

TWO-LOOP PART OF THE RATIONAL HOMOTOPY OF SPACES OF LONG EMBEDDINGS

JIM CONANT, JEAN COSTELLO, VICTOR TURCHIN, AND PATRICK WEED

ABSTRACT. Arone and Turchin defined graph-complexes computing the rational homotopy of the spaces of long embeddings. The graph-complexes split into a direct sum by the number of loops in graphs. In this paper we compute the homology of its two-loop part.

1. INTRODUCTION

Let $\text{Emb}_c(\mathbb{R}^m, \mathbb{R}^N)$ denote the space of smooth embeddings $\mathbb{R}^m \hookrightarrow \mathbb{R}^N$ coinciding with a fixed linear embedding outside a compact subset of \mathbb{R}^m . By $\overline{\text{Emb}}_c(\mathbb{R}^m, \mathbb{R}^N)$ we denote the homotopy fiber of the inclusion $\text{Emb}_c(\mathbb{R}^m, \mathbb{R}^N) \hookrightarrow \text{Imm}_c(\mathbb{R}^m, \mathbb{R}^N)$, where $\text{Imm}_c(\mathbb{R}^m, \mathbb{R}^N)$ is the space of immersions with the same behavior at infinity. In [1, 2] Arone and Turchin study the rational homology and homotopy of $\text{Emb}_c(\mathbb{R}^m, \mathbb{R}^N)$ and $\overline{\text{Emb}}_c(\mathbb{R}^m, \mathbb{R}^N)$. In particular in [2] the authors define graph-complexes $\mathcal{E}_\pi^{m,N}$ computing the rational homotopy $\pi_* \overline{\text{Emb}}_c(\mathbb{R}^m, \mathbb{R}^N) \otimes \mathbb{Q}$, $N \geq 2m + 2$. The graph-complexes up to a regrading depend only on the parities of m and N . Thus there are only 4 cases to consider. The connection between $\pi_* \overline{\text{Emb}}_c(\mathbb{R}^m, \mathbb{R}^N) \otimes \mathbb{Q}$ and $\pi_* \text{Emb}_c(\mathbb{R}^m, \mathbb{R}^N) \otimes \mathbb{Q}$, $N \geq 2m + 2$, is established by [2, Corollary 4.3]. The graph-complexes $\mathcal{E}_\pi^{m,N}$ split into a direct sum of complexes by the number of univalent vertices (this number is called *Hodge degree*) and the number of loops (first Betti number) in the graphs. This paper computes the homology of the part of $\mathcal{E}_\pi^{m,N}$ spanned by two-loop graphs. The homology of the part spanned by trees and of the part spanned by one-loop graphs was computed in [2, Section 3].

Theorems 5.5, 5.8, 5.14, 5.16 describe generating functions of the homology ranks and of the Euler characteristics in each of the four cases. Theorems 6.1-6.4 give explicit formulae for those ranks. We can make two conclusions from our computations. Firstly, for all parities of m and N , the homology of the two-loop part for any Hodge degree is concentrated in only one grading. Thus the Euler characteristics completely determine the homology ranks. Secondly, the ranks grow linearly with the Hodge degree.

Our method is borrowed from similar computations of the dimensions of the space of 2-loop uni-trivalent graphs modulo *IHX* and *AS* relations [6], which appears in the study of finite type knot invariants, see also [4, 5]. The latter space is the bottom degree part of our homology for the case when both m and N are odd. The field of coefficients for the considered complexes is always \mathbb{Q} .

2. COMPLEXES OF UNI- ≥ 3 -VALENT GRAPHS

In this section we recall definition of the complex $\mathcal{E}_\pi^{m,N}$ of uni- ≥ 3 -valent graphs from [2]. The homology of this complex is naturally isomorphic to $\pi_* \overline{\text{Emb}}_c(\mathbb{R}^m, \mathbb{R}^N) \otimes \mathbb{Q}$, $N \geq 2m + 2$.

The complex $\mathcal{E}_\pi^{m,N}$ is spanned by abstract connected graphs having a non-empty set of non-labeled *external vertices* of valence 1, and a possibly empty set of non-labeled *internal vertices* of valence ≥ 3 . The graphs are allowed to have edges joining a vertex to itself and multiple edges. For such graph define its *orientation set* as the union of the set of its external vertices (considered as elements of degree $-m$), the set of its internal vertices (considered as elements of degree $-N$), and the set of its edges (considered as elements of degree $(N-1)$). By an *orientation* of a graph we will understand ordering of its orientation set together with an orientation of all its edges. Two such graphs are *equivalent* if there is a bijection between their sets of vertices and edges respecting the adjacency structure of the graphs, orientation of the edges, and the order of the orientation sets. The space of $\mathcal{E}_\pi^{m,N}$ is the quotient space of the vector space freely spanned by equivalence classes of such oriented graphs modulo the orientation relations:

- (1) $\Upsilon_1 = (-1)^n \Upsilon_2$ if Υ_1 differs from Υ_2 by reversing the orientation of an edge.
- (2) $\Upsilon_1 = \pm \Upsilon_2$, where Υ_2 is obtained from Υ_1 by a permutation of the orientation set. The sign here is the Koszul sign of the permutation taking into account the degrees of the elements.

Notice that if a graph has a symmetry that produces a negative sign, then such a graph is zero modulo orientation relations (1-2). The total degree of a graph is the sum of degrees of all the elements from its orientation set. The differential $\partial\Upsilon$ of a graph $\Upsilon \in \mathcal{E}_\pi^{m,N}$ is defined as the sum of expansions of its internal vertices. An expanded vertex is replaced by an edge. The set of edges adjacent to the expanded vertex splits into two sets – one containing the edges that go to one vertex of the new edge and the other set containing the edges that go to the other vertex. An expansion of a vertex of valence ℓ is a sum of $\frac{2^\ell - 2\ell - 2}{2} = 2^{\ell-1} - \ell - 1$ graphs obtained in such way. One subtracts $2\ell + 2$ to exclude graphs with internal vertices of valence < 3 , and one divides by 2 because of the symmetry. The orientation set of a new graph is obtained by adding the new vertex and the new edge as the first and second elements to the orientation set, and by orienting the new edge from the old vertex to the new one. There is a freedom which of 2 vertices of the new edge is considered as a new one and which as an old one, but regardless of this choice, the orientation of the boundary graph is the same. All the graphs in the differential appear with positive sign (the sign is hidden in the way we order the orientation set and orient the new edge).

Notice that the differential preserves the number of external vertices t referred as *Hodge degree* and also the first Betti number L (number of loops in a graph). The first Betti number s of the graph obtained by gluing together all univalent vertices will be called *complexity*. It is also preserved by the differential. One obviously has

$$L = s - t + 1.$$

The part of $\mathcal{E}_\pi^{m,N}$ of Hodge degree t and complexity s is denoted by $\mathcal{E}_\pi^{m,N}(s, t)$.

We will also define *defect* of a graph as the sum $\sum_v (|v| - 3)$, where v runs through the internal vertices of the graph, and $|v|$ is the valence of v . The defect measures how much the graph is different from a uni-trivalent one. The differential decreases the defect by one.

3. HAIRY GRAPH-COMPLEXES

In this section we define a quasi-isomorphic subcomplex $\mathcal{H}_\pi^{m,N}$ of $\mathcal{E}_\pi^{m,N}$ which will require a few definitions.

An edge in a connected graph is called *tree-type* if when one removes it, the graph becomes disconnected, whose exactly one connected part is a tree. By the *frame* of a graph we will mean

its subgraph generated by edges which are not tree-type. Notice that the frame of a connected graph is always connected. A tree-type edge is called a *hair* if one of its vertices is univalent and the other one belongs to the frame and has no other tree-type edges adjacent to it. A vertex is called *cut* if when one removes it the graph becomes disconnected.

One defines the *complex* $\mathcal{H}_\pi^{m,N}$ of hairy graphs as the subcomplex of $\mathcal{E}_\pi^{m,N}$ spanned by trees with ≤ 3 external vertices and by graphs whose frame has no cut-vertices and whose tree-type edges are all hairs.

