On the toric ideal of a matroid

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ABSTRACT. Describing minimal generating set of a toric ideal is a well-studied and difficult problem. In 1980 White conjectured that the toric ideal associated to a matroid is equal to the ideal generated by quadratic binomials corresponding to symmetric exchanges.

We prove White's conjecture up to saturation, that is that the saturations of both ideals are equal. In the language of algebraic geometry this means that both ideals define the same projective scheme. Additionally we prove the full conjecture for strongly base orderable matroids.

1. Introduction

Let M be a matroid on a ground set E with the set of bases $\mathfrak{B} \subset \mathcal{P}(E)$ (the reader is referred to [16] for background of matroid theory). For a fixed field \mathbb{K} let $S_M := \mathbb{K}[y_B : B \in \mathfrak{B}]$ be a polynomial ring. Let φ_M be the \mathbb{K} -homomorphism:

$$\varphi_M : S_M \ni y_B \to \prod_{e \in B} x_e \in \mathbb{K}[x_e : e \in E].$$

The toric ideal of a matroid M, denoted by I_M , is the kernel of the map φ_M . For a realizable matroid M the toric variety associated with the ideal I_M has a very nice embedding as a subvariety of a Grassmannian [8]. It is the closure of the torus orbit of the point of the Grassmannian corresponding to the matroid M.

The family \mathfrak{B} of bases, from the definition of a matroid, is nonempty and satisfies *exchange property* — for every bases B_1, B_2 and $e \in B_1 \setminus B_2$ there exists $f \in B_2 \setminus B_1$, such that $(B_1 \setminus e) \cup f$ is also a basis.

Brualdi [3] showed that bases of a matroid satisfy also symmetric exchange property — for every bases B_1, B_2 and $e \in B_1 \setminus B_2$ there exists $f \in B_2 \setminus B_1$, such that both $(B_1 \setminus e) \cup f$ and $(B_2 \setminus f) \cup e$ are bases.

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Surprisingly, even a stronger property, known as multiple symmetric exchange property, is true — for every bases B_1, B_2 and $A_1 \subset B_1$ there exists $A_2 \subset B_2$, such that both $(B_1 \setminus A_1) \cup A_2$ and $(B_2 \setminus A_2) \cup A_1$ are bases (for simple proofs see [14, 23], and [12, 13] for more exchange properties).

Suppose that a pair of bases D_1, D_2 is obtained from a pair of bases B_1, B_2 by a symmetric exchange. That is, $D_1 = (B_1 \setminus e) \cup f$ and $D_2 = (B_2 \setminus f) \cup e$ for some $e \in B_1$ and $f \in B_2$. Then we say that the quadratic binomial $y_{B_1}y_{B_2} - y_{D_1}y_{D_2}$ corresponds to symmetric exchange. It is clear that such binomials belong to the ideal I_M . White conjectured that they generate this ideal.

CONJECTURE 1 (White 1980, [21]). For every matroid M its toric ideal I_M is generated by quadratic binomials corresponding to symmetric exchanges.

Since every toric ideal is generated by binomials it is not hard to rephrase the above conjecture in the combinatorial language. It asserts that if two multisets of bases of a matroid have equal union (as a multiset), then one can pass between them by a sequence of symmetric exchanges. In fact this is the original formulation due to White. We immediately see that the conjecture does not depend on the field \mathbb{K} .

The most significant partial result is due to Blasiak [1], who confirmed the conjecture for graphical matroids. Kashiwabara [11] checked the case of matroids of rank at most 3. Schweig [18] proved the case of lattice path matroids, which are a subclass of transversal matroids. Recently, Bonin [2] confirmed the conjecture for sparse paving matroids.

A matroid is strongly base orderable if for any two bases B_1 and B_2 there is a bijection $\pi : B_1 \to B_2$ satisfying the multiple symmetric exchange property, that is: $(B_1 \setminus A) \cup \pi(A)$ is a basis for every $A \subset B_1$. This implies that π restricted to the intersection $B_1 \cap B_2$ is the identity. Moreover, $(B_2 \setminus \pi(A)) \cup A$ is a basis for every $A \subset B_1$ (by the multiple symmetric exchange property for $B_1 \setminus A$). The class of strongly base orderable matroids is closed under taking minors. It is a large class of matroids, characterized by a matroid property instead of a specific presentation, contrary to the case of graphical, transversal or lattice path matroids.

We prove White's conjecture for strongly base orderable matroids. As a consequence it is true for gammoids (every gammoid is strongly base orderable [17]), and in particular for transversal matroids (every transversal matroid is a gammoid [16]). So far, for transversal matroids, it was known only that the toric ideal I_M is generated by quadratic binomials [4].

