional "nilpotent" group $B_0^{\mathbb{Z}}$, first steps.** Take the group $B_0^{\mathbb{Z}}$, fix some its Hilbert completion i.e. a Hilbert-Lie group $B_2(a)$, $a\in\mathfrak{A}$ and the corresponding Hilbert-Lie algebra $\mathfrak{g}=\mathfrak{b}_2(a)$. The corresponding dual space $\mathfrak{g}^*=\mathfrak{b}_2^*(a)$ has the form

$$\mathfrak{b}_2^*(a)=\{y=\sum_{(k,n)\in\mathbb{Z}^2,k>n}y_{kn}E_{kn}\mid \|y\|^2_{\mathfrak{b}_2^*(a)}=\sum_{(k,n)\in\mathbb{Z}^2,k>n}\mid y_{kn}\mid^2 a_{kn}^{-1}<\infty\}. \tag{3.8}$$

The *adjoint action* $B_2(a)\to\mathrm{Aut}(\mathfrak{b}_2(a))$ of the group $B_2(a)$ on its Lie algebra $\mathfrak{b}_2(a)$ is:

$$\mathfrak{b}_2(a)\ni x\mapsto \mathrm{Ad}_t(x):=txt^{-1}\in\mathfrak{b}_2(a),\quad t\in B_2(a). \tag{3.9}$$

The *pairing* between $\mathfrak{g}=\mathfrak{b}_2(a)$ and $\mathfrak{g}^*=\mathfrak{b}_2^*(a)$ is correctly defined by the trace:

$$\mathfrak{g}^*\times\mathfrak{g}\ni(y,x)\mapsto\langle y,x\rangle:=tr(xy)=\sum_{(k,n)\in\mathbb{Z}^2,k<n}x_{kn}y_{nk}\in\mathbb{R}. \tag{3.10}$$

The *coadjoint action* of the group $B_2(a)$ on the dual $\mathfrak{g}^*=\mathfrak{b}_2^*(a)$ to $\mathfrak{g}=\mathfrak{b}_2(a)$ is as follows: for $t\in B_2(x)$ and $y\in\mathfrak{b}_2^*(a)$

$$t=I+\sum_{(k,n)\in\mathbb{Z}^2,k<n}t_{kn}E_{kn},\ y=\sum_{(k,n)\in\mathbb{Z}^2,k>n}y_{kn}E_{kn},\ t^{-1}:=I+\sum_{(k,n)\in\mathbb{Z}^2,k<n}t^{-1}_{kn}E_{kn}$$

we have

$$(t^{-1}yt)_{pq}=\sum_{m=-\infty}^{q}(t^{-1}y)_{pm}t_{mq}=\sum_{m=-\infty}^{q}\sum_{r=p}^{\infty}t^{-1}_{pr}y_{rm}t_{mq},\ (p,q)\in\mathbb{Z}^2,p>q,$$

hence

$$\mathrm{Ad}^*_x(y)=(t^{-1}yt)_-:=I+\sum_{(p,q)\in\mathbb{Z}^2,p>q}(t^{-1}yt)_{pq}E_{pq}. \tag{3.11}$$

We consider four different type of orbits with respect to the coadjoint action of the group $B_2(a)$ in the dual space $\mathfrak{b}_2^*(a)$.

Case 1) The finite-dimensional orbits corresponding to a *finite points* $y=\sum_{(k,n)\in\mathbb{Z},k>n}y_{kn}E_{kn}\in\mathfrak{b}_2^*(a)$ (finiteness of y means that only finite number of y_{kn} are nonzero). This orbits leads to the induced representations of an appropriate finite-dimensional groups G_n^m, $m\in\mathbb{Z}$, $n\in\mathbb{N}$ defined by (2.41). All irreducible unitary representations of the groups G_n^m are completely described by the Kirillov orbit method hence the finite-dimensional orbits gives us the set $\bigcup_{n\in\mathbb{N}}\widehat{G_n^m}\subset\widehat{B_0^{\mathbb{Z}}}$ (see subsection 3.9, Remark 3.17 for embedding $\widehat{G_n^m}\subset\widehat{G_{n+1}^m}$).

Case 2) 0-dimensional orbits are of the form:

$$\mathcal{O}_0=y,\ y\in\mathfrak{b}_2^*(a),\quad y=\sum_{k\in\mathbb{Z}}y_{k+1,k}E_{k+1,k}.$$

The Lie algebra $\mathfrak{b}_2(a)$ is subordinate to the functional y, $\langle y,[\mathfrak{b}_2(a),\mathfrak{b}_2(a)]\rangle=0$ since

$$[\mathfrak{b}_2(a),\mathfrak{b}_2(a)]=\{x\in\mathfrak{b}_2(a)\mid x=\sum_{(k,n)\in\mathbb{Z}^2,k+1<n}x_{kn}E_{kn}\}.$$

The one-dimensional representation of the Lie algebra $\mathfrak{b}_2(a)$ are

$$\mathfrak{b}_2(a)\ni x\mapsto\langle y,x\rangle=\sum_{k\in\mathbb{Z}}x_{k,k+1}y_{k+1,k}\in\mathbb{R}.$$

Corresponding one-dimensional representations of the group $B_2(a)$ are as follows:

$$B_2(a)\ni\exp(x)\mapsto\exp(2\pi i(\langle y,x\rangle))=\exp(2\pi i\sum_{k\in\mathbb{Z}}x_{k,k+1}y_{k+1,k})\in S^1. \tag{3.12}$$

They are all *irreducible and nonequivalent* for different $y=\sum_{k\in\mathbb{Z}}y_{k+1,k}E_{k+1,k}\in\mathfrak{b}_2^*(a)$.

Case 3) Generic orbit is generated for an arbitrary $m\in\mathbb{Z}$ by a point $y\in\mathfrak{b}_2^*(a)$

$$y=\sum_{p=0}^{\infty}y_{m+p+1,m-p}E_{m+p+1,m-p}\in\mathfrak{b}_2^*(a),\quad\text{with } y_{m+p+1,m-p}\neq 0,\ p+1\in\mathbb{N}. \tag{3.13}$$

Sections 3.7 and 3.8 are devoted to the study of this case.

Case 4) General orbits generated by an arbitrary non finite points

$$y=\sum_{(k,n)\in\mathbb{Z},k>n}y_{kn}E_{kn}\in\mathfrak{b}_2^*(a).$$

Problem. How to construct the induced representations for general orbits and study their irreducibility?

3.7. **Construction of the induced representations of the group $B_0^{\mathbb{Z}}$ corresponding to a generic orbits.** Consider more carefully the case 3). The irreducibility we shall study in the following subsection. Take as before the group $B_0^{\mathbb{Z}}$, fix some its Hilbert completion i.e. a Hilbert-Lie group $B_2(a),\ a\in\mathfrak{A}$, the corresponding Hilbert-Lie algebra $\mathfrak{g}=\mathfrak{b}_2(a)$ and its dual $\mathfrak{g}^*=\mathfrak{b}_2^*(a)$ as in the previous subsection.

