QUASICONVEXITY IN THE RELATIVELY HYPERBOLIC GROUPS

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ABSTRACT. We study different notions of quasiconvexity for a subgroup H of a relatively hyperbolic group G. The first result establishes equivalent conditions for H to be relatively quasiconvex. As a corollary we obtain that the relative quasiconvexity is equivalent to the dynamical quasiconvexity. This answers to a question posed by D. Osin [Os06].

In the second part of the paper we prove that a subgroup H of a finitely generated relatively hyperbolic group G acts cocompactly outside its limit set if and only if it is (absolutely) quasiconvex and every its infinite intersection with a parabolic subgroup of G has finite index in the parabolic subgroup.

We then obtain a list of different subgroup properties and establish relations between them.

1. INTRODUCTION

1.1. **Results and history.** Let G be a discrete group acting by homeomorphisms on a compactum X with the *convergence property*, i.e. the induced action on the space $\mathbf{\Theta}^3X$ of subsets of X of cardinality 3 is properly discontinuous. We say in this case that G acts 3*-discontinuously* on X. Denote by T the limit set $\mathbf{\Lambda}G$ and by $A = \mathbf{\Omega}G$ the discontinuity domain for the action $G \curvearrowright X$. We have $X = T \sqcup A$.

It is well-known that the action of a word-hyperbolic group on its Gromov boundary has convergence property [Gr87], [Tu94]. However there are convergence actions of groups that are not Gromov hyperbolic: the actions of non-geometrically finite Kleinian groups or those containing parabolic subgroups of rank at least 2; the actions of finitely generated groups on the space of ends and on their Floyd boundaries [Ka03]; the actions of the groups of homeomorphisms of spheres, discontinuous outside a zero-dimensional set [GM87].

An important class of groups form *relatively hyperbolic groups* (RHG for short). B. Bowditch [Bo97] proposed a construction of the "boundary" for such groups. This is a compactum where the group acts with the convergence property. A. Yaman proved that a group is RHG if it acts on a metrisable compactum X *geometrically finitely*, i.e. every point of X is either conical or bounded parabolic [Ya04].

We call an action $G \curvearrowright X$ *2-cocompact* if the induced action on the space $\mathbf{\Theta}^2X$ of distinct pairs is cocompact.

It follows from [Ge09], [Tu98] that an action of a finitely generated group G on a compactum X is geometrically finite if and only if it admits a 3-discontinuous and 2-cocompact action on X. So we will further regard the existence of a 3-discontinuous and 2-cocompact action as the definition of RHG. Note that an advantage of this definition is that many results known for finitely generated RHG remain valid for non-finitely generated ones [GP10].

Recall that a subset F of the Cayley graph of a group G is called (absolute) *quasiconvex* if every geodesic with the endpoints in F belong to a uniform neighborhood of F [Gr87]. Similarly

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a subset F of the Cayley graph of G is called *relatively quasiconvex* if every geodesic in the relative Cayley graph with endpoints in F belongs to a uniformly bounded neighborhood of F in the absolute Cayley graph (see Subsection 6.2).

B. Bowditch [Bo99] characterized the quasiconvex subgroups of Gromov hyperbolic groups in terms of their action on the Gromov boundary of the group. He proved that a subgroup H of a hyperbolic group G is quasiconvex if and only if for any two disjoint closed subsets K and L of T there are at most finitely many distinct elements of G such that the images of the limit set of H under them intersect both L and K (see Subsection 4.3). He calls the latter property *dynamical quasiconvexity*. A natural question arises: can the dynamical quasiconvexity for relatively hyperbolic groups be expressed in geometrical terms as it occurred for hyperbolic groups. D. Osin has conjectured [Os06, Problem 5.3] that, for RHG, the relative quasiconvexity is equivalent to the Bowditch dynamical quasiconvexity. This conjecture follows from our first main result.

We consider other "relative quasiconvexity" properties. Partially dynamical, partially geometrical. One of them is called *visible quasiconvexity* and means that the set of points of A such that a given set $F \subset X$ has sufficiently big diameter with respect to a shortcut metric (see 2.5) based at points which must belong to a bounded neighborhood of F with respect to the graph distance (see 4.3).

Generalizing the notion of relative quasiconvexity we call a subset $F \subset A$ α*-relatively quasiconvex* for some distortion function α if every α-distorted path with endpoints in F and outside the system of horospheres belongs to a bounded neighborhood of F (see 6.2).

Our first main result shows that all these notions of the relative quasiconvexity are equivalent.

Theorem A. *Let a finitely generated discrete group G act 3-discontinuously and 2-cocompactly on a compactum X. The following properties of a subset F of the discontinuity domain of the action are equivalent:*
— F is relatively quasiconvex;
— F is visibly quasiconvex;
— F is relatively α-quasiconvex where α is a quadratic polynomial with big enough coefficients.

Moreover, if H is a subgroup of G acting cofinitely on F then the visible quasiconvexity of F is equivalent to the dynamical quasiconvexity of H with respect to the action $G \curvearrowright X$. ♦

Note that in the first assertion of Theorem A we do not require that F is acted upon by a subgroup of G. If in particular $F = H$ is a subset of the Cayley graph of G then the second assertion of the Theorem implies the following Corollary answering affirmatively the above question of Osin.

Corollary. *A finitely generated subgroup H of a relatively hyperbolic group G is relatively quasiconvex if and only if it is dynamically quasiconvex.*

A graph Γ acting upon by a group G is called G-cofinite if it has at most finitely many G-non-equivalent edges; it is called *fine* if for every $n \in \mathbb{N}$ and for every edge e the set of simple loops in Γ passant par e of length n is finite. As an application of the above methods we obtain a generalization of the following result known for finitely generated groups [Ya04] to the case of infinitely generated (in general uncountable) RHG.

Proposition (7.1.2). *Let G be a group acting 2-cocompactly and 3-discontinuously on a compactum T. Then there exists a hyperbolic, G-cofinite graph Γ whose vertex stabilizers are all finite except the vertices corresponding to the parabolic points for the action $G \curvearrowright T$. Furthermore the graph Γ is fine.*

The aim of the second part of the paper is to relate different notions of the (absolute) quasiconvexity of the subgroups of a relatively hyperbolic group. We call a finitely generated subgroup H of a such group G *weakly α-quasiconvex* if H acts properly on A (i.e. the point stabilizers are finite) and there exists an orbit of H for which every two points can be joined by an α-distorted path lying in a uniformly bounded neighborhood of the orbit. This is a priori a partial case of a more general definition according to which H is α-quasiconvex if **every** α-distorted path in A connecting two points of an H-invariant and H-finite set E (i.e. $|E/H| < \infty$) is contained in a bounded neighborhood of E.

The following result describes the case when both these conditions are equivalent to a stronger property to have cocompact action outside the limit set.

Theorem B. *Let a finitely generated group G act 3-discontinuously and 2-cocompactly on a compactum X. Let* Par *be the set of the parabolic points for this action. Suppose that $A{=}X\setminus T{\neq}\varnothing$ where $T{=}\mathbf{\Lambda}G$ is the limit set for the action. Then there exists a constant $\lambda_0 \in]1, +\infty[$ such that the following properties of a subgroup H of G are equivalent:*

a :*H is weakly α-quasiconvex for some distortion function α for which $\alpha(n){\leqslant}\lambda_0^n$ $(n \in \mathbb{N})$, and for every $\mathfrak{p}{\in}\mathsf{Par}$ the subgroup $H{\cap}\mathsf{St}_G\mathfrak{p}$ is either finite or has finite index in $\mathsf{St}_G\mathfrak{p}$;*

b : *the space $(X \setminus \mathbf{\Lambda}H)/H$ is compact;*

c : *for every distortion function α bounded by λ_0^n $(n \in \mathbb{N})$, every H-invariant H-finite set $E{\subset}A$ is α-quasiconvex and for every $\mathfrak{p}{\in}\mathsf{Par}$ the subgroup $H{\cap}\mathsf{St}_G\mathfrak{p}$ is either finite or has finite index in* $\mathsf{St}_G\mathfrak{p}$. ♦

The choice of the above constant λ_0 will be discussed in 2.5. In particular every subexponential function satisfies our hypothesis. We also note that Theorem B shows that the cocompactness outside the limit set is a stronger condition than the usual quasiconvexity as it requires to preserve the parabolic subgroups in the above sense. One of the applications of the method used in the proof is the following.

Proposition (9.2.1). *Let G be a group acting 3-discontinuously and 2-cocompactly on a compactum X. Suppose H is a subgroup of G acting cocompactly on $X \setminus \mathbf{\Lambda}H$. If G is finitely presented then H also is.*

Note that every maximal parabolic subgroup of a RHG acts cocompactly outside its limit point on X. The above Proposition is known in the case when H is maximal parabolic [DG10]. However it is easy to construct an example of a quasiconvex subgroup H which cannot be parabolic for any geometrically finite action of the ambiant group G such that H still admits a cocompact action outside its limit set (Example 1, Subsection 9.2). We provide a direct proof of the Proposition in this more general case.

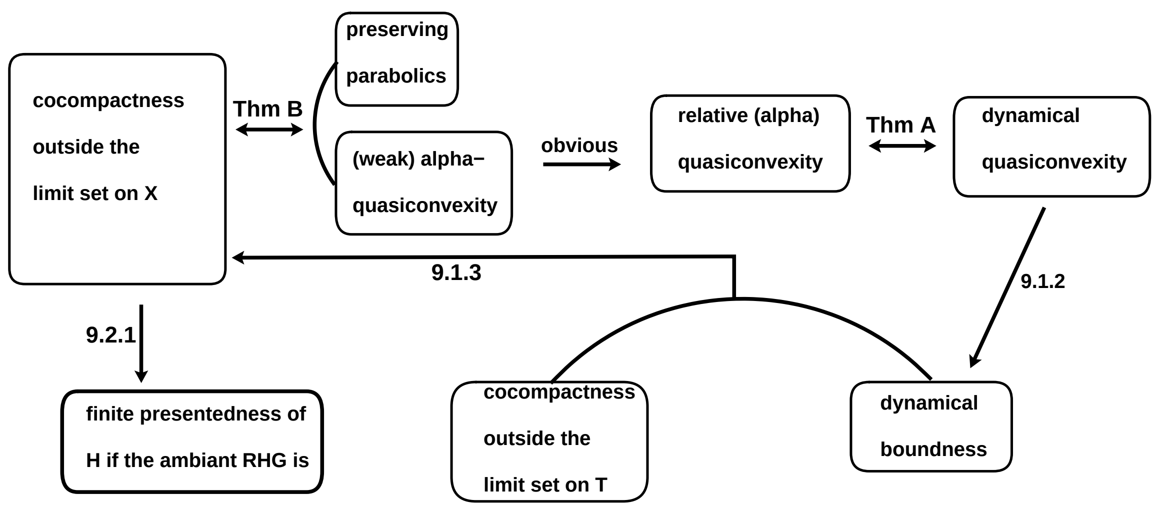
We note also that the cocompactness on the space $X \setminus \mathbf{\Lambda}H$ differs from the cocompactness on the “thinner" space $T \setminus \mathbf{\Lambda}H$ where $T{=}\mathbf{\Lambda}G$. There exist examples of finitely generated discrete (Kleinian) subgroups of the isometry group $\mathrm{Isom}\mathbb{H}^3$ of the real hyperbolic space $\mathbb{H}^3$ acting non-geometrically finitely on $\mathbb{H}^3$ and cocompactly outside their limit sets on $\mathbb{S}^2$ (so called *totally degenerate groups*). L. Bers proved that they appear on the boundary of the classical Teichmüller space of a closed surface [Be70].

A subgroup H of a group G acting 3-discontinuously on X is called *dynamically bounded* if every infinite set S of pairwise distinct elements of G modulo H contains an infinite subset S_0 such that $T \setminus \bigcup_{s\in S_0} s(\mathbf{\Lambda}H)$ has a non-empty interior (see Proposition 9.1.1 for equivalent definitions). Our next result is the following.

Proposition (9.1.2). *Let G act 3-discontinuously on $X{=}T{\sqcup}A$. Suppose H is a dynamically bounded subgroup of G acting cocompactly on $T \setminus \mathbf{\Lambda}H$. Then H acts cocompactly on $\widetilde{T} \setminus \mathbf{\Lambda}H$.*

We note that we do not assume in the Proposition that the action of G on X is 2-cocompact nor that G is finitely generated. We note also that the property opposite to the dynamical boundness was studied by C. McMullen [McM96]. If a discrete subgroup H is not dynamically bounded in the full isometry group $G{=}\mathrm{Isom}\mathbb{H}^3$ then he says that the limit set of H contains *deep points* (i.e. *repeller* points for infinite sequences of elements of G converging to a limit cross and not satisfying the above definition). It turns out that the limit set of a totally degenerate group contains uncountably many deep points [McM96, Corollary 3.15]. It seems to be an intriguing question to know whether such an example of a finitely generated subgroup of a relatively hyperbolic (or even geometrically finite Kleinian) group could exist.

To summarize we obtain a list of different properties of subgroups of a relatively hyperbolic group. The following diagram illustrates a natural order relation between them established in the paper.



Main Results and their Corollaries

1.2. **The structure of the paper.** In Section 2 we generalize a useful lemma by A. Karlsson [Ka03] about the Floyd length of "far" geodesics to the α-distorted curves for some appropriate scalar function α (Section 2). Using this lemma we obtain in Section 3 a uniform bound for the size of the projections of subsets of X.

Generalizing the ideas of [GP09] we prove in Section 4 that the (α-)convex hull of a closed set in X is itself closed in X (4.1.3). As a corollary we obtain that the subgroups of relatively hyperbolic groups acting cocompactly on X outside their limit sets are undistorted (Corollary 4.2.1) and α-quasiconvex (Proposition 4.2). We then introduce a notion of visible quasiconvexity and prove that it is equivalent to the dynamical quasiconvexity (4.3.3).

In Section 5 we discuss the notion of a general system of horospheres. In particular we obtain a uniform bound for the size of the projection of one horosphere onto another one (5.1.2). This result is used in the sequel.

The proof of Theorem A is completed in Section 6. We first prove that a lift of a geodesic path from the relative Cayley graph to the absolute Cayley graph is α-distorted for a quadratic polynomial α (6.1.1). We use it to prove that a relatively quasiconvex subset is visibly quasiconvex (6.2.1). These results imply Theorem A.

To illustrate the effectiveness of our methods we give in the Section 7 simple independent proofs of some known results about RHGs which use heavy techniques and require heavy references. A new result obtained here is the above Proposition 7.1.2.