THEOREM 2. If M is a strongly base orderable matroid, then the toric ideal I_M is generated by quadratic binomials corresponding to symmetric exchanges.

Our argument uses an idea from the proof presented in [17] of a theorem of Davies and McDiarmid [6]. Suppose two strongly base orderable matroids on the ground set E have the same rank. The theorem of Davies and McDiarmid asserts that if E can be partitioned into bases in each matroid, then there exists also a common partition.

Let \mathfrak{m} be the ideal generated by all variables in the polynomial ring S_M (socalled *irrelevant ideal*). Recall that $I : \mathfrak{m}^{\infty} = \{a \in S_M : a\mathfrak{m}^n \subset I \text{ for some } n \in \mathbb{N}\}$ is called the *saturation* of an ideal I with respect to the ideal \mathfrak{m} . Let J_M be the ideal generated by quadratic binomials corresponding to symmetric exchanges. Clearly, $J_M \subset I_M$ and White's conjecture asserts that the ideals J_M and I_M are equal. We prove for arbitrary matroid M that the ideals J_M and I_M are equal up to saturation with respect to the irrelevant ideal \mathfrak{m} . In fact the ideal I_M , as a prime ideal, is saturated $I_M : \mathfrak{m}^{\infty} = I_M$.

Ideals are central objects of commutative algebra. From the point of view of algebraic geometry one is interested in schemes defined by them. A homogeneous ideal (I_M and J_M are homogeneous) defines two schemes – affine and projective (we refer the reader to [7, 5] for background of toric geometry). Two ideals define the same affine scheme if and only if they are equal. Thus White's conjecture asserts equality of affine schemes defined by I_M and J_M . Homogeneous ideals define the same projective scheme if and only if their saturations with respect to the irrelevant ideal are equal. Thus we prove equality of projective schemes defined by I_M and J_M . More information on distinctions between sets and affine or projective schemes in the case of toric varieties can be found in the last part of Section 4 and in Section 5 of [15].

The projective toric variety $\operatorname{Proj}(S_M/I_M)$ has been studied before (see [8, 10]). It is often required that a projective toric variety is normal. Indeed, White proved the stronger property that the variety $\operatorname{Proj}(S_M/I_M)$ is projectively normal [22].

THEOREM 3. White's conjecture is true up to saturation. That is, for every matroid M we have $J_M : \mathfrak{m}^{\infty} = I_M$. In other words the projective schemes $\operatorname{Proj}(S_M/I_M)$ and $\operatorname{Proj}(S_M/J_M)$ are equal.

As a corollary we get that both ideals have equal radicals and the same affine set of zeros (since both I_M and J_M are contained in \mathfrak{m}). Moreover, it follows that in order to prove White's conjecture it is enough to show that the ideal J_M is saturated, radical or prime.

Conjecture 1 is an algebraic reformulation (cf. [20]) of the original conjecture due to White expressed in the combinatorial language. Actually, White stated three conjectures of growing difficulty. In the algebraic language the weakest asserts that the toric ideal I_M is generated by quadratic binomials. The second one is Conjecture 1, and the most difficult is an analog of Conjecture 1 for the noncommutative polynomial ring S_M . We discuss them in details in the last section. We prove that Conjecture 1 holds for the direct sum $M \oplus M$ if and only if its noncommutative version holds for M. In particular we get that the strongest version holds for all strongly base orderable, graphical, and cographical matroids. We mention also how to extend Theorems 2 and 3 to discrete polymatroids.

2. White's conjecture for strongly base orderable matroids

PROOF OF THEOREM 2. Recall that J_M is the ideal generated by quadratic binomials corresponding to symmetric exchanges. The ideal I_M , as a toric ideal, is generated by binomials. Thus it is enough to prove that all binomials of I_M belong to the ideal J_M .

Fix $n \geq 2$. We are going to show by decreasing induction on the overlap function

$$d(y_{B_1} \cdots y_{B_n}, y_{D_1} \cdots y_{D_n}) := \max_{\pi \in S_n} \sum_{i=1}^n |B_i \cap D_{\pi(i)}|$$

that a binomial $y_{B_1} \cdots y_{B_n} - y_{D_1} \cdots y_{D_n} \in I_M$ belongs to J_M . Clearly, the biggest possible value of d is r(M)n, where r(M) denotes the rank of matroid M.

If $d(y_{B_1}\cdots y_{B_n}, y_{D_1}\cdots y_{D_n}) = r(M)n$, then there exists a permutation $\pi \in S_n$ such that $B_i = D_{\pi(i)}$ for each *i*. Hence $y_{B_1}\cdots y_{B_n} - y_{D_1}\cdots y_{D_n} = 0 \in J_M$.