We shall write the analog of the induced representation of the group $B_0^{\mathbb{Z}}$ for generic orbits (see Examples 2.7, 2.8 and 2.14) corresponding to the point $y\in\mathfrak{b}_2^*(a)$ defined by (3.13) following steps 1)–3) of Definition 3.4.

Step 1) *Extension of the representation* $S:H\to U(V)$. For fixed $m\in\mathbb{Z}$, consider the decomposition

$$B^{\mathbb{Z}}=B_mB(m)B^{(m)}$$

similar to the decomposition (2.19), where $B^{\mathbb{Z}}=\{I+\sum_{k,n\in\mathbb{Z},\,k<n}x_{kn}E_{kn}\}$,

$$B_m=\{I+\sum_{(k,r)\in\Delta_m}x_{kr}E_{kr}\},\ B(m)=\{I+\sum_{(k,r)\in\Delta(m)}x_{kr}E_{kr}\},\ B^{(m)}=\{I+\sum_{(k,r)\in\Delta^{(m)}}x_{kr}E_{kr}\},$$

$$\Delta_m=\{(k,r)\in\mathbb{Z}^2\mid m+1\le k<r\},\quad \Delta(m)=\{(k,r)\in\mathbb{Z}^2\mid k\le m<r\},$$

and $\Delta^{(m)}=\{(k,r)\in\mathbb{Z}^2\mid k<r\le m\}$.

Since the algebras $\mathfrak{h}_0(m),\ m\in\mathbb{Z}$ defined as follows $\mathfrak{h}_0(m)=\{t-I\mid t\in B_0(m)\}$, where $B_0(m)=B(m)\cap B_0^{\mathbb{Z}}$, are commutative, so $\langle y,[\mathfrak{h}_0(m),\mathfrak{h}_0(m)]\rangle=0$, hence they are subordinate to the functional $y\in\mathfrak{g}^*=\mathfrak{b}_2^*(a)$. The corresponding one-dimensional representation of the algebra $\mathfrak{h}_0(m)=\mathfrak{h}(m)\bigcap\mathfrak{g}_0^{\mathbb{Z}}$ is

$$\mathfrak{h}_0(m)\ni x\mapsto\langle y,x\rangle=\sum_{p=0}^{\infty}x_{m-p,m+p+1}y_{m+p+1,m-p}\in\mathbb{R}.$$

The unitary representation of the corresponding group $H_0(m)$ is

$$H_0(m)\ni\exp(x)\mapsto S(\exp(x))=\exp(2\pi i\langle y,x\rangle)\in S^1.$$

This representation can be extended to representation of the corresponding Hilbert-Lie group $\tilde H=H_2(m,a)=B(m)\bigcap B_2(a)$ (we note that $t=\exp(t-1)$):

$$H_2(m,a)\ni\exp(x)\mapsto S(\exp(x))=\exp(2\pi i\langle y,x\rangle)\in S^1.$$

In what follows we shall use a notation $B_2(m,a)$ for the group $H_2(m,a)$.

Step 2 a) *Construction of the completion* $\tilde X=\tilde H\backslash\tilde G$ of the space $X=H\backslash G$. It is difficult to construct an appropriate measure on the space $X_{m,0}=B_0(m)\backslash B_0^{\mathbb{Z}}$ since it is isomorphic to the space $\mathbb{R}_0^\infty\subset\mathbb{R}_0^\infty$. That is why we consider two homogeneous spaces, an appropriate completions of the space $X_{m,0}$:

$$X_{m,2}(a)=B_{m,2}(a)\backslash B_2(a),\quad X_m=B(m)\backslash B^{\mathbb{Z}}.$$

Since the decompositions holds

$$B_0^{\mathbb{Z}}=B_{m,0}B_0(m)B_0^{(m)},\quad B_2(a)=B_{m,2}(a)B_2(m,a)B_2^{(m)}(a),\quad B^{\mathbb{Z}}=B_mB(m)B^{(m)},$$

(see Remark 2.9), we have the following inclusions: $X_{m,0}\subset X_{m,2}(a)\subset X_m$, where

$$X_{m,0}\simeq B_{m,0}\times B_0^{(m)},\quad X_{m,2}(a)\simeq B_{m,2}(a)\times B_2^{(m)}(a),\quad X_m=B(m)\backslash B^{\mathbb{Z}}\simeq B_m\times B^{(m)}.$$

Step 2 b) We *construct a measure* μ_b on the space X_m with support $X_{m,2}(a)$ i.e. such that $\mu_b(X_{m,2}(a))=1$. That is we take $\tilde X=\tilde H\backslash\tilde G=B_2(m,a)\backslash B_2(a)$.

Remark 3.10. *On the space X_m we can take any $B_0^{\mathbb{Z}}$-quasi-invariant ergodic measure, construct the induced representation and study the irreducibility. We consider the simplest case of the Gaussian measure, the infinite product of one-dimensional Gaussian measure.*

We construct the measure μ_b on the space $X_m \simeq B_m \times B^{(m)}$ as a product-measure $\mu_b = \mu_{b,m} \otimes \mu_b^{(m)}$, where $\mu_{b,m}$ (resp. $\otimes\mu_b^{(m)}$) is Gaussian product measure on the group B_m (resp. $B^{(m)}$) defined as follows:

$$d\mu_{b,m}(x_m) = \otimes_{(k,n)\in\Delta_m} d\mu_{b_{kn}}(x_{kn}) = \otimes_{(k,n)\in\Delta_m} \sqrt{\frac{b_{kn}}{\pi}} \exp(-b_{kn}x_{kn}^2)dx_{kn}, \tag{3.14}$$

$$d\mu_b^{(m)}(x^{(m)}) = \otimes_{(k,n)\in\Delta^{(m)}} d\mu_{b_{kn}}(x_{kn}) = \otimes_{(k,n)\in\Delta^{(m)}} \sqrt{\frac{b_{kn}}{\pi}} \exp(-b_{kn}x_{kn}^2)dx_{kn}. \tag{3.15}$$

The corresponding Hilbert space is

$$\mathcal{H}^m = L^2(X_m, \mu_b) = L^2(B_m \times B^{(m)}, \mu_{b,m} \otimes \mu_b^{(m)}).$$

Lemma 3.11 (Kolmogorov's zero-one law, [17])**.** *We have $\mu_{b,m}\otimes\mu_b^{(m)}(B_{m,2}(a)\times B_2^{(m)}(a)) = 1$ if and only if*

$$\sum_{(k,n)\in\Delta(m)\cup\Delta^{(m)}} \frac{a_{kn}}{b_{kn}} < \infty.$$

Lemma 3.12 ([9, 10])**.** *The measure $\mu_b = \mu_{b,m} \otimes \mu_b^{(m)}$ is $B_{m,0} \times B_0^{(m)}$-right-quasi-invariant i.e. $(\mu_b)^{R_t} \sim \mu_b$ for all $t \in B_{m,0} \times B_0^{(m)}$ if and only if*

$$S_{kn}^R(\mu_b) = \sum_{r=-\infty}^{k-1} \frac{b_{rn}}{b_{rk}} < \infty, \quad \textit{for all, } k < n \le m.$$

