PROJECTIVE CENTER POINT AND TVERBERG THEOREMS

ROMAN KARASEV* AND BENJAMIN MATSCHKE**

ABSTRACT. We present projective versions of the center point theorem and Tverberg's theorem, interpolating between the original and the so-called "dual" center point and Tverberg theorems.

Furthermore we give a common generalization of these and many other known (transversal, constraint, dual, and colorful) Tverberg type results in a single theorem, as well as some essentially new results about partitioning measures in projective space.

1. INTRODUCTION

In this paper we focus on two classical topics in discrete geometry: the center point theorem from Neumann [19] and Rado [21] (see also Grünbaum [10]) and Tverberg's theorem [23]. Many deep generalizations of these classical results have been made in the last three decades, starting from the topological generalization by Bárány, Shlosman, and Szűcs [1]. A good review on this topic and numerous references are given in Matoušek's book [18]. After this book was published, new achievements were made by Hell [11, 12], Engström [8] and Engström–Norén [9], K. [15], and Blagojević–M.–Ziegler [3, 4], establishing "constrained", "dual", and "optimal colorful" Tverberg type theorems.

The discrete center point theorem states the following: For any finite set $X \subset \mathbb{R}^d$ there exists a *center point* $c \in \mathbb{R}^d$ such that any closed half-space $H \ni c$ contains at least $\left\lceil \frac{|X|}{d+1} \right\rceil$ points of X. In K. [15] a dual center point theorem and a dual Tverberg theorem for families of hyperplanes were proved. The dual center point theorem states: For any family of n hyperplanes in general position in $\mathbb{R}^d$ there exists a point c such that any ray starting at c intersects at least $\left\lceil \frac{n}{d+1} \right\rceil$ hyperplanes.

Here the use of the adjective "dual" is rather descriptive, it does not refer to projective duality. Thus it is interesting to dualize it once more projectively and compare it with the original center point theorem.

Definition 1.1. Any two distinct hyperplanes H_1 and H_2 partition $\mathbb{R}P^d$ into two pieces. In this paper, we always consider the pieces as being closed. If H_1 and H_2 coincide then we consider $H_1 = H_2$ as one piece and the whole $\mathbb{R}P^d$ as the other.

The projective dual of the "dual center point theorem" reads: Assume that X is a family of n points in $\mathbb{R}P^d$ and $c \in \mathbb{R}P^d$ is another point such that the family $X \cup c$ is in general position. Then there exists a hyperplane $W \subseteq \mathbb{R}P^d$ such that together with any hyperplane $H_1 \ni c$ it partitions $\mathbb{R}P^d$ into two parts each containing at least $\left\lceil \frac{n}{d+1} \right\rceil$ points of X. From the proof of this theorem we can assure that W does not contain c;

however if we omit the general position assumption then the theorem remains true by a compactness argument but W may happen to contain c.

Now we are going to interpolate between the original center point theorem and the latter "dual to dual" version (they appear as special cases when V is the hyperplane at infinity or when V is a point):

Theorem 1.2 (Projective center point theorem). *Suppose that $V \subset \mathbb{R}P^d$ is a projective subspace of dimension v and X is a finite point set with $|X| = n$. Put*

$$r = \left\lceil \frac{n}{(d - v)(v + 1) + 1} \right\rceil .$$

Then there exists a projective subspace $W \subset \mathbb{R}P^d$ of dimension $d - v - 1$ such that any pair of hyperplanes $H_1 \supseteq V$ and $H_2 \supseteq W$ partitions $\mathbb{R}P^d$ into two parts each containing at least r points of X.

If we require the general position assumption, that no r points of X together with V are contained in a hyperplane, then W may be chosen disjoint from V.

The ordinary center point theorem [19, 21] is usually stated for measures, which follows from the discrete version by an approximation argument. Here is the corresponding version:

Theorem 1.3 (Projective center point theorem for measures). *Suppose that $V \subset \mathbb{R}P^d$ is a projective subspace of dimension v and μ is a probability measure on $\mathbb{R}P^d$. Then there exists a projective subspace $W \subset \mathbb{R}P^d$ of dimension $d - v - 1$ such that any pair of hyperplanes $H_1 \supseteq V$ and $H_2 \supseteq W$ partitions $\mathbb{R}P^d$ into two parts P_1 and P_2 so that*

$$\mu(P_1), \mu(P_2) \geq \frac{1}{(d - v)(v + 1) + 1}.$$

Remark 1.4. If μ is absolutely continuous then the case $W \cap V \neq \emptyset$ is automatically excluded, because H_1 and H_2 must not coincide in this case.

The rest of the paper is organized as follows: We start with the proof of Theorem 1.2 in Section 2. In Section 3 we analogously interpolate between Tverberg's theorem and its dual. In Section 4 we state a very general theorem incorporating almost all that we know about (dual, transversal, constrained, colorful) Tverberg type theorems. In particular this implies a projective center transversal theorem, which generalizes Theorem 1.3, see Section 5. Exchanging quantors can possibly lead to other interesting theorems. As an instance of this we prove another projective Tverberg theorem and a transversal generalization in Sections 6 and 7.