Suppose the assertion holds for all binomials with the overlap function greater than d < r(M)n. Let $y_{B_1} \cdots y_{B_n} - y_{D_1} \cdots y_{D_n}$ be a binomial of I_M with the overlap function equal to d. Without loss of generality we can assume that the identity permutation realizes the maximum in the definition of the overlap function. Then for some i there exists $e \in B_i \setminus D_i$. Clearly, $y_{B_1} \cdots y_{B_n} - y_{D_1} \cdots y_{D_n} \in I_M$ if and only if $B_1 \cup \cdots \cup B_n = D_1 \cup \cdots \cup D_n$ as multisets. Thus there exists $j \neq i$ such that $e \in D_j \setminus B_j$. Without loss of generality we can assume that i = 1, j = 2. Since M is a strongly base orderable matroid, there exist bijections $\pi_B : B_1 \to B_2$ and $\pi_D : D_1 \to D_2$ with the multiple symmetric exchange property. Recall that π_B is the identity on $B_1 \cap B_2$, and similarly that π_D is the identity on $D_1 \cap D_2$.

Let G be a graph on a vertex set $B_1 \cup B_2 \cup D_1 \cup D_2$ with edges $\{b, \pi_B(b)\}$ for all $b \in B_1 \setminus B_2$ and $\{d, \pi_D(d)\}$ for all $d \in D_1 \setminus D_2$. Graph G is bipartite since it is a sum of two matchings. Split the vertex set of G into two independent (in the graph sense) sets S and T. Define:

$$B'_1 = (S \cap (B_1 \cup B_2)) \cup (B_1 \cap B_2), \ B'_2 = (T \cap (B_1 \cup B_2)) \cup (B_1 \cap B_2),$$

$$D'_1 = (S \cap (D_1 \cup D_2)) \cup (D_1 \cap D_2), \ D'_2 = (T \cap (D_1 \cup D_2)) \cup (D_1 \cap D_2).$$

By the multiple symmetric exchange property of π_B sets B'_1, B'_2 are bases obtained from the pair B_1, B_2 by a sequence of symmetric exchanges. Therefore the binomial $y_{B_1}y_{B_2}y_{B_3}\cdots y_{B_n} - y_{B'_1}y_{B'_2}y_{B_3}\cdots y_{B_n}$ belongs to J_M . Analogously the binomial $y_{D_1}y_{D_2}y_{D_3}\cdots y_{D_n} - y_{D'_1}y_{D'_2}y_{D_3}\cdots y_{D_n}$ belongs to J_M . Moreover, since S and Tare disjoint we have that

$$d(y_{B'_1}y_{B'_2}y_{B_3}\cdots y_{B_n}, y_{D'_1}y_{D'_2}y_{D_3}\cdots y_{D_n}) > d(y_{B_1}y_{B_2}\cdots y_{B_n}, y_{D_1}y_{D_2}\cdots y_{D_n}).$$

By the inductive assumption $y_{B'_1}y_{B'_2}y_{B_3}\cdots y_{B_n} - y_{D'_1}y_{D'_2}y_{D_3}\cdots y_{D_n}$ also belongs to J_M . By adding the first and the third and subtracting the second of the above binomials we get the inductive assertion.

3. Projective scheme-theoretic version of White's conjecture for arbitrary matroids

PROOF OF THEOREM 3. Since $J_M \subset I_M$ we get that $J_M : \mathfrak{m}^{\infty} \subset I_M : \mathfrak{m}^{\infty} = I_M$.

We prove the opposite inclusion $I_M \subset J_M : \mathfrak{m}^{\infty}$. As I_M is toric, it is enough to prove that any binomial $y_{B_1} \ldots y_{B_n} - y_{D_1} \ldots y_{D_n} \in I_M$ belongs to $J_M : \mathfrak{m}^{\infty}$. Hence it is enough to show that for each basis $B \in \mathfrak{B}$ we have

$$y_B^{(r(M)-1)n}(y_{B_1}\cdots y_{B_n}-y_{D_1}\cdots y_{D_n})\in J_M,$$

since then

$$(y_{B_1}\cdots y_{B_n}-y_{D_1}\cdots y_{D_n})\mathfrak{m}^{(r(M)-1)n|\mathfrak{B}|}\subset J_M$$

Let $B \in \mathfrak{B}$ be a basis. The polynomial ring S_M has a natural grading given by the degree function $\deg(y_{B'}) = 1$, for each variable $y_{B'}$. We define the Bdegree by $\deg_B(y_{B'}) = |B' \setminus B|$, and extend this notion also to bases $\deg_B(B') = |B' \setminus B|$. Notice that the ideal I_M is homogeneous with respect to both gradings. Additionally *B*-degree of y_B is zero, thus multiplying by y_B does not change *B*degree of a polynomial. Observe that if $\deg_B(B') = 1$, then *B'* differs from *B* only by a single element. We call such a basis, and the corresponding variable, balanced. A monomial or a binomial is called balanced if all its variables are balanced. We will prove by induction on the *B*-degree of a binomial the following claim. As argued before this will finish the proof.