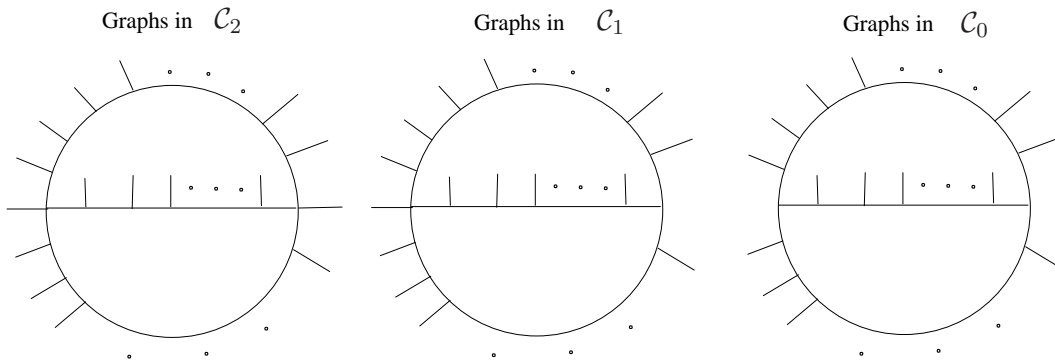
Theorem 3.1. *The inclusion $\mathcal{H}_\pi^{m,N} \hookrightarrow \mathcal{E}_\pi^{m,N}$ is a quasi-isomorphism.*

Proof. For the part $\mathcal{H}_\pi^{m,N}(t+1, t) \hookrightarrow \mathcal{E}_\pi^{m,N}(t+1, t)$, i.e. the part which is spanned by trees, see [2, Proposition 3.2]. For the case when the first Betti number is positive the proof follows from the argument of [3, Theorem 1.1]. In that paper, graphs do not have hairs, the orientation data is not as general, and the differential contracts, rather than expands edges. However the argument still works with a little modification. One writes $\partial = \partial_s + \partial_{ns}$, where ∂_s expands cut vertices into separating edges and $\partial_{ns} = \partial - \partial_s$. Consider the subspace of graphs which have a cut vertex which is not a trivalent vertex incident to a hair, which is a complex with respect to ∂_s . By the same argument as in that paper, the homology of this complex is trivial. So in the spectral sequence for the double complex, only graphs without cut vertices survive, and the surviving boundary operator ∂_{ns} is the standard boundary restricted to the hairy graph subcomplex. \square

4. TWO-LOOP PART OF $\mathcal{H}_\pi^{m,N}$

From Theorem 3.1 one can immediately conclude that the loopless part of $\mathcal{H}_\pi^{m,N}$ is always one-dimensional. It is spanned by the tree with two vertices when m and N are of the same parity and by the tree with three external and one internal vertices when m and N are of opposite parity. The differential is obviously zero. The one-loop part of $\mathcal{H}_\pi^{m,N}$ is spanned by wheels – graphs obtained from a circle by attaching several hairs. [2, Proposition 3.3] tells when such wheels survive their dihedral symmetry. The differential is also zero in this situation.

Let \mathcal{C}_i denote the space of two-loop hairy graphs of defect i . Notice that the frame of any such graph has the shape of the Θ -graph. The defect i can be either 0, 1, or 2, see the figure below.



One gets the complex

$$0 \xrightarrow{\partial_3} \mathcal{C}_2 \xrightarrow{\partial_2} \mathcal{C}_1 \xrightarrow{\partial_1} \mathcal{C}_0 \xrightarrow{\partial_0} 0. \tag{4.1}$$

We will see that the differential $\partial_2: \mathcal{C}_2 \rightarrow \mathcal{C}_1$ is injective. Which means that the homology must be concentrated in defect 0 and 1. Let a_k , respectively b_k , be the rank of the homology of defect 0, respectively 1, in Hodge degree k . Let χ_k denote the Euler characteristic of (4.1) in Hodge degree k . We will compute the generating functions

$$h_0(t) = \sum_{k=1}^{\infty} a_k t^k, \quad h_1(t) = \sum_{k=1}^{\infty} b_k t^k, \quad \chi(t) = \sum_{k=1}^{\infty} \chi_k t^k.$$

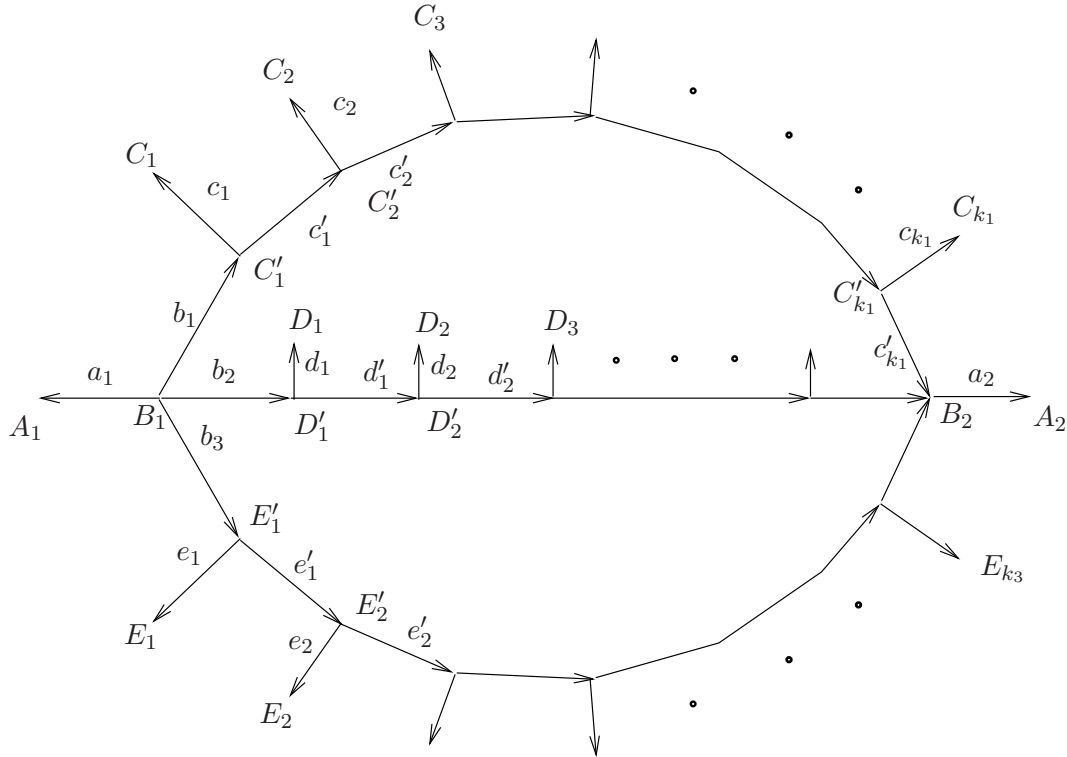
Obviously, h_0 and h_1 depend on the parities of m and N . The total degree of a graph with k hairs in \mathcal{C}_0 is $k(N - m - 2) + N - 3$. This implies the following lemma.

Lemma 4.1. *One has*

$$\chi(t) = (-1)^{N-1} [h_0((-1)^{N-m}t) - h_1((-1)^{N-m}t)].$$

Proof. Follows from injectivity of ∂_2 , see Section 5. \square

The main idea for our computations is that the spaces of graphs will be encoded as certain spaces of polynomials. Denote by $x_1^{k_1} x_2^{k_2} x_3^{k_3}$ the graph from \mathcal{C}_2 that has k_1 hairs on the upper edge of Θ , k_2 hairs on the middle one, and k_3 hairs on the lower one. Its edges are oriented as in the figure below.



The ordering of its orientation set is as follows. First one has $A_1 A_2 a_1 a_2 B_1 B_2 b_1 b_2 b_3$. Then one puts $(C_1 c_1 C_1' c_1')(C_2 c_2 C_2' c_2') \dots (C_{k_1} c_{k_1} C_{k_1}' c_{k_1}')$, then a similar product of $(D_i d_i D_i' d_i')$, $i = 1 \dots k_2$, corresponding to the middle edge, then a similar product of $(E_i e_i E_i' e_i')$, $i = 1 \dots k_3$.