Step 3) The corresponding induced representation of the group $B_0^{\mathbb{Z}}$ we defined as follows:

$$(T_t^{m,y} f)(x) = S(h(x,t)) \left(\frac{d\mu_b(xt)}{d\mu_b(x)}\right)^{1/2} f(xt), \ x \in X_m, \ t \in G, \tag{3.16}$$

where (see (3.21))

$$S(h(x,t)) = \exp(2\pi i\langle y, h(x,t) - 1\rangle) = \exp\Big(2\pi i \mathrm{tr}\left((t-I)B(x,y)\right)\Big).$$

3.8. **Irreducibility of the induced representations of the group $B_0^{\mathbb{Z}}$ corresponding to a generic orbits.** Consider the induced representation $T^{m,y}$ of the group $B_0^{\mathbb{Z}}$ corresponding to a generic orbit $\mathcal{O}_y$, generated by the point
$y = \sum_{r=0}^{\infty} y_{m+r+1,m-r} E_{m+r+1,m-r} \in \mathfrak{b}_2^*(a)$ defined by (3.16). Set for $(k,r) \in \Delta(m)$
(3.17)

$$S_{kr}(t_{kr}) := \langle y, (h(x, E_{kr}(t_{kr})) - I)\rangle, \quad \text{then} \quad A_{kr} = \frac{d}{dt}\exp(2\pi i S_{kr}(t))|_{t=0} = 2\pi i S_{kr}(1).$$

Let us denote by $\mathbb{S}^{(m)} = \mathbb{S}$ the following matrix (compare with (2.23) and (2.24)):

$$\mathbb{S} = (S_{kr})_{(k,r)\in\Delta(m)}, \quad \text{where} \quad S_{kr} = S_{kr}(1). \tag{3.18}$$

We calculate now the matrix $\mathbb{S}(t) = (S_{kr}(t_{kr}))_{(k,r)\in\Delta(m)}$ and the matrix $\mathbb{S} = (S_{kr}(1))_{(k,r)\in\Delta(m)}$ using analog of the Lemma 2.11. As in (2.22) we have

$$\langle y, h(x,t) - I\rangle = \mathrm{tr}\left(H(x,t)y\right) = \mathrm{tr}\left(x^{(m)}t_0x_m^{-1}y\right) = \mathrm{tr}\left(t_0x_m^{-1}yx^{(m)}\right) = \mathrm{tr}\left(t_0B(x,y)\right),$$

where $t_0=t-I$ and for $x_m\in B_m,\ x^{(m)}\in B^{(m)}$ we denote

$$B(x,y)=x_m^{-1}yx^{(m)}\cong\begin{pmatrix}1&0\\0&x_m^{-1}\end{pmatrix}\begin{pmatrix}0&0\\y&0\end{pmatrix}\begin{pmatrix}x^{(m)}&0\\0&1\end{pmatrix}=\begin{pmatrix}0&0\\x_m^{-1}yx^{(m)}&0\end{pmatrix}. \tag{3.19}$$

By definition we have (recall that $E_{kn}(t_{kn})=I+t_{kn}E_{kn}$)

$$S_{kn}(t_{kn})=\langle y,(h(x,E_{kn}(t_{kn}))-I)\rangle=\mathrm{tr}(t_{kn}E_{kn}B(x,y)),$$

hence by analog of the Lemma 2.11 we conclude that

$$\mathbb{S}=(S_{kn}(1))_{k,r}=(\mathrm{tr}\,(E_{kr}B(x,y)))_{k,r}=B^T(x,y)=(x^{(m)})^Ty^T(x_m^{-1})^T=\begin{pmatrix}0&(x^{(m)})^Ty^T(x_m^{-1})^T\\0&0\end{pmatrix}. \tag{3.20}$$

So, we have

$$S(h(x,t))=\exp(2\pi i\langle y,(h(x,t)-I)\rangle)=\exp\Big(2\pi i\mathrm{tr}\,((t-I)B(x,y))\Big). \tag{3.21}$$

Using results of [12] we conclude that the following lemma holds.

Lemma 3.13. *The measure $\mu_b=\mu_{b,m}\otimes\mu_b^{(m)}$ is $B_{m,0}\times B_0^{(m)}$-right-ergodic if*

$$E(\mu_b)=\sum_{k<n\le m}\frac{S_{kn}^R(\mu_b)}{b_{kn}}<\infty.$$

Theorem 3.14. *The induced representation $T^{m,y}$ of the group $B_0^{\mathbb{Z}}$ defined by formula (3.16), corresponding to generic orbit $\mathcal{O}_y$, generated by the point $y=\sum_{r=0}^{\infty}y_{m+r+1,m-r}E_{m+r+1,m-r}\in\mathfrak{b}_2^*(a)$ is irreducible if the measure $\mu_{b,m}\otimes\mu_b^{(m)}$ on the group $B_m\times B^{(m)}$ is right $B_{m,0}\times B_0^{(m)}$-ergodic. Moreover the generators of one-parameter groups $A_{kr}=\frac{d}{dt}T^{m,y}_{I+tE_{kr}}\mid_{t=0}$ are as follows*

$$A_{kr}=\sum_{s=-\infty}^{k-1}x_{ks}D_{rs}+D_{kr},\ (k,r)\in\Delta^{(m)},\quad A_{kr}=\sum_{s=m+1}^{k-1}x_{ks}D_{rs}+D_{kr},\ (k,r)\in\Delta_m,$$

$$(2\pi i)^{-1}\big(A_{kr}\big)_{(k,r)\in\Delta(m)}=\mathbb{S}^{(m)}=(S_{kr})_{(k,r)\in\Delta(m)}=\big(x_m^{-1}yx^{(m)}\big)^T.$$

Here we denote by $D_{kn}=D_{kn}(\mu_b)$ the operator of the partial derivative corresponding to the shift $x\mapsto x+tE_{kn}$ and the measure μ_b on the group $B_m\times B^{(m)}\ni x=I+\sum x_{kr}E_{kr}$:

$$(D_{kn}(\mu_b)f)(x)=\frac{d}{dt}\left(\frac{d\mu_b(x+tE_{kn})}{d\mu_b(x)}\right)^{1/2}f(x+tE_{kn})\mid_{t=0},\quad D_{kn}(\mu_b)=\frac{\partial}{\partial x_{kn}}-b_{kn}x_{kn}. \tag{3.22}$$

The irreducibility of the induced representation of the group $B_0^{\mathbb{Z}}$ follows from the following lemma.

