ON THE EQUIVARANT DE RHAM COHOMOLOGY FOR NON-COMPACT LIE GROUPS

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Abstract. Let G be a connected and non-necessarily compact Lie group acting on a connected manifold M. In this short note we announce the following result: for a G-invariant closed differential form on M, the existence of a closed equivariant extension in the Cartan model for equivariant cohomology is equivalent to the existence of an extension in the homotopy quotient.

1. Introduction

The Cartan model for the equivariant cohomology of the manifold M

$$\Omega^*_G M := (S(\mathfrak{g}^*) \otimes \Omega^* M)^G, \quad d_G = d + \Omega^a \iota_{X_a}$$

can be seen as the de Rham version for the equivariant cohomology. Whenever the Lie group G is compact, Cartan proved an equivariant version of the DeRahm Theorem, stating that the cohomology of the Cartan complex is canonically isomorphic to the cohomology with real coefficients of the homotopy quotient $H * (\Omega^*_G M) \cong H^*(M \times_G EG; \mathbb{R})$ [1] cf. [4, Thm. 2.5.1]. When the Lie group G is not compact, the cohomology of the complex $\Omega^*_G M$ (which we also call the Cartan complex) fails in many situations to be isomorphic to the cohomology of the homotopy quotient, and the explicit relation between the two has been very scarcely addressed.

Nevertheless, the Cartan complex is very well suited for studying equivariant conditions at the infinitesimal level. Of particular interest is the study of the conditions under which there is absence of anomalies in gauged WZW actions on Lie groups. In [5] Witten showed that the absence of anomalies in gauged WZW actions on compact Lie groups was equivalent to the existence of closed equivariant extension of the WZW term on the Cartan complex, further showing that the existence or absence of anomalies is purely topological. The arguments of Witten could be extended without trouble to the non-compact case (see [2, Chapter 4]), and together with the main result of this paper, we conclude that the absence or existence of anomalies is purely topological fact, independent of the compacity of the Lie group.

In this short note we investigate the relation between the cohomology of the G-equivariant Cartan complex of M and the cohomology of the homotopy quotient

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$M \times_G EG$, and we show that indeed there is a surjective map from the former to the latter. In particular this result implies that for a G-invariant closed differential form on M, the existence of a closed equivariant extension in the Cartan model for equivariant cohomology is equivalent to the existence of an extension in the homotopy quotient.

2. Equivariant Cartan complex for connected Lie groups

Let G be a connected Lie group with lie algebra $\mathfrak{g}$. Let $K \subset G$ be a maximal compact subgroup of G and denote by $\mathfrak{k}$ its Lie algebra. The inclusion of Lie algebras $\mathfrak{k} \hookrightarrow \mathfrak{g}$ induces a dual map $\mathfrak{g}^* \to \mathfrak{k}^*$ which is $\mathfrak{k}$-equivariant. Therefore we have the K-equivariant map

$$S(\mathfrak{g}^*) \to S(\mathfrak{k}^*)$$

from the symmetric algebra on $\mathfrak{g}^*$ to the symmetric algebra on $\mathfrak{k}^*$.

Consider a manifold M endowed with an action of G. The Cartan complex associated to the G-manifold M is

$$\Omega^*_G M := (S(\mathfrak{g}^*) \otimes \Omega^* M)^G, \quad d_G = d + \Omega^a \iota_{X_a}$$

where a runs over a base of $\mathfrak{g}$, Ω^a denotes the element in $\mathfrak{g}^*$ dual to a and X_a is the vector field on M that defines the element $a \in \mathfrak{g}$.

Remark 1. In the literature, whenever the Cartan complex is used, it is assumed that the Lie group is compact. In this note we extend the notation of Cartan to the non-compact case.