In Section 8 we prove all statements of Theorem B in the cyclic order. The most difficult part is to prove the implication 'a⇒b'. This is done by constructing a discrete analog of the Dirichlet fundamental polyhedron for a discrete group acting on $\mathbb{H}^n$. The main step is to prove that this set, denoted by F_v $(v{\in}A)$, is compact in $X \setminus \mathbf{\Lambda}H$. The proof is based on the methods developed in Section 4.

In the last Section 9 we study subgroups of convergence groups which admit the dynamically boundness property. We prove here Propositions 9.1.2 and 9.2.1 mentioned above. The dynamical boundness turns out to be the weakest subgroup property studied in the paper: all other quasiconvexity properties imply it (see the table above). At the end of the Section we provide some examples of dynamically bounded subgroups which are not (relatively) quasiconvex and not finitely presented. We finish the paper by stating several questions which seem to be open and intriguing.

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2. Karlsson functions for generalized quasigeodesics

2.1. **Notations and definitions.** We keep some notations and terminology of [GP09] and [Ge10]. The canonical distance function on the set Γ^0 of vertices of a graph Γ is denoted by d. By Γ^1 we denote the set of pairs of vertices joined by edges.

For a subset S of a metric space $(M;\boldsymbol{\delta})$ and a nonnegative number r we consider the r-*neighborhood* $\mathsf{N}_r^{\boldsymbol{\delta}}S{\leftrightharpoons}\{\mathfrak{p}{\in}M\ :\ \boldsymbol{\delta}(S,\mathfrak{p}){\leqslant}r\}$. For a set S of vertices of a graph we sometimes write N_rS instead of $\mathsf{N}_r^{\mathsf{d}}S$.

For a path $\gamma : I \to \Gamma^0$ in a graph Γ we call I its *domain* $\mathsf{Dom}\gamma$ and the set $\gamma(I)$ its *image* $\mathsf{Im}\gamma$. The diameter of $\mathsf{Dom}\gamma$ is the *length* of γ. If $|I|{<}\infty$ we define $\partial\gamma{\leftrightharpoons}\gamma(\partial I)$. We extend naturally the meaning of $\partial\gamma$ over the half-infinite and bi-infinite paths in the case when Γ^0 is a discrete subset of a Hausdorff topological space X and the corresponding infinite branches of γ converge to points of X.

By $\mathsf{length}_{\boldsymbol{\delta}}\gamma$ we denote the length of a path γ with respect to a path-metric $\boldsymbol{\delta}$.

Considering a function f defined on a subset of $\mathbb{Z}$ we sometimes write f_n instead of $f(n)$.

By $|S|$ we denote the cardinality of a set S. By $\mathbf{\Theta}^nS$ we denote the set of all subsets of S of cardinality n. When S is a topological space then $\mathbf{\Theta}^nS$ is considered with the induced topology. By S^nS we denote the set of "generalized unordered n-tuples": formally this is the quotient of the Cartezian power S^n by the action of the permutation group.

If S is acted upon by a group G it is called G-set. A subset M of G-set S is called G-*finite* if M meets finitely many G-orbits. In this case the image of M in S/G is finite.

Recall that a limit point $p \in \mathbf{\Lambda} G$ for the convergence action of G on a compactum S is called *parabolic* if it is the unique limit point for the action of its stabilizer $\mathsf{St}_G p=\{g \in G \ : \ gp=p\}$ on S. A parabolic limit point $p \in \mathbf{\Lambda} G$ is called *bounded parabolic* if $S \setminus \{p\}/\mathsf{St}_G p$ is compact.

2.2. **Distorted paths.** A nondecreasing function $\alpha : \mathbb{N} \to \mathbb{R}_{>0}$ such that $\forall n\ \alpha_n \geqslant n$ is called a *distortion function*. Thus, the minimal distortion function is the function $\mathsf{id} : n \mapsto n$.

Let α be a distortion function. A path $\gamma : I \to \Gamma^0$ in a graph Γ is said to be α-*distorted* if $\mathsf{diam} J \leqslant \alpha(\mathsf{diam}\gamma(\partial J))$ for every **finite** interval $J \subset I$.

If α has one of the following forms (1) $n \mapsto n$, (2)$n \mapsto Cn$, (3) $n \mapsto Cn{+}D$, the notion 'α-distorted' means respectively 'geodesic', 'Lipschitz', 'large-scale Lipschitz' ('quasigeodesic'). The case when α is a quadratic polynomial will be of our particular interest.

2.3. **Scaling of the graph metric.** Recall the notions related to the Floyd metrics.

A function $f : \mathbb{N} \to \mathbb{R}$ is said to be a (Floyd) *scaling function* if $\sum_{n \geqslant 0} f_n < \infty$ and there exists a positive λ such that $1 \geqslant f_{n+1}/f_n \geqslant \lambda$ for all $n \in \mathbb{N}$. The supremum of such numbers λ is called the *decay rate* of f.

Let f be a scaling function and let Γ be a connected graph. For each vertex $v \in \Gamma^0$ we define on Γ^0 a new metric $\boldsymbol{\delta}_{v,f}$ as the maximal among the metrics ϱ on Γ^0 such that $\varrho(x, y) \leqslant f(\mathsf{d}(v, \{x, y\}))$ for each $\{x, y\} \in \Gamma^1$. We say that $\boldsymbol{\delta}_{v,f}$ is the *Floyd metric* (with respect to the scaling function f) *based* at v.

When f is fixed we write $\boldsymbol{\delta}_v$ instead of $\boldsymbol{\delta}_{v,f}$. When v is also fixed we write $\boldsymbol{\delta}$ instead of $\boldsymbol{\delta}_v$.

One verifies that $\boldsymbol{\delta}_u/\boldsymbol{\delta}_v \geqslant \lambda^{\mathsf{d}(u,v)}$ for $u, v \in \Gamma^0$. Thus the Cauchy completion $\overline{\Gamma}_f$ of Γ^0 with respect to $\boldsymbol{\delta}_{v,f}$ does not depend on v. The *Floyd boundary* is the space $\partial_f \Gamma \leftrightharpoons \overline{\Gamma}_f \setminus \Gamma^0$. Every d-isometry of Γ extends to a homeomorphism $\overline{\Gamma}_f \to \overline{\Gamma}_f$. The Floyd metrics extend continuously onto the Floyd completion $\overline{\Gamma}_f$.

2.4. **Karlsson functions.** Let f be a Floyd function and let α be a distortion function.
A non-increasing function $\mathsf{K} : \mathbb{R}_{>0} \to \mathbb{N}$ is called *Karlsson function* for the pair (f, α) if

$$\mathsf{d}(v, \mathsf{Im}\gamma) \leqslant \mathsf{K}(\mathsf{length}_{\boldsymbol{\delta}_{v,f}} \gamma) \tag{2.4.1}$$

for each α-distorted path γ in a connected graph with a vertex v. A pair (f, α) where f is a scaling function and α is a distortion function is said to be *appropriate* if it possesses a Karlsson function. It is proved in [Ka03] that every pair of the form (f, id) is appropriate. A similar agrument can be applied to show that (f, α) is appropriate for $\alpha : n \mapsto Cn{+}D$.

We need one more class of appropriate pairs. Actually, all pairs considered in this article belong to this class.

Proposition 2.4.1. *If* $\sum_{n \geqslant 0} \alpha_{2n+1} f_n < \infty$ *then the pair* (f, α) *is appropriate.*

Proof. Let v be the reference point of some graph Γ. Denote $|x| \leftrightharpoons \mathsf{d}(v, x)$.

Let $\gamma : I \to \Gamma^0$ be an α-distorted path. We can assume that $\mathsf{d}(v, \mathsf{Im}\gamma) = |\gamma(0)| \leftrightharpoons r$. It suffices to prove that $\mathsf{length}_{\boldsymbol{\delta}}(\gamma|_{I \cap \mathbb{N}})$ is small enough whenever r is big enough. So we can assume that I is an initial segment of $\mathbb{N}$.

By induction we define a strictly increasing sequence $x_s \in I$ for $s \geqslant r$ such that $|\gamma(x_s)| = s$ and $\gamma([x_s, x_{s+1}]) \cap \mathsf{N}^{\mathsf{d}}_{s-1} v = \varnothing$. Indeed let $x_r \leftrightharpoons 0$. If x_s is already defined and different from $\mathsf{max} I$, put $x_{s+1} \leftrightharpoons 1{+}\mathsf{max}\{x \in I \ : \ x \geqslant x_s$ and $\mathsf{d}(v, \gamma(x)) = s\}$. Now the interval I has subdivided into the segments $I_s \leftrightharpoons [x_s, x_{s+1}]$. By the $\triangle$-inequality we have $\mathsf{length}_d \gamma|_{I_s} = \mathsf{diam}_{\mathsf{d}} I_s = x_{s+1} - x_s \leqslant \alpha_{2s+1}$, hence $\mathsf{length}_{\boldsymbol{\delta}}\gamma|_{I_s} \leqslant f(d(v, I_s)) \cdot \mathsf{length}_d\gamma|_{I_s} \leqslant \alpha_{2s+1}f_s$. Thus $\mathsf{length}_{\boldsymbol{\delta}}\gamma \leqslant \sum_{s=r}^{k-1}\alpha_{2s+1}f_s + \alpha_{2k}f_k$ and $k < +\infty$ only if I is finite and $|\gamma(x_k)|{=}|\gamma(\mathsf{max}I)|$. In any case we have

$$\mathsf{length}_{\boldsymbol{\delta}}\gamma \leqslant \sum_{s=r}^{\infty}\alpha_{2s+1}f_s \text{ where } r{=}\mathsf{d}(v, \mathsf{Im}\gamma). \tag{2.4.2}$$

Thus the function $\varepsilon \mapsto \min\{r : \sum_{s=r}^{\infty}\alpha_{2s+1}f_s \leqslant \varepsilon/2\}$ is a Karlsson function for (f, α). Indeed if not then $r{=}\mathsf{d}(v, \mathsf{Im}\gamma) > \mathsf{K}(\mathsf{length}_{\boldsymbol{\delta}}\gamma){=}r_0$ and $\sum_{s=r}^{\infty}\alpha_{2s+1}f_s \leqslant \sum_{s=r_0}^{\infty}\alpha_{2s+1}f_s < \mathsf{length}_{\boldsymbol{\delta}}\gamma$ contradicting 2.4.2. □

It follows immediately from 2.4.1 that every α-distorted ray converges to a point at the Floyd boundary. So, an α-distorted ray extends to a continuous map $\overline{\mathbb{N}}{\leftrightharpoons}\mathbb{N}{\cup}\infty \to \overline{\Gamma}_f$. Similarly, any α-distorted line extends to a continuous map $\overline{\mathbb{Z}}{\leftrightharpoons}\mathbb{Z}{\cup}\{\pm\infty\} \to \overline{\Gamma}_f$ (see [GP09, Proposition 2.5] for the case of affine distortion).

Now $\partial\gamma{\leftrightharpoons}\gamma(\partial I)$ is well-defined for finite, half-infinite and bi-infinite α-distorted paths. It is a subset of $\overline{\Gamma}_\lambda$.

2.5. **Floyd map.** From now on we fix a compactum which we denote for the sake of convenience by $\widetilde{T}$. We also fix a 3-discontinuous action of a discrete group G on $\widetilde{T}$. If the opposite is not stated we will also suppose that the action is 2-cocompact. We have $\widetilde{T}{=}T{\sqcup}A$ where $T{=}\mathbf{\Lambda}G$ is the limit set and $A{=}\mathbf{\Omega}\widetilde{T}$ is the discontinuity set for the action $G{\curvearrowright}\widetilde{T}$. Up to adding a discrete G-orbit to the space T we can always assume that A is a non-empty, discrete and G-finite set (i.e. $|A/G| < \infty$), and the compactum $\widetilde{T}$ contains at least 3 points (i.e. $\mathbf{\Theta}^3\widetilde{T}{\neq}\emptyset$).

Let Γ^1 be a G-finite subset of $\mathbf{\Theta}^2A$ such that the graph Γ with $\Gamma^0{=}A$ is connected. Such subset exists if and only if G is finitely generated (see e.g. [GP09] or [GP10] where A is the vertex set of the Cayley graph of G or a G-orbit of an entourage of T).

Convention 1. Since now on by default we assume that G is a finitely generated group acting on a locally finite, G-finite and connected graph Γ such that $\Gamma^0{=}A$. We will always assume that $|\widetilde{T}| > 2$.

It is proved in [Ge10] that there exists an exponential scaling function $f_0(n){=}\mu_0^n$ ($\mu_0{\in}]0, 1[,\ n{\in}\ \mathbb{N}$), and a metric ϱ on $\widetilde{T}$ determining the topology of $\widetilde{T}$ such that $\boldsymbol{\delta}_{v,f_0}{\geqslant}\varrho$ on A where $\boldsymbol{\delta}_{v,f_0}$ is the Floyd metric on $\overline{\Gamma}_{f_0}$ at a point $v \in A$. Thus the inclusion map $A \hookrightarrow \widetilde{T}$ extends continuously to the map $\overline{\Gamma}_{f_0} \to \widetilde{T}$ called *Floyd map*.

The Floyd map induces a set of shortcut metrics $\overline{\boldsymbol{\delta}}_v$ on $\widetilde{T}$ $(v{\in}A)$, where every $\overline{\boldsymbol{\delta}}_v$ is the maximal among all metrics ρ on $\widetilde{T}$ [GP09].

We denote by $\lambda_0 \in]1, +\infty[$ the maximal constant for which the distortion function $\alpha_n{=}\lambda_0^n$ $(n \in \mathbb{N})$ is appropriate for the above Floyd function f_0.

Convention 2. We will always consider a Floyd function f satisfying $f(n){\geqslant}f_0(n)$ $(n \in \mathbb{N})$, so the Floyd map also exists for f. For a fixed Floyd function f we will always choose an appropriate distortion function α $(\alpha_n{\leqslant}\lambda_0^n\ (n \in \mathbb{N}))$.

For every appropriate pair (f, α) we fix a Karlsson function denoted by $\mathsf{K}_{f,\alpha}$. We also write K_α instead of $\mathsf{K}_{f,\alpha}$ and K instead of K_{id}.

3. Projections

3.1. **Boundary equivalence.** For a set $E{\subset}A$ define $\partial E{\leftrightharpoons}T{\cap}\overline{E}$. This "boundary" is nonempty if and only if E is infinite. Since A is a discrete open subset of a compactum, for any neighborhood N of ∂E in $\widetilde{T}$ the set $E\setminus N$ is finite. In particular $\overline{E}{=}E{\cup}\partial E$. Thus, for $a{\in}A$ and $\varepsilon{>}0$ the number

(3.1.1) $$\mathsf{C}_{E,a}(\varepsilon){\leftrightharpoons}\mathsf{min}\{r : E\setminus \mathsf{N}^{\mathsf{d}}_r a \subset \mathsf{N}^{\overline{\boldsymbol{\delta}}_a}_{\varepsilon}\partial E\}$$

is finite, where $\mathsf{N}^{\mathsf{d}}_r$ and $\mathsf{N}^{\overline{\boldsymbol{\delta}}_a}_{\varepsilon}$ are r and ε-neighborhoods with respect to the metrics d on A and $\overline{\boldsymbol{\delta}}_a$ on $\widetilde{T}$ respectively.