Lemma 3.15. *Two von Neumann algebra $\mathfrak{A}^S$ and $\mathfrak{A}^x$ in the space $\mathcal{H}^m=L^2(X_m,\mu_b)$ generated respectively by the sets of unitary operators $U_{kr}(t)$ and $V_{kr}(t)$ coincides, where*

$$(U_{kr}(t)f)(x)=\exp(2\pi iS_{kr}(t))f(x),\quad (V_{kr}(t)f)(x):=\exp(2\pi itx_{kr})f(x), \tag{3.23}$$

$$\mathfrak{A}^S=\big(U_{kr}(t)=T^{m,y}_{I+tE_{kr}}=\exp(2\pi iS_{kr}(t))\mid t\in\mathbb{R},\ (k,r)\in\Delta(m)\big)'',$$

$$\mathfrak{A}^x=\big(V_{kr}(t)=\exp(2\pi itx_{kr})\mid t\in\mathbb{R},\ (k,r)\in\Delta_m\bigcup\Delta^{(m)}\big)''.$$

Proof. Using the decomposition (3.20)

$$\mathbb{S}^{(m)} = B(x,y)^T = (x_m^{-1}yx^{(m)})^T = (x^{(m)})^T y^T (x_m^{-1})^T$$

we conclude that $\mathfrak{A}^S \subseteq \mathfrak{A}^x$ (see the proof of Lemma 2.17).

To prove that $\mathfrak{A}^S \supseteq \mathfrak{A}^x$ it is sufficient to find the expressions of the matrix element of the matrix $x^{(m)} \in B^{(m)}$ and $x_m^{-1} \in B_m$ in terms of the matrix elements of the matrix $\mathbb{S}^{(m)} = (S_{kr})_{(k,r)\in\Delta(m)}$. To do this we connect the above decomposition $\mathbb{S}^{(m)} = B(x,y)^T$ (see (3.19)) and the Gauss decomposition $C = LDU$ for infinite matrices (see Theorem 4.2). By (3.19) we get $B(x,y) = x_m^{-1}yx^{(m)}$.

To find a matrix connected with the matrix $\mathbb{S}^{(m)}$, for which an appropriate decomposition LDU holds we *recall the expressions for* $B(x,y)$ *for small* n and finite-dimensional groups G_n^m (see Example (2.14)). We note that $J_m^2 = I$, where

$$J_m \in \mathrm{Mat}(\infty,\mathbb{R}), \quad J_m = \sum_{r\in\mathbb{Z}} E_{m+r+1,m-r}.$$

For G_3^3 we get

$$B(x,y) = x_m^{-1}yx^{(m)} = \begin{pmatrix} 1 & x_{45}^{-1} & x_{46}^{-1} & x_{47}^{-1} \\ 0 & 1 & x_{56}^{-1} & x_{57}^{-1} \\ 0 & 0 & 1 & x_{67}^{-1} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & y_{43} \\ 0 & 0 & y_{52} & 0 \\ 0 & y_{61} & 0 & 0 \\ y_{70} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & x_{01} & x_{02} & x_{03} \\ 0 & 1 & x_{12} & x_{13} \\ 0 & 0 & 1 & x_{23} \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$B(x,y)J = \begin{pmatrix} 1 & x_{45}^{-1} & x_{46}^{-1} & x_{47}^{-1} \\ 0 & 1 & x_{56}^{-1} & x_{57}^{-1} \\ 0 & 0 & 1 & x_{67}^{-1} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{43} & 0 & 0 & 0 \\ 0 & y_{52} & 0 & 0 \\ 0 & 0 & y_{61} & 0 \\ 0 & 0 & 0 & y_{70} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ x_{23} & 1 & 0 & 0 \\ x_{13} & x_{12} & 1 & 0 \\ x_{03} & x_{02} & x_{01} & 1 \end{pmatrix}. \tag{3.24}$$

We use the infinite-dimensional analog of the latter presentation, i.e. instead of the group $G_n = B(n,\mathbb{R})$ consider the infinite-dimensional group $B_0^{\mathbb{Z}}$ and do the same. Let

$$x_m \in B_m,\ x^{(m)} \in B^{(m)},\ y = \sum_{r=0}^{\infty} y_{m+r+1,m-r}E_{m+r+1,m-r} \in \mathfrak{g}_2^*(a)$$

and $J = J_m = \sum_{r\in\mathbb{Z}} E_{m+r+1,m-r}$. Then we get $\mathbb{S}^T = B(x,y) = x_m^{-1}yx^{(m)}$.

Set $C = C(x,y) = B(x,y)J$ then $C = UDL$, more precisely we have:

$$B(x,y)J = x_m^{-1}yJ_mJ_mx^{(m)}J_m = UDL, \text{ where } U = x_m^{-1},\ D = yJ_m,\ L = J_mx^{(m)}J_m, \tag{3.25}$$

$$C = B(x,y)J = \begin{pmatrix} 1 & x_{45}^{-1} & x_{46}^{-1} & x_{47}^{-1} & \dots \\ 0 & 1 & x_{56}^{-1} & x_{57}^{-1} & \dots \\ 0 & 0 & 1 & x_{67}^{-1} & \dots \\ 0 & 0 & 0 & 1 & \dots \\ & & & & \dots \end{pmatrix} \begin{pmatrix} y_{43} & 0 & 0 & 0 & \dots \\ 0 & y_{52} & 0 & 0 & \dots \\ 0 & 0 & y_{61} & 0 & \dots \\ 0 & 0 & 0 & y_{70} & \dots \\ & & & & \dots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ x_{23} & 1 & 0 & 0 & \dots \\ x_{13} & x_{12} & 1 & 0 & \dots \\ x_{03} & x_{02} & x_{01} & 1 & \dots \\ & & & & \dots \end{pmatrix}, \tag{3.26}$$

$$C = \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1n} & \dots \\ c_{21} & c_{22} & \dots & c_{2n} & \dots \\ & & \dots & & \dots \\ c_{n1} & c_{n2} & \dots & c_{nn} & \dots \\ & & \dots & & \dots \end{pmatrix} = \begin{pmatrix} 1 & u_{12} & \dots & u_{1n} & \dots \\ 0 & 1 & \dots & u_{2n} & \dots \\ & & \dots & & \dots \\ 0 & 0 & \dots & 1 & \dots \\ & & \dots & & \dots \end{pmatrix} \begin{pmatrix} d_1 & 0 & \dots & 0 & \dots \\ 0 & d_2 & \dots & 0 & \dots \\ & & \dots & & \dots \\ 0 & 0 & \dots & d_n & \dots \\ & & \dots & & \dots \end{pmatrix} \begin{pmatrix} 1 & 0 & \dots & 0 & \\ l_{21} & 1 & \dots & 0 & \dots \\ & & \dots & & \dots \\ l_{n1} & l_{n2} & \dots & 1 & \dots \\ & & \dots & & \dots \end{pmatrix}.$$

To finish the proof of the Lemma it is sufficient to find the decomposition (3.26) $C = UDL$.