CLAIM 4. If $b \in I_M$ is a binomial, then $y_B^{\deg_B(b)-\deg(b)}b \in J_M$.

If $\deg_B(b) - \deg(b) < 0$, then by $y_B^{\deg_B(b) - \deg(b)} b \in J_M$ we mean that $y_B^{\deg(b) - \deg_B(b)}$ divides b, and the quotient belongs to J_M .

Let $n = \deg_B(b)$. If n = 0, then the claim is obvious, since 0 is the only binomial in I_M with B-degree equal to 0. Suppose n > 0. As we would like to work with balanced variables, we begin with the following lemma.

LEMMA 5. For every basis $B' \in \mathfrak{B}$ there exist balanced bases $B_1, \ldots, B_{\deg_B(B')}$ such that

$$y_B^{\deg_B(B')-1}y_{B'} - y_{B_1} \cdots y_{B_{\deg_B(B')}} \in J_M.$$

PROOF. The proof goes by induction on $\deg_B(B')$. If $\deg_B(B') = 0, 1$, then the assertion is clear. Suppose that $\deg_B(B') > 1$. From the symmetric exchange property for $e \in B' \setminus B$ there exists $f \in B \setminus B'$ such that both $B_1 = (B \setminus f) \cup e$ and $B'' = (B' \setminus e) \cup f$ are bases. Now $\deg_B(B_1) = 1$ and $\deg_B(B'') = \deg_B(B') - 1$. Applying the inductive assumption to B'' we obtain balanced bases $B_2, \ldots, B_{\deg_B(B')}$ satisfying

$$y_B^{\deg_B(B')-2} y_{B''} - y_{B_2} \cdots y_{B_{\deg_B(B')}} \in J_M.$$

Hence, since $y_B y_{B'} - y_{B_1} y_{B''}$ corresponds to symmetric exchange, we get

$$y_B^{\deg_B(B')-1}y_{B'} - y_{B_1} \cdots y_{B_{\deg_B(B')}} = y_B^{\deg_B(B')-2} \left(y_B y_{B'} - y_{B_1} y_{B''} \right) + y_{B_1} \left(y_B^{\deg_B(B')-2} y_{B''} - y_{B_2} \cdots y_{B_{\deg_B(B')}} \right) \in J_M.$$

Lemma 5 allows us to replace each factor $y_B^{\deg_B(y_{B'})-\deg(y_{B'})}y_{B'}$ of a monomial $y_B^{\deg_B(m)-\deg(m)}m$ by a product of balanced variables (modulo the ideal J_M). Notice that the *B*-degree is preserved. Hence for binomials of fixed *B*-degree equal to *n* Claim 4 is equivalent to the following one.

CLAIM 6. If $b = y_{B_1} \cdots y_{B_n} - y_{D_1} \cdots y_{D_n} \in I_M$ is a balanced binomial, then $b \in J_M$.

With a balanced monomial $m = y_{B_1} \cdots y_{B_n}$ we associate a bipartite multigraph G(m). The vertex classes of G(m) are B and $E \setminus B$ (where E is the ground set of matroid M). Each edge corresponds to a variable y_{B_i} of the monomial m. Namely, if $B_i = (B \setminus f) \cup e$ for some $f \in B, e \in E \setminus B$ we put an edge $\{e, f\}$ in G(m). In this way G(m) is a multigraph with deg(m) edges.

Let $b = y_{B_1} \cdots y_{B_n} - y_{D_1} \cdots y_{D_n} \in I_M$ be a balanced binomial of *B*-degree equal to *n*. Observe that *b* belongs to I_M if and only if each vertex from *E* has the same degree with respect to graphs $G(y_{B_1} \cdots y_{B_n})$ and $G(y_{D_1} \cdots y_{D_n})$. Thus we can apply the following lemma (we leave the proof as an easy exercise).

LEMMA 7. Let G and H be bipartite multigraphs with the same vertex classes. Suppose that each vertex has the same degree with respect to G and H. Then the symmetric difference of multisets of edges of G and H can be partitioned into alternating cycles. That is simple cycles of even length with consecutive edges from different graphs.