The composition of the natural maps

$$(S(\mathfrak{g}^*) \otimes \Omega^* M)^G \hookrightarrow (S(\mathfrak{g}^*) \otimes \Omega^* M)^K \to (S(\mathfrak{k}^*) \otimes \Omega^* M)^K$$

induces a homomorphism of Cartan complexes

$$\Omega^*_G M \to \Omega^*_K M.$$

Theorem 1. *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, let $\mathfrak{k}$ be the Lie algebra of the maximal compact subgroup K of G and consider a G-manifold M. Then the map*

$$\Omega^*_G M \to \Omega^*_K M$$

induces a surjective map in cohomology

$$H^*(\Omega^*_G M, d_G) \twoheadrightarrow H^*(\Omega^*_K M, d_K).$$

Since there are canonical isomorphisms $H^*(\Omega^*_K M, d_K) \cong H^*(M \times_K EK, \mathbb{R}) \cong H^*(M \times_G EG, \mathbb{R})$, we conclude that the canonical map

$$H^*(\Omega^*_G M, d_G) \twoheadrightarrow H^*(M \times_G EG, \mathbb{R})$$

is surjective.

Proof. Consider the complex $C^k(G, S(\mathfrak{g}^*) \otimes \Omega^\bullet M)$ defined in [3, Section 2.1] whose elements are smooth maps

$$f(g_1, \dots, g_k | X) : G^k \times \mathfrak{g} \to \Omega^\bullet M,$$

which vanish if any of the arguments g_i equals the identity of G. The differentials d and ι are defined by the formulas

$$\begin{aligned} (df)(g_1, \dots, g_k | X) &= (-1)^k df(g_1, \dots, g_k | X) \quad \text{and} \\ (\iota f)(g_1, \dots, g_k | X) &= (-1)^k \iota(X) f(g_1, \dots, g_k | X), \end{aligned}$$

as in the case of the differentials in Cartan's model for equivariant cohomology [1,4].

The differential $\bar{d} : C^k \to C^{k+1}$ is defined by the formula

$$(\bar{d}f)(g_0, \dots, g_k|X) = f(g_1, \dots, g_k|X) + \sum_{i=1}^{k}(-1)^i f(g_0, \dots, g_{i-1}g_i, \dots, g_k|X) + (-1)^{k+1} g_k f(g_0, \dots, g_{k-1}|\mathrm{Ad}(g_k^{-1})X),$$

and the fourth differential $\bar{\iota} : C^k \to C^{k-1}$ is defined by the formula

$$(\bar{\iota}f)(g_1, \dots, g_{k-1}|X) = \sum_{i=0}^{k-1}(-1)^i \frac{\partial}{\partial t} f(g_1, \dots, g_i, e^{tX_i}, g_{i+1} \dots, g_{k-1}|X),$$

where $X_i = \mathrm{Ad}(g_{i+1} \dots g_{k-1})X$.

If the map

$$f : G^k \to S(\mathfrak{g}^*) \otimes \Omega^\bullet M$$

has for image a homogeneous polynomial of degree l, then the total degree of the map f is $deg(f) = k + l$. The structural maps $d, \iota, \bar{d}$ and $\bar{\iota}$ are all of degree 1, and the operator

$$d_G = d + \iota + \bar{d} + \bar{\iota}$$

becomes a degree 1 map that squares to zero.

The cohomology of the complex

$$(C^*(G, S(\mathfrak{g}^*) \otimes \Omega^\bullet M), d_G)$$

will be denoted by

$$H^*(G, S(\mathfrak{g}^*) \otimes \Omega^\bullet M)$$

and in [3, Thm. 2.2.3] it was shown that there is a canonical isomorphism of rings

$$H^*(G, S(\mathfrak{g}^*) \otimes \Omega^\bullet M) \cong H^*(M \times_G EG; \mathbb{R})$$

Note that there are natural maps of complexes

$$C^*(G, S(\mathfrak{g}^*) \otimes \Omega^* M) \to C^*(K, S(\mathfrak{k}^*) \otimes \Omega^* M)$$

inducing an isomorphism on cohomology groups

$$H^*(G, S(\mathfrak{g}^*) \otimes \Omega^* M) \xrightarrow{\cong} H^*(K, S(\mathfrak{k}^*) \otimes \Omega^* M).$$

This isomorphism follows from the fact that the inclusion $K \subset G$ is a homotopy equivalence inducing a homotopy equivalence

$$M \times_K EK \simeq M \times_G EG$$

and the fact that

$$H^*(M \times_G EG, \mathbb{R}) \cong H^*(G, S(\mathfrak{g}^*) \otimes \Omega^* M)$$

for any connected Lie group G.