Let us *suppose that we can find the inverse matrix* C^{-1}. Then by (3.25) holds $C^{-1} = L^{-1}D^{-1}U^{-1}$ and we can use Theorem 4.2 to find

$$L^{-1} = J_m(x^{(m)})^{-1}J_m, \quad D^{-1} = y^{-1}J_m, \quad U^{-1} = x_m.$$

Hence, we can find the matrix elements of the matrix $(x^{(m)})^{-1} \in B^{(m)}$ and $x_m \in B_m$ in terms of the matrix elements of the matrix $C^{-1} = (\mathbb{S}^TJ)^{-1} = (B(x,y)J)^{-1}$. Finally, we can also find the matrix elements of the matrix $x^{(m)} \in B^{(m)}$ using formulas (2.31). This finish the proof of the lemma since in this case we have $x_{kr}\ \eta\ \mathfrak{A}^S$ for $(k,r)\in\Delta_m\bigcup\Delta^{(m)}$. Hence $\mathfrak{A}^S\subseteq\mathfrak{A}^x$.

1) To find the inverse matrix C^{-1} we write two decompositions:

$$C=L_1D_1U_1=UDL,\quad C^{-1}=(U_1)^{-1}(D_1)^{-1}(L_1)^{-1}=L^{-1}D^{-1}U^{-1}. \tag{3.27}$$

2) Using (3.27) we can find L_1, D_1 and U_1 by Theorem 4.2. More precisely, for all $x\in\Gamma_G$, where

$$\Gamma_C=\{x\in B_m\times B^{(m)}\mid M^{12\dots k}_{12\dots k}(C(x))\neq 0,\ k\in\mathbb{N}\}$$

holds the decomposition $C(x)=L_1D_1U_1$ and the matrix elements of the matrix $L_1,\ D_1$ and U_1 are rational functions in $c_{kn}(x)$.
3) We can find $(L_1)^{-1}$ and $(U_1)^{-1}$ using formulas (2.31). Note that J_mLJ_m, U, and $J_mL^{-1}J_m$, $U^{-1}\in B_2(a)$.
4) Using identity (3.27) we can calculate $C^{-1}=(U_1)^{-1}(D_1)^{-1}(L_1)^{-1}$, since $L^{-1},\ D^{-1}$ and U^{-1} are well defined.
5) Using equality (3.27) we can find the decomposition $C^{-1}=L^{-1}D^{-1}U^{-1}$ of the matrix C^{-1} by Theorem 4.2. In other words, the decompositions holds $C^{-1}=L^{-1}D^{-1}U^{-1}$ for all $x\in\Gamma_{G^{-1}}$, where

$$\Gamma_{C^{-1}}=\{x\in B_m\times B^{(m)}\mid M^{12\dots k}_{12\dots k}(C^{-1}(x))\neq 0,\ k\in\mathbb{N}\}$$

and the matrix elements of the matrix L^{-1}, D^{-1} and U^{-1} are rational functions in matrix elements $c^{-1}_{kn}(x)$ of the matrix C^{-1}.

We make the last remark. Let us denote $(L_1)^{-1}=(L^{-1}_{1;kn})_{kn}, (D_1)^{-1}=\mathrm{diag}(d^{-1}_{1;k})_k$ and $(U_1)^{-1}=(U^{-1}_{1;kn})_{kn}$. The decompositions $C=L_1D_1U_1$ and $C^{-1}=(U_1)^{-1}(D_1)^{-1}\times(L_1)^{-1}$ hold for $x\in\Gamma_C\cap\Gamma_{C^{-1}}$, i.e. almost for all $x\in B_m\times B^{(m)}$ with respect to the measure μ_b since $\mu_b(\Gamma_C\cap\Gamma_{C^{-1}})=1$. We conclude that the convergence

$$c^{-1}_{kn}(x)=\sum_{m\in\mathbb{N}}U^{-1}_{1;km}d^{-1}_{1;m}L^{-1}_{1;mn},\ \ k,n\in\mathbb{N}$$

holds pointwise almost everywhere $x\in B_m\times B^{(m)}$ (mod μ_b). Since $U^{-1}_{1;km},\ d^{-1}_{1;m}$ and $L^{-1}_{1;mn}\,\eta\,\mathfrak{A}^S$ by 2) and 3), we conclude by Lemma 5.1 that $c^{-1}_{kn}(x)\,\eta\,\mathfrak{A}^S$. This finish the proof of the lemma. □

Proof. of the Theorem 3.14. To prove the irreducibility of the induced representation consider the restriction $T^{m,y}\mid_{B_0(m)}$ of this representation to the commutative subgroup $B_0(m)$ of the group $B_0^{\mathbb{Z}}$. Note that

$$\mathfrak{A}^x=\big(\exp(2\pi itx_{kr})\mid t\in\mathbb{R},\ (k,r)\in\Delta_m\bigcup\Delta^{(m)}\big)''=L^\infty(B_m\times B^{(m)},\mu_{b,m}\otimes\mu_b^{(m)}).$$

By Lemma 3.15 the von Neumann algebra $\mathfrak{A}^S$ generated by this restriction coincides with $\mathfrak{A}^x=L^\infty(B_m\times B^{(m)},\mu_{b,m}\otimes\mu_b^{(m)})$. Let now a bounded operator A in the Hilbert space $\mathcal{H}^m$ commute with the representation $T^{m,y}$. Then A commute by the above arguments with $L^\infty(B_m\times B^{(m)},\mu_{b,m}\otimes\mu_b^{(m)})$, therefore the operator A itself is an operator of multiplication by some essentially bounded function $a\in L^\infty$ i.e. $(Af)(x)=a(x)f(x)$ for $f\in\mathcal{H}^m$. Since A commute with the representation $T^{m,y}$ i.e. $[A,T^{m,y}_t]=0$ for all $t\in B_{m,0}\times B_0^{(m)}$, where $B_{m,0}=B_m\cap B_0^{\mathbb{Z}}$ and $B_0^{(m)}=B^{(m)}\cap B_0^{\mathbb{Z}}$, we conclude that

$$a(x)=a(xt)\ (\mathrm{mod}\ \mu_{b,m}\otimes\mu_b^{(m)})\quad\text{for}\quad\text{all}\quad t\in B_{m,0}\times B_0^{(m)}.$$

Since the measure $\mu_{b,m}\otimes\mu_b^{(m)}$ on the group $B_m\times B^{(m)}$ is right $B_{m,0}\times B_0^{(m)}$-ergodic we conclude that $a(x)=const$ (mod $dx_m\otimes dx^{(m)}$). □