We choose one alternating cycle, and denote its consecutive vertices by $f_1, e_1, f_2, e_2, \ldots, f_r, e_r, f_1$. For each $i \in \mathbb{Z}/r\mathbb{Z}$ the sets $B'_i = (B \setminus f_i) \cup e_i$ and $D'_i = (B \setminus f_i) \cup e_{i-1}$ are bases. Notice that $y_{B'_1} \cdots y_{B'_r}$ divides $y_{B_1} \cdots y_{B_n}$, let m_1 be the quotient. Analogously let m_2 be the quotient of $y_{D_1} \cdots y_{D_n}$ by $y_{D'_1} \cdots y_{D'_r}$.

Suppose r < n. Then the balanced binomial $b' = y_{B'_1} \cdots y_{B'_r} - y_{D'_1} \cdots y_{D'_r}$ belongs to I_M and has B-degree less than n. From the inductive assumption we get that $b' \in J_M$. Observe that

$$b = y_{B_1} \cdots y_{B_n} - y_{D_1} \cdots y_{D_n} = m_1 b' - y_{D'_1} \cdots y_{D'_n} (m_2 - m_1)$$

and $m_2 - m_1 \in I_M$. The balanced binomial $b'' = m_2 - m_1 \in I_M$ has B-degree less than n. By the inductive assumption $b'' \in J_M$, and as a consequence $b \in J_M$.

Suppose now that r = n. We can assume that $E = \{f_1, e_1, \dots, f_n, e_n\}$, since otherwise we can contract $B \setminus \{f_1, \ldots, f_n\}$ and restrict our matroid to the set $\{f_1, e_1, \ldots, f_n, e_n\}$. Obviously the assertion of the claim extends from such a minor to the matroid.

We say that a monomial m_3 is *achievable* from a monomial m_4 if $m_3 - m_4 \in J_M$. In this situation we say also that variables of m_3 are *achievable* from m_4 . Observe that if there is a variable different from y_B that is achievable from both monomials $y_{B_1}\cdots y_{B_n}$ and $y_{D_1}\cdots y_{D_n}$, then the assertion follows by induction. Indeed, if a variable $y_{B'}$ is achievable from both, then there exist monomials m_5, m_6 such that

$$b = (y_{B_1} \cdots y_{B_n} - y_{B'}m_5) + (y_{B'}m_6 - y_{D_1} \cdots y_{D_n}) + y_{B'}(m_5 - m_6)$$

The binomial $b' = m_5 - m_6 \in I_M$ has B-degree less than n, thus by the inductive assumption $y_B^{\deg_B(b')-\deg(b')}b' \in J_M$. Hence $b' \in J_M$ because

 $\deg_B(b') - \deg(b') = \deg_B(b) - \deg(b) - \deg_B(y_{B'}) + \deg(y_{B'}) \le 0.$

Suppose contrary – no variable different from y_B is achievable from both monomials of b. We will exclude this case by reaching a contradiction. For $k, i \in \mathbb{Z}/n\mathbb{Z}$ we define:

$$S_k^i := B \cup \{e_k, e_{k+1}, \dots, e_{k+i-1}\} \setminus \{f_k, f_{k+1}, \dots, f_{k+i-1}\}, \\T_k^i := B \cup \{e_{k-1}, e_k, \dots, e_{k+i-2}\} \setminus \{f_k, f_{k+1}, \dots, f_{k+i-1}\}, \\U_k^i := B \cup \{e_{k-i}\} \setminus \{f_k\}.$$

The sets S_k^i and T_k^i differ only on the set $\{e_1, \ldots, e_n\}$ by a shift by one. Notice that $S_k^1 = U_k^0 = B'_k$, $T_k^1 = U_k^1 = D'_k$ and $S_k^n = T_{k'}^n$ for arbitrary $k, k' \in \mathbb{Z}/n\mathbb{Z}$. Hence $m_7 := y_{B'_1} \cdots y_{B'_n} = y_{S_1^1} \cdots y_{S_n^1}$ and $m_8 := y_{D'_1} \cdots y_{D'_n} = y_{T_1^1} \cdots y_{T_n^1}$ are the monomials of *b*, that is $b = m_7 - m_8$.

LEMMA 8. Suppose that for a fixed 0 < i < n and every $k \in \mathbb{Z}/n\mathbb{Z}$ the following conditions are satisfied:

- (1) the sets S_k^i and T_k^i are bases, (2) the monomial $y_B^{i-1}y_{S_k^i}\prod_{j\neq k,...,k+i-1}y_{S_j^1}$ is achievable from m_7 ,
- (3) the monomial $y_B^{i-1}y_{T_k^i}\prod_{j\neq k,...,k+i-1}y_{T_j^1}$ is achievable from m_8 .