Abusing notation we will denote by $x_1^{k_1}x_2^{k_2}x_3^{k_3}$ similar graphs in \mathcal{C}_1 and \mathcal{C}_0 . The ordering of the orientation sets will start in these cases as $A_1a_1B_1B_2b_1b_2b_3$, $B_1B_2b_1b_2b_3$ respectively.

The two-loop graphs have two possible types of symmetry: symmetry with respect to the vertical line and transpositions of edges in the Θ -graph. If a graph has a symmetry that produces a negative sign, it means the graph is zero modulo orientation relations. The lemma below computes the signs arising from these symmetries.

Lemma 4.2. (1) *The symmetry of the graph $x_1^{k_1}x_2^{k_2}x_3^{k_3} \in \mathcal{C}_2$ with respect to the vertical line produces the sign*

$$(-1)^{m+N+1}(-1)^{k_1+k_2+k_3}(-1)^{(m+N)\sum_{i=1}^3\frac{k_i(k_i-1)}{2}}.$$

(2) *The symmetry of the graph $x_1^{k_1}x_2^{k_2}x_3^{k_3} \in \mathcal{C}_0$ with respect to the vertical line produces the sign*

$$(-1)^{k_1+k_2+k_3}(-1)^{(m+N)\sum_{i=1}^3\frac{k_i(k_i-1)}{2}}.$$

(3) *The graph $x_1^{k_1}x_2^{k_2}x_3^{k_3} \in \mathcal{C}_i$, $i = 0 \dots 2$, is equal to $x_1^{k_2}x_2^{k_1}x_3^{k_3}$ with the sign*

$$(-1)^{N-1}(-1)^{k_1k_2(m+N)}.$$

Proof. Notice that the orientation of $x_1^{k_1}x_2^{k_2}x_3^{k_3}$ that we defined matters only for (3). In the latter case the sign $(-1)^{N-1}$ arises from the transposition of b_1 and b_2 . The sign $(-1)^{k_1k_2(m+N)}$ is due to the permutation of the factors $(D_id_iD'_id'_i)$ with $(E_je_jE'_je'_j)$.

The sign in (1) is the product of

- $(-1)^m(-1)^{N-1}(-1)^N$ – corresponding to the transpositions of A_1 with A_2 , a_1 with a_2 , B_1 with B_2 ;
- $(-1)^{(N-1)\sum_{i=1}^3\frac{k_i(k_i+1)}{2}}$ – corresponding to permutation of b_i 's, c_i 's, d_i 's, e_i 's;
- $(-1)^{(m+1)\sum_{i=1}^3\frac{k_i(k_i-1)}{2}}$ – corresponding to permutation of C_i 's, c_i 's, C'_i 's, D_i 's, d_i 's, D'_i 's, E_i 's, e_i 's, E'_i 's;
- $(-1)^{N(k_1+k_2+k_3+3)}$ – corresponding to the change of orientation of edges.

The sign in (2) is the same as in (1) except that one should not count the transpositions of A_1 with A_2 and of a_1 with a_2 . \square

5. TWO-LOOP HOMOLOGY COMPUTATIONS

5.1. When both N and m are odd. Let $\mathbb{Q}[x_1, x_2, x_3]$ denote the the free commutative algebra generated by x_1, x_2, x_3 .

Definition 5.1. *We say that a polynomial is even (odd, respectively) if all of its monomials are of even (odd, respectively) degree.*

In the previous section we encoded two-loop graphs by monomials $x_1^{k_1}x_2^{k_2}x_3^{k_3}$. The symmetric group S_3 acts on the space $\mathbb{Q}[x_1, x_2, x_3]$ by permuting the variables x_1, x_2, x_3 , which geometrically correspond to the reordering of the edges in the theta-graph. Lemma 4.2 (3) tells us what sign should be used to obtain an equivalent monomial-graph. Instead of taking a quotient by this action we will be taking invariants since in characteristic zero, invariants coincide with coinvariants. When both m and N are odd, a graph with k_i , $i = 1 \dots 3$, hairs respectively, will be denoted instead by $\frac{1}{6}\sum_{\sigma \in S_3}x_{\sigma(1)}^{k_1}x_{\sigma(2)}^{k_2}x_{\sigma(3)}^{k_3}$. It follows from Lemma 4.2 that the space \mathcal{C}_2 in this case

is represented by odd symmetric polynomials, the space \mathcal{C}_1 is represented by the space of all symmetric polynomials, and the space \mathcal{C}_0 is represented by even symmetric polynomials whose all monomials are of strictly positive degree:

$$0 \xrightarrow{\partial_3} \text{Sym}^{\text{odd}}[x_1, x_2, x_3] \xrightarrow{\partial_2} \text{Sym}[x_1, x_2, x_3] \xrightarrow{\partial_1} \text{Sym}_{>0}^{\text{even}}[x_1, x_2, x_3] \xrightarrow{\partial_0} 0.$$

The following result is well known.

Theorem 5.2. *The algebra $\text{Sym}[x_1, x_2, x_3]$ is isomorphic to the free commutative algebra generated by three letters: $\mathbb{Q}[e_1, e_2, e_3]$, where $e_1 = x_1 + x_2 + x_3$, $e_2 = x_1x_2 + x_1x_3 + x_2x_3$, and $e_3 = x_1x_2x_3$.*

So, we have as a basis for $\text{Sym}[x_1, x_2, x_3]$ polynomials of the form $e_1^\alpha e_2^\beta e_3^\gamma$, where $\alpha, \beta, \gamma \in \mathbb{N}_0$. Also, every such symmetric polynomial has degree $\alpha + 2\beta + 3\gamma$, since e_1 is of degree 1, e_2 is of degree 2, and e_3 is of degree 3. Our next corollary follows, which we use freely throughout this section.

Corollary 5.3.

$$\text{Sym}^{\text{even}}[x_1, x_2, x_3] = \langle e_1^\alpha e_2^\beta e_3^\gamma \mid \alpha + \gamma \equiv 0 \pmod{2} \rangle$$

and

$$\text{Sym}^{\text{odd}}[x_1, x_2, x_3] = \langle e_1^\alpha e_2^\beta e_3^\gamma \mid \alpha + \gamma \equiv 1 \pmod{2} \rangle.$$

Now we will describe the differentials. Let $A: \mathbb{Q}[x_1, x_2, x_3] \rightarrow \mathbb{Q}[x_1, x_2, x_3]$ be the automorphism of $\mathbb{Q}[x_1, x_2, x_3]$ sending each generator x_i to $-x_i$, $i = 1 \dots 3$. Geometrically this automorphism corresponds to the symmetry with respect to the vertical line that our hairy Θ -graphs have. Notice that the projection to the space of even, respectively odd, monomials is described by $f \mapsto \frac{1}{2}(f + A(f))$, respectively $f \mapsto \frac{1}{2}(f - A(f))$. Sometimes the even, respectively odd, part of f will also be denoted by $[f]_{\text{even}}$, respectively $[f]_{\text{odd}}$.

Lemma 5.4. *In the case both m and N are odd, the differentials of the hairy two-loop graph-complex are defined as follows*

$$\begin{aligned} \partial_2 f &= -2e_1 f, \\ \partial_1 f &= -\frac{1}{2}(e_1 f + A(e_1 f)) = -[e_1 f]_{\text{even}}. \end{aligned}$$

Proof. The coefficient 2 in the definition of ∂_2 is due to the fact that there are two 4-valent vertices to expand, which produce the same result due to the symmetry. Such an expansion produces a new hair on one of the three edges of Θ which is algebraically expressed as a multiplication by $e_1 = x_1 + x_2 + x_3$. The sign minus¹ in both cases is due to the rule of signs for the differential, see Section 2. See also Remark 5.12 where general formulas for the differentials in all the four cases are given. \square

We now wish to compute homology. Notice

$$H_i := \frac{\ker(\partial_i)}{\text{im}(\partial_{i+1})}.$$

¹These signs as well as the coefficient 2 are not important for the homology computations. For this reason we do not show how these signs are calculated.