Remark 3.16. *We would like to show that* $T^{m,y} = \lim_n T^{m,y_n}$*. To be more precise consider the projection* $B_0^{\mathbb{Z}} \mapsto G_n^m$ *of the group* $B_0^{\mathbb{Z}}$ *on the subgroup* G_n^m *and all other projections: homogeneous spaces, measures, Hilbert spaces and representations:*

$$X_m = B_m \times B^{(m)} \mapsto X_{m,n} = B_{m,n} \times B^{(m,n)}, \quad \mu_{b,m} \otimes \mu_b^{(m)} \mapsto \mu_{b,m,n} \otimes \mu_b^{(m,n)}$$

$$\mathcal{H}^m = L^2(B_m \times B^{(m)}, \mu_{b,m} \otimes \mu_b^{(m)}) \mapsto L^2(B_{m,n} \times B^{(m,n)}, \mu_{b,m,n} \otimes \mu_b^{(m,n)})$$
$$\cong L^2(B_{m,n} \times B^{(m,n)}, dx_{m,n} \otimes dx^{(m,n)}) = \mathcal{H}^{m,n}$$
$$T^{m,y} \mapsto T^{m,y_n}, \quad n \in \mathbb{N}.$$

Since the measure $\mu_{b,m,n} \otimes \mu_b^{(m,n)}$ is equivalent with the Haar measure (compare (2.43) and (3.14)) we conclude that the corresponding representations T^{μ,m,y_n} in the spaces $L^2(B_{m,n} \times B^{(m,n)}, \mu_{b,m,n} \otimes \mu_b^{(m,n)})$ and T^{m,y_n} in the space $L^2(B_{m,n} \times B^{(m,n)}, dx_{m,n} \otimes dx^{(m,n)})$ are equivalent. This implies $T^{m,y} = \lim_n T^{m,y_n}$.

3.9. **Dual description of the groups $B_0^{\mathbb{N}}$ and $B_0^{\mathbb{Z}}$. First steps.** Let $\hat{G}$ be the dual of the group G. Our *aim* is to describe $\hat{G}$ for $G = \varinjlim_n G_n$ where $G_n = B(n,\mathbb{R})$ is the group of all $n \times n$ upper triangular real matrices with units on the principal diagonal, i.e. we would like to describe the dual of the group $B_0^{\mathbb{N}}$ of infinite in one direction and $B_0^{\mathbb{Z}}$ infinite in both directions matrices. Consider the inductive limit $G = \varinjlim_n G_n$ of nilpotent groups $G_n = B(n,\mathbb{R})$. The symmetric (resp. nonsymmetric) imbedding gives us two infinite-dimensional analog of “nilpotent” groups $B_0^{\mathbb{Z}}$ (resp. $B_0^{\mathbb{N}}$).

We do not know the description of all $\hat{G}$. We only know that the set $\hat{G}$ contains the following three classes of representations.

1) The set $\hat{G}$ contains $\bigcup_n \hat{G}_n$ i.e. $\hat{G} \supset \bigcup_n \hat{G}_n$. One may use Kirillov's orbit method [4, 7] to describe $\hat{G}_n$. The embedding $\hat{G}_n \subset \widehat{G_{n+1}}$ is described in Remark 3.17.

2) We have $\hat{G} \setminus \bigcup_n \hat{G}_n \neq \emptyset$. Namely $\hat{G} \setminus \bigcup_n \hat{G}_n$ contains ”regular” $T^{R,\mu}$ and ”quasiregular” $\pi^{R,\mu,X}$ representations of the group G (see subsection 3.1).

3) Induced representations (see subsection 3.6).

It is natural together with the group $B_0^{\mathbb{N}}$ (resp. $B_0^{\mathbb{Z}}$) consider all Hilbert-Lie completion $B_2^{\mathbb{N}}(a)$ (resp. $B_2^{\mathbb{Z}}(a)$) and the group of all upper-triangular matrices $B^{\mathbb{N}}$ (resp. $B^{\mathbb{Z}}$) (see subsections 3.5, 3.4)

$$G_n \to B_0^{\mathbb{N}} \to B_2^{\mathbb{N}}(a) \to B^{\mathbb{N}} \to G_n.$$
$$G_n^m \to B_0^{\mathbb{Z}} \to B_2^{\mathbb{Z}}(a) \to B^{\mathbb{Z}} \to G_n^m.$$

Together with all imbedding and projections of all mentioned groups $G_n = B(n,\mathbb{R})$ we have:

$$B(n,\mathbb{R}) \stackrel{i_n^{n+1}}{\to} B(n+1,\mathbb{R}) \stackrel{i_n^{\infty}}{\to} B_0^{\mathbb{N}} \to B_2(a) \to B^{\mathbb{N}} \to B(n+1,\mathbb{R}) \stackrel{p_{n+1}^n}{\to} B(n,\mathbb{R}),$$

where the imbedding i_n^{n+1} and the projections p_{n+1}^n are defined as follows:

$$B(n,\mathbb{R}) \ni x \mapsto i_n^{n+1}(x) = x + E_{n+1,n+1} \in B(n+1,\mathbb{R}),$$
$$B(n+1,\mathbb{R}) \ni x = x^{n+1}x_n \mapsto p_{n+1}^n(x) = x_n \in B(n,\mathbb{R}),$$

$$\text{where} \quad x^{n+1} = I + \sum_{k=1}^n x_{kn+1}E_{kn+1}, \quad x_n = I + \sum_{1\leq k<m\leq n} x_{km}E_{km}.$$

For groups $G_n^m \simeq B(2n,\mathbb{R})$ defined by (2.41) consider the homomorphism $p_{n+1}^{s,m,n}$: $G_{n+1}^m \mapsto G_n^m$ defined as follows (for simplicity we define $p_{n+1}^{s,m,n}$ for $m = 0$)

$$G_{n+1}^0 \ni x = x_{\uparrow}^{n+1} x_n x_{\rightarrow}^n \mapsto p_{n+1}^{s,0,n}(x) = x_n \in G_n^0,$$

where

$$x_{\uparrow}^{n+1} = I + \sum_{-n<k<n+1} x_{k,n+1}E_{k,n+1}, \quad x_{\rightarrow}^{n} = I + \sum_{-n<k\leq n+1} x_{-n,k}E_{-n,k}.$$

Remark 3.17. *The embedding* $\widehat{B(n,\mathbb{R})} \mapsto \widehat{B(n+1,\mathbb{R})}$ *(resp.* $\widehat{G_n^m} \mapsto \widehat{G_{n+1}^m}$*) is induced by the homomorphism (3.9)* $p_{n+1}^n : B(n+1,\mathbb{R}) \mapsto B(n,\mathbb{R})$ *(resp. by the homomorphism (3.9)* $p_{n+1}^{s,m,n} : G_{n+1}^m \mapsto G_n^m$ *). So for* $m \in \mathbb{Z}$ *we get* $\bigcup_{n\in\mathbb{N}} \widehat{G_n^{(m)}} \subset \widehat{B_0^{\mathbb{Z}}}$*. Similarly, we have* $\cup_{n\in\mathbb{N}}\widehat{B(n,\mathbb{N})} \subset \widehat{B_0^{\mathbb{N}}}$