Then for every $k \in \mathbb{Z}/n\mathbb{Z}$ neither of the sets U_k^{-i} , U_k^{i+1} is a basis.

PROOF. Suppose contrary that U_k^{-i} is a basis. Then $y_{S_k^i}y_{S_{k+1}^i} - y_{T_{k+1}^i}y_{U_k^{-i}}$, by the definition, belongs to J_M . Thus, by the assumption, the variable $\tilde{y}_{T_{k+1}}^{*+}$ would be achievable from both m_7 and m_8 , which is a contradiction. The argument for U_k^{i+1} is analogous.

LEMMA 9. Suppose that for a fixed 0 < i < n and every $k \in \mathbb{Z}/n\mathbb{Z}$ the following conditions are satisfied:

- (1) the set S_k^i is a basis,
- (2) the monomial $y_B^{i-1}y_{S_k^i}\prod_{j\neq k,\ldots,k+i-1}y_{S_k^i}$ is achievable from m_7 ,
- (3) the set U_k^{-j} is not a basis for any $0 < j \le i$.

Then for every $k \in \mathbb{Z}/n\mathbb{Z}$ the set S_k^{i+1} is a basis and $y_B^i y_{S_k^{i+1}} \prod_{j \neq k, \dots, k+i} y_{S_j^1}$ is a monomial achievable from m_7 .

PROOF. From the symmetric exchange property for $e_k \in S_k^1 \setminus S_{k+1}^i$ it follows that there exists $x \in S_{k+1}^i \setminus S_k^1$ such that $\tilde{S}_{k+1}^i = (S_{k+1}^i \setminus x) \cup e_k$ and $\tilde{S}_k^1 = (S_k^1 \setminus e_k) \cup x$ are also bases. Thus $x \in \{f_k, e_{k+1}, e_{k+2}, \ldots, e_{k+i}\}$. Notice that if $x = e_{k+j}$ for some j, then $\tilde{S}_k^1 = U_k^{-j}$ contradicting condition (3). Thus $x = f_k$. Hence $\tilde{S}_{k+1}^i = S_k^{i+1}$ and $\tilde{S}_k^1 = B$. In particular the binomial $y_{S_{k+1}^i} y_{S_k^1} - y_B y_{S_k^{i+1}}$ belongs to J_M (condition (1) guarantees that the variable $y_{S_{k+1}^i}$ exists). Thus the assertion follows from condition (2).

Analogously we get the following shifted version of Lemma 9.

LEMMA 10. Suppose that for a fixed 0 < i < n and every $k \in \mathbb{Z}/n\mathbb{Z}$ the following conditions are satisfied:

- (1) the set T_k^i is a basis,
- (2) the monomial $y_B^{i-1} y_{T_k^i} \prod_{j \neq k, \dots, k+i-1} y_{T_i^1}$ is achievable from m_8 ,
- (3) the set U_k^{j+1} is not a base for any $0 < j \le i$.

Then for every $k \in \mathbb{Z}/n\mathbb{Z}$ the set T_k^{i+1} is a basis and $y_B^i y_{T_k^{i+1}} \prod_{j \neq k, \dots, k+i} y_{T_j^1}$ is a monomial achievable from m_8 .

We are ready to reach a contradiction by an inductive argument. First we verify that for i = 1 the assumptions of Lemma 8 are satisfied. Suppose now that for some $1 \leq i < n$ the assumptions of Lemma 8 are satisfied for every $1 \leq j \leq i$. Then, by Lemma 8 the assumptions of both Lemma 9 and Lemma 10 are satisfied for every $1 \leq j \leq i$. Thus by the assertions of Lemmas 9 and 10, the assumptions of Lemma 8 are satisfied for all $1 \leq j \leq i + 1$. We obtain that the assumptions and the assertions of Lemmas 8, 9 and 10 are satisfied for every $1 \leq i < n$. For i = n - 1 we get that the monomial $y_B^{n-1}y_{S_1^n} = y_B^{n-1}y_{T_1^n}$ is achievable from both m_7 and m_8 , this gives a contradiction.

4. Remarks

We begin with the original formulation of conjectures stated by White in [21].

Two sequences of bases $\mathcal{B} = (B_1, \ldots, B_n)$ and $\mathcal{D} = (D_1, \ldots, D_n)$ are compatible if $B_1 \cup \cdots \cup B_n = D_1 \cup \cdots \cup D_n$ as multisets (that is if $y_{B_1} \cdots y_{B_n} - y_{D_1} \cdots y_{D_n} \in I_M$). White defines three equivalence relations. Two sequences of bases \mathcal{B} and \mathcal{D} of equal length are in relation:

 \sim_1 if \mathcal{D} may be obtained from \mathcal{B} by a composition of symmetric exchanges. That is \sim_1 is the transitive closure of the relation which exchanges a pair of bases B_i, B_j in a sequence into a pair obtained by a symmetric exchange.