Open problems. Some of our theorems need technical assumptions in order to assure that certain topological obstructions do not vanish and such that we are able to prove that. These include the usual prime power assumption in Tverberg type theorems, as well as more specific assumptions in the results of Sections 6 and 7. It would be very interesting to know in which cases these assumptions are necessary and in which cases they can be avoided.

Moreover, Sections 6 and 7 already indicate that interesting versions and generalizations of our projective center point and Tverberg theorems might be obtained by exchanging quantors and by including additional subspaces on top of V and W. Is there a reasonable natural generalization incorporating all results of this paper?

2. Proof of the projective center point theorem

In order to prove Theorem 1.2 we first dualize it projectively (and interchange v and $d - v - 1$): First note that dual to Definition 1.1, any two points $x, y \in \mathbb{R}P^d$ partition the space of hyperplanes in $\mathbb{R}P^d$ into two pieces, which are characterized by where a hyperplane intersects the (or a) line ℓ through x and y, since ℓ gets partitioned into two closed parts by x and y as well.

Theorem 2.1. *Suppose that $V \subset \mathbb{R}P^d$ is a projective subspace of dimension v and Ξ is a family of hyperplanes with $|\Xi| = n$. Put*

$$r = \left\lceil \frac{n}{(d-v)(v+1)+1} \right\rceil.$$

Then there exists a projective subspace $W \subset \mathbb{R}P^d$ of dimension $d - v - 1$ such that the following condition holds: For a pair of points $x \in V$ and $y \in W$ and any line $\ell \supseteq \{x, y\}$ the pair (x, y) partitions the line ℓ into two parts and any of these parts intersects at least r members of Ξ.

The general position assumption now becomes: No r hyperplanes of Ξ have a common point in V. In the following proof it will guarantee that V and W have no common point and the line ℓ is uniquely determined by the pair of points x and y.

Let us go to the homogeneous coordinate space $\mathbb{R}^{d+1}$ with the corresponding subspace $\widehat{V}$ of dimension $v+1$ and a family of hyperplanes $\widehat{\Xi}$ given by homogeneous linear equations $\lambda_i(x) = 0$ for $i = 1, \ldots, n$. We have to find an appropriate subspace $\widehat{W}$ of dimension $d-v$.

Consider the formal simplex Δ^{n-1} with vertices indexed by $\{1, \ldots, n\}$. A point $p \in \Delta^{n-1}$ is an n-tuple of nonnegative reals $p = (t_1, \ldots, t_n)$ that sum to 1. Choose an $\varepsilon > 0$ and consider the following quadratic form on $\mathbb{R}^{d+1}$:

$$q_{p,\varepsilon}(x) = \sum_{i=1}^{n} t_i \lambda_i(x)^2 + \varepsilon\langle x, x\rangle,$$

where $\langle x, x \rangle$ is the standard quadratic form on $\mathbb{R}^{d+1}$. We would prefer to put $\varepsilon = 0$ but have to consider positive ε in order to make sure that $q_{p,\varepsilon}(x)$ is positive definite for any $p \in \Delta^{n-1}$.

Let $\widehat{W}_{p,\varepsilon} \subset \mathbb{R}^{d+1}$ be the subspace that is orthogonal to $\widehat{V}$ with respect to the scalar product defined by $q_{p,\varepsilon}(x)$. Obviously $\widehat{W}_{p,\varepsilon}$ depends continuously on $p \in \Delta^{n-1}$ and $\varepsilon > 0$ and $\widehat{W}_{p,\varepsilon}$ always has trivial intersection with $\widehat{V}$. Note that the space of all possible linear subspaces $\widehat{W}$ of dimension $d - v$ having trivial intersection with $\widehat{V}$ is a Schubert cell in the Grassmannian $G(d + 1, d - v)$, which is homeomorphic to $\mathbb{R}^D$, $D = (d - v)(v + 1)$. Thus for a fixed $\varepsilon > 0$ we get a map $\Delta^{n-1} \to \mathbb{R}^D$ that sends p to $\widehat{W}_{p,\varepsilon}$.

Now we invoke the topological center point theorem from K. [16], which states that any continuous map from Δ^{n-1} to a metric space of covering dimension D, such as $\mathbb{R}^D$, has the property that the intersection of the images of all faces of codimension $\left\lceil \frac{n}{D+1} \right\rceil - 1$ in Δ^{n-1} is non-empty.

Thus in our situation we find a subspace $\widehat{W}_\varepsilon$ with the following property: For any face $F \in \Delta^{n-1}$ of dimension at least $n - r$ (i.e. spanned by $n - r + 1$ vertices of Δ^{n-1}) there exists $p_{F,\varepsilon} \in F$ such that $\widehat{W}_\varepsilon = \widehat{W}_{p_{F,\varepsilon},\varepsilon}$.