Definition. Two sets $E, F{\subset}A$ are said to be ∂-*equivalent* (notation $E{\sim_\partial}F$) if $\partial E{=}\partial F$.

Proposition 3.1.2. *$E{\sim_\partial}\mathsf{N}^{\mathsf{d}}_r E$ for every $E{\subset}A$, $r{\in}\mathbb{N}$.*

Proof. It suffices to prove the statement for $r{=}1$. The result follows from the fact that the metric $\overline{\boldsymbol{\delta}}_a$ determines the topology of $\widetilde{T}$ and that the $\overline{\boldsymbol{\delta}}_a$-length of an edge e tends to zero while $\mathsf{d}(a, e) \to \infty$. □

3.2. **Projections of subsets of A.** For a vertex $a{\in}A$ define the *projection set* $\mathsf{Pr}_E a{\leftrightharpoons}\{v{\in}E : \mathsf{d}(a, v){=}\mathsf{d}(a, E)\}$. For a nonempty set $B{\subset}A$ define $\mathsf{Pr}_E B{\leftrightharpoons}\cup\{\mathsf{Pr}_E b : b{\in}B\}$.

Proposition 3.2.1. *If $\partial E{\cap}\partial B{=}\varnothing$ $(E, B{\subset}A)$ then $\mathsf{Pr}_E B$ is finite.*

Proof. Suppose that $\partial E{\cap}\partial B{=}\varnothing$ for $E, B{\subset}A$. We can assume that E is infinite and hence $\partial E{\neq}\varnothing$. Since $\partial E{\cap}\overline{B}{=}\varnothing$ the number

(3.2.2) $$\rho{\leftrightharpoons}\rho(E, B){\leftrightharpoons}\mathsf{sup}\{\overline{\boldsymbol{\delta}}_v(\partial E, \overline{B}) : v{\in}E\}$$

is positive. Let $0{<}\delta{<}\varepsilon{<}\rho$ and let $a{\in}E$ be such that $\overline{\boldsymbol{\delta}}_a(\partial E, \overline{B}){>}\varepsilon$. We will show that $\mathsf{Pr}_E B$ is within a bounded distance from a.

Denote $r{\leftrightharpoons}\mathsf{C}_{E,a}(\varepsilon{-}\delta)$. If $b{\in}B$, $v{\in}\mathsf{Pr}_E b$ then either $v{\in}\mathsf{N}^{\mathsf{d}}_r a$ or $v{\in}\mathsf{N}^{\overline{\boldsymbol{\delta}}_a}_{\varepsilon-\delta}\partial E$. In the latter case we have $\overline{\boldsymbol{\delta}}_a(b, v){\geqslant}\overline{\boldsymbol{\delta}}_a(\partial E, \overline{B}) - \overline{\boldsymbol{\delta}}_a(v, \partial E){>}\varepsilon - \varepsilon + \delta{=}\delta$. Thus for a geodesic segment γ between b and v by 2.4.1 we have $\mathsf{d}(a, \mathsf{Im}\gamma){\leqslant}s{\leftrightharpoons}\mathsf{K}(\delta)$. Therefore for $c{\in}\mathsf{Im}\gamma$ such that $\mathsf{d}(a, c){=}\mathsf{d}(a, \mathsf{Im}\gamma)$ we obtain $\mathsf{d}(b, c){+}\mathsf{d}(c, v){=}\mathsf{d}(b, v){\leqslant}\mathsf{d}(b, a){\leqslant}\mathsf{d}(b, c){+}\mathsf{d}(c, a)$. Thus $\mathsf{d}(c, v){\leqslant}\mathsf{d}(c, a)$ and $\mathsf{d}(a, v){\leqslant}\mathsf{d}(a, c){+}\mathsf{d}(c, v){\leqslant}2s$. It yields

(3.2.3) $$d(a, v){\leqslant}2 \cdot \max\{r, \mathsf{K}(\delta)\}.$$

□

3.3. **Projection of the subsets of $\widetilde{T}$.** For a set $F{\subset}\widetilde{T}$ denote by $\mathsf{Loc}_{\widetilde{T}}F$ the set of all neighborhoods of F in $\widetilde{T}$.

Let $E{\subset}A$. A $\widetilde{T}$-neighborhood P of a point $\mathfrak{p}{\in}T{\setminus}\partial E$ is called E-*stable* if $\mathsf{Pr}_E(P{\cap}A){=}\mathsf{Pr}_E(Q{\cap}A)$ for every $Q{\in}\mathsf{Loc}_{\widetilde{T}}\mathfrak{p}$ such that $Q{\subset}P$. By 3.2.1 every point $\mathfrak{p}{\in}T \setminus \partial E$ possesses a E-stable neighborhood since otherwise we would have a strictly decreasing infinite sequence of sets of the form $\mathsf{Pr}_E(P{\cap}A)$, $P{\in}\mathsf{Loc}_{\widetilde{T}}\mathfrak{p}$. If P, Q are E-stable neighborhood of $\mathfrak{p}$ then $P{\cap}Q$ is also an E-stable neighborhood and $\mathsf{Pr}_E(P{\cap}A){=}\mathsf{Pr}_E(P{\cap}Q{\cap}A){=}\mathsf{Pr}_E(Q{\cap}A)$.

Now we can extend the projection map over $\widetilde{T} \setminus \partial E$: the projection $\mathsf{Pr}_E\mathfrak{p}$ of a point $\mathfrak{p}{\in}T \setminus \overline{E}$ is the projection of any its E-stable neighborhood.

We need a uniform estimate for the size of the projection. To this end we put

(3.3.1) $$C_E(\varepsilon){\leftrightharpoons}\mathsf{sup}\{\mathsf{C}_{E,a}(\varepsilon) : a{\in}E\}.$$

Let us call an **infinite** set $E{\subset}A$ *weakly homogeneous* if $C_E(\varepsilon) < \infty$ for every $\varepsilon{>}0$.

The following example is motivating. Let H be an infinite subgroup of G and let E be an H-finite subset of A. Since $\mathsf{C}_{E,a}(\varepsilon)=\mathsf{C}_{gE,ga}(\varepsilon)$ $(g\in G)$ the set $\{C_{E,a}(\varepsilon) : a\in E\}$ is finite for every $\varepsilon > 0$. Hence E is weakly homogeneous.

We call C_E the *convergence function* for $E\subset A$. Its role is similar to that ot Karlsson functions.

Assuming that the constants ε and δ from the proof of 3.2.1 satisfy $\delta\geqslant\rho/4$ and $\varepsilon-\delta=\rho/4$ we have

Proposition 3.3.2. *For a weakly homogeneous set E the d-diameter of $\mathsf{Pr}_E B$ depends only on the number $\rho=\rho(E,B)$ of 3.2.2 and the function C_E. More precisely,*

$$\mathsf{diam}_{\mathsf{d}}\mathsf{Pr}_E B \leqslant 2\cdot\mathsf{max}\{\mathsf{C}_E(\rho/4), \mathsf{K}(\rho/4)\}. \tag{3.3.3}$$

□

We extend the distance function d over $\widetilde{T}^2$ by setting $\mathsf{d}(\mathfrak{p},\mathfrak{q})\leftrightharpoons\infty$ for $\mathfrak{q}\neq\mathfrak{p}$ and $\mathfrak{p}\in T$. So, for $F\subset\widetilde{T}$ we have $\mathsf{N}^{\mathsf{d}}_r F=\partial F\cup\mathsf{N}^{\mathsf{d}}_r(F\cap A)$.

4. Hulls and convexity

4.1. **α-quasiconvexity and α-hull.** Let α be a distortion function. The *α-hull* of set $F\subset\widetilde{T}$ is $\mathsf{H}_\alpha F\leftrightharpoons\bigcup\{\mathsf{Im}\gamma : \gamma$ is an α-distorted path in A and $\partial\gamma\subset F\}$.
A set $F\subset\widetilde{T}$ is said to be *α-quasiconvex* if $\mathsf{H}_\alpha F \subset \mathsf{N}_r F$ for some $r<\infty$. In the case when $\alpha=\mathsf{id}$ "α-quasiconvex" means "quasiconvex".

In the sequel we will always assume that α satisfies the hypothesis of Proposition 2.4.1. Since $\sum_{n\geqslant 0} f_n < +\infty$ the function $\alpha+1$ also satisfies it. On the other hand, $\mathsf{N}_r\mathsf{H}_\alpha E\subset\mathsf{H}_{\alpha+2r}E$. This implies that

$$A \subset \cup_{r\geqslant 0}\mathsf{H}_{\alpha+r}E \tag{4.1.1}$$

for every $E\subset A$.

Proposition 4.1.2. *For every $\varepsilon>0$ there exists a number $s=s(\varepsilon,\alpha)$ such that $\mathsf{H}_\alpha F\subset\mathsf{N}^{\overline{\boldsymbol{\delta}}_a}_\varepsilon F\cup\mathsf{N}^{\mathsf{d}}_s a$ for every $F\subset\widetilde{T}$ and $a\in A$.*

Proof. It is similar to that of [GP09, Main Lemma].

Define $r\leftrightharpoons\mathsf{K}_\alpha(\varepsilon)$, $s\leftrightharpoons r+\frac{1}{2}\alpha(2r)$.

Let $v\in\mathsf{H}_\alpha F\setminus\mathsf{N}^{\overline{\boldsymbol{\delta}}_a}_\varepsilon F$ and let $\gamma : I \to A$ be an α-distorted path with $\gamma(0)=v$ and $\partial\gamma\subset F$. Denote $\gamma_+\leftrightharpoons\gamma|_{I\cap\mathbb{N}}$, $\gamma_-\leftrightharpoons\gamma|_{I\cap(-\mathbb{N})}$. Since $\mathsf{length}_{\boldsymbol{\delta}_a}\gamma_\pm \geqslant \boldsymbol{\delta}_a(F,v) \geqslant \overline{\boldsymbol{\delta}}_a(F,v) > \varepsilon$ by 2.4.1 we have $\mathsf{d}(a,\mathsf{Im}\gamma_\pm)\leqslant r$. Let $J\ni 0$ be a subsegment of I with $\gamma(\partial J)\subset\mathsf{N}^{\mathsf{d}}_r a$. So $\mathsf{diam}_{\mathsf{d}}\gamma(\partial J)\leqslant 2r$ and $\mathsf{length}_{\mathsf{d}}\gamma|_J\leqslant\alpha(2r)$. Hence $d(v,a)\leqslant d(a,\mathsf{Im}\gamma_\pm)+\frac{1}{2}\cdot\mathsf{length}_{\mathsf{d}}\gamma|_J\leqslant r+\frac{1}{2}\alpha(2r)=s$. So $v\in\mathsf{N}^{\mathsf{d}}_s a$. □

Proposition 4.1.3. *$E\sim_\partial \mathsf{H}_\alpha\overline{E}$ for every $E\subset A$.*

Proof. Suppose by contradiction that $\mathfrak{p}\in\partial\mathsf{H}_\alpha\overline{E}\setminus\partial E$. If $0<\varepsilon<\overline{\boldsymbol{\delta}}_a(\partial E,\mathfrak{p})$ for $a\in A$ then, by 4.1.2, $\mathsf{H}_\alpha E$ is contained in the closed set $\mathsf{N}^{\overline{\boldsymbol{\delta}}_a}_\varepsilon E\cup\mathsf{N}^{\mathsf{d}}_s a$ that does not contain $\mathfrak{p}$. A contradiction. □

4.2. **Subgroups acting cocompactly outside its limit set.** We provide below several properties of a subgroup H of G acting cocompactly on the complement $\widetilde{T}\setminus\boldsymbol{\Lambda}H$ of its limit set $\Lambda(H)$. In particular the group H can be a parabolic subgroup of G for the action on $\widetilde{T}$. However there are a lot of examples of subgroups satisfying this property and which are essentially non-parabolic (see Example 1 in Subsection 9.2.1). In the following Proposition we use the projection map Pr_E on a subset $E\subset A$ introduced in 3.2.

Proposition 4.2.1. *Let H be a subgroup of G acting cocompactly on $\widetilde{T} \setminus \mathbf{\Lambda} H$ and let E be a nonempty H-finite H-invariant set. Then the multivalued map Pr_E is quasi-isometric i.e. there exists a constant C such that $\mathsf{diam}(\mathsf{Pr}_E e) \leqslant C$ for every edge $e{\in}\Gamma^1$.*

Proof. By 3-discontinuity, $\partial E{=}\mathbf{\Lambda} H$. Let K be compact set such that $HK{=}\widetilde{T} \setminus \mathbf{\Lambda} H$. By 3.1.2 the set $K_1{\leftrightharpoons}\mathsf{N}^{\mathsf{d}}_1 K$ is closed and disjoint from ∂E. For every edge $e{\in}\Gamma^1$ there exists $h{\in}H$ such that $he{\subset}K_1$. Then $\mathsf{diam}(\mathsf{Pr}_E e){=}\mathsf{diam}(\mathsf{Pr}_E(he)){\leqslant}C{\leftrightharpoons}\mathsf{diam}(\mathsf{Pr}_E K_1)$.

Since E is weakly homogeneous by 3.3.3 we have $C < \infty$. □

Corollary. *If H acts cocompactly on $\widetilde{T} \setminus \mathbf{\Lambda} H$ then H is undistorted in G.* □

If, for a subgroup $H{<}G$, the space $(\widetilde{T} \setminus \mathbf{\Lambda} H)/H$ is compact then every closed H-invariant set $E{\subset}A$ is H-finite. Indeed, its image in $(\widetilde{T} \setminus \mathbf{\Lambda} H)/H$ is a closed discrete subset of a compact space.

In [GP09, Lemma 3.3] we make use of this observation for parabolic subgroups. The following Proposition show that the stronger quasiconvexity property is true for such subgroups.

Proposition 4.2.2. *For every subgroup $H{<}G$ acting cocompactly on $\widetilde{T} \setminus \mathbf{\Lambda} H$ and every H-finite H-invariant set E the set $\mathsf{H}_\alpha \overline{E}$ is H-finite and $\overline{E}$ is α-quasiconvex. In particular, H is an α-quasiconvex subgroup for any appropriate distortion function α.*

Proof. By 4.1.3 the set $\partial E{\cup}\mathsf{H}_\alpha \overline{E}$ is closed. Hence the H-invariant set $\mathsf{H}_\alpha \overline{E}$ is closed in $\widetilde{T} \setminus \partial E$. By the above observation it is H-finite. Thus $\mathsf{H}_\alpha \overline{E}{\subset}\mathsf{N}^{\mathsf{d}}_r E$ for some $r > 0$. So $\overline{E}$ is α-quasiconvex.