Let us denote by $B_2^{\mathbb{N}}(a)$ (resp. $B_2^{\mathbb{Z}}(a)$) the completion of the subgroup $B_0^{\mathbb{N}} \subset \mathrm{GL}_0(2\infty,\mathbb{R})$ (resp. $B_0^{\mathbb{Z}} \subset \mathrm{GL}_0(2\infty,\mathbb{R})$) in the Hilbert-Lie group $\mathrm{GL}_2(a)$. Since (see [8])

$$B_0^{\mathbb{N}} = \bigcap_{a\in\mathfrak{A}} B_2^{\mathbb{N}}(a) \quad (\text{resp.} \quad B_0^{\mathbb{Z}} = \bigcap_{a\in\mathfrak{A}} B_2^{\mathbb{Z}}(a))$$

we conclude that

$$\widehat{B_0^{\mathbb{N}}} = \bigcup_{a\in\mathfrak{A}} \widehat{B_2^{\mathbb{N}}(a)} \quad (\text{resp.} \quad \widehat{B_0^{\mathbb{Z}}} = \bigcup_{a\in\mathfrak{A}} \widehat{B_2^{\mathbb{Z}}(a)}).$$

It leaves to describe $\widehat{B_2^{\mathbb{N}}(a)}$ (resp. $\widehat{B_2^{\mathbb{Z}}(a)}$) for all $a \in \mathfrak{A}$. The problem of developing the orbit method for the Hilbert-Lie group $B_2^{\mathbb{N}}(a)$ (resp. $B_2^{\mathbb{Z}}(a)$) could be easier, since the corresponding Lie algebra $\mathfrak{b}_2^{\mathbb{N}}(a)$ (resp. $\mathfrak{b}_2^{\mathbb{Z}}(a)$) is a Hilbert-Lie algebra, the dual $(\mathfrak{b}_2^{\mathbb{N}}(a))^*$ (resp. $(\mathfrak{b}_2^{\mathbb{Z}}(a))^*$) and the pairing between $\mathfrak{b}_2^{\mathbb{N}}(a)$ (resp. $\mathfrak{b}_2^{\mathbb{Z}}(a)$) and $(\mathfrak{b}_2^{\mathbb{N}}(a))^*$ (resp. $(\mathfrak{b}_2^{\mathbb{Z}}(a))^*$) are well defined (see subsection 3.6).

Using (3.9) we conclude

$$B_0^{\mathbb{N}} = \varinjlim_{n,i} B(n,\mathbb{R}), \quad B_0^{\mathbb{N}} = \varprojlim_{a} B_2^{\mathbb{N}}(a), \quad B^{\mathbb{N}} = \varprojlim_{n,p} B(n,\mathbb{R}), \tag{3.28}$$

$$\widehat{B_0^{\mathbb{N}}} \supset \widehat{B_2^{\mathbb{N}}(a)} \supset \widehat{B^{\mathbb{N}}},$$

finally we conclude that

$$\widehat{B_0^{\mathbb{N}}} = \bigcup_{a\in\mathfrak{A}} \widehat{B_2^{\mathbb{N}}(a)}, \quad \widehat{B^{\mathbb{N}}} = \bigcup_{n\in\mathbb{N}} \widehat{G_n} = \bigcup_{n\in\mathbb{N}} \widehat{B(n,\mathbb{R})}. \tag{3.29}$$

The similar relations holds also for groups $B_0^{\mathbb{Z}} \subset B_2^{\mathbb{Z}}(a) \subset B^{\mathbb{Z}}$.

Definition 3.18. *We call the representation of the group* $G = \varinjlim_n G_n$ local *if it depends only on the elements of the subgroup* G_n *for some fixed* $n \in \mathbb{N}$.

The last relation in (3.28) and (3.29) we can reformulated as follows:

Theorem 3.19. *(V.L. Ostrovsky, PhD dissertation, 1986). The class of all irreducible unitary* local representations *of the group* $B_0^{\mathbb{N}} = \varinjlim_n B(n,\mathbb{R})$ *coincides with the class* $\bigcup_n \hat{G}_n$.

4. Appendix 1. Gauss decompositions

4.1. **Gauss decomposition of $n \times n$ matrices.** We need some decomposition of the matrix $C \in \mathrm{Mat}(n,\mathbb{C})$. Let us denote by

$$M_{j_1 j_2 \ldots j_r}^{i_1 i_2 \ldots i_r}(C), \ 1 \leq i_1 < \ldots < i_r \leq n, \ 1 \leq j_1 < \ldots < j_r \leq n$$

the minors of the matrix C with $i_1, i_2, \ldots, i_r$ rows and $j_1, j_2, \ldots, j_r$ columns.

Theorem 4.1 (Gauss decomposition, [2]). *A matrix $C\in\mathrm{Mat}(n,\mathbb{C})$ admits the following decomposition $C=LDU$ (Gauss decomposition),*

(4.1)
$$\begin{pmatrix} c_{11} & c_{12} & \dots & c_{1n}\\ c_{21} & c_{22} & \dots & c_{2n}\\ & & \dots & \\ c_{n1} & c_{n2} & \dots & c_{nn}\end{pmatrix}=\begin{pmatrix} 1 & 0 & \dots & 0\\ l_{21} & 1 & \dots & 0\\ & & \dots & \\ l_{n1} & l_{n2} & \dots & 1\end{pmatrix}\begin{pmatrix} d_1 & 0 & \dots & 0\\ 0 & d_2 & \dots & 0\\ & & \dots & \\ 0 & 0 & \dots & d_n\end{pmatrix}\begin{pmatrix} 1 & u_{12} & \dots & u_{1n}\\ 0 & 1 & \dots & u_{2n}\\ & & \dots & \\ 0 & 0 & \dots & 1\end{pmatrix}$$

where L (resp. U) is lower (resp. upper) triangular matrix and D a diagonal matrix if and only if all principal minors of the matrix C are different from zeros i.e. $M^{1,2,\dots,k}_{1,2,\dots,k}(C)\neq 0,\ 1\leq k\leq n$. Moreover the matrix elements of the matrices L, U and D are given by the formulas (see [2, Ch.II, §4, (44), (45)]*)*

$$l_{mk}=\frac{M^{1,2,\dots,k-1,m}_{1,2,\dots,k-1,k}(C)}{M^{1,2,\dots,k-1,k}_{1,2,\dots,k-1,k}(C)},\quad u_{km}=\frac{M^{1,2,\dots,k-1,k}_{1,2,\dots,k-1,m}(C)}{M^{1,2,\dots,k-1,k}_{1,2,\dots,k-1,k}(C)},1\leq k<m\leq n,\tag{4.2}$$

$$d_1=M^1_1(C),\quad d_k=\frac{M^{1,2,\dots,k}_{1,2,\dots,k}(C)}{M^{1,2,\dots,k-1}_{1,2,\dots,k-1}(C)},\quad 2\leq k\leq n.\tag{4.3}$$