By a compactness argument we can choose a sequence $\varepsilon_m \to +0$ and corresponding points p_{F,ε_m} such that the respective $\widehat{W}_{\varepsilon_m}$ converge to some $\widehat{W}$ in the Grassmannian topology, the points p_{F,ε_m} tend to some points p_F, and so do the quadratic forms $q_{p_{F,\varepsilon_m},\varepsilon_m} \to q_F$.

Consider a pair of lines $\widehat{x} \subseteq \widehat{V}$ and $\widehat{y} \subseteq \widehat{W}$ (all lines and subspaces in $\mathbb{R}^{d+1}$ are linear and pass through the origin). Let $\widehat{\ell}$ be a two-dimensional plane containing them both. The lines $\widehat{x}$ and $\widehat{y}$ break $\widehat{\ell}$ projectively in two closed parts: P_1 and P_2. We have to show that any of them intersects at least r members of $\widehat{\Xi}$ nontrivially. Assume the contrary: a subfamily of $n - r + 1$ hyperplanes (denote it F) intersect P_1 only at the origin; this means that every corresponding λ_i is nonzero on $\widehat{\ell}$ and its zero set (minus the origin) is contained in the interior of P_2.

Consider now F as a face of Δ^{n-1} and take its corresponding p_F and q_F. The lines $\widehat{x}$ and $\widehat{y}$ must be orthogonal with respect to q_F because the relation "orthogonal" depends continuously on everything. The quadratic form q_F may be degenerate but it is nonzero on $\widehat{\ell}$, because all the corresponding hyperplanes intersect $\widehat{\ell}$ transversally. Consider two cases:

- Case 1. The lines $\widehat{x}$ and $\widehat{y}$ are different. Let them be the x and the y axes and let the interior of P_2 be defined by $xy < 0$. Then $q_F|_{\widehat{\ell}}$ is a convex combination of the forms $(a_i x + b_i y)^2$ with positive a_i and b_i, $i \in F$. The corresponding scalar product between $(1, 0)$ and $(0, 1)$ is therefore $\sum_{i \in F} t_i a_i b_i > 0$, which is a contradiction.
- Case 2. The lines $\widehat{x}$ and $\widehat{y}$ coincide. Since $\widehat{x} \perp_{q_F} \widehat{y}$, every λ_i whose square comes with a positive coefficient t_i in q_F must be zero on $\widehat{x} = \widehat{y}$, and thus the corresponding hyperplane in $\widehat{\Xi}$ intersects both parts $P_1 = \widehat{x} = \widehat{y}$ and $P_2 = \widehat{\ell}$ nontrivially.

It remains to verify that the general position assumption implies $\widehat{V} \cap \widehat{W} = 0$. Assuming the contrary, there are lines $\widehat{x} \subseteq \widehat{V}$ and $\widehat{y} \subseteq \widehat{W}$ with $\widehat{x} = \widehat{y}$. For any $(n-r)$-face F of Δ^{n-1} the above Case 2 implies that at least one hyperplane of $\widehat{\Xi}$ corresponding to a vertex of F contains $\widehat{x} = \widehat{y}$. Hence at least r hyperplanes of $\widehat{\Xi}$ contain $\widehat{x} \subseteq \widehat{V}$ and this is exactly what we exclude by the general position assumption. $\qquad \square$

3. A PROJECTIVE TVERBERG THEOREM

Instead of the topological center point theorem from [16] we could invoke the topological Tverberg theorem from [1, 20, 25] (see also [18]) to obtain the following:

Theorem 3.1. *Suppose that $V \subset \mathbb{R}P^d$ is a projective subspace of dimension v and X is a finite point set with $|X| = (D+1)(r-1)+1$, where $D = (d-v)(v+1)$ and r is a prime power.*

Then there exists a projective subspace $W \subset \mathbb{R}P^d$ of dimension $d-v-1$ and a partition of X into r subfamilies X_j $(j = 1, \ldots, r)$ such that any pair of hyperplanes $H_1 \supseteq V$ and $H_2 \supseteq W$ partitions $\mathbb{R}P^d$ into two parts so that each part has nonempty intersection with every X_j.

Remark 3.2. An additional general position assumption "no r points of X together with V are contained in a hyperplane" will again guarantee that W may be chosen disjoint from V.

Remark 3.3. The number of partitions of X into r subfamilies such that there exist a W as in the theorem can be bounded from below as in Vučić–Živaljević [27] (if r is a prime) and in Hell [11] (for $r = p^\ell$) by $\frac{1}{(r-1)!} \left(\frac{r}{\ell+1} \right)^{\lceil (r-1)(d+1)/2 \rceil}$.

Remark 3.4. There is also a colorful extension as in Blagojević–M.–Ziegler [3] (compare with Remark 4.4) and "constraint Tverberg"-extensions as in Hell [12] and Engström–Norén [9].

Proof. We start as in Section 2 by dualizing this statement and considering the continuous map from Δ^{n-1} ($n = (D+1)(r-1)+1$) to subspaces $\widehat{W}_{p,\varepsilon}$. By the topological Tverberg theorem [1, 20, 25] there exist r points $p_1, \ldots, p_r \in \Delta^{n-1}$ with disjoint supports (corresponding to subfamilies Ξ_j) such that their corresponding subspaces $\widehat{W}_{p_j,\varepsilon}$ coincide.