In the case when E is a single H-orbit this means the last statement of 4.2.2. □

4.3. **Dynamical and visible quasiconvexity.** For a set $F{\subset}\widetilde{T}$ define its *ε-hull* as

$$\mathsf{V}_\varepsilon F{\leftrightharpoons}\{a{\in}A : \mathsf{diam}_{\overline{\boldsymbol{\delta}}_a} F{\geqslant}\varepsilon\}.$$

Note that $\mathsf{V}_\varepsilon F{=}\mathsf{V}_\varepsilon \overline{F}$ for every $F{\subset}\widetilde{T}$.

Definition. A set $F{\subset}\widetilde{T}$ is said to be *visibly quasiconvex* if for every $\varepsilon{>}0$ there exists $r{=}r(\varepsilon){<}\infty$ such that $\mathsf{V}_\varepsilon F \subset \mathsf{N}^{\mathsf{d}}_r F$. We call the function $\mathsf{Q}_F : \varepsilon \mapsto r(\varepsilon)$ *visible quasiconvexity function*.

Proposition 4.3.1. *$\partial \mathsf{V}_\varepsilon \overline{E} \subset \partial E$ for every $E{\subset}A$ and $\varepsilon{>}0$.*

Proof. Let $v{\in}\mathsf{V}_\varepsilon \overline{E}{=}\mathsf{V}_\varepsilon E$. Let $\gamma : I \to A$ is a d-geodesic with $\partial\gamma{\subset}E$ and $\mathsf{diam}_{\overline{\boldsymbol{\delta}}_v}\partial\gamma \geqslant \varepsilon$. We have $\mathsf{d}(v, \mathsf{H}_{\mathsf{id}} E) \leqslant \mathsf{d}(v, \mathsf{Im}\gamma) \leqslant r{\leftrightharpoons}\mathsf{K}(\varepsilon)$. Thus $\mathsf{V}_\varepsilon \overline{E} \subset \mathsf{N}_r \mathsf{H}_{\mathsf{id}} E$. Since $\mathsf{N}_r \mathsf{H}_{\mathsf{id}} E \overset{\text{by 3.1.2}}{\sim_\partial} \mathsf{H}_{\mathsf{id}} E \overset{\text{by 4.1.3}}{\sim_\partial} E$ it yields $\partial \mathsf{V}_\varepsilon \overline{E} \subset \partial E$. □

Corollary. *Every quasiconvex set is visibly quasiconvex.*
Proof. It follows immediately from the inclusions $\mathsf{V}_\varepsilon \overline{E} \subset \mathsf{N}_r \mathsf{H}_{\mathsf{id}} E$ and $\mathsf{H}_{\mathsf{id}} E \subset \mathsf{N}_{r_0}(E)$ where $r{=}\mathsf{K}(\varepsilon)$ and r_0 is the quasiconvexity constant of E. □

We now recall few facts needed in the sequel. Let X be a compactum. A neighborhood of the diagonal $\mathbf{\Delta}^2 X$ of the space X^2 is called *entourage*. The set of all entourages on X is denoted by $\mathsf{Ent}X$. For an entourage $\mathbf{u}$ a subset S of X is called $\mathbf{u}$-*small* if $S^2 \subset \mathbf{u}$. The set of all $\mathbf{u}$-small sets is denoted by $\mathsf{Small}(\mathbf{u})$ (see [Ge09] and [GP10] for more details).

Definition [Bo99]. A subgroup H of a discrete group G acting 3-discontinuously on a compactum X is said to be *dynamically quasiconvex* if for every entourage $\mathbf{u}$ of X the set $G_{\mathbf{u}}{=}\{g{\in}G : g(\mathbf{\Lambda} H){\notin}\mathsf{Small}(\mathbf{u})\}$ is H-finite with respect to the H-action from the **right** .

Remarks. a) The above definition coincides with the notion of the dynamical quasiconvexity proposed in [Bo99]. The latter one states that the set of the left cosets

(4.3.2) $$\{gH\ :\ gS\cap L{\neq}\emptyset \text{ and } gS\cap K{\neq}\emptyset\}$$

is finite, whenever K and L are disjoint closed subsets of X and $S{=}\mathbf{\Lambda}H$. Note first that one can consider here only the entourages of a special form $\mathbf{u}_{P,Q}{=}\mathsf{S}^2T\setminus P{\times}Q$ where P and Q are disjoint closed sets is not a restriction since the set of entourages of this form generates the filter $\mathsf{Ent}T$ of entourages.

In order to see that (4.3.2) is equivalent to the definition above suppose first that 4.3.2 is true. Let us assume by contradiction that there exists an open entourage $\mathbf{u}\in\mathsf{Ent}X$ for which $G_{\mathbf{u}}/H$ is an infinite set. Then there exists a sequence $\{x_n,y_n\}\subset g_nS$ such that $<x_n,y_n>\in\mathbf{v}$, where $\mathbf{v}$ is the closed complement of $\mathbf{u}$ in Θ^2X. Up to passing to a subsequence we obtain $x_n\to x$ and $y_n\to y$ and $<x,y>\in\mathbf{v}$. So we can choose closed disjoint neighborhoods L and K of the points x and y such that $g_nS\cap L\neq\emptyset$ and $g_nS\cap K\neq\emptyset$ for infinitely many n what is impossible.

If, conversely K and L are disjoint closed subsets of X, let $\mathbf{u}{=}(K\times L)'$. Then the Definition implies that the set 4.3.2 is at most finite.

b) Note that the definitions remain equivalent if one restricts to the entourages of the form $\mathbf{u}_\varepsilon{\leftrightharpoons}\{\{\mathfrak{p},\mathfrak{q}\}:\boldsymbol{\delta}(\mathfrak{p},\mathfrak{q}){<}\varepsilon\}$ $(\varepsilon{>}0)$ where $\boldsymbol{\delta}$ is a metric determining the topology of X. In our case $\mathbf{\Theta}^2X/G$ is compact hence we can restrict ourselves to the entourages that belong to a fixed G-orbit that generates the filter $\mathsf{Ent}X$ [Ge09, Prop E].

Proposition 4.3.3. *Let G act 3-discontinuously on a compactum $\widetilde{T}$. An orbit $F{\subset}A$ of a subgroup H of G is visibly quasiconvex if and only if H is dynamically quasiconvex in G.*

Proof. Since H acts on $\widetilde{T}$ 3-discontinuously we have $\partial F{=}\mathbf{\Lambda}H$. For $g{\in}G$ the set gF is an orbit of the group gHg^{-1}. Thus $\partial(gF){=}\mathbf{\Lambda}(gHg^{-1}){=}g\mathbf{\Lambda}H$. So, the dynamical quasiconvexity of H is equivalent to the **right** H-finiteness of the sets of the form $\{g{\in}G:\partial(gF){\notin}\mathsf{Small}(\mathbf{u})\}$ $(\mathbf{u}{\in}\mathsf{Ent}\widetilde{T})$.

We fix a reference vertex $a{\in}A$ and consider the generating set of entourages of the form $\mathbf{u}_\varepsilon$ with respect to the metric $\overline{\boldsymbol{\delta}}_a$. Since $\overline{\boldsymbol{\delta}}_a(gx,gy){=}\overline{\boldsymbol{\delta}}_{g^{-1}a}(x,y)$, the following property is equivalent to the dynamical quasiconvexity of H:

$(*)$ for every $\varepsilon{>}0$ the set $\mathcal{G}_\varepsilon{\leftrightharpoons}\{g{\in}G:\mathsf{diam}_{\overline{\boldsymbol{\delta}}_{ga}}\partial F{\geqslant}\varepsilon\}$ is **left** H-finite.

Since the left H-action preserves the d-distance from F, if $\mathcal{G}_\varepsilon/H$ is finite then $\mathsf{d}(\mathcal{G}_\varepsilon a,F)$ is bounded. On the other hand, if $\mathsf{d}(\mathcal{G}_\varepsilon a,F)$ is bounded then $\mathcal{G}_\varepsilon a$ is H-finite. Since the action $G\curvearrowright A$ is properly discontinuous the set $\mathcal{G}_\varepsilon$ is also H-finite. So $(*)$ is equivalent to:

$(**)$ $\mathsf{d}(\mathcal{G}_\varepsilon a,F)$ is bounded for every $\varepsilon{>}0$.

Thus if F is visibly quasiconvex then (**) is true for every $a{\in}A$, so H is dynamically quasiconvex.

Conversely, suppose that $\mathsf{d}(\mathcal{G}_\delta a,F){\leqslant}R_\delta$ for every $\delta{>}0$. Let $S\subset A$ be a finite set containing a and intersecting each G-orbit in A. Since change of the reference point is a bilipschitz transformation, the ratio $\overline{\boldsymbol{\delta}}_x/\overline{\boldsymbol{\delta}}_y$ $(x,y{\in}S)$ is bounded.

Let $v{\in}\mathsf{V}_\varepsilon F$, i.e, $\mathsf{diam}_{\overline{\boldsymbol{\delta}}_v}F{\geqslant}\varepsilon$. Then $v{\in}gS$ for some $g{\in}G$. So $\mathsf{diam}_{\overline{\boldsymbol{\delta}}_{ga}}F\geqslant\delta{\leftrightharpoons}\frac{\varepsilon}{C}$ for some uniform constant C. We have $g{\in}\mathcal{G}_\delta$, $\mathsf{d}(ga,F){\leqslant}R_\delta$, $\mathsf{d}(v,F)\leqslant R_\delta{+}\mathsf{diam}_{\mathsf{d}}S$. So F is visibly quasiconvex. □

4.4. **Horocycles. Definition**. A bi-infinite α-distorted path $\gamma : \mathbb{Z} \to A$ is called α-*horocycle* at $\mathfrak{p} \in T$ if $\lim\limits_{n\to\pm\infty} \gamma(n){=}\mathfrak{p}$. We call the unique limit point $\mathfrak{p}$ of γ *base* of the horocycle.

Recall that a limit point $x \in \mathbf{\Lambda}G$ is called *conical* if there exists an infinite sequence of distinct elements $g_n \in G$ and distinct points $a, b \in S$ such that $g_n(y) \to a$ for all $y{\neq}x$ and $g_n(x) \to b$.

Proposition 4.4.1. *There is no α-horocycle at conical point.*

The proof of this fact for quasigeodesic horocycles [GP09, Lemma 3.6] works for α-horocycles too. □

Proposition 4.4.2. *Suppose that the action $G{\curvearrowright}\widetilde{T}$ is 3-discontinuous and 2-cocompact. Then there exists $\varepsilon{>}0$ such that if α-horocycles γ, δ with* **distinct** *bases $\mathfrak{p}, \mathfrak{q}$ meet $a{\in}A$ then $\overline{\boldsymbol{\delta}}_a(\mathfrak{p}, \mathfrak{q}) \geqslant \varepsilon$.*

Remark. This statement could be easily deduced from the results of [Ge09]. However this should require the theory of linkness and betweenness relation developed in [Ge09]. We prefer to give a simple independent proof. Here for the first time we use the 2-cocompactness of the action $G{\curvearrowright}\widetilde{T}$.

Proof. By 2-cocompactness there exists $\varrho{>}0$ such that for every different $x, y{\in}T$ one has $\overline{\boldsymbol{\delta}}_v(x, y){>}\varrho$ for some $v{\in}A$. Let v be such a vertex for $\mathfrak{p}$ and $\mathfrak{q}$. Then the vertex a does not belong to at least one of the sets $\mathsf{N}^{\overline{\boldsymbol{\delta}}_v}_{\varrho/2}\mathfrak{p}$, $\mathsf{N}^{\overline{\boldsymbol{\delta}}_v}_{\varrho/2}\mathfrak{q}$. So by 4.1.2 $\mathsf{d}(v, a){\leqslant}s{\leftrightharpoons}s(\varrho/2, \alpha)$ and $\overline{\boldsymbol{\delta}}_a(\mathfrak{p}, \mathfrak{q}){>}\lambda^s\varrho$. □

Corollary. *Every $a \in A$ can belong to a uniformly bounded number of α-horocycles at different bases.*

Proof. Since A is G-finite it is enough to prove that every $a \in A$ can belong to at most finitely many α-horocycles with different bases. Suppose not and $a \in \bigcap_{i\in I} \gamma_i$ where γ_i is an α-horocycle at p_i and $|I|{=}\infty$. Since $\widetilde{T}$ is a compactum the infinite set $P{=}\{p_i \mid i \in I\}$ must contain a convergent subsequence which is impossible by Proposition 4.4.2 . □

5. Horospheres

5.1. **Systems of horospheres.** Let $\mathsf{St}_G a$ denote the stabilizer $\{g{\in}G : ga{=}a\}$ of a point $a \in \widetilde{T}$ in G. We make use of the following obvious property of the actions of a group G on sets:

Proposition 5.1.1. *For G-finite G-sets A, B the following properties of a G-set $S{\subset}A{\times}B$ are equivalent:*
a : *S is G-finite;*
b : *for every $a{\in}A$ the set $S \cap \{a\}{\times}B$ is $\mathsf{St}_G a$-finite;*
c : *for every $b{\in}B$ the set $S \cap A{\times}\{b\}$ is $\mathsf{St}_G b$-finite.*

□

We apply 5.1.1 to the case when A is as above and $B{\leftrightharpoons}\mathsf{Par}$ is the set of parabolic points. Taking into account that $\mathsf{St}_G a$ is finite for each $a{\in}A$ we have the following corollary:

Proposition 5.1.2. *The following properties of G-set $S{\subset}A{\times}\mathsf{Par}$ are equivalent:*
a : *S is G-finite;*
b : *for every $a{\in}A$ the set $S \cap \{a\}{\times}\mathsf{Par}$ is finite;*
c : *for every $\mathfrak{p}{\in}\mathsf{Par}$ the set $S \cap A{\times}\{\mathfrak{p}\}$ is $\mathsf{St}_G\mathfrak{p}$-finite.*

□

Definition. Any G-invariant G-finite subset S of $A{\times}\mathsf{Par}$ determines a *system of horospheres*. For such S each set $S_{\mathfrak{p}}\leftrightharpoons\{a{\in}A : (a,\mathfrak{p}){\in}S\}$ is called horosphere *at* the parabolic point $\mathfrak{p}$. The entire set S is completely defined by the family $\mathsf{Par}{\ni}\mathfrak{p}\mapsto S_{\mathfrak{p}}{\subset}A$. So such a family also *determines a system of horospheres*. To satisfy the conditions of 5.1.2 this map should be G-equivariant and each $S_{\mathfrak{p}}$ should be $\mathsf{St}_G\mathfrak{p}$-finite.