 \sim_2 if \mathcal{D} may be obtained from \mathcal{B} by a composition of symmetric exchanges and permutations of the order of the bases.

 \sim_3 if \mathcal{D} may be obtained from \mathcal{B} by a composition of multiple symmetric exchanges.

Let TE(i) denote the class of matroids for which every two compatible sequences of bases \mathcal{B}, \mathcal{D} are in relation $\mathcal{B} \sim_i \mathcal{D}$ (the notion TE(i) is the same as the original one in [21]). An algebraic meaning of the property TE(3) is that the toric ideal I_M is generated by quadratic binomials. A matroid M belongs to TE(2)if and only if the toric ideal I_M is generated by quadratic binomials corresponding to symmetric exchanges. The property TE(1) is an analog of TE(2) for the noncommutative polynomial ring S_M .

We are ready to formulate the original conjecture [21, Conjecture 12] of White.

CONJECTURE 11. The following equalities hold:

- (1) TE(1) = the class of all matroids,
- (2) TE(2) = the class of all matroids,
- (3) TE(3) = the class of all matroids.

Clearly, Conjecture 1 coincides with Conjecture 11 (2). It is straightforward [21, Proposition 5] that:

(1) $TE(1) \subset TE(2) \subset TE(3)$,

(2) classes TE(1), TE(2) and TE(3) are closed under taking minors and dual,
(3) classes TE(1) and TE(3) are closed under direct sum.

White also claims that the class TE(2) is closed under direct sum, however unfortunately there is a gap in his proof. We believe that it is an open question. Corollary 14 will show some consequences of TE(2) being closed under direct sum for the relation between classes TE(1) and TE(2).

LEMMA 12. For any matroid M the following conditions are equivalent:

- (1) $M \in TE(1)$,
- (2) $M \in TE(2)$ and for any two bases $(B_1, B_2) \sim_1 (B_2, B_1)$ holds.

PROOF. Implication $(1) \Rightarrow (2)$ is clear from the definition. To get the opposite implication it is enough to recall that any permutation is a composition of transpositions.

PROPOSITION 13. For any matroid M the following conditions are equivalent:

- (1) $M \in TE(1)$,
- (2) $M \oplus M \in TE(1)$,
- (3) $M \oplus M \in TE(2)$.

PROOF. Implications $(1) \Rightarrow (2) \Rightarrow (3)$ were already discussed. To get $(3) \Rightarrow (1)$ suppose that a matroid M satisfies $M \oplus M \in TE(2)$. By [B', B''] we denote a basis of $M \oplus M$ consisting of a basis B' of M on the first copy and B'' on the second.

First we prove that $M \in TE(2)$. Let $\mathcal{B} = (B_1, \ldots, B_n)$ and $\mathcal{D} = (D_1, \ldots, D_n)$ be compatible sequences of bases of M. If B is a basis of M, then $\mathcal{B}' = ([B_1, B], \ldots)$ and $\mathcal{D}' = ([D_1, B], \ldots)$ are compatible sequences of bases of $M \oplus M$. From the assumption we have $\mathcal{B}' \sim_2 \mathcal{D}'$. Notice that any symmetric exchange in $M \oplus M$ restricted to the first coordinate is either trivial or a symmetric exchange. Thus, the same symmetric exchanges certify that $\mathcal{B} \sim_2 \mathcal{D}$ in M.

Due to Lemma 12, in order to prove $M \in TE(1)$ it is enough to show that for any two bases B_1, B_2 of M the relation $(B_1, B_2) \sim_1 (B_2, B_1)$ holds. Sequences of bases $([B_1, B_1], [B_2, B_2])$ and $([B_2, B_1], [B_1, B_2])$ in $M \oplus M$ are compatible. Thus by the assumption one can be obtained from the other by a composition of symmetric exchanges and permutations. By the symmetry, without loss of generality we can assume that permutations are not needed. Now by projecting these symmetric exchanges to the first coordinate we get that $(B_1, B_2) \sim_1 (B_2, B_1)$ in M.

As a corollary we obtain that for reasonable classes of matroids the 'standard' version of White's conjecture is equivalent to the 'strong' one.

COROLLARY 14. If a class of matroids \mathfrak{C} is closed under direct sums, then $\mathfrak{C} \subset TE(1)$ if and only if $\mathfrak{C} \subset TE(2)$. In particular:

- (1) strongly base orderable, graphical, cographical matroids belong to TE(1),
- (2) Conjectures 11 (1) and (2) are equivalent,
- (3) the class TE(2) is closed under direct sum if and only if TE(1) = TE(2).