After that we pass to the limit $\varepsilon_m \to 0$ and repeat the same arguments. The degenerate case $\widehat{V} \cap \widehat{W} \neq 0$ is possible only when every Ξ_j has a hyperplane containing the same common line $\widehat{x} \in \widehat{V} \cap \widehat{W}$. So again it is excluded by the general position assumption. $\quad\square$

4. A TRANSVERSAL PROJECTIVE TVERBERG THEOREM

Tverberg and Vrećica [24] conjectured a common generalization of the ham sandwich theorem, the center transversal theorem, and Tverberg's theorem, which they proved in some special cases. Later Živaljević [29], Vrećica [26], K. [14], and Blagojević–M.–Ziegler [4] proved further special cases and colorful extensions.

The following theorem and the subsequent Remark 4.4 generalize all those results (which accumulate to the case $v = d-1$ below) and Theorem 3.1 (which is the special case $m = 1$).

Theorem 4.1. *Let $d > v \geq 0$, $m \geq 1$, $d > w := m(d-v)-1$, $D := (d-v)(d-w)$, and let p be a prime. Suppose that $p = 2$ or $m = 1$ or $d - w$ is even. Let $r_1 = p^{\alpha_1}, \ldots, r_m = p^{\alpha_m}$ be powers of p.*

Suppose that $X^1, \ldots, X^m$ are m point sets in $\mathbb{R}P^d$ of size $|X^j| = (D+1)(r_j-1)+1$ and $V \subset \mathbb{R}P^d$ is a projective subspace of dimension v.

Then there exists a projective subspace $W \subset \mathbb{R}P^d$ of dimension w and partitions of each X^j into r_j pieces respectively,

$$X^j = X_1^j \,\dot\cup\, \ldots \,\dot\cup\, X_{r_j}^j,$$

such that any pair of hyperplanes $H_1 \supseteq V$ and $H_2 \supseteq W$ partitions $\mathbb{R}P^d$ into two closed parts so that each part has nonempty intersection with every X_i^j.

Remark 4.2. It is natural to conjecture the same statement also for all positive integers r_j and without the technical assumption of $p = 2$ or $m = 1$ or $d - w$ being even.

Remark 4.3. The assumption that no r points of any X^i together with V are contained in a hyperplane will guarantee that W and V intersect transversally.

Remark 4.4. In the case when all $r_j = p$, Theorem 4.1 has a colorful extension analogously to the colorful Tverberg–Vrećica theorem of [4]: If for every j we color the point set X^i so that every color appears at most $r_j - 1$ times, then the partitions of the sets X^j can be made rainbow colored in the sense that every part X_i^j uses every color at most once.

Proof. Let $n_j := |X^j| = (D+1)(r_j-1)+1$ ($1 \leq j \leq m$). As above we regard V as a linear $(v+1)$-dimensional subspace of $\mathbb{R}^{d+1}$, denoted by V'. Let $\widehat{V} \subseteq (\mathbb{R}^{d+1})^*$ be its dual space of dimension $d - v$ (this is the same as going to the projective dual statement of the theorem). We have to find a certain $(w+1)$-dimensional subspace $W' \subseteq \mathbb{R}^{d+1}$. We will instead find its dual space $\widehat{W} \subseteq (\mathbb{R}^{d+1})^*$ of dimension $d - w$.

Note that the dimension of $\widehat{V}$ and $\widehat{W}$ will sum up to $2d - v - w$. Let B be the set of all $(2d - v - w)$-dimensional subspaces $U \subseteq (\mathbb{R}^{d+1})^*$ that contain $\widehat{V}$. Sending U to its quotient $U/\widehat{V}$ gives us a bijection $B \cong G(v+1, d-w)$, which we use to topologize B. The Grassmannian here may be considered oriented or not oriented; we defer this choice till the end of the proof.

Let us fix a subspace $U \in B$, $j \in \{1, \ldots, m\}$ and an $\varepsilon > 0$. As in the proof of Theorem 3.1, we want to find r_j quadratic forms $q_{p_i,\varepsilon}$ on U such that $p_1, \ldots, p_{r_j} \in \Delta^{n_j-1}$ have

pairwise disjoint supports and such that the orthogonal complement of $\widehat{V}$ in U with respect to any $q_{p_i,\varepsilon}$ is the same $(d-w)$-subspace of U. As in the usual configuration space/test map scheme for topological Tverberg-type theorems, see for example Matoušek [18], the r_j-tuples of coefficient vectors $(p_i)_{1 \le i \le r_j}$ of those quadratic forms are the preimages of the thin diagonal of a test-map

$$f_{U,j,\varepsilon} : (\Delta^{n_j-1})^{r_j}_{\Delta(2)} \longrightarrow_{\Sigma_{r_j}} (S_U)^{r_j},$$
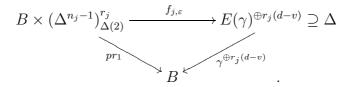
where S_U is the Schubert cell in $G(2d-w-v, d-w)$ that consists of all $(d-w)$-subspaces of U that intersect $\widehat{V}$ trivially; and $(\Delta^{n_j-1})^{r_j}_{\Delta(2)}$ denotes the r_j-fold pairwise deleted product of Δ^{n_j-1}, which is defined as the union of products $F_1 \times \ldots \times F_{r_j}$ of pairwise disjoint faces of Δ^{n_j-1}. The test-map is naturally Σ_{r_j}-equivariant, where Σ_{r_j} denotes the symmetric group on r_j elements.