Examples:
1. The set $\{(\mathrm{a},\mathfrak{p}) : \mathrm{a}\underset{A,k}{\#}\mathfrak{p}\}$ studied in [Ge09, 6.10–7.2] for fixed $k{\geqslant}2$ determines a system of horospheres.
2. (most important for this paper) For a distortion function α, the family $\mathfrak{p}\mapsto\mathsf{H}_\alpha\mathfrak{p}$ is a system of horospheres. The condition (b) of 5.1.2 follows from Corollary of 4.4.2. This family has been studied in [GP09] for affine functions α.
3. If $\mathfrak{p}\mapsto S_{\mathfrak{p}}$ is a system of horospheres and r is a positive integer then the family $\mathfrak{p}\mapsto\mathsf{N}^{\mathsf{d}}_r S_{\mathfrak{p}}$ is also a system of horospheres.
4. If $\mathfrak{p}\mapsto S_{\mathfrak{p}}$ is a system of horospheres and α is an appropriate distortion function then the family $\mathfrak{p}\mapsto\mathsf{H}_\alpha S_{\mathfrak{p}}$ is also a system of horospheres.
5. The union of two systems of horospheres is a system of horospheres.

Proposition 5.1.3. *Let S be a system of horospheres. Then for every $r{\geqslant}0$ the set $\{(\mathfrak{p},\mathfrak{q}){\in}\mathsf{Par}^2 : \mathsf{d}(S_{\mathfrak{p}},S_{\mathfrak{q}}){\leqslant}r\}\}$ is G-finite.*

Proof. By passing to the system of horospheres $\mathfrak{p}\mapsto\mathsf{N}_r S_{\mathfrak{p}}$ the problem reduces to the case $r{=}0$. In this case consider the set $\{(a,\mathfrak{p},\mathfrak{q}){\in}A{\times}\mathsf{Par}^2 : a{\in}S_{\mathfrak{p}}{\cap}S_{\mathfrak{q}}\}$. It is G-finite by 5.1.2. The set $\{(\mathfrak{p},\mathfrak{q}){\in}\mathsf{Par}^2 : S_{\mathfrak{p}}{\cap}S_{\mathfrak{q}}{\neq}\varnothing\}$ is the image of the latter one by the G-equivariant map of forgetting the a-component. So it is also G-finite. □

Proposition 5.1.4. *Given a system of horospheres $\mathfrak{p}\mapsto S_{\mathfrak{p}}$ there exists a positive C such that $\mathsf{diam}_{\mathsf{d}}\mathsf{Pr}_{S_{\mathfrak{p}}}S_{\mathfrak{q}}\leqslant C$ for each pair $\{\mathfrak{p},\mathfrak{q}\}$ of distinct parabolic points.*

Proof. Let $\mathfrak{q}{\in}\mathsf{Par}$. Since the action $\mathsf{St}_G\mathfrak{q}{\curvearrowright}(\widetilde{T}\setminus\mathfrak{q})$ is cocompact, the subgroup $S_{\mathfrak{q}}{=}\mathsf{St}_G\mathfrak{q}$ is quasiconvex by 4.2.2 and hence visibly quasiconvex by Corollary of 4.3.1. Since Par/G is finite, the visible quasiconvexity function $\mathsf{Q}_{S_{\mathfrak{q}}}$ (see 4.3) can be chosen independently of $\mathfrak{q}$. We denote any such function by Q_S. That is: $\forall\mathfrak{q}{\in}\mathsf{Par}\forall\varepsilon{>}0\exists r{\leftrightharpoons}\mathsf{Q}_S(\varepsilon) : \{a{\in}A : \mathsf{diam}_{\overline{\boldsymbol{\delta}}_a}S_{\mathfrak{q}}{\geqslant}\varepsilon\}\subset\mathsf{N}_r S_{\mathfrak{q}}$.

Since Par/G is finite it suffices to find C for a particular $\mathfrak{p}$. We thus fix it and denote $H{\leftrightharpoons}\mathsf{St}_G\mathfrak{p}$, $\Sigma{\leftrightharpoons}S_{\mathfrak{p}}$.

Let K be a compact fundamental set for $H{\curvearrowright}(T\setminus\mathfrak{p})$. So $K{\cap}\overline{\Sigma}{=}\varnothing$. Since $\overline{\Sigma}$ is weakly homogeneous by 3.2.1 the set $\mathsf{Pr}_\Sigma K$ is finite and $\varrho{\leftrightharpoons}\mathsf{min}\{\overline{\boldsymbol{\delta}}_v(\Sigma,K) : v{\in}\mathsf{Pr}_\Sigma K\}>0$. By 5.1.3 the set $P{\leftrightharpoons}\{\mathfrak{q}{\in}\mathsf{Par} : \mathsf{d}(\Sigma,S_{\mathfrak{q}}){\leqslant}r{\leftrightharpoons}\mathsf{Q}_S(\varrho/2)\}$ is H-finite. So $C_1{\leftrightharpoons}\mathsf{sup}\{\mathsf{diam}_{\mathsf{d}}(\mathsf{Pr}_\Sigma S_{\mathfrak{q}}) : \mathfrak{q}{\in}P\}<\infty$.

If now $\mathfrak{q}{\notin}K\setminus P$ then up to applying an element from H we can assume that $\mathfrak{q}{\in}K$. For $v\in\mathsf{Pr}_\Sigma\mathfrak{q}$ we have $d(v,S_{\mathfrak{q}})>r$ and $\mathsf{diam}_{\overline{\boldsymbol{\delta}}_v}(S_{\mathfrak{q}}){<}\varrho/2$. Thus

$$\overline{\boldsymbol{\delta}}_v(\mathfrak{p},S_{\mathfrak{q}}){\geq}\overline{\boldsymbol{\delta}}_v(\mathfrak{p},\mathfrak{q})-\overline{\boldsymbol{\delta}}_v(\mathfrak{q},S_{\mathfrak{q}}){\geq}\overline{\boldsymbol{\delta}}_v(\Sigma,K)-\mathsf{diam}_{\overline{\boldsymbol{\delta}}_v}(S_{\mathfrak{q}}){\geq}\varrho{-}(\varrho/2){=}\varrho/2.$$

Hence for the number $\rho(S_{\mathfrak{q}},\Sigma)$, defined in 3.2.2, we have $\rho(S_{\mathfrak{q}},\Sigma){\geq}\varrho/2$. By 3.3.2 we obtain $C_2{\leftrightharpoons}\mathsf{sup}\{\mathsf{diam}_{\mathsf{d}}(\mathsf{Pr}_\Sigma S_{\mathfrak{q}}) : \mathfrak{q}{\notin}P\}<\infty$. So we put $C{\leftrightharpoons}\mathsf{max}\{C_1,C_2\}$. □

Corollary. *Given a system S of horospheres there exists a positive number C such that $\mathsf{diam}_{\mathsf{d}}(S_{\mathfrak{p}}{\cap}S_{\mathfrak{q}}){\leqslant}C$ for each pair $\{\mathfrak{p},\mathfrak{q}\}$ of distinct parabolic points.* □

5.2. **Horospherical depth. Definition**. Let α be a distortion function and let $\gamma : I \to A$ be a path. For $i{\in}I$ we define the *horospherical depth* of i as

(5.2.1) $$\mathsf{depth}_\alpha(i,\gamma) \leftrightharpoons \mathsf{sup}\{r{\in}\mathbb{N} : \mathsf{N}_r i{\subset}I \text{ and } \exists\mathfrak{p}{\in}\mathsf{Par}\ \gamma(\mathsf{N}_r i) \subset \mathsf{H}_\alpha\mathfrak{p}\}.$$

To take into account multiple points we put for $v{\in}\mathsf{Im}\gamma$

(5.2.2) $$\mathsf{depth}_\alpha(v,\gamma){\leftrightharpoons}\mathsf{inf}\{\mathsf{depth}_\alpha(i,\gamma) : \gamma(i){=}v\}.$$

Applying the above Corollary to the system of the horospheres $\mathfrak{p} \mapsto \mathsf{H}_\alpha\mathfrak{p}$, we obtain that there exists a constant h such that if γ is α-distorted and $\mathsf{depth}_\alpha(i,\gamma){\geqslant}h$ then there is exactly one $\mathfrak{p}{\in}\mathsf{Par}$ such that $\gamma(\mathsf{N}_h i){\subset}\mathsf{H}_\alpha\mathfrak{p}$. We call such h the *critical depth value* for α.

Until the end of this subsection we fix an appropriate distortion function α.

For a vertex $v{\in}A$ denote by $\mathsf{NH}_{v,e,\alpha}$ the set of **finite** α-distorted paths γ of length $>\alpha_0{=}\alpha(1)$ such that $\mathsf{depth}_\alpha(v,\gamma){\leqslant}e$. Note that $\partial\gamma$ is a proper pair for $\gamma{\in}\mathsf{NH}_{v,e,\alpha}$.

Proposition 5.2.3. *The set* $\{\partial\gamma : \gamma{\in}\mathsf{NH}_{v,e,\alpha}\}$ *is bounded in* $\mathbf{\Theta}^2\widetilde{T}$*.*

Proof. Otherwise there is a limit point $\{\mathsf{p},\mathsf{p}\}$ for this set. Since A is discrete we have $\mathfrak{p}{\in}T$. By compactness of the Tikhonoff topology there exists an α-horocycle $\gamma : \mathbb{Z} \to A$ at $\mathfrak{p}$ such that $\gamma(0){=}v$ and for every finite segment $I{\subset}\mathbb{Z}$ there exists $\delta{\in}\mathsf{NH}_{v,e,\alpha}$ such that $\gamma|_I{=}\delta|_I$.

We have $\mathsf{Im}\gamma{\subset}\mathsf{H}_\alpha\mathfrak{p}$ hence $\mathsf{depth}_\alpha(0,\gamma){=}\infty$ contradicting with the boundness of $\mathsf{depth}_\alpha(0,*)$ on $\mathsf{NH}_{v,e,\alpha}$. ☐

Corollary. $\exists\varepsilon{>}0\ \forall v{\in}A\ \forall\gamma{\in}\mathsf{NH}_{v,e,\alpha}\ :\ \mathsf{diam}_{\overline{\boldsymbol{\delta}}_v}(\partial\gamma){>}\varepsilon.$ ☐

6. Relative geodesics

6.1. **Lifts.** Let S be a system of horospheres. We attach to our graph Γ new edges joining by an edge of length 1 each pair of points that belong to an horosphere. The new graph is called *relative graph* and is denoted by Δ. The corresponding *relative distance* function is denoted by $\overline{\mathsf{d}}$. The edges of $\Delta^1 \setminus \Gamma^1$ are called *horospherical* and those belonging to $S_\mathfrak{p}$ are called $\mathfrak{p}$*-horospherical*. A change of the system of horospheres yields a quasi-isometry of the relative graphs. To distinguish pathes in Γ and Δ we speak of Γ*-paths* and Δ*-paths*.

A Γ-path γ is called a *lift* of a Δ-path δ if these pathes have the same non-horospherical edges, and, instead of any horospherical edge of δ in γ one has a d-geodesic segment with the same endpoints. Every subpath $\gamma|_I$ (and the interval I) of γ coming from an edge of δ we call δ*-piece* of γ. So, to each edge of δ (called δ-edge), horospherical or not, there corresponds exactly one δ-piece of γ. Note that a lift of a $\overline{\mathsf{d}}$-geodesic Δ-path is not necessarily injective.

A subpath $\gamma|_I$ (and the interval I) of γ is said to be *integral* if it is a lift of some subpath of δ.

Proposition 6.1.1. *There exists a quadratic polynomial* α *such that any lift* γ *of any* $\overline{\mathsf{d}}$*-geodesic* Δ*-path* δ *is* α*-distorted. Moreover,* $\mathsf{depth}_\alpha(v,\gamma)$ *is uniformely bounded for every* $v{\in}\mathsf{Im}\delta$*.*

Proof. Consider a lift γ of a $\overline{\mathsf{d}}$-geodesic path $\delta : [j_-,j_+] \to A$. Let $I{=}[i_-,i_+]$ be the corresponding subinterval of $\mathsf{Dom}\gamma$. We must prove that $\mathsf{diam} I{\leqslant}\alpha_n$ where $n{\leftrightharpoons}\mathsf{diam}_\mathsf{d}\gamma(\partial I)$ and α is a quadratic polynomial that does not depend on γ and δ.

Denote $P{\leftrightharpoons}\{\mathfrak{p}{\in}\mathsf{Par} :$ there is a $\mathfrak{p}$-horospherical edge in $\delta\}$. Since δ is $\overline{\mathsf{d}}$-geodesic it has exactly one $\mathfrak{p}$-horospherical edge for each $\mathfrak{p}{\in}P$. Denote by $\gamma_\mathfrak{p} : I_\mathfrak{p} \to A$ the corresponding d-geodesic segment of γ. Note that $\mathsf{Im}\gamma_\mathfrak{p} \subset \mathsf{H}_\mathsf{id}S_\mathfrak{p}$. By the quasiconvexity of horospheres (see 4.2.2) there exists r such that $\mathsf{H}_\mathsf{id}S_\mathfrak{p} \subset \mathsf{N}_r S_\mathfrak{p}$ for each $\mathfrak{p}{\in}\mathsf{Par}$. We fix such r. Let C_1 be a maximum of the constants determined in Proposition 4.2.1 for $E \coloneqq \mathsf{N}_r S_\mathfrak{p}$ ($\mathfrak{p}\in\mathsf{Par}$). Let C_2 be the constant determined by 5.1.4 for the system of horospheres $\mathfrak{p} \mapsto \mathsf{N}_r S_\mathfrak{p}$.

We can assume that the points i_- and i_+ belong respectively to the first and to the last intervals of the set $\{I_\mathfrak{p} : \mathfrak{p}\in P\}$. So $\overline{\mathsf{d}}(\delta(j_\pm), \gamma(i_\pm)) \leqslant r{+}1$. Since $\mathsf{length}_{\overline{\mathsf{d}}}\delta{=}\mathsf{diam}_{\overline{\mathsf{d}}}\partial\delta \leqslant 2r{+}2{+}\mathsf{diam}_\mathsf{d}\partial\gamma{=}n{+}2r{+}2$ we have $0 \leqslant r_1 \coloneqq |\{\text{non-horospherical edges of } \delta\}| \leqslant n{+}2r{+}2{-}|P|$.