Since $\partial_0 : C_0 \rightarrow 0$,

$$\ker(\partial_0) = \langle e_1^\alpha e_2^\beta e_3^\gamma \mid \alpha + \gamma \equiv 0 \pmod{2}, \alpha + \beta + \gamma > 0 \rangle.$$

Secondly, we have

$$\text{im}(\partial_1) = \langle e_1^{\alpha+1} e_2^\beta e_3^\gamma \mid \alpha + \gamma \equiv 1 \pmod{2} \rangle.$$

So, we have

$$\begin{aligned} H_0 &= \frac{\ker(\partial_0)}{\text{im}(\partial_1)} \\ &= \frac{\langle e_1^\alpha e_2^\beta e_3^\gamma \mid \alpha + \gamma \equiv 0 \pmod{2}, \alpha + \beta + \gamma > 0 \rangle}{\langle e_1^{\alpha+1} e_2^\beta e_3^\gamma \mid \alpha + \gamma \equiv 1 \pmod{2} \rangle} \\ &= \langle e_2^\beta e_3^\gamma \mid \gamma \equiv 0 \pmod{2}, \beta + \gamma > 0 \rangle. \end{aligned}$$

We will now compute H_1 . We have

$$\ker(\partial_1) = \langle e_1^\alpha e_2^\beta e_3^\gamma \mid \alpha + \gamma \equiv 0 \pmod{2} \rangle.$$

In other words this kernel consists of symmetric even polynomials. Similarly we have

$$\text{im}(\partial_2) = \langle e_1^{\alpha+1} e_2^\beta e_3^\gamma \mid \alpha + \gamma \equiv 1 \pmod{2} \rangle.$$

Thus,

$$\begin{aligned} H_1 &= \frac{\langle e_1^\alpha e_2^\beta e_3^\gamma \mid \alpha + \gamma \equiv 0 \pmod{2} \rangle}{\langle e_1^{\alpha+1} e_2^\beta e_3^\gamma \mid \alpha + \gamma \equiv 1 \pmod{2} \rangle} \\ &= \langle e_2^\beta e_3^\gamma \mid \gamma \equiv 0 \pmod{2} \rangle. \end{aligned}$$

We see that $H_0 = \mathbb{Q}_{>0}[e_2, e_3^2]$ and $H_1 = \mathbb{Q}[e_2, e_3^2]$. As a consequence we get the following result.

Theorem 5.5. *In case both m and N are odd, the generating functions of the dimensions of the two-loop homology H_0 and H_1 , and of the Euler characteristics, are*

$$\begin{aligned} h_0(t) &= \frac{1}{(1-t^2)(1-t^6)} - 1, \\ h_1(t) &= \frac{t}{(1-t^2)(1-t^6)}, \\ \chi(t) &= \frac{1}{(1+t)(1-t^6)} - 1. \end{aligned}$$

Although the underlying spaces of H_0 and H_1 are the same, we add the factor t in $h_1(t)$ because of the additional hair growing from the left vertex of the Θ -graph. The formula for $\chi(t)$ was obtained using Lemma 4.1.

5.2. When both N and m are even. This is the second case where m and N have the same parity. We consider the same space of polynomials $\mathbb{Q}[x_1, x_2, x_3]$ as in our previous case. However, we obtain a negative sign when we transpose two edges, see Lemma 4.2 (3). Thus, we can no longer use symmetric polynomials to describe our space; instead, we turn to antisymmetric polynomials. An antisymmetric polynomial is defined as a polynomial on n variables such that

any transposition of variables results in a negative sign. The two-loop hairy graph-complex has now the following form:

$$0 \xrightarrow{\partial_3} \text{ASym}^{\text{odd}}[x_1, x_2, x_3] \xrightarrow{\partial_2} \text{ASym}[x_1, x_2, x_3] \xrightarrow{\partial_1} \text{ASym}^{\text{even}}[x_1, x_2, x_3] \xrightarrow{\partial_0} 0$$

Lemma 5.6. *In the case both m and N are even the differentials in the two-loop hairy graph-complex are defined as follows*

$$\begin{aligned} \partial_2 f &= 2e_1 f, \\ \partial_1 f &= \frac{1}{2}(e_1 f + A(e_1 f)) = [e_1 f]_{\text{even}}. \end{aligned}$$

See Remark 5.12 for a general formula. One can clearly see that $\partial^2 = 0$. The following result is well known.

Theorem 5.7. *The space $\text{ASym}[x_1, x_2, x_3]$ of antisymmetric polynomials is a free module over the algebra $\text{Sym}[x_1, x_2, x_3]$ of symmetric polynomials generated by the only element*

$$\Delta = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3).$$

The terms \mathcal{C}_i of our complex are generated by the bases:

$$\begin{aligned} \mathcal{C}_2 &= \langle \Delta e_1^\alpha e_2^\beta e_3^\gamma \mid \alpha + \gamma \equiv 0 \pmod{2} \rangle; \\ \mathcal{C}_1 &= \langle \Delta e_1^\alpha e_2^\beta e_3^\gamma \rangle; \\ \mathcal{C}_0 &= \langle \Delta e_1^\alpha e_2^\beta e_3^\gamma \mid \alpha + \gamma \equiv 1 \pmod{2} \rangle. \end{aligned}$$

We then compute H_0 and H_1 :

$$\begin{aligned} H_0 &= \frac{\ker(\partial_0)}{\text{im}(\partial_1)} \\ &= \frac{\langle \Delta e_1^\alpha e_2^\beta e_3^\gamma \mid \alpha + \gamma \equiv 1 \pmod{2} \rangle}{\langle \Delta e_1^{\alpha+1} e_2^\beta e_3^\gamma \mid \alpha + \gamma \equiv 0 \pmod{2} \rangle} \\ &= \langle \Delta e_2^\beta e_3^\gamma \mid \gamma \equiv 1 \pmod{2} \rangle \end{aligned}$$

Following a similar process for H_1 :

$$\begin{aligned} H_1 &= \frac{\ker(\partial_1)}{\text{im}(\partial_2)} \\ &= \frac{\langle \Delta e_1^\alpha e_2^\beta e_3^\gamma \mid \alpha + \gamma \equiv 1 \pmod{2} \rangle}{\langle \Delta e_1^{\alpha+1} e_2^\beta e_3^\gamma \mid \alpha + \gamma \equiv 0 \pmod{2} \rangle} \\ &= \langle \Delta e_2^\beta e_3^\gamma \mid \gamma \equiv 1 \pmod{2} \rangle \end{aligned}$$

Thus we immediately obtain the following

Theorem 5.8. *In case both m and N are even, the generating functions of the dimensions of the two-loop homology H_0 and H_1 , and of the Euler characteristics, are*

$$\begin{aligned} h_0(t) &= \frac{t^6}{(1-t^2)(1-t^6)}, \\ h_1(t) &= \frac{t^7}{(1-t^2)(1-t^6)}, \\ \chi(t) &= -\frac{t^6}{(1+t)(1-t^6)}. \end{aligned}$$

Proof. Notice that both H_0 and H_1 are free modules over $\mathbb{Q}[e_2, e_3^2]$ generated by Δe_3 . The degree of Δe_3 is 6 and the graded dimension of $\mathbb{Q}[e_2, e_3^2]$ is $\frac{1}{(1-t^2)(1-t^6)}$. The result follows. Again to compute $\chi(t)$ we used Lemma 4.1. \square

5.3. When N is odd and m is even. When m and N are of different parities, Lemma 4.2 (3) tells us that the generators x_i should be odd to make the action of S_3 well defined. To emphasize that they are odd we will denote them by ξ_i , $i = 1 \dots 3$ instead. Let $\mathbb{Q}\langle \xi_1, \xi_2, \xi_3 \rangle$ be a super graded algebra generated by odd ξ_i 's, $i = 1 \dots 3$, that anti-commute with each other, but not with themselves. To be precise we require $\xi_i \xi_j = -\xi_j \xi_i$ when $i \neq j$, but not $\xi_i^2 = 0$. Notice that this algebra is neither commutative nor supercommutative. The group S_3 acts on it by renumbering the variables. We will denote by $\text{Sym}\langle \xi_1, \xi_2, \xi_3 \rangle$ its S_3 -invariant part. Notice however that a symmetrization of a monomial $\xi_1^{k_1} \xi_2^{k_2} \xi_3^{k_3}$ can be zero if for example $k_1 = k_2$ and they are odd. Again this algebra is neither commutative nor supercommutative. It is not hard to find its set of generators and relations, but we omit it here since it unfortunately does not help with our homology computations. We will still denote by $e_1 := \xi_1 + \xi_2 + \xi_3$.