Elements $\widehat{W} \in S_U$ can be uniquely written as kernels of linear projections $U \to \widehat{V}$. When choosing an inner product on U, those projections are in bijection with maps $U/\widehat{V} \to \widehat{V}$, which are elements in $(U/\widehat{V})^* \otimes \widehat{V}$. Thus,

$$(4.1) \qquad\qquad S_U \cong (U/\widehat{V})^* \otimes \widehat{V} \cong \mathbb{R}^D$$

where the first isomorphism is natural in $U \in B$ if the inner products on the U's are consistently the restriction of a fixed inner product on $\mathbb{R}^{d+1}$. The maps $f_{U,j,\varepsilon}$ depend continuously on $U \in B$. Let $\gamma : E(\gamma) \to B$ denote the tautological vector bundle over $B \cong G(v+1, d-w)$ of rank $d-w$, its fibers being $U/\widehat{V}$. The union of the Schubert cells S_U forms in a natural way a vector bundle over B (S_U being the fiber over $U \in B$), which by (4.1) is isomorphic to $(\gamma^*)^{\oplus(d-v)} \cong \gamma^{\oplus(d-v)}$.

Hence the collection of maps $f_{U,j,\varepsilon}$ ($U \in B$) forms a Σ_{r_j}-equivariant bundle map $f_{j,\varepsilon}$ over B,

$$B \times (\Delta^{n_j-1})^{r_j}_{\Delta(2)} \xrightarrow{\quad f_{j,\varepsilon} \quad} E(\gamma)^{\oplus r_j(d-v)} \supseteq \Delta$$

with pr_1 going to B and $\gamma^{\oplus r_j(d-v)}$ going to B.

Here, $\Delta \cong E(\gamma^{\oplus(d-v)})$ denotes the thin diagonal subbundle, and we are interested in the preimages $f_{j,\varepsilon}^{-1}(\Delta)$. In particular we have to show that

$$(4.2) \qquad\qquad f_{1,\varepsilon}^{-1}(\Delta) \cap \ldots \cap f_{m,\varepsilon}^{-1}(\Delta) \neq \emptyset.$$

This can be proved exactly as in [14] or [4] using a parametrized Borsuk–Ulam type theorem. The only topological assumption that we need is that the mod-p-Euler class of Δ, $e(\Delta) = e(\gamma)^{d-v} \in H^D(B; \mathbb{F}_p)$, satisfies

$$(4.3) \qquad\qquad e(\Delta)^{m-1} \neq 0.$$

We know that $e(\Delta)^{m-1} = e(\gamma)^{(v+1)-(d-w)} \neq 0$ if $p = 2$ and we consider the nonoriented Grassmannian (see Hiller [13]) or if $p > 2$, $\dim \gamma = d-w$ is even, and we consider the oriented Grassmannian (see [14, Lemma 8]). The same is trivially true if $m = 1$, since then B is a single point pt and the "Euler class" is the generator of $H^0(\mathrm{pt}; \mathbb{F}_p)$. Finally, (4.3) is holds true if $p = 2$ or $d-w$ is even or $m = 1$.

As in the previous proofs, letting s go to zero finishes the proof. $\square$

5. A PROJECTIVE CENTER TRANSVERSAL THEOREM

As an immediate corollary of Theorem 4.1 we obtain a projective generalization of Dol'nikov's [5, 6, 7] and Živaljević's [28] center transversal theorem (which is the special case $v = d - 1$):

Corollary 5.1 (Projective center transversal theorem). *Let $d > v \geq 0$, $m \geq 1$, $d > w := m(d - v) - 1$, and $D := (d - v)(d - w)$.*

Suppose that $\mu_1, \ldots, \mu_m$ are m probability measures on $\mathbb{R}P^d$ and $V \subset \mathbb{R}P^d$ is a projective subspace of dimension v.