We now claim that $\mathsf{diam}(I{\cap}I_\mathfrak{p}) \leqslant C(2n{+}2r{+}1)$ where $C{\coloneqq}\max\{C_1, C_2\}$. Indeed let β be a d-geodesic segment between the endpoints of γ. Consider the path ω that joins the endpoints of $\gamma(\partial(I{\cap}I_\mathfrak{p}))$ formed by β and the two pieces of γ between the endpoints of γ and the corresponding endpoints of $\gamma_\mathfrak{p}$ (one of the pieces can be empty). We then "project" ω onto $\mathsf{N}_r S_\mathfrak{p}$ as follows: for each vertex $v \in \mathsf{Im}\beta{\cup}\mathsf{Im}\delta{\cup}\partial\gamma$ we choose a vertex in $\mathsf{Pr}_{\mathsf{N}_r S_\mathfrak{p}} v$ and join them by d-geodesic segments. Each edge of β and each non-horospherical edge of δ gives at most C_1 edges of the projection. Each piece $\gamma_\mathfrak{q}$ ($\mathfrak{q}\in P\setminus\mathfrak{p}$), corresponding to an horospherical edge of δ gives at most C_2 edges in the projection. The curve ω does not contain $\mathfrak{p}$-horospherical edges. Thus the d-distance between the endpoints of $\gamma|_{I\cap I_\mathfrak{p}}$ is at most $C_1{\cdot}(n{+}r_1) + C_2(|P|{-}1) \leqslant C(2n{+}2r{+}1)$ and our claim is proved.

Since δ has at most $n{+}2r{+}2$ edges (either horospherical or not) we have the following estimate $\mathsf{length}_\mathsf{d}\gamma \leqslant C(n{+}2r{+}2)(2n{+}2r{+}1){=}\alpha_n$ where α is a polynomial of degree 2. Thus the lift of a Δ-geodesic path δ is an α-distorted path γ in Γ proving the first part of the Proposition.

To estimate the α-horospherical depth of the vertices of δ in γ we fix a number s (see 4.2.2) such that $\mathsf{H}_\alpha\mathfrak{p}{\subset}\mathsf{H}_\alpha S_\mathfrak{p} \subset \mathsf{N}^\mathsf{d}_s S_\mathfrak{p}$ for every $\mathfrak{p}{\in}\mathsf{Par}$.

Let $v{\in}\mathsf{Im}\delta$. Assume that $\gamma(0){=}v$.

Let $K{=}[k_-, k_+]$ be a maximal subinterval of $I{\coloneqq}\mathsf{Dom}\gamma$ containing 0 such that $\gamma(\partial K){\subset}\mathsf{H}_\alpha\mathfrak{p}$ for some $\mathfrak{p}$. We fix such $\mathfrak{p}$.

We have $\mathsf{depth}_\alpha(0, \gamma){=}\min\{|k_-|, |k_+|\}$. Since each two points of γK can be joined by a $\overline{\mathsf{d}}$-geodesic path of length at most $2s{+}1$ through $S_\mathfrak{p}$ we have $\mathsf{diam}_{\overline{\mathsf{d}}}(\delta^{-1}\gamma(K)) \leqslant 2s{+}1$.

Let $L{=}[l_-, l_+]$ be the largest integral subinterval of K and let $M{=}[m_-, m_+]$ be the smallest integral interval containing K. So we have $0{\in}L{\subset}K{\subset}M{\subset}I$.

Since δ is $\overline{\mathsf{d}}$-geodesic, at least one of the integral intervals $M_-{\coloneqq}[m_-, 0]$, $M_+{\coloneqq}[0, m_+]$ does not contain a $S_\mathfrak{p}$-edge of δ. Let us assume that it is M_+. Note that $k_+ > l_+$ only if $\gamma|_{[l_+,k_+]}$ belongs to $\mathsf{N}_s S_\mathfrak{q}$ for some $\mathfrak{q}{\in}\mathsf{Par} \setminus \mathfrak{p}$. Thus $k_+{-}l_+ \leqslant c{\coloneqq}\max\{\mathsf{diam}_\mathsf{d}(\mathsf{N}_s S_\mathfrak{p}{\cap}\mathsf{N}_s S_\mathfrak{q}) : \{\mathfrak{p}, \mathfrak{q}\}{\in}\Theta^2\mathsf{Par}\}$ (see 5.1.3). Since each δ-edge in $[0, l_+]$ yields at most c edges in $\gamma \cap \mathsf{N}_s S_\mathfrak{p}{\cap}\mathsf{N}_s S_\mathfrak{q}$ we obtain $l_+{\leqslant}(2s{+}1)c$. Hence $\mathsf{depth}_\alpha(0, \gamma) \leqslant (2s{+}2)c$. □

6.2. **Relative hull.** For a set $F{\subset}A$ define its *relative hull* $\mathsf{H}_{\rm rel}F{\coloneqq}\cup\{\mathsf{Im}\delta : \delta \text{ is a } \overline{\mathsf{d}}\text{-geodesic } \Delta\text{-path with } \partial\delta{\subset}F\}$. A set is said to be *relatively quasiconvex* if $\mathsf{H}_{\rm rel}F \subset \mathsf{N}_r F$ for some $r{<}\infty$.

From now on we suppose that $\sum_{n\geqslant 0} n^2 f_n{<}\infty$ for our scaling function f (2.3). For example we can take $f_n{=}(n{+}1)^{-3-\varepsilon}$ for any $\varepsilon{>}0$. Thus any pair (f, α) where α is a quadratic polynomial is appropriate.

Denote by λ the decay rate (2.3) of f.

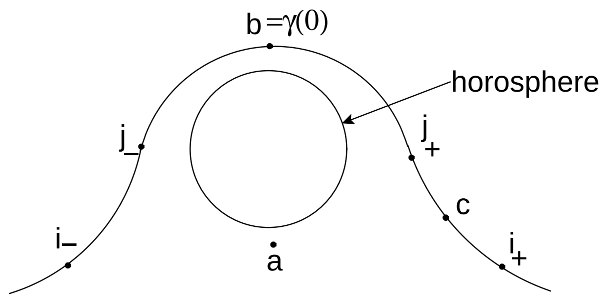
Proposition 6.2.1. *There exists a function $r{=}r(\varepsilon)$ such that for every $F{\subset}A$, $\mathsf{V}_\varepsilon F{\subset}\mathsf{N}^\mathsf{d}_r\mathsf{H}_{\rm rel}F$. In particular, every relatively quasiconvex set is visibly quasiconvex (see 4.3).*

Proof. Let α be the distortion function from 6.1.1. For $\varepsilon{>}0$ we will find r that depends only on α, ε, λ, the Karlsson functions, and the convergence functions (see 3.3) of the horospheres.

Let $a{\in}\mathsf{V}_\varepsilon F$, i.e, $\mathsf{diam}_{\overline{\boldsymbol{\delta}}_a} F{\geqslant}\varepsilon$. Connect a pair of points of $\overline{\boldsymbol{\delta}}_a$-diameter $\geqslant\frac{\varepsilon}{2}$ in F with a $\overline{\mathsf{d}}$-geodesic Δ-path δ and consider its lift $\gamma : I{=}[i_-, i_+] \to A$. By Proposition 6.1.1 γ is α-distorted. Since the pair (f, α) is appropriate we can assume that $\mathsf{d}(a, b) \leqslant d_1{\coloneqq}\mathsf{K}_\alpha(\frac{\varepsilon}{2})$ where $b{\coloneqq}\gamma(0)$.

If $b \in \delta^0$ we are done, so suppose not. We have $\mathsf{diam}_{\overline{\boldsymbol{\delta}}_b}\partial\gamma \geqslant \rho \leftrightharpoons \lambda^{d_1}\frac{\varepsilon}{2}$. Let $J{=}[j_-, j_+]$ be the δ-piece (see 6.1) of I containing 0 and let $J_-{=}[i_-, j_-]$, $J_+{=}[j_+, i_+]$ be the complementary subintervals of I. By the $\triangle$-inequality, at least one of the numbers $\overline{\boldsymbol{\delta}}_b(\gamma(j_+), \gamma(i_+))$, $\overline{\boldsymbol{\delta}}_b(\gamma(j_-), \gamma(j_+))$, $\overline{\boldsymbol{\delta}}_b(\gamma(i_-), \gamma(j_-))$ should be $\geqslant\frac{\rho}{3}$. Respectively, consider these three cases. The third one reduces obviously to the first.

In the first case we have $\mathsf{d}(b, c) \leqslant d_2 \leftrightharpoons \mathsf{K}_\alpha(\frac{\rho}{3})$ for some $c{=}\gamma(k)$, $k{\in}J_+$. As γ is α-distorted we have $k{\leqslant}\alpha(d_2)$. Since $j_+{\in}[0, k] \cap \delta^0$ it follows that $\mathsf{d}(b, \mathsf{Im}\delta){\leqslant}\alpha(d_2)$.



Lift of a relative geodesic

In the second case since $b \notin \delta^0$ there exists an horosphere $S_\mathfrak{p}$ ($\mathfrak{p}{\in}\mathsf{Par}$) such that $\gamma(\partial J){\subset}S_\mathfrak{p}$, $\gamma|_J$ is geodesic, and $b{\in}E{\leftrightharpoons}\mathsf{H}_{\mathsf{id}}S_\mathfrak{p}$. We claim that $\mathsf{d}(b, \mathsf{Im}\delta) \leqslant d_3{\leftrightharpoons}\mathsf{C}_E(\frac{\rho}{6})$ (see 3.3.1). If not, since $\gamma(j_\pm) \in \mathsf{Im}\delta$, we have $d(b, \gamma(j_\pm)){>}d_3$. By 3.1.1 it follows that $\overline{\boldsymbol{\delta}}_b(\gamma(j_\pm), p{=}\partial E){\leq}\frac{\rho}{6}$ and so $\overline{\boldsymbol{\delta}}_b(\gamma(j_+), \gamma(j_-)) \leq \frac{\rho}{3}$ which is impossible. Since the set S of the horospheres is G-finite the above constant d_3 is uniform for every $\mathfrak{p} \in \mathsf{Par}$.

So, in either case we have the uniform bound

$$\mathsf{d}(a, \mathsf{H}_{\mathrm{rel}}F) \leq \mathsf{d}(a, b) + \mathsf{d}(b, \mathsf{H}_{\mathrm{rel}}F) \leq d_1 + \max\{d_3, \alpha(d_2)\}$$

as claimed. □

Definition. Let α be a distortion function and let e be a positive integer. Define the (α, e)-*hull* of a set $F{\subset}A$ as

$\mathsf{H}_{\alpha,e}F{\leftrightharpoons}\{\gamma(i) : \gamma$ is an α-distorted path with $\partial\gamma{\subset}F$ and $\mathsf{depth}_\alpha(i, \gamma){\leqslant}e\}$.

A set $F{\subset}A$ is said to be *relatively α-quasiconvex* if its (α, e)-hull is within a bounded distance from F.

It follows from the corollary of 5.1.4 that this notion of quasiconvexity does not depend on e when e is sufficiently large.

Proposition 6.2.2. *There exists a function $\varepsilon{=}\varepsilon(\alpha, e)$ such that $\mathsf{H}_{\alpha,e}F{\subset}\mathsf{V}_\varepsilon F$ for every $F{\subset}A$. In particular, every visibly quasiconvex set is α-quasiconvex for any appropriate distortion function α.*

Proof. For $v{\in}A$ the number $\mathsf{inf}\{\mathsf{diam}_{\overline{\boldsymbol{\delta}}_v}\partial\gamma : \gamma{\in}\mathsf{NH}_{v,e,\alpha}\}$ is positive by the Corollary of 5.2.3. Since A is G-finite the number $\varepsilon{\leftrightharpoons}\mathsf{inf}\{\mathsf{diam}_{\overline{\boldsymbol{\delta}}_v}\partial\gamma : v{\in}A, \gamma{\in}\mathsf{NH}_{v,e,\alpha}\}$ is also positive. It follows that if $v{\in}\mathsf{H}_{\alpha,e}F$ then $v{\in}\mathsf{V}_\varepsilon F$. □

Proposition 6.2.3. *Let α be a distortion function from 6.1.1. Then there exists a number v such that $\mathsf{H}_{\mathrm{rel}}F{\subset}\mathsf{H}_{\alpha,e}F$ for arbitrary set $F{\subset}A$. In particular every relative α-quasiconvex set F is relatively quasiconvex.*

Proof. The result follows immediately from 6.1.1. □

Summing up the results of Subsection 6.2 and Proposition 4.3.3, we obtain.

Theorem A. *Let a finitely generated discrete group G act 3-discontinuously and 2-cocompactly on a compactum $\widetilde{T}$. The following properties of a subset F of the discontinuity domain of the action are equivalent:*
— F is relatively quasiconvex;
— F is visibly quasiconvex;
— F is relatively α-quasiconvex where α is a quadratic polynomial with big enough coefficients.

Moreover, if H is a subgroup of G and F is H-finite then the visible quasiconvexity of F is equivalent to the dynamical quasiconvexity of H with respect to the action $G\curvearrowright\widetilde{T}$. □

7. The lifts of geodesics from the relative graph and some applications.

7.1. **Hyperbolicity of the relative graph.** As one of the applications of our methods we give an easy proof of the main results of [Ya04]. We first suppose that G is a finitely generated relatively hyperbolic group admitting a geometrically finite convergence action $G\curvearrowright T$, or equivalently the action is 2-cocompact. Let Γ be a locally finite, connected, G-graph. Denote by Δ the corresponding relative graph with respect to the system of horospheres (see 6.1).

Our first aim is to show that the relative graph is Gromov hyperbolic.

Proposition 7.1.1. *There exists a constant r such that, for every $\overline{\mathsf{d}}$-geodesic triangle in Δ, every its side is within the r-neighborhood in Δ of the union of the other two sides.*

Proof. Let α be the distortion function from 6.1.1 and let e be the upper bound for $\mathsf{depth}_\alpha(v,\gamma)$ from 6.1.1. Let $\varepsilon \leftrightharpoons \varepsilon(\alpha,e)$ be the number from 6.2.2.

Consider a $\overline{\mathsf{d}}$-geodesic triangle with edges δ,δ',δ''. Let γ,γ',γ'' be the lifts. We can assume that $\delta(0){=}\gamma(0){=}v$. By Proposition 6.1.1 we have $\gamma{\in}\mathsf{NH}_{v,e,\alpha}$. Hence by Corollary of 5.2.3 $\mathsf{diam}_{\overline{\boldsymbol{\delta}}_v}\partial\gamma\geqslant\varepsilon$. Thus one of the numbers $\mathsf{diam}_{\overline{\boldsymbol{\delta}}_v}\partial\gamma'$, $\mathsf{diam}_{\overline{\boldsymbol{\delta}}_v}\partial\gamma''$ is $\geqslant\frac{\varepsilon}{2}$. It follows that $\mathsf{d}(v,\mathsf{Im}\gamma'{\cup}\mathsf{Im}\gamma'')\leqslant\mathsf{K}_\alpha(\frac{\varepsilon}{2})$. So putting $r{=}\mathsf{K}_\alpha(\frac{\varepsilon}{2}){+}1$ we obtain $\overline{\mathsf{d}}(v,\mathsf{Im}\delta'{\cup}\mathsf{Im}\delta'')\leqslant r$. □

Remark. One of the equivalent definitions of (strong) relative hyperbolicity of a group was proposed by B. Bowditch. It claims that a group is relatively hyperbolic if and only if it possesses a cofinite action on a Gromov hyperbolic graph Δ ('cofinite' means that Δ^1 is G-finite) which is *fine*, that is for every n and every edge e of Δ the set of simple loops of length n that path through e is finite.