In the same way as we associate the toric ideal with a matroid one can associate a toric ideal I_P with a discrete polymatroid P. Herzog and Hibi [9] extend White's conjecture to discrete polymatroids. They also ask if the toric ideal I_P of a discrete polymatroid possesses a quadratic Gröbner basis (we refer the reader to [19]).

REMARK 15. Theorem 2 and Theorem 3 are true for discrete polymatroids.

There are several ways to prove that our results hold also for discrete polymatroids. One possibility is to use Lemma 5.4 from [9]. It reduces a question if a binomial is generated by quadratic binomials corresponding to symmetric exchanges from a discrete polymatroid to a certain matroid. Another possibility is to associate to a discrete polymatroid $P \subset \mathbb{Z}^n$ a matroid M_P on the ground set $\{1, \ldots, r(P)\} \times \{1, \ldots, n\}$. A set I is independent if there is $v \in P$ such that $|I \cap \{1, \ldots, r(P)\} \times \{i\}| \leq v_i$ holds for all i. It is straightforward that compatibility of sequences of bases and generation are the same in P and in M_P . Moreover, one can easily show that a symmetric exchange in M_P corresponds to at most two symmetric exchanges in P.

References

- J. Blasiak, The toric ideal of a graphic matroid is generated by quadrics, Combinatorica 28 (2008), 283-297.
- J. Bonin, Basis-exchange properties of sparse paving matroids, Adv. in Appl. Math. 50 (2013), no. 1, 6-15.
- [3] R.A. Brualdi, Comments on bases in dependence structures, Bull. Austral. Math. Soc. 1 (1969), 161-167.
- [4] A. Conca, Linear spaces, transversal polymatroids and ASL domains, J. Algebraic Combin. 25 (2007), 25-41.
- [5] D. Cox, J. Little, H. Schenck, Toric varieties, Graduate Studies in Mathematics 124, American Mathematical Society, Providence, 2011.
- [6] J. Davies and C. McDiarmid, Disjoint common transversals and exchange structures, J. London Math. Soc. 14 (1976), 55-62.
- [7] W. Fulton, Introduction to toric varieties, Annals of Mathematics Studies 131, Princeton University Press, Princeton, 1993.
- [8] I.M. Gelfand, R.M. Goresky, R.D. MacPherson, V.V. Serganova, Combinatorial Geometries, Convex Polyhedra and Schubert Cells, Advances in Math. 63 (1987), 301-316.
- [9] J. Herzog, T. Hibi, Discrete Polymatroids, J. Algebraic Combin. 16 (2002), 239-268.
- [10] M.M. Kapranov, B. Sturmfels, A.V. Zelevinsky, Chow polytopes and general resultants, Duke Math. J. 67 (1992), no. 1, 189-218.
- [11] K. Kashiwabara, The toric ideal of a matroid of rank 3 is generated by quadrics, Electron. J. Combin. 17 (2010), RP28, 12pp.
- [12] J. Kung, Basis-Exchange Properties, in: N.White (ed) Theory of matroids, Encyclopedia Math. Appl. 26, Cambridge University Press, Cambridge, 1986.
- [13] M. Lasoń, The coloring game on matroids, arXiv:1211.2456.

- [14] M. Lasoń, W. Lubawski, On-line list coloring of matroids, arXiv:1302.2338.
- [15] M. Michałek, Constructive degree bounds for group-based models, J. Combin. Theory Ser. A 120 (2013), no. 7, 1672-1694.
- [16] J.G. Oxley, Matroid Theory, Oxford Science Publications, Oxford University Press, Oxford, 1992.
- [17] A. Schrijver, Combinatorial Optimization, Polyhedra and Efficiency, Springer-Verlag, New York, 2003.
- [18] J. Schweig, Toric ideals of lattice path matroids and polymatroids, J. Pure Appl. Algebra 215 (2011), 2660-2665.
- [19] B. Sturmfels, Gröbner Bases and Convex Polytopes, Univ. Lecture Series 8, American Mathematical Society, Providence, 1995.
- [20] B. Sturmfels, Equations defining toric varieties, Proc. Sympos. Pure Math. 62 (1997), 437-449.
- [21] N. White, A unique exchange property for bases, Linear Algebra and its App. 31 (1980), 81-91.
- [22] N. White, The basis monomial ring of a matroid, Advances in Math. 24 (1977), 292-297.
- [23] D.R. Woodall, An exchange theorem for bases of matroids, J. Combin. Theory Ser. B 16 (1974), 227-228.