Then there exists a projective subspace $W \subset \mathbb{R}P^d$ of dimension w such that any pair of hyperplanes $H_1 \supseteq V$ and $H_2 \supseteq W$ partitions $\mathbb{R}P^d$ into two closed parts P_1 and P_2 so that for any $j \in \{1, \ldots, m\}$,

$$\mu_j(P_1), \mu_j(P_2) \geq \frac{1}{D + 1}.$$

Proof. We approximate the measures μ_j by finite point sets X^j (as in the proof of Tverberg and Vrećica [24] that their conjecture implies the center transversal theorem). We apply Theorem 4.1, avoiding the parity assumption of $d - w$ being even by simply taking $p = 2$ and r_j equal to appropriate powers of two. Both parts P_1 and P_2 will contain at least one point from every X_i^j and thus at least r_j points from every X^j. Since $r_j/|X^j| \geq 1/(D+1)$ and $G(d + 1, w + 1)$ is compact, a limit argument finishes the proof. □

Remark 5.2. The case $m = 1$ could be called a projective center point theorem in analogy with Rado's center point theorem [21], which is the special case $m = 1$ and $v = d - 1$. Similarly, the case $m = d$ and $v = w = d - 1$ is the ham sandwich theorem; compare with Blagojević–K. [2, Thm. 10] for an actual projective mass partition theorem.

6. ANOTHER PROJECTIVE TVERBERG THEOREM

We could also ask what happens if we allow in Theorem 3.1, for example, to choose V arbitrarily. A partial answer is given by the following:

Theorem 6.1. *Let $0 \leq v < d$, r_1 and r_2 be the powers of the same prime p, and X^1 and X^2 be finite point sets with $|X^j| = (D + 1)(r_j - 1) + 1$, where $D = (v + 1)(d - v)$. We further assume that D is even and $\binom{\lfloor \frac{d+1}{2} \rfloor}{\lfloor \frac{v+1}{2} \rfloor}$ is nonzero mod p.*

Then there exist projective subspaces $V, W \subset \mathbb{R}P^d$ of dimensions v and $d - v - 1$ and partitions of every X^j into r_j pieces

$$X^j = X_1^j \dot\cup \ldots \dot\cup X_{r_j}^j,$$

such that any pair of hyperplanes $H_1 \supseteq V$ and $H_2 \supseteq W$ partitions $\mathbb{R}P^d$ into two parts so that each part has nonempty intersection with every X_i^j.

Remark 6.2. It was already known to Kummer that $\binom{n}{k}$ is nonzero mod p if and only if for all $i \geq 0$ the i'th digit of n in the p-ary representation is greater or equal to the i'th digit of k.

Remark 6.3. As a corollary we obtain a corresponding center point theorem for partitioning two measures under the assumption that D is even. The proof is analogous to the one of Corollary 5.1.

Proof. We follow the proof of Theorem 3.1 for X^j and describe the set of appropriate $\widehat{W}_j$ for every $\widehat{V}$ as a preimage. We have to find some $\widehat{V}$ such that $\widehat{W}_1$ and $\widehat{W}_2$ are equal. Their difference is an element of $(\mathbb{R}^{d+1}/\widehat{V}) \otimes \widehat{V}$. So (the formal argument is similar to the proof

of Theorem 4.1, using a parameterized testmap) we observe that the spaces $(\mathbb{R}^{d+1}/\widehat{V}) \otimes \widehat{V}$ form a vector bundle over the Grassmannian $G(d+1, v+1)$ of all possible V and this bundle is isomorphic to $\gamma \otimes \gamma^{\perp}$, where γ is the tautological bundle over the Grassmannian and $\gamma^{\perp}$ its orthogonal complement bundle.

It remains to check that the Euler class $e(\gamma \otimes \gamma^{\perp}) \in H^D(G(d+1, v+1); \mathcal{F}_p)$ is nonzero (the coefficients are $\mathbb{F}_p$ with possible twist corresponding to the orientation character of $\gamma \otimes \gamma^{\perp}$). An open neighborhood (and thus the tangent space) of $G(d+1, v+1)$ at a point $\widehat{V}$ can be identified with

$$\widehat{V}^* \otimes \widehat{V}^{\perp} = \mathrm{Hom}(\widehat{V}, \widehat{V}^{\perp}),$$

because adding such maps to the inclusion $\widehat{V} \hookrightarrow \mathbb{R}^{d+1}$ gives linear maps $\widehat{V} \to \mathbb{R}^{d+1}$ with all possible images intersecting $\widehat{V}^{\perp}$ trivially. Thus the images of such maps constitute the Schubert cell neighborhood of $\widehat{V}$. Since the identifications are natural and $\gamma^* \cong \gamma$, we have:

$$\gamma \otimes \gamma^{\perp} \cong TG(d+1, v+1).$$

Thus the Euler class $e(\gamma \otimes \gamma^{\perp})$ is Poincaré dual to the element of $H_0(G(d+1, v+1); \mathbb{F}_p)$ equal to the Euler characteristic of $G(d+1, v+1) \bmod p$; the cohomology twist is eliminated because it has to be tensor multiplied by the orientation sheaf of $TG(d+1, v+1)$. The Euler characteristic of the Grassmannian is equal to the value of the q-binomial coefficient $\binom{d+1}{v+1}_q$ at $q = -1$. This value can be calculated by the inductive formula:

$$\binom{n}{k}_{-1} = \binom{n-1}{k-1}_{-1} + (-1)^k \binom{n-1}{k}_{-1}, \quad \binom{n}{0}_{-1} = \binom{n}{n}_{-1} = 1,$$

and it finally turns out to be zero if D is odd and equal to

$$\binom{\lfloor \frac{d+1}{2} \rfloor}{\lfloor \frac{v+1}{2} \rfloor}$$

if D is even. $\qquad\square$

7. A corresponding "transversal" generalization

In this section we extend Theorem 6.1 under some algebraic assumptions to the case of more than two masses in the same way as Theorem 4.1 extends Theorem 3.1. The pay-off for establishing this is that the dimension of V and W must increase.