To describe the symmetry with respect to the vertical line and applying the signs from Lemma 4.2 (1-2), we define a linear map $A: \mathbb{Q}\langle \xi_1, \xi_2, \xi_3 \rangle \rightarrow \mathbb{Q}\langle \xi_1, \xi_2, \xi_3 \rangle$ by

$$A(\xi_1^{k_1} \xi_2^{k_2} \xi_3^{k_3}) = (-1)^{\sum_{i=1}^3 \frac{k_i(k_i+1)}{2}} \xi_1^{k_1} \xi_2^{k_2} \xi_3^{k_3}.$$

Given a super graded algebra \mathcal{A} one can define its opposite algebra \mathcal{A}^{op} by taking the same space and defining the new product \star as follows

$$a \star b := (-1)^{|a||b|} ba.$$

A homomorphism $\mathcal{A} \rightarrow \mathcal{B}^{op}$ is called an antihomomorphism, and an isomorphism $\mathcal{A} \rightarrow \mathcal{A}^{op}$ is called an antiautomorphism. The following is easy to check.

Lemma 5.9. *The linear map $A: \mathbb{Q}\langle \xi_1, \xi_2, \xi_3 \rangle \rightarrow \mathbb{Q}\langle \xi_1, \xi_2, \xi_3 \rangle$ can also be described as the antiautomorphism of $\mathbb{Q}\langle \xi_1, \xi_2, \xi_3 \rangle$ that sends the generators ξ_i to $-\xi_i$, $i = 1 \dots 3$.*

Since $A^2 = \text{id}$ this antiautomorphism is an *anti-involution*.

Definition 5.10. *We say that an element $f \in \mathbb{Q}\langle \xi_1, \xi_2, \xi_3 \rangle$ is **even* (**odd*, respectively) if $A(f) = f$ ($A(f) = -f$, respectively).*

With this, we may describe our graph complex for this case:

$$0 \xrightarrow{\partial_3} \text{Sym}^{*\text{even}}\langle \xi_1, \xi_2, \xi_3 \rangle \xrightarrow{\partial_2} \text{Sym}\langle \xi_1, \xi_2, \xi_3 \rangle \xrightarrow{\partial_1} \text{Sym}_{>0}^{*\text{even}}\langle \xi_1, \xi_2, \xi_3 \rangle \xrightarrow{\partial_0} 0$$

The superscript **even* says that we take only subspace of **even* polynomials; the subscript > 0 means that we take polynomials without constant term.

Lemma 5.11. *For the case m even and N odd, the differentials in the two-loop hairy complex are described as follows:*

$$\begin{aligned}\partial_2 f &= -(-1)^{|f|} 2f e_1, \\ \partial_1 f &= \frac{1}{2}(e_1 f + A(e_1 f)) = \frac{1}{2}(e_1 f - (-1)^{|f|} A(f) e_1) = [e_1 f]_{*even}.\end{aligned}$$

Remark 5.12. *In all the four cases the differentials in the two loop hairy graph-complex can be described as follows*

$$\begin{aligned}\partial_2 f &= (-1)^{N+|f|} 2f e_1; \\ \partial_1 f &= (-1)^{mN} \frac{1}{2}(e_1 f + A(e_1 f)).\end{aligned}$$

In the above $|f|$ refers to the supergrading of f . In the case m and N of the same parity, $|f|$ is always even.

The signs $(-1)^N$ and $(-1)^{mN}$ in the aforementioned formulas are not important for the homology computations for which reason we don't show how they were calculated.

The following lemma proves that $\partial^2 = 0$.

Lemma 5.13. *Let $\omega = \alpha\xi_1 + \beta\xi_2 + \gamma\xi_3$ be a linear element of $\mathbb{Q}\langle\xi_1, \xi_2, \xi_3\rangle$. If $f \in \mathbb{Q}\langle\xi_1, \xi_2, \xi_3\rangle$ is **even*, respectively **odd*, then $\omega f \omega$ is **odd*, respectively **even*.*

Proof. Using the fact that A is an antiautomorphism and that $A(\omega) = -\omega$, we get

$$A(\omega f \omega) = (-1)^{|f|+1} A(f \omega) A(\omega) = (-1)^{|f|+1} (-1)^{|f|} A(\omega) A(f) A(\omega) = -\omega A(f) \omega.$$

The result follows. □

Unfortunately the property of being **even* or **odd* is not preserved by a product. Thus in particular $\text{Sym}^{*even}\langle\xi_1, \xi_2, \xi_3\rangle$ is not an algebra. This makes our computations more difficult for the case when $m + N$ is odd. In order to compute the homology we will introduce a good basis in our spaces \mathcal{C}_i , $i = 0 \dots 2$. First denote by (k_1, k_2, k_3) the symmetrization of $\xi_1^{k_1} \xi_2^{k_2} \xi_3^{k_3}$ that has coefficient 1 in front of $\xi_1^{k_1} \xi_2^{k_2} \xi_3^{k_3}$. For example,

$$(1, 0, 0) = \xi_1 + \xi_2 + \xi_3; \quad (2, 2, 2) = \xi_1^2 \xi_2^2 \xi_3^2.$$

The elements (k_1, k_2, k_3) satisfying $k_1 \geq k_2 \geq k_3$ and if $k_i = k_{i+1}$, then k_i is even, $i = 1, 2$, form a basis of $\text{Sym}\langle\xi_1, \xi_2, \xi_3\rangle$ that will be denoted by \mathcal{B} . This set comes with a natural order – first one compares the degree $k_1 + k_2 + k_3$, then within the same degree one orders the elements lexicographically. Since the above elements form a basis, any symmetric polynomial f is a unique linear combination of them. The maximal element of \mathcal{B} that has a non-zero coefficient in the above sum, will be called the leading term of f . The subset of **even*, respectively **odd*, elements in \mathcal{B} will be denoted by \mathcal{B}^{*even} , respectively \mathcal{B}^{*odd} . For a basis of \mathcal{C}_2 we will simply choose the set

$\mathcal{B}^{\text{even}}$. For a basis of \mathcal{C}_1 we will take a union of the following sets

$$\begin{aligned} & \{fe_1 \mid f \in \mathcal{B}\}, \\ & \{(k_2 + 1, k_2, k_3) \mid k_2 \equiv 1 \pmod{2}, k_2 > k_3\}, \\ & \{(k_2, k_2, k_3) \mid k_2 \equiv 0 \pmod{2}, k_2 \geq k_3\}. \end{aligned}$$

The above union of sets forms a basis of $\mathcal{C}_1 = \text{Sym}\langle \xi_1, \xi_2, \xi_3 \rangle$ since every element of \mathcal{B} appears exactly once as a leading term of one of the elements in the union.

Since ∂_2 up to a sign is a multiplication by $2e_1$ on the right, we have

$$\text{im}(\partial_2) = \langle fe_1 \mid f \in \mathcal{B}^{\text{even}} \rangle.$$

Thus $\text{im}(\partial_2)$ is a space spanned by a subset of basis elements.

Now, we compute ∂_1 on the basis elements. Applying Lemmas 5.11, 5.13,

$$\begin{aligned} \partial_1(fe_1) &= 0, \text{ for } f \in \mathcal{B}^{\text{even}}; \\ \partial_1(fe_1) &= e_1fe_1, \text{ for } f \in \mathcal{B}^{\text{odd}}. \end{aligned}$$

Notice that the second line produces elements whose leading terms form the subset of $\mathcal{B}^{\text{even}}$ of elements (k_1, k_2, k_3) such that $k_1 > k_2 + 2$, or $k_1 = k_2 + 2$ and k_2 is even.