In our case the action $G\curvearrowright\Delta$ is not cofinite, but the metric space Δ^0 can be isometrically and equivariantly embedded into a G-cofinite hyperbolic graph by the following well-known construction: let $\widetilde{\Delta}$ be the graph whose set of vertices is $A{\cup}\mathsf{Par}$ and the set of edges is $\Gamma^1{\cup}S$ (where S is a G-finite subset of horospheres in $A{\times}\mathsf{Par}$ see 5.1). We consider on $\widetilde{\Delta}^0$ the path-metric in which the Γ-edges have length 1 and the S-edges have length $\frac{1}{2}$. The inclusion $\Delta^0{\hookrightarrow}\widetilde{\Delta}^0$ is an **isometry** with respect to the path-metrics. Thus we can denote the distance in $\widetilde{\Delta}$ by $\overline{\mathsf{d}}$. Since $A{=}\Delta^0{\subset}\widetilde{\Delta}^0{\subset}\mathsf{N}^{\overline{\mathsf{d}}}_{1/2}A$, by Proposition 7.1.1 the graph $\widetilde{\Delta}$ is hyperbolic. The action $G\curvearrowright\widetilde{\Delta}$ is cofinite. To prove the finess of the graph $\widetilde{\Delta}$ we need the following lemma motivated by [Bo97, Lemma 7.1]:

Lemma. *There exists a quadratic polynomial α such that for every simple loop δ in $\widetilde{\Delta}$ and every its lift γ one has*

$$\mathsf{length}_{\mathsf{d}}\gamma\leqslant\alpha(\mathsf{length}_{\overline{\mathsf{d}}}\delta).$$

Proof. Let δ be a simple loop in the graph $\widetilde{\Delta}$ of length n. It can path at most once through a parabolic point $\mathfrak{p}\in\mathsf{Par}$. So we can suppose that δ is a simple loop in the graph Δ having at most one $\mathfrak{p}$-horospherical edge for each $\mathfrak{p}\in\mathsf{Par}$. The argument now repeats the proof of the first part of 6.1.1 with obvious simplification. For the sake of completeness we include it.

Let $P\leftrightharpoons\{\mathfrak{p}\in\mathsf{Par}$: there is a $\mathfrak{p}$-horospherical edge in $\delta\}$. Consider a lift γ of δ (see 6.1). We can regard γ as a map from the vertex set Ξ^0 of a simplicial circle Ξ taking edges to Γ-edges. For every δ-piece $\gamma_{\mathfrak{p}}$ denote by $\omega_{\mathfrak{p}}$ the "complementing path" i.e. the restriction of γ onto $\Xi^0\setminus\mathsf{Dom}\gamma_{\mathfrak{p}}$. By our assumption $\omega_{\mathfrak{p}}$ does not path through $\gamma_{\mathfrak{p}}$ anymore. Thus $\omega_{\mathfrak{p}}$ consists of $n{-}1$ δ-pieces each piece is either $\gamma_{\mathfrak{q}}$ where $\mathfrak{q}\in P\setminus\mathfrak{p}$ or a Γ-edge.

By projecting $\omega_{\mathfrak{p}}$ onto $S_{\mathfrak{p}}$ and comparing the length of the resulting curve with the geodesic segment $\gamma_{\mathfrak{p}}$ we have $\mathsf{length}_{\mathsf{d}}\gamma_{\mathfrak{p}}\leqslant C(n{-}1)$ where C is the constant from the proof of 6.1.1. Thus $\mathsf{length}_{\mathsf{d}}\gamma\leqslant Cn(n{-}1)$. □

Corollary[Ya04]. *For a finitely generated relatively hyperbolic group G the graph $\widetilde{\Delta}$ is fine.*

Proof. The graph Γ is locally finite. So by the above Lemma there are at most finitely many lifts of a simple loop of length n in $\widetilde{\Delta}$ passing through a given edge. It follows that $\widetilde{\Delta}$ is fine. □

The result of Yaman remains valid for relatively hyperbolic groups without assuming their finite generatedness and even their countability (cf with [Hr10]).

Proposition 7.1.2. *Let G be a group acting 2-cocompactly and 3-discontinuously on a compactum T. Then there exists a hyperbolic, G-cofinite graph $\widetilde{\Delta}$ whose vertex stabilizers are all finite except the vertices corresponding to the parabolic points for the action $G\curvearrowright T$. Furthermore the graph $\widetilde{\Delta}$ is fine.*

Proof. We will use few facts from [GP10]. The group G satisfying the above assumptions acts discontinuously on a G-finite graph of entourages $\mathcal{G}$. Denote by P_i $(i{=}1,...,n)$ the system of non-conjugate maximal parabolic subgroups of G for the action $G\curvearrowright T$. It is shown in [GP10, Theorem A] that there is a graph $\widetilde{\mathcal{G}}$ obtained by refinement of $\mathcal{G}$ such that all its connected components are G-equivalent; and if $\mathcal{G}_0$ is a component of $\widetilde{\mathcal{G}}$ then its stabilizer G_0 is a finitely generated relatively hyperbolic subgroup of G with respect to the system $Q_i{=}P_i\cap G_0$. The connected components of $\widetilde{\mathcal{G}}$ are adjacent along the set of parabolic points $\mathfrak{p}\in\mathsf{Par}$ (not belonging to $\widetilde{\mathcal{G}}$).

Let now $\widetilde{\Delta}$ be the graph obtained by joining every vertex of $\widetilde{\mathcal{G}}$ belonging to an horosphere $S_{\mathfrak{p}}\in S$ with the parabolic point $\mathfrak{p}$ by an edge of length $\frac{1}{2}$. The graph $\widetilde{\Delta}$ is G-cofinite. Denote by $\widetilde{\Delta}_0$ the subgraph of $\widetilde{\Delta}$ corresponding to the component $\mathcal{G}_0$ of $\widetilde{\mathcal{G}}$. By the above Corollary the graph $\widetilde{\Delta}_0$ is hyperbolic and fine.

There is an induced action of G on a bipartite graph $\mathcal{T}$ whose vertices are of two types $\mathcal{H}$ and $\mathcal{C}$ corresponding respectively to the horospheres of $\widetilde{\mathcal{G}}$ (of *horospherical* type) and to the connected components of $\widetilde{\mathcal{G}}$ (of *non-horospherical* type). Two vertices $H\in\mathcal{H}$ and $C\in\mathcal{C}$ are connected by an edge in $\mathcal{T}$ if the corresponding horosphere H and the component C intersect. One can also obtain $\mathcal{T}$ from the graph $\widetilde{\Delta}$ by contracting every component $g(\widetilde{\Delta}_0)$ $(g\in G)$ into a vertex of $\mathcal{C}$-type and every parabolic vertex $\mathfrak{p}\in\mathsf{Par}$ into a vertex of $\mathcal{H}$-type. By [GP10, Lemma 3.36] the graph $\mathcal{T}$ is a tree. So every loop in $\widetilde{\Delta}$ is contained in $g(\widetilde{\Delta}_0)$ for some $g\in G$. It follows that the graph $\widetilde{\Delta}$ is itself a cofinite, hyperbolic and fine. □

7.2. **The lifts of $\overline{\mathsf{d}}$-geodesics are d-quasigeodesics.** Let S be a system of horospheres. It follows from the definition of a system of horospheres that the value $\mathsf{C}_S(\varepsilon)\leftrightharpoons\mathsf{sup}\{\mathsf{C}_{S_{\mathfrak{p}}}(\varepsilon)$: $\mathfrak{p}\in\mathsf{Par}\}$ (see 3.3) is finite for every $\varepsilon{>}0$.

Proposition 7.2.1. *Given a system S of horospheres there exists a number d such that if $\gamma : I \to A$ is α-distorted, with $\partial\gamma\subset S_{\mathfrak{p}}$ then $\gamma(I \setminus \mathsf{N}_d\partial I)\subset\mathsf{H}_\alpha\mathfrak{p}$.*

Proof. If such d were not exist one could find a sequence of α-distorted paths $\gamma_n : [i_n^-, i_n^+] \to A$ with $\partial\gamma_n\subset S_{\mathfrak{p}_n}$, $\gamma_n(0)\notin\mathsf{H}_\alpha\mathfrak{p}_n$ ($\mathfrak{p}_n \in \mathsf{Par}$), and $|i_n^\pm| \to \infty$. Since the set Par is G-finite by applying G and passing to a subsequence we can suppose that $\mathfrak{p}_n{=}\mathfrak{p}$. As $\mathsf{St}_G\mathfrak{p}$ acts cocompactly on $\widetilde{T} \setminus \mathfrak{p}$ we can also assume that $\gamma_n(0){=}v$ do not depend on n. So by passing to a subsequence once more we can find a sequence of paths that converges in the Tikhonoff topology to an infinite α-distorted path γ. Since $\partial S_{\mathfrak{p}}{=}\mathfrak{p}$, it is an α-horocycle and $\gamma(0)\notin\mathsf{H}_\alpha\mathfrak{p}$. A contradiction. □

Let α be a distortion function and let e be a positive integer.

Definition. An α-distorted path $\gamma : I \to A$ is called *e-piecewise geodesic* if every subpath consisting of points of α-depth $\geqslant e$ (see 5.2) is geodesic.

It follows from 6.1.1 that for any system of horospheres S there exists e such that every lift γ of a $\overline{\mathsf{d}}$-geodesic path δ is e-piecewise geodesic α-distorted path for a quadratic polynomial α_n.

Proposition 7.2.2. *There exists a function $c{=}c(\alpha, e)$ such that every e-piecewise geodesic α-distorted path is β-distorted where $\beta(n){=}cn{+}c$ ($n \in \mathbb{N}$).*

Proof. Consider a e-piecewise geodesic α-distorted path $\gamma : I{\leftrightharpoons}[0, i_+] \to A$ and a geodesic path $\omega : J{\leftrightharpoons}[0, j_+] \to A$ with $\gamma(0){=}\omega(0){=}a$, $\gamma(i_+){=}\omega(j_+){=}b$. Let h be the critical depth value (see 5.2) for geodesics. Denote
$N{\leftrightharpoons}\{j{\in}J : \mathsf{depth}_{\mathsf{id}}(j, \omega){\leqslant}h\}$. By Corollary of 5.2.3 there exists $\varepsilon{>}0$ such that, for $j{\in}N$, one has $\overline{\boldsymbol{\delta}}_{\omega(j)}(a, b){\geqslant}\varepsilon$ and hence $\mathsf{d}(\omega(j), \mathsf{Im}\gamma) \leqslant r{\leftrightharpoons}\mathsf{K}_\alpha(\varepsilon)$. Denote $s{\leftrightharpoons}\mathsf{K}_\alpha(\frac{\varepsilon}{2})$, $t{\leftrightharpoons}2r{+}\alpha(r{+}s)$.

Lemma. *If $x, y{\in}N$ and $y{-}x > t$ and $\gamma(x_1){\in}\mathsf{N}_r(\omega(x))$, $\gamma(y_1){\in}\mathsf{N}_r(\omega(y))$ then $x_1 < y_1$.*

Proof. Suppose not and $x_1{\geqslant}y_1$. We have $d(\omega(x), [\gamma(y_1), \omega(y)]) > t{-}r{=}r{+}\alpha(r{+}s){\geqslant}s$ where $[\gamma(y_1), \omega(y)] \subset A$ is a geodesic of length at most r between $\gamma(y_1)$ and $\omega(y)$.

By 2.4.1 $\overline{\boldsymbol{\delta}}_{\omega(x)}(\omega(y), \gamma(y_1)) \leqslant \frac{\varepsilon}{2}$. By Corollary of 5.2.3 we also have $\overline{\boldsymbol{\delta}}_{\omega(x)}(a, \omega(y)) \geqslant \varepsilon$, so $\overline{\boldsymbol{\delta}}_{\omega(x)}(a, \gamma(y_1)) \geqslant \frac{\varepsilon}{2}$. Applying again 2.4.1 we obtain $\mathsf{d}(\omega(x), \gamma([0, y_1])) \leqslant s$.

Let now $x_2{\in}[0, y_1]$ be such that $\mathsf{d}(\omega(x), \gamma(x_2)){\leqslant}s$. Thus $\mathsf{d}(\gamma(x_1), \gamma(x_2)) \leqslant r{+}s$ and $x_1{-}x_2 \leqslant \alpha(r{+}s)$. Since $y_1{\in}[x_2, x_1]$ we have
$\mathsf{d}(\omega(x), \omega(y)) \leqslant \mathsf{d}(\omega(x), \gamma(x_1)){+}\mathsf{d}(\gamma(x_1), \gamma(y_1)){+}\mathsf{d}(\gamma(y_1), \omega(y)) \leqslant 2r{+}\alpha(r{+}s){=}t$. Since ω is a geodesic we obtain $y - x{\leqslant}t$. A contradiction. □

We continue the proof of 7.2.2. Subdivide the interval J into segments $J_k{\leftrightharpoons}[j_k, j_{k+1}]$ using the following inductive rule. Put $j_0{\leftrightharpoons}0$. After the choice of j_k if $j_k{=}j_+$ then we finish. If not, define $j_{k+1}{\leftrightharpoons}j_+$ if $j_+{-}j_k \leqslant t$. Otherwise
$j_{k+1}{\leftrightharpoons}\mathsf{min}\{j{\in}N : j{>}j_k{+}t\}$.

Let m be the biggest k for which j_k is defined.

By the above argument, there exist $i_k{\in}[0, i_+]$ such that $\mathsf{d}(\omega(j_k), \gamma(i_k)) \leqslant r$ for $k{\in}[0, m{+}1]$. By Lemma, the indices i_k form an increasing sequence. So I gets subdivided into the segments $I_k{\leftrightharpoons}[i_k, i_{k+1}]$. It suffices to find a linear polynomial β such that $\mathsf{diam}I_k \leqslant \beta(\mathsf{diam}J_k)$ for all k.

If $\mathsf{diam}J_k \leqslant t{+}1$ then $\mathsf{diam}\partial(\gamma|_{I_k}) \leqslant t{+}1{+}2r$ hence
$\mathsf{diam}I_k \leqslant \alpha(t{+}1{+}2r) \leqslant \alpha(t{+}1{+}2r){\cdot}\mathsf{diam}J_k$.