Let $0 \leq v, w < d$, $m \geq 1$, $r_j = 2^{\alpha_j}$ $(1 \leq j \leq m)$, $\widehat{v} := d - v$, $\widehat{w} := d - w$, $D := \widehat{v}\widehat{w}$. Suppose that $X^1, \ldots, X^m$ are m point sets in $\mathbb{R}P^d$ of size $|X^j| = (D+1)(r_j - 1) + 1$. We will find a condition under which there exist projective subspaces $V, W \subset \mathbb{R}P^d$ of dimensions v and w and partitions of every X^j into r_j pieces

$$X^j = X_1^j \dot{\cup} \ldots \dot{\cup} X_{r_j}^j,$$

such that any pair of hyperplanes $H_1 \supseteq V$ and $H_2 \supseteq W$ partitions $\mathbb{R}P^d$ into two parts so that each part has nonempty intersection with every X_i^j.

Similarly to the above sections, the topological setup for this is the following. Let $B := F(\widehat{v}, \widehat{v} + \widehat{w}, d+1)$ be the partial flag manifold of all pairs $(\widehat{V}, U)$ of linear subspaces of $\mathbb{R}^{d+1}$ such that $\dim(\widehat{V}) = \widehat{v}$, $\dim U = \widehat{v} + \widehat{w}$, and $\widehat{V} \subset U$. Let η be the rank D vector bundle over B whose fiber over $(\widehat{V}, U)$ is $\widehat{V} \otimes (U/\widehat{V})$. Let $w_i(\eta) \in H^i(B; \mathbb{F}_2)$ denote its Stiefel–Whitney classes. The condition we look for is

$$(7.1) \qquad\qquad\qquad w_D(\eta)^{m-1} \neq 0.$$

We can put this condition into purely algebraic terms as follows using a particular instance of the splitting principle. Let $F := \mathrm{Flag}(\mathbb{R}^{d+1}) = F(1, 2, \ldots, d+1)$ be the complete real flag manifold in $\mathbb{R}^{d+1}$. There is a canonical projection $p : F \to B$. Let $\ell_1, \ldots, \ell_{d+1}$ be the tautological line bundles over F. Their first Stiefel–Whitney classes $e_i := w_1(\ell_i)$ generate the ring $H^*(F; \mathbb{F}_2)$. More exactly,

$$(7.2) \qquad H^*(F; \mathbb{F}_2) \cong \mathbb{F}_2[e_1, \ldots, e_{d+1}] / \left(\prod (1 + e_i) = 1 \right).$$

The relation means geometrically that the sum $\ell_1 \oplus \ldots \oplus \ell_{d+1}$ is trivial and algebraically that all elementary symmetric polynomials in $e_1, \ldots, e_{d+1}$ vanish. The reader may find the latter fact and other standard facts about vector bundles and their characteristic classes in [17].

Now,

$$
\begin{aligned}
w(p^*\eta) &= w\big((\ell_1 \oplus \ldots \oplus \ell_{\widehat{v}}) \otimes (\ell_{\widehat{v}+1} \oplus \ldots \oplus \ell_{\widehat{v}+\widehat{w}})\big) \\
&= \prod_{i,j}^* w(\ell_i \otimes \ell_j) \;=\; \prod_{i,j}^* \big(1 + w_1(\ell_i) + w_1(\ell_j)\big),
\end{aligned}
$$

where $\prod_{i,j}^*$ denotes that the product runs over all pairs (i, j) with $1 \le i \le \widehat{v}$ and $\widehat{v}+1 \le j \le \widehat{v}+\widehat{w}$. In particular,

$$(7.3) \qquad w_D(p^*\eta) = \prod_{i,j}^* (e_i + e_j).$$

Thus (7.1) is equivalent to $\prod_{i,j}^* (e_i + e_j)^{m-1} \ne 0$ in $H^*(F; \mathbb{F}_2)$. This proves:

Theorem 7.1. *Let $0 \le v, w < d$, $m \ge 1$, $r_j = 2^{\alpha_j}$ $(1 \le j \le m)$, $D := (d - v)(d - w)$, such that the product (7.3) is non-zero in (7.2).*

Suppose that $X^1, \ldots, X^m$ are m point sets in $\mathbb{R}P^d$ of size $|X^j| = (D+1)(r_j - 1) + 1$.

Then there exist projective subspaces $V, W \subset \mathbb{R}P^d$ of dimensions v and w and partitions of every X^j into r_j pieces

$$X^j = X_1^j \,\dot\cup\, \ldots \,\dot\cup\, X_{r_j}^j,$$

such that any pair of hyperplanes $H_1 \supseteq V$ and $H_2 \supseteq W$ partitions $\mathbb{R}P^d$ into two parts so that each part has nonempty intersection with every X_i^j.