We continue our computation of ∂_1 on the basis elements:

X	$\partial_1 X$	
$(k_2 + 1, k_2, k_3)$	$2(k_2 + 1, k_2 + 1, k_3)$	k_2 odd, $k_3 \equiv 0 \pmod{4}$, $k_2 > k_3$
$(k_2 + 1, k_2, k_3)$	$(k_2 + 2, k_2, k_3)$	k_2 odd, $k_3 \equiv 1 \pmod{4}$, $k_2 > k_3$
$(k_2 + 1, k_2, k_3)$	$(k_2 + 2, k_2, k_3) - (k_2 + 1, k_2, k_3 + 1)$	k_2 odd, $k_3 \equiv 2 \pmod{4}$, $k_2 > k_3$
$(k_2 + 1, k_2, k_3)$	$2(k_2 + 1, k_2 + 1, k_3) - (k_2 + 1, k_2, k_3 + 1)$	k_2 odd, $k_3 \equiv 3 \pmod{4}$, $k_2 > k_3$
(k_2, k_2, k_3)	0	k_2 even, $k_3 \equiv 0 \pmod{4}$, $k_2 \geq k_3$
(k_2, k_2, k_3)	$(k_2 + 1, k_2, k_3)$	k_2 even, $k_3 \equiv 1 \pmod{4}$, $k_2 > k_3$
(k_2, k_2, k_3)	$(k_2 + 1, k_2, k_3)$	k_2 even, $k_3 \equiv 2 \pmod{4}$, $k_2 \geq k_3$
(k_2, k_2, k_3)	$(k_2, k_2, k_3 + 1)$	k_2 even, $k_3 \equiv 3 \pmod{4}$, $k_2 > k_3 + 1$
$(k_3 + 1, k_3 + 1, k_3)$	$3(k_3 + 1, k_3 + 1, k_3 + 1)$	$k_3 \equiv 3 \pmod{4}$

It is easy to see that the images from the first line are twice the images of the next to the last one. If one excludes the elements $\{(k_2 + 1, k_2, k_3) \mid k_2 \equiv 1 \pmod{2}, k_3 \equiv 0 \pmod{4}, k_2 > k_3\}$ from the basis of \mathcal{C}_1 (i.e. elements that appear on the first line above), then all the non-zero images of such set have different leading terms. Thus these images are linearly independent. To complete the obtained set to a basis of \mathcal{C}_0 we add elements

$$\{(k_2 + 1, k_2, k_3) \mid k_2 \equiv 1 \pmod{2}, k_3 \equiv 0, 3 \pmod{4}, k_2 > k_3\} \cup \{(k_3 + 1, k_3 + 1, k_3) \mid k_3 \equiv 3 \pmod{4}\}.$$

This immediately implies that the above set is a basis of cycles in H_0 . In other words,

$$(5.1) \quad H_0 = \frac{\ker(\partial_0)}{\text{im}(\partial_1)} = \langle (k_2 + 1, k_2, k_3) \mid k_2 \equiv 1 \pmod{2}, k_3 \equiv 0, 3 \pmod{4}, k_2 > k_3 \rangle \oplus \langle (k_3 + 1, k_3 + 1, k_3) \mid k_3 \equiv 3 \pmod{4} \rangle.$$

We will now compute H_1 . It is also easy to see from above that

$$\begin{aligned} \ker(\partial_1) = \text{im}(\partial_2) \oplus \langle (k_2, k_2, k_3) \mid k_2 \equiv 0 \pmod{2}, k_3 \equiv 0 \pmod{4}, k_2 \geq k_3 \rangle \oplus \\ \langle 2(k_2 + 1, k_2 + 1, k_3) - (k_2 + 1, k_2, k_3 + 1) \mid k_2 \equiv 1 \pmod{2}, k_3 \equiv 3 \pmod{4}, k_2 > k_3 \rangle. \end{aligned}$$

Thus we get

$$(5.2) \quad H_1 = \langle (k_2, k_2, k_3) \mid k_2 \equiv 0 \pmod{2}, k_3 \equiv 0 \pmod{4}, k_2 \geq k_3 \rangle \oplus \\ \langle (k_2 + 1, k_2 + 1, k_3) - (k_2 + 1, k_2, k_3 + 1) \mid k_2 \equiv 1 \pmod{2}, k_3 \equiv 3 \pmod{4}, k_2 > k_3 \rangle.$$

Theorem 5.14. *In case m is even and N is odd, the generating functions of the dimensions of the two-loop graph-homology H_0 and H_1 , and of the Euler characteristics, are*

$$\begin{aligned} h_0(t) &= \frac{t^3 + t^{11} + t^{14} - t^{15}}{(1 - t^4)(1 - t^{12})}, \\ h_1(t) &= \frac{t + t^{16}}{(1 - t^4)(1 - t^{12})}, \\ \chi(t) &= \frac{t - t^{11} - t^{13} + t^{14}}{(1 + t^2)(1 - t^{12})}. \end{aligned}$$

Proof. We start with h_0 . The first summand of (5.1) has the generating functions of dimensions

$$\frac{t^3 + t^{14}}{(1 - t^4)(1 - t^{12})}.$$

In this formula t^3 and t^{14} in the numerator correspond to $(2, 1, 0)$ and $(6, 5, 3)$ respectively. Then we can simultaneously increase k_2 and k_3 by 4, or just increase k_2 by 2. The first action gives the factor

$$1 + t^{12} + t^{24} + t^{36} + \dots = \frac{1}{1 - t^{12}}.$$

to our generating function. The second action gives the factor

$$1 + t^4 + t^8 + \dots = \frac{1}{1 - t^4}.$$

Similarly the second summand of (5.1) has the generating function of dimensions

$$\frac{t^{11}}{1 - t^{12}},$$

where t^{11} corresponds to $(4, 4, 3)$.

For $h_1(t)$, the first summand of (5.2) has the generating function of dimensions

$$\frac{t}{(1 - t^4)(1 - t^{12})}.$$

Here the numerator t corresponds to $(0, 0, 0)$. Notice that the corresponding graph has exactly 1 hair growing from the left vertex of the Θ graph. For the second sum we get the function

$$\frac{t^{16}}{(1 - t^4)(1 - t^{12})}.$$

Here t^{16} states for the element $2(6, 6, 3) - (6, 5, 4)$.

To compute $\chi(t)$, again we use Lemma 4.1. □

5.4. **When N is even and m is odd.** This case is similar to the one considered in the previous subsection. From Lemma 4.2 one can see that our graph-complex in this situation can be described as follows:

$$0 \xrightarrow{\partial_3} \text{ASym}^{*\text{even}}\langle \xi_1, \xi_2, \xi_3 \rangle \xrightarrow{\partial_2} \text{ASym}\langle \xi_1, \xi_2, \xi_3 \rangle \xrightarrow{\partial_1} \text{ASym}^{*\text{even}}\langle \xi_1, \xi_2, \xi_3 \rangle \xrightarrow{\partial_0} 0.$$

In the above ASym denotes the space of anti-symmetric polynomials in $\mathbb{Q}\langle \xi_1, \xi_2, \xi_3 \rangle$. The superscript **even* indicates that we only take the subspace of **even* ones, see Definition 5.10. The subscript > 0 indicates that the generating monomials should be of strictly positive degree. The space of anti-symmetric polynomials $\text{ASym}\langle \xi_1, \xi_2, \xi_3 \rangle$ is a bimodule over the algebra $\text{Sym}\langle \xi_1, \xi_2, \xi_3 \rangle$. It is generated by

$$\Delta_2 = \xi_1\xi_2 + \xi_2\xi_3 + \xi_3\xi_1,$$

and

$$\Delta_3 = \xi_1\xi_2\xi_3$$

as a left (or right) module (and by Δ_2 as a bimodule). But contrary to the classical case this action is not free.

Lemma 5.15. *For the case m even and N odd, the differentials in the two-loop hairy complex are described as follows:*

$$\begin{aligned} \partial_2 f &= (-1)^{|f|} 2f e_1, \\ \partial_1 f &= \frac{1}{2}(e_1 f + A(e_1 f)) = \frac{1}{2}(e_1 f - (-1)^{|f|} A(f) e_1) = [e_1 f]_{*\text{even}}. \end{aligned}$$

See Remark 5.12 for a general rule of signs.