Proof. If we write $L^{-1}C=DU$, we get
$$M^{1,2,\dots,k-1,k}_{1,2,\dots,k-1,k}(C)=M^{1,2,\dots,k-1,k}_{1,2,\dots,k-1,k}(L^{-1}C)=M^{1,2,\dots,k-1,k}_{1,2,\dots,k-1,k}(DU)=d_1\dots d_k,$$
this implies (4.3). Moreover, we get also
$$M^{1,2,\dots,k-1,k}_{1,2,\dots,k-1,m}(L^{-1}C)=M^{1,2,\dots,k-1,k}_{1,2,\dots,k-1,m}(C)=M^{1,2,\dots,k-1,k}_{1,2,\dots,k-1,m}(DU)=d_1\dots d_ku_{km},\ k<m,$$
this implies the second formula in (4.2). Similarly if we write $CU^{-1}=LD$ we get
$$M^{1,2,\dots,k-1,m}_{1,2,\dots,k-1,k}(CU^{-1})=M^{1,2,\dots,k-1,m}_{1,2,\dots,k-1,k}(C)=M^{1,2,\dots,k-1,m}_{1,2,\dots,k-1,k}(LD)=d_1\dots d_kl_{mk},\ k<m,$$
this implies the first formula in (4.2). □

4.2. **Gauss decomposition of infinite order matrices.** Let us consider the infinite matrix $C,L,D,U\in\mathrm{Mat}(\infty,\mathbb{C})$.

Theorem 4.2 (Gauss decomposition $C=LDU$). *A matrix $C\in\mathrm{Mat}(\infty,\mathbb{C})$ admits the following decomposition $C=LDU$ (Gauss decomposition),*

$$\begin{pmatrix} c_{11} & c_{12} & \dots & c_{1n} & \dots\\ c_{21} & c_{22} & \dots & c_{2n} & \dots\\ & & \dots & & \dots\\ c_{n1} & c_{n2} & \dots & c_{nn} & \dots\\ & & \dots & & \dots\end{pmatrix}=\begin{pmatrix} 1 & 0 & \dots & 0 & \\ l_{21} & 1 & \dots & 0 & \dots\\ & & \dots & & \dots\\ l_{n1} & l_{n2} & \dots & 1 & \dots\\ & & \dots & & \dots\end{pmatrix}\begin{pmatrix} d_1 & 0 & \dots & 0 & \dots\\ 0 & d_2 & \dots & 0 & \dots\\ & & \dots & & \dots\\ 0 & 0 & \dots & d_n & \dots\\ & & \dots & & \dots\end{pmatrix}\begin{pmatrix} 1 & u_{12} & \dots & u_{1n} & \dots\\ 0 & 1 & \dots & u_{2n} & \dots\\ & & \dots & & \dots\\ 0 & 0 & \dots & 1 & \dots\\ & & \dots & & \dots\end{pmatrix}\tag{4.4}$$

where L (resp. U) is lower (resp. upper) triangular matrix and D a diagonal matrix of infinite order if and only if all principal minors of the matrix C are different from zeros i.e. $M^{1,2,\dots,k}_{1,2,\dots,k}(C)\neq 0,\ k\in\mathbb{N}$. Moreover the matrix elements of the matrices L, U and D are given by the same formulas as in the Theorem 4.1:

$$l_{mk}=\frac{M^{1,2,\dots,k-1,m}_{1,2,\dots,k-1,k}(C)}{M^{1,2,\dots,k-1,k}_{1,2,\dots,k-1,k}(C)},\quad u_{km}=\frac{M^{1,2,\dots,k-1,k}_{1,2,\dots,k-1,m}(C)}{M^{1,2,\dots,k-1,k}_{1,2,\dots,k-1,k}(C)},\quad k,m\in\mathbb{N},\ k<m,\tag{4.5}$$

$$d_1=M^1_1(C),\quad d_k=\frac{M^{1,2,\dots,k}_{1,2,\dots,k}(C)}{M^{1,2,\dots,k-1}_{1,2,\dots,k-1}(C)},\quad k\in\mathbb{N},\ k>1.\tag{4.6}$$

Proof. The proof repeat word by word the proof of the Theorem 4.1. □

5. Appendix 2. One elementary fact concerning abelian von Neumann algebras

Let $(X, \mathcal{F}, \mu)$ be a *measurable space*, with a finite measure $\mu(X) < \infty$, where $\mathcal{F}$ is a sigma-algebra. Consider the set $(f_n) = (f_n)_{n\in\mathbb{N}}$ of measurable real valued functions on X i.e. $f_n : X \mapsto \mathbb{R}$. Denote by $B(H)$ the von Neumann algebra of all bounded operators in the Hilbert space $H = L^2(X, \mu)$ and let $\mathfrak{A}^{(f_n)}(\in B(H))$ be a von Neumann algebra generated by operators $U_n(t)$ of multiplication by functions $\exp(itf_n(x)),\ n \in \mathbb{N}$

$$\mathfrak{A}^{(f_n)} = \big(U_n(t) = e^{itf_n} \mid n \in \mathbb{N},\, t \in \mathbb{R}\big)''.$$

We are interesting in the following *question*. Let $f_n \to f$ as $n \to \infty$ in some sense. When $U(t) = e^{itf} \in \mathfrak{A}^{(f_n)}$ for all $t \in \mathbb{R}$?

Since $\mathfrak{A}^{(f_n)}$ is a von Neumann algebra it is sufficient to find when the strong convergence of the unitary operators in the space H holds i.e. $s.\lim_n U_n(t) = U(t)$, where the operators $U_n(t),\ n \in \mathbb{N}$ and $U(t)$ are defined as follows

$$(U_n(t)g)(x) = e^{itf_n(x)}g(x), \quad (U(t)g)(x) = e^{itf(x)}g(x), \quad g \in L^2(X, \mu),\ t \in \mathbb{R}.$$

Lemma 5.1. *Let $f_n \to f$ as $n \to \infty$ pointwise almost everywhere, then $s.\lim_n U_n(t) = U(t)$ hence $U(t) = e^{itf} \in \mathfrak{A}^{(f_n)}$.*

Proof. For $g \in H$ we get

$$\|(U_n(t) - U(t))g\|^2 = \int_X \mid \big(e^{itf_n(x)} - e^{itf(x)}\big)g(x) \mid^2 d\mu(x) =$$

$$\int_X \mid e^{itf_n(x)-itf(x)} - 1 \mid^2 \mid g(x) \mid^2 d\mu(x) = \int_X \mid e^{it\alpha_n(x)} - 1 \mid^2 \mid g(x) \mid^2 d\mu(x) \to 0$$

as $n \to \infty$, if $\alpha_n(x) := f_n(x) - f(x) \to 0$ pointwise almost everywhere by Lebesgue's dominated convergence theorem. □

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