References

[1] I. Bárány, S.B. Shlosman, A. Szűcz. On a topological generalization of a theorem of Tverberg. // J. London Math. Soc., II. Ser. 23 (1981), 158–164.

[2] P.V.M. Blagojević, R.N. Karasev. Extensions of theorems of Rattray and Makeev. // arXiv:1011.0869 (2010).

[3] P.V.M. Blagojević, B. Matschke, G.M. Ziegler. Optimal bounds for the colored Tverberg problem. // arXiv:0910.4987 (2009).

[4] P.V.M. Blagojević, B. Matschke, G.M. Ziegler. Optimal bounds for a colorful Tverberg–Vrećica type problem. // Adv. Math. 226 (2011), 5198–5215.

[5] V.L. Dol'nikov. Common transversals for families of sets in $\mathbb{R}^n$ and connections between theorems of Helly and Borsuk. // (In Russian) Dokl. Akad. Nauk USSR 297(4), (1987), 777–780.

[6] V.L. Dol'nikov. A generalization of the ham sandwich theorem. // (In Russian) Mat. Zametki 52 (1992), 27–37. English translation in Math. Notes 52 (1993), 771–779.

[7] V.L. Dol'nikov. Transversal of families of sets in $\mathbb{R}^n$ and a connection between the Helly and Borsuk theorems. // Russian Academy of Sciences. Sbornik Mathematics 79:1 (1994), 93–107.

[8] A. Engström. A local criterion for Tverberg graphs. // Combinatorica 31:3 (2011), 321–332.

[9] A. Engström, P. Norén. Tverberg's theorem and graph coloring. // arXiv:1105.1455 (2011).

[10] B. Grünbaum. Partitions of mass-distributions and of convex bodies by hyperplanes. // Pacific J. Math. 10 (1960), 1257–1261.

[11] S. Hell. On the number of Tverberg partitions in the prime power case. // European J. Comb. 28:1 (2007), 347–355.

[12] S. Hell. Tverberg's theorem with constraints. // J. Comb. Theory, Series A 115(8), (2008), 1402–1416.

[13] H.L. Hiller. On the cohomology of real Grassmannians. // Trans. Amer. Math. Soc. 257 (1980), 521–533.

[14] R.N. Karasev. Tverberg's transversal conjecture and analogues of nonembeddability theorems for transversals. // Discrete Comput. Geom. 38 (2007), 513–525.

[15] R.N. Karasev. Dual theorems on central points and their generalizations. // Sbornik: Mathematics 199:10 (2008), 1459–1479; corrected English version available at arXiv:0909.4915 (2009).

[16] R.N. Karasev. A topological central point theorem. // arXiv:1011.1802 (2010).

[17] G. Luke, A.S. Mishchenko. Vector bundles and their applications. Springer Verlag, 1998.

[18] J. Matoušek. Using the Borsuk–Ulam theorem. 2nd printing. // Berlin-Heidelberg, Springer Verlag, 2007.

[19] B.H. Neumann. On an invariant of plane regions and mass distributions. // J. London Math. Soc. 20 (1945), 226–237.

[20] M. Özaydın. Equivariant maps for the symmetric group. // Preprint (1987), 17 pages.

[21] R. Rado. A theorem on general measure. // J. London Math. Soc. 21 (1947), 291–300.

[22] R.E. Stong. Cup products in Grassmannians. // Topology Appl. 13 (1982), 103–113.

[23] H. Tverberg. A generalization of Radon's theorem. // J. London Math. Soc. 41 (1966), 123–128.

[24] H. Tverberg, S. Vrećica. On generalizations of Radon's theorem and the ham sandwich theorem. // Europ. J. Combinatorics 14 (1993), 259–264.

[25] A.Yu. Volovikov. On a topological generalization of the Tverberg theorem. // Mathematical Notes 59:3 (1996), 324–326.

[26] S. Vrećica. Tverberg's conjecture. // Discrete Comput. Geometry 29 (2003), 505–510.

[27] A. Vućić, R.T. Živaljević. Notes on a conjecture of Sierksma. // Discrete Comput. Geom. 9 (1993), 339–349.

[28] R.T. Živaljević, S.T. Vrećica. An extension of the ham sandwich theorem. // Bull. London Math. Soc. 22 (1990), 183–186.

[29] R.T. Živaljević. The Tverberg–Vrećica problem and the combinatorial geometry on vector bundles. // Israel J. Math. 111 (1999), 53–76.

Roman Karasev, Dept. of Mathematics, Moscow Institute of Physics and Technology, Institutskiy per. 9, Dolgoprudny, Russia 141700; and Laboratory of Discrete and Computational Geometry, Yaroslavl' State University, Sovetskaya st. 14, Yaroslavl', Russia 150000

E-mail address: r_n_karasev@mail.ru

Benjamin Matschke, Institute for Advanced Study, Princeton

E-mail address: matschke@ias.edu