Now that we have fully described our complex, we may compute H_1 and H_0 . We do this in a similar manner to the previous case: we construct a basis for each space by first defining the image of our map ∂ , then adding in other elements in our space in a way that each possible leading term appears exactly once. For a monomial $\xi_1^{k_1} \xi_2^{k_2} \xi_3^{k_3}$ its anti-symmetrization with coefficient one in front of $\xi_1^{k_1} \xi_2^{k_2} \xi_3^{k_3}$ will be denoted $[k_1, k_2, k_3]$. For example $\Delta_2 = [1, 1, 0]$, $\Delta_3 = [1, 1, 1]$. The set

$$\mathcal{B} = \{[k_1, k_2, k_3] \mid k_1 \geq k_2 \geq k_3, \text{ if } k_i = k_{i+1}, \text{ then } k_i \text{ is odd, } i = 1, 2\}$$

is a basis of $\text{ASym}\langle \xi_1, \xi_2, \xi_3 \rangle$. Its subset consisting of **even* monomials will be denoted by $\mathcal{B}^{*\text{even}}$. The latter set will be used as a basis for \mathcal{C}_2 . For a basis for \mathcal{C}_1 we will take a union of the following sets

$$\begin{aligned} &\{f e_1 \mid f \in \mathcal{B}\}, \\ &\{[k_2 + 1, k_2, k_3] \mid k_2 \equiv 0 \pmod{2}, k_2 > k_3\}, \\ &\{[k_2, k_2, k_3] \mid k_2 \equiv 1 \pmod{2}, k_2 \geq k_3\}. \end{aligned}$$

Since ∂_2 is up to a sign multiplication by $2e_1$ on the right, we have

$$\text{im}(\partial_2) = \langle f e_1 \mid f \in \mathcal{B}^{*\text{even}} \rangle.$$

Thus, $\text{im}(\partial_2)$ is a space spanned by a subset of basis elements.

Now, we compute ∂_1 on the basis elements. Applying Lemma 5.15,

$$\begin{aligned} \partial_1(f e_1) &= 0 \text{ for } f \in \mathcal{B}^{*\text{even}}; \\ \partial_1(f e_1) &= e_1 f e_1 \text{ for } f \in \mathcal{B}^{*\text{odd}}. \end{aligned}$$

We continue our computation of ∂_1 on the basis elements:

X	$\partial_1(X)$	
$[k_2 + 1, k_2, k_3]$	$-2[k_2 + 1, k_2 + 1, k_3] - [k_2 + 1, k_2, k_3 + 1]$	k_2 even, $k_3 \equiv 0 \pmod{4}$, $k_2 > k_3$
$[k_2 + 1, k_2, k_3]$	$[k_2 + 2, k_2, k_3] - [k_2 + 1, k_2, k_3 + 1]$	k_2 even, $k_3 \equiv 1 \pmod{4}$, $k_2 > k_3$
$[k_2 + 1, k_2, k_3]$	$[k_2 + 2, k_2, k_3]$	k_2 even, $k_3 \equiv 2 \pmod{4}$, $k_2 > k_3$
$[k_2 + 1, k_2, k_3]$	$2[k_2 + 1, k_2 + 1, k_3]$	k_2 even, $k_3 \equiv 3 \pmod{4}$, $k_2 > k_3$
$[k_2, k_2, k_3]$	$[k_2 + 1, k_2, k_3]$	k_2 odd, $k_3 \equiv 0 \pmod{4}$, $k_2 \geq k_3$
$[k_2, k_2, k_3]$	0	k_2 odd, $k_3 \equiv 1 \pmod{4}$, $k_2 \geq k_3$
$[k_2, k_2, k_3]$	$[k_2, k_2, k_3 + 1]$	k_2 odd, $k_3 \equiv 2 \pmod{4}$, $k_2 > k_3 + 1$
$[k_3 + 1, k_3 + 1, k_3]$	$3[k_3 + 1, k_3 + 1, k_3 + 1]$	$k_3 \equiv 2 \pmod{4}$
$[k_2, k_2, k_3]$	$[k_2 + 1, k_2, k_3] + [k_2, k_2, k_3 + 1]$	k_2 odd, $k_3 \equiv 3 \pmod{4}$, $k_2 > k_3$
$[k_3, k_3, k_3]$	$[k_3 + 1, k_3, k_3]$	$k_3 \equiv 3 \pmod{4}$

It is easy to see that the images from the fourth line are twice the images of the seventh line. If one excludes the elements $\{[k_2 + 1, k_2 + 1, k_3] \mid k_2 \equiv 0 \pmod{2}, k_3 \equiv 3 \pmod{4}\}$ from the basis of \mathcal{C}_1 (i.e. the elements that appear on the fourth line above), then all the non-zero images of such set have different leading terms. Thus these images are linearly independent. To complete the obtained set to a basis of \mathcal{C}_0 we add elements

$$\begin{aligned} & \{[k_2 + 1, k_2, k_3] \mid k_2 \equiv 0 \pmod{2}, k_3 \equiv 1 \pmod{4}, k_2 > k_3\} \cup \\ & \{[k_2 + 1, k_2, k_3] \mid k_2 \equiv 0 \pmod{2}, k_3 \equiv 2 \pmod{4}, k_2 > k_3 + 1\} \cup \{[k_3 + 1, k_3 + 1, k_3] \mid k_3 \equiv 0 \pmod{4}\}. \end{aligned}$$

This immediately implies the above set is a basis of cycles in H_0 . In other words,

$$(5.3) \quad H_0 = \frac{\ker(\partial_0)}{\text{im}(\partial_1)} = \langle [k_2 + 1, k_2, k_3] \mid k_2 \equiv 0 \pmod{2}, k_3 \equiv 1 \pmod{4}, k_2 > k_3 \rangle \oplus \langle [k_2 + 1, k_2, k_3] \mid k_2 \equiv 0 \pmod{2}, k_3 \equiv 2 \pmod{4}, k_2 > k_3 \rangle \oplus \langle [k_3 + 1, k_3 + 1, k_3] \mid k_3 \equiv 0 \pmod{4} \rangle.$$

We will now compute H_1 . It is easy to see from above that

$$\begin{aligned} \ker(\partial_1) = \text{im}(\partial_2) \oplus \langle 2[k_2 + 1, k_2 + 1, k_3] - [k_2 + 1, k_2, k_3 + 1] \mid k_2 \equiv 0 \pmod{2}, k_3 \equiv 2 \pmod{4}, k_2 > k_3 \rangle \oplus \\ \langle [k_2, k_2, k_3] \mid k_2 \equiv 1 \pmod{2}, k_3 \equiv 1 \pmod{4}, k_2 \geq k_3 \rangle. \end{aligned}$$

Thus we get

$$(5.4) \quad H_1 = \langle 2[k_2 + 1, k_2 + 1, k_3] - [k_2 + 1, k_2, k_3 + 1] \mid k_2 \equiv 0 \pmod{2}, k_3 \equiv 2 \pmod{4}, k_2 > k_3 \rangle \oplus \langle [k_2, k_2, k_3] \mid k_2 \equiv 1 \pmod{2}, k_3 \equiv 1 \pmod{4}, k_2 \geq k_3 \rangle.$$

Theorem 5.16. *In case m is odd and N is even, the generating functions of the two-loop graph-homology H_0 and H_1 , and of the Euler characteristics, are*

$$\begin{aligned} h_0(t) &= \frac{t^2 + t^{11}}{(1 - t^4)(1 - t^{12})}, \\ h_1(t) &= \frac{t^4 + t^{13}}{(1 - t^4)(1 - t^{12})}, \\ \chi(t) &= \frac{-t^2 + t^{11}}{(1 + t^2)(1 - t^{12})}. \end{aligned}$$

Proof. We start with h_0 . The first summand in (5.3) has the generating functions of dimensions

$$\frac{t^6}{(1-t^4)(1-t^{12})}.$$

In this formula t^6 in the numerator corresponds to $[3, 2, 1]$. The denominator is obtained exactly as described in the proof of Theorem 5.14. The second summand has the generating function of the dimensions

$$\frac{t^{11}}{(1-t^4)(1-t^{12})},$$

where t^{11} comes from $[5, 4, 2]$. The last summand of (5.3) has the generating function of dimensions

$$\frac{t^2}{1-t^{12}},$$

where t^2 corresponds to $[1, 1, 0]$.