Filtering the double complex $C^*(G, S(\mathfrak{g}^*) \otimes \Omega^* M)$ by the degree of the elements in $S(\mathfrak{g}^*) \otimes \Omega^* M$ we obtain a spectral sequence whose first page is

$$E_1 = H^*_d(G, S(\mathfrak{g}^*) \otimes \Omega^* M),$$

the differentiable cohomology of G with values in the graded representation $S(\mathfrak{g}^*) \otimes \Omega^* M$. Note that in the 0-th row we obtain

$$E_1^{*,0} = (S(\mathfrak{g}^*) \otimes \Omega^* M)^G = \Omega^*_G M.$$

The same degree filtration applied to the complex $C^*(K, S(\mathfrak{k}^*)\otimes\Omega^* M)$ produces a spectral sequence which at the first page is $\overline{E}_1 = H^*_d(K, S(\mathfrak{k}^*)\otimes\Omega^* M)$, and since K is compact this simply becomes

$$\overline{E}_1^{*,0} = (S(\mathfrak{k}^*)\otimes\Omega^* M)^K = \Omega^*_K M$$

with $\overline{E}_1^{p,q} = 0$ for $q \neq 0$.

The first differential of the spectral sequence once restricted to the 0-th row $E_1^{*,0} = \Omega^*_G M$ is precisely the differential of the Cartan complex; therefore we obtain

$$E_2^{*,0} = H^*(\Omega^*_G M).$$

Equivalently we obtain

$$\overline{E}_2^{*,0} = H^*(\Omega^*_K M) \cong H^*(M \times_K EK, \mathbb{R}),$$

but in this case the spectral sequence collapses at the second page and the only non zero elements in $\overline{E}_\infty$ appear on the 0-th row $\overline{E}_\infty^{*,0} \cong H^*(M \times_K EK, \mathbb{R})$.

The canonical map between the complexes

$$C^*(G, S(\mathfrak{g}^*)\otimes\Omega^* M) \to C^*(K, S(\mathfrak{k}^*)\otimes\Omega^* M)$$

induces a map of spectral sequences $E_\bullet \to \overline{E}_\bullet$, and we know that at the pages at infinity it should induce an isomorphism $E_\infty^{*,\star} \stackrel{\cong}{\to} \overline{E}_\infty^{*,\star}$. Therefore the map

$$E_2^{*,0} \to \overline{E}_2^{*,0}$$

must be a surjective map, and hence we have the canonical map

$$\Omega^*_G M = E_1^{*,0} \to \overline{E}_1^{*,0} = \Omega^*_K M$$

inducing the desired surjective map in cohomology

$$H^*(\Omega^*_G M, d_G) \twoheadrightarrow H^*(\Omega^*_K M, d_K).$$

□

Finally, from the previous theorem we may conclude:

Corollary 1. *For a G-invariant closed differential form on M, the existence of a closed equivariant extension in the Cartan model for equivariant cohomology is equivalent to the existence of an extension in the homotopy quotient.*

Proof. A G-invariant closed differential form on M may be extended to a closed form in the Cartan complex if and only if its cohomology class lies in the image of the projection map

$$H^*(\Omega^*_G M, d_G) \to H^*(M).$$

This projection map can be seen as the composition of the maps

$$H^*(\Omega^*_G M, d_G) \twoheadrightarrow H^*(\Omega^*_K M, d_K) \to H^*(M).$$

Since the left hand side map is surjective, a G-invariant closed differential form on M may be extended to a closed form in the Cartan complex if and only if its cohomology class lies in the image of the right hand side map. The canonical isomorphisms

$$H^*(M \times_G EG; \mathbb{R}) \cong H^*(M \times_K EK; \mathbb{R}) \cong H^*(\Omega^*_K M, d_K)$$

imply the result. □

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