For $h_1(t)$, the first summand of (5.4) has the generating function of dimensions

$$\frac{t^{13}}{(1-t^4)(1-t^{12})}.$$

Here the numerator t^{13} corresponds to $2[5, 5, 2] - [5, 4, 3]$. For the second summand we get the function

$$\frac{t^4}{(1-t^4)(1-t^{12})}.$$

Here t^4 stands for the element $[1, 1, 1]$.

To compute $\chi(t)$, again we use Lemma 4.1. □

6. CONCENTRATION OF HOMOLOGY

We will now examine the concentration of homology in all four cases.

We will first focus on the case where both N and m are odd. Recall that the formula for the graded dimension of H_0 is

$$h_0(t) = \frac{1}{(1-t^2)(1-t^6)} - 1$$

and the graded dimension for H_1 is

$$h_1(t) = \frac{t}{(1-t^2)(1-t^6)}.$$

Theorem 6.1. *Let a_k represent the sequence of coefficients generated by h_0 , and let b_k represent the sequence of coefficients generated by h_1 . When N and m are odd, we have*

$$a_k = \begin{cases} \lceil \frac{k+1}{6} \rceil & k \equiv 0 \pmod{2} \\ 0 & k \equiv 1 \pmod{2} \end{cases},$$

and

$$b_k = \begin{cases} 0 & k \equiv 0 \pmod{2} \\ \lceil \frac{k}{6} \rceil & k \equiv 1 \pmod{2} \end{cases}.$$

Proof. Follows from the fact that the expansion of $\frac{1}{(1-t^2)(1-t^6)}$ has coefficient zero in front of any odd exponent t^k and coefficient $\lceil \frac{k+1}{6} \rceil$ in front of any even exponent. \square

The next case we examine is when both N and m are even. We have

$$h_0(t) = \frac{t^6}{(1-t^2)(1-t^6)},$$

$$h_1(t) = \frac{t^7}{(1-t^2)(1-t^6)}.$$

Theorem 6.2. *As before, let a_k represent the sequence of coefficients generated by h_0 , and let b_k represent the sequence of coefficients generated by h_1 . Then, when N and m are even,*

$$a_k = \begin{cases} \lfloor \frac{k}{6} \rfloor & k \equiv 0 \pmod{2} \\ 0 & k \equiv 1 \pmod{2} \end{cases}$$

$$b_k = \begin{cases} 0 & k \equiv 0 \pmod{2} \\ \lfloor \frac{k}{6} \rfloor & k \equiv 1 \pmod{2} \end{cases}$$

Proof. Similar to the previous theorem. \square

We now turn to the cases where $N+m$ is odd, beginning with N odd and m even. Our formulas for graded dimension are

$$h_0(t) = \frac{t^3 + t^{11} + t^{14} - t^{15}}{(1-t^4)(1-t^{12})},$$

and the graded dimension for H_1 is

$$h_1(t) = \frac{t + t^{16}}{(1-t^4)(1-t^{12})}.$$

Theorem 6.3. *When N is odd and m is even, using the same notation as above, we have*

$$a_k = \begin{cases} 0 & k \equiv 1, 0 \pmod{4} \\ \lfloor \frac{k}{12} \rfloor & k \equiv 2 \pmod{4} \\ \lceil \frac{k+2}{12} \rceil & k \equiv 3 \pmod{4} \end{cases}$$

and

$$b_k = \begin{cases} 0 & k \equiv 2, 3 \pmod{4} \\ \lfloor \frac{k-1}{12} \rfloor & k \equiv 0 \pmod{4} \\ \lceil \frac{k}{12} \rceil & k \equiv 1 \pmod{4} \end{cases}$$

Proof. Follows from the fact that $\frac{1}{(1-t^4)(1-t^{12})}$ has coefficient zero in front of any exponent t^k with k non-divisible by 4, and coefficient $\lceil \frac{k+1}{12} \rceil$ for k divisible by 4. \square

Finally, when N is even and m is odd, recall that the graded dimension of H_0 is

$$h_0(t) = \frac{t^2 + t^{11}}{(1-t^4)(1-t^{12})},$$

and the graded dimension for H_1 is

$$h_1(t) = \frac{t^4 + t^{13}}{(1-t^4)(1-t^{12})}.$$

Again, we expand these series with respect to t to find a general formula to compute the coefficient for a given exponent.

Theorem 6.4. *When N is even and m is odd, we have*

$$a_k = \begin{cases} 0 & k \equiv 1, 0 \pmod{4} \\ \lfloor \frac{k+1}{12} \rfloor & k \equiv 3 \pmod{4} \\ \lceil \frac{k}{12} \rceil & k \equiv 2 \pmod{4} \end{cases},$$

where a_k represents the coefficient of H_0 , and

$$b_k = \begin{cases} 0 & k \equiv 2, 3 \pmod{4} \\ \lfloor \frac{k}{12} \rfloor & k \equiv 1 \pmod{4} \\ \lceil \frac{k}{12} \rceil & k \equiv 0 \pmod{4} \end{cases}.$$

where b_k represents the coefficient of H_1 .

Proof. Similar to the previous theorem. □

Remark 6.5. *Notice that, for a given Hodge grading, in all the four cases the homology is concentrated in a single degree.*

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TABLES

In this appendix section we present tables of ranks of the 2-loops graph-homology in small Hodge degrees. Notice that the results of our computations of Euler characteristics (the third column in each table) confirm previous computations — the diagonal $s = t + 1$ in [2, Tables 1, 3, 5, 7].

Hodge Degree	$h_0(t)$	$h_1(t)$	$\chi(t)$
1	0	1	-1
2	1	0	1
3	0	1	-1
4	1	0	1
5	0	1	-1
6	2	0	2
7	0	2	-2
8	2	0	2
9	0	2	-2
10	2	0	2
11	0	2	-2
12	3	0	3
13	0	3	-3
14	3	0	3
15	0	3	-3
16	3	0	3
17	0	3	-3
18	4	0	4
19	0	4	-4
20	4	0	4
21	0	4	-4
22	4	0	4
23	0	4	-4

TABLE 1. Concentration of homology when N and m are odd.

Hodge Degree	$h_0(t)$	$h_1(t)$	$\chi(t)$
1	0	0	0
2	0	0	0
3	0	0	0
4	0	0	0
5	0	0	0
6	1	0	-1
7	0	1	1
8	1	0	-1
9	0	1	1
10	1	0	-1
11	0	1	1
12	2	0	-2
13	0	2	2
14	2	0	-2
15	0	2	2
16	2	0	-2
17	0	2	2
18	3	0	-3
19	0	3	3
20	3	0	-3
21	0	3	3
22	3	0	-3
23	0	3	3

TABLE 2. Concentration of homology when N and m are even.

Hodge Degree	$h_0(t)$	$h_1(t)$	$\chi(t)$
1	0	1	1
2	0	0	0
3	1	0	-1
4	0	0	0
5	0	1	1
6	0	0	0
7	1	0	-1
8	0	0	0
9	0	1	1
10	0	0	0
11	2	0	-2
12	0	0	0
13	0	2	2
14	1	0	1
15	2	0	-2
16	0	1	-1
17	0	2	2
18	1	0	1
19	2	0	-2
20	0	1	-1
21	0	2	2
22	1	0	1
23	3	0	-3

TABLE 3. Concentration of homology when N is odd and m is even.

Hodge Degree	$h_0(t)$	$h_1(t)$	$\chi(t)$
1	0	0	0
2	1	0	-1
3	0	0	0
4	0	1	1
5	0	0	0
6	1	0	-1
7	0	0	0
8	0	1	1
9	0	0	0
10	1	0	-1
11	1	0	1
12	0	1	1
13	0	1	-1
14	2	0	-2
15	1	0	1
16	0	2	2
17	0	1	-1
18	2	0	-2
19	1	0	1
20	0	2	2
21	0	1	-1
22	2	0	-2
23	0	2	2

TABLE 4. Concentration of homology when N is even and m is odd.

UNIVERSITY OF TENNESSEE
E-mail address: jconant@math.utk.edu

UNIVERSITY OF MINNESOTA
E-mail address: coste085@umn.edu

KANSAS STATE UNIVERSITY
E-mail address: turchin@math.ksu.edu

UNIVERSITY OF CALIFORNIA, DAVIS
E-mail address: psweed@math.ucdavis.edu