If $\mathsf{diam}J_k > t{+}1$ then J_k contains a piece of id-depth ${>}h$. Hence, for a uniquely determined $\mathfrak{p}{\in}\mathsf{Par}$, the endpoints of $\omega|_{J_k}$ belong to the $t{+}1$-neighborhood of $\mathsf{H}_{\mathsf{id}}\mathfrak{p}$.

Let d be the constant from 7.2.1 for the distortion function α and for the system of horospheres $\mathfrak{p} \mapsto \mathsf{N}_{t+1+r}\mathsf{H}_{\mathsf{id}}\mathfrak{p}$.

If $\mathsf{diam}I_k > 2(d{+}e{+}r{+}t)$ then, by 7.2.1, the interval I_k contains a nonempty subinterval $I_k^{\mathrm{geo}} \leftrightharpoons I_k \setminus \mathsf{N}_{d+e}\partial I_k \subset H_\alpha p$ of depth $\geqslant e$. By the hypothesis $\gamma|_{I_k^{\mathrm{geo}}}$ is a geodesic subpath. Thus $\mathsf{length}_{\mathsf{d}}\gamma|_{I_k^{\mathrm{geo}}} {=} \mathsf{diam}I_k^{\mathrm{geo}} \leqslant \mathsf{diam}J_k + 2(r{+}d{+}e)$. Hence $\mathsf{diam}I_k \leqslant \mathsf{diam}J_k + 2r{+}4d{+}4e$.

If $\mathsf{diam}I_k \leqslant 2(d{+}e{+}r{+}t)$ then also $\mathsf{diam}I_k \leqslant 2(d{+}e{+}r{+}t){\cdot}\mathsf{diam}J_k$.

We have $[a,b] \subset \bigcup_k I_k$, so $b - a \leqslant cd(\gamma(a), \gamma(b)) + c$ where $c{=}\max\{2(d+e+r+t), 2r + 4d + 4e, \alpha(t+1+2r)\}$. □

As a direct consequence of the above Proposition we obtain.

Corollary [DS05, Thm. 1.12(4)]**.** *The lift of every* $\overline{\mathsf{d}}$*-geodesic is* d*-quasigeodesic.* □

8. Criteria for the subgroup quasiconvexity in RHG.

8.1. **Statement of the result.** The aim of this Section is to prove Theorem B giving criteria for a subgroup of a relatively hyperbolic group to be quasiconvex.

Let q be a positive integer and let α be a distortion function (see Subsection 2.2).

Definition. A subset E of a metric space M is called *weakly α-quasiconvex* if there is a positive integer q such that for each $x, y{\in}E$ there exists an α-distorted path γ such that $x, y{\in}\mathsf{N}_q\mathsf{Im}\gamma{\subset}\mathsf{N}_q E$.

A subgroup H of a finitely generated group G is said to be *weakly α-quasiconvex* if there is a proper (i.e. stabilizers are finite) action of G on a connected graph Γ such that some H-orbit $\subset\Gamma^0$ is weakly α-quasiconvex.

We precise that the word 'weakly' appears in the above definition since we do not request the above property to be true for every path having endpoints in E (in which case it is called α-quasiconvex).

The main result of this Section relates the (weak) α-quasiconvexity (see 4.1) with the existence of cocompact action outside of the limit set (see 4.2). The constant λ_0 below is fixed in our Convention 2 (see 2.5).

Theorem B. *Let a finitely generated group G act 3-discontinuously and 2-cocompactly on a compactum $\widetilde{T}$. Let* Par *be the set of the parabolic points for this action. Suppose that $A{=}\widetilde{T}\setminus T{\neq}\varnothing$ where $T{=}\mathbf{\Lambda}G$ is the limit set for the action. Then there exists a constant $\lambda_0 \in]1, +\infty[$ such that the following properties of a subgroup H of G are equivalent:*

a :*H is weakly α-quasiconvex for some distortion function α for which $\alpha(n){\leqslant}\lambda_0^n$ $(n \in \mathbb{N})$, and for every $\mathfrak{p}{\in}\mathsf{Par}$ the subgroup $H{\cap}\mathsf{St}_G\mathfrak{p}$ is either finite or has finite index in $\mathsf{St}_G\mathfrak{p}$;*

b : *the space $(\widetilde{T} \setminus \mathbf{\Lambda}H)/H$ is compact;*

c : *for every distortion function α bounded by λ_0^n $(n \in \mathbb{N})$, every H-invariant H-finite set $E{\subset}A$ is α-quasiconvex and for every $\mathfrak{p}{\in}\mathsf{Par}$ the subgroup $H{\cap}\mathsf{St}_G\mathfrak{p}$ is either finite or has finite index in* $\mathsf{St}_G\mathfrak{p}$. ⧫

Note that the implication '$\mathsf{c}{\Rightarrow}\mathsf{a}$' is trivial. The implication '$\mathsf{b}{\Rightarrow}\mathsf{c}$' is rather simple (see 8.3 below). The Section is mainly devoted to the proof of '$\mathsf{a}{\Rightarrow}\mathsf{b}$'.

8.2. **Preliminary results.** We start with the following obvious:

Proposition 8.2.1. *Let a group G act properly on a set M. Let A_0, A_1 be subgroups of G and let E_ι be A_ι-finite* **non-disjoint** *subsets of M for $\iota{=}0{,}1$. Then $|E_0{\cap}E_1|{=}\infty \Leftrightarrow |A_0{\cap}A_1|{=}\infty$.* □

Since now on we fix a discrete finitely generated group G, a compactum $\widetilde{T}$ and a 3-discontinuous 2-cocompact action $G{\curvearrowright}\widetilde{T}$. Denote by $T{=}\mathbf{\Lambda}G$ the limit set and suppose that $A{\leftrightharpoons}\widetilde{T} \setminus \mathbf{\Lambda}G{\neq}\varnothing$.

Since G is finitely generated and A is G-finite there is a G-finite set $\Gamma^1{\subset}\mathbf{\Theta}^2A$ such that the graph Γ with $\Gamma^0{=}A$ is connected (see e.g. [GP10, Lemma 3.11]). We fix the graph Γ.

For $x,y{\in}A$ denote by $[x,y]$ the geodesic $\{a{\in}A : \mathsf{d}(x,a){+}\mathsf{d}(a,y){=}\mathsf{d}(x,y)\}$ between x and y.

Proposition 8.2.2. *Let E be any subset of A and let $x{\in}A, z{\in}E, y{\in}\mathsf{Pr}_E x$ (see Subsection 3.2). Then $\mathsf{d}(z,[x,y]) \geqslant \frac{1}{2}\mathsf{d}(z,y)$.*

Proof. If $t{\in}[x,y]$ and $\mathsf{d}(z,t){=}r$ then $\mathsf{d}(t,y){\leqslant}r$ by the definition of Pr_E.
By $\triangle$-inequality, $\mathsf{d}(z,y){\leqslant}\mathsf{d}(z,t){+}\mathsf{d}(t,y){\leqslant}2r$. □

Proposition 8.2.3. *Let O be a weakly α-quasiconvex H-orbit in A. A parabolic point $\mathfrak{p}{\in}\mathsf{Par}$ belongs to $\overline{O}$ if and only if $H{\cap}P$ is infinite where $P{=}\mathsf{St}_G\mathfrak{p}$.*

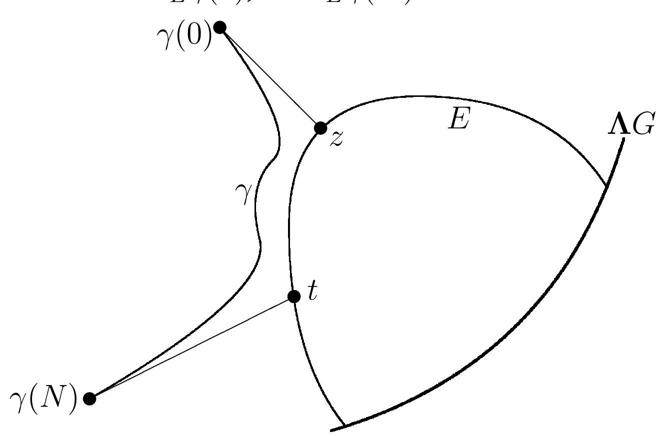
Proof. If $H{\cap}P$ is infinite then obviously $p \in \overline{O}$. Suppose $p \in \overline{O}$. Let us fix $v{\in}O$. By compactness argument there is an α-ray $\gamma : \mathbb{Z}_{\geqslant 0} \to A$ starting in $\mathsf{N}_q v$ and converging to $\mathfrak{p}$ whose image is contained in $\mathsf{N}_q O$. Let S be a system of horospheres and let $d{\coloneqq}\mathsf{d}(\gamma(0), S_\mathfrak{p})$. So $\gamma(0)$ belongs to the P-finite set $\mathsf{N}_d S_\mathfrak{p}$. Hence $\mathsf{Im}\gamma$ is contained in a P-finite set $\mathsf{H}_\alpha\mathsf{N}_d S_\mathfrak{p}$. Thus the intersection of O with the P-finite set $\mathsf{N}_q\mathsf{H}_\alpha\mathsf{N}_d S_\mathfrak{p}$ is infinite. By 8.2.1, $H{\cap}P$ is infinite. □

For a fixed Floyd function f we fix an appropriate distortion function α (see 2.4).

Proposition 8.2.4. *Let H be a subgroup of G acting cocompactly on $\widetilde{T} \setminus \mathbf{\Lambda}H$, and let E be an H-invariant H-finite subset of A. Then there exist constants R, d such that, for every α-distorted path γ in A, if the distance between the E-projections of its endpoints is greater than R then these projections are contained in $\mathsf{N}_d\mathsf{Im}\gamma$.*

Proof. Let K be a compact fundamental set for the action of H on $\widetilde{T} \setminus \mathbf{\Lambda}H$. By 3.2.1 the set $\mathsf{Pr}_E K$ is finite so the number $\varepsilon_0{\coloneqq}\frac{1}{3}\mathsf{min}\{\overline{\boldsymbol{\delta}}_z(K,\partial E) : z{\in}\mathsf{Pr}_E K\}$ is positive.
We take $\varepsilon{\in}(0,\varepsilon_0)$ and $d{\coloneqq}\mathsf{K}_\alpha(\varepsilon)$ where K_α is the Karlsson function (2.4.1) corresponding to α.
Let $R{\coloneqq}\mathsf{max}\{\mathsf{C}_E(\varepsilon), 2d\}$ where $C_E(\varepsilon)$ is finite as E is weakly homogeneous (see Subsection 3.3).

Let $\gamma : [0,N] \to A$ be an α-distorted path. Up to applying an element of H we can suppose that $\gamma(0){\in}K$. Let $z{\in}\mathsf{Pr}_E\gamma(0)$, $t{\in}\mathsf{Pr}_E\gamma(N)$.



Since $d(z,t) > C_E(\varepsilon)$ so by 3.1.1 and 3.3.1 we obtain $\overline{\boldsymbol{\delta}}_z(t,\partial E){\leqslant}\varepsilon$. We have $d(z,t) \geq R \geq 2d$ so $d(z,[t,\gamma(N)]) \geq d$ by Lemma 8.2.2. Thus $\overline{\boldsymbol{\delta}}_z(\gamma(N),t){\leqslant}\varepsilon$. If by contradiction $d(z,\mathsf{Im}\gamma) > d$ then we would have $\overline{\boldsymbol{\delta}}_z(\gamma(0),\gamma(N)){\leqslant}\varepsilon$. So summing all these three inequalities we would have $\overline{\boldsymbol{\delta}}_z(K,\partial E){\leqslant}\overline{\boldsymbol{\delta}}_z(\gamma(0),\partial E){\leqslant}\overline{\boldsymbol{\delta}}_z(\gamma(N),\gamma(0)){+}\overline{\boldsymbol{\delta}}_z(t,\gamma(N)){+}\overline{\boldsymbol{\delta}}_z(t,\partial E){\leqslant}3\varepsilon{<}3\varepsilon_0$ contradicting to the choice of ε_0. □

As an immediate corollary we obtain the following

Proposition 8.2.5. *Given a system S of horospheres there exist constants R, d such that for every $\mathfrak{p}\in\mathsf{Par}$ and every α-distorted path γ in A if the distance between the $S_\mathfrak{p}$-projections of its endpoints is greater than R then these projections are contained in $\mathsf{N}_d\mathsf{Im}\gamma$.*

Proof. Since Par is G-finite the problem reduces to the case of a fixed $\mathfrak{p}\in\mathsf{Par}$. The subgroup $P\leftrightharpoons\mathsf{St}_G\mathfrak{p}$ acts cocompactly on $\widetilde{T}\setminus\mathfrak{p}$. So the assertion follows from 8.2.4 applied to $H{=}\mathsf{St}_G\mathfrak{p}$ and $E{=}S_\mathfrak{p}$. □

8.3. **Implication '**$\mathsf{b}\Rightarrow\mathsf{c}$**'.** We fix a subgroup H satisfying property 'b' of Theorem B. Let E be an H-finite H-invariant subset E of A. Denote by $\overline{E}$ the closure of E in $\widetilde{T}$ and $\partial E\leftrightharpoons\overline{E}\setminus E$. It follows from the the convergence property that $\partial E{=}\mathbf{\Lambda}H$. The α-quasiconvexity of E follows from Proposition 4.2.2. So the only thing to prove is the last part of the statement.

We fix a compact fundamental domain K for the action $H\curvearrowright(\widetilde{T}\setminus\partial E)$. Consider a system of horospheres $\{S_\mathfrak{p} : \mathfrak{p}\in\mathsf{Par}\}$ (see 5.1). The set $\mathcal{P}_H{=}\{\mathfrak{p}\in\mathsf{Par} : |S_\mathfrak{p}\cap E|{=}\infty\}$ is H-invariant. As $K\cap\partial E{=}\emptyset$ there exists an entourage $\mathbf{u}\in\Theta^2\widetilde{T}$ such that $\mathbf{u}\cap(K\times\partial E){=}\varnothing$. Since every parabolic subgroup $\mathsf{St}_G\mathfrak{p}$ is quasiconvex (see 4.2.2), it is dynamically quasiconvex (Theorem A). Furthermore there are at most finitely many G-non-equivalent parabolic points [Ge09, Main Theorem, a]. Hence the set $\mathcal{P}_\mathbf{u}\leftrightharpoons\{\mathfrak{p}\in\mathsf{Par} : S_\mathfrak{p}$ is not $\mathbf{u}$-small$\}$ is finite. We have $K\cap\cup\{S_\mathfrak{p} : \mathfrak{p}\in\mathcal{P}_H\}{=}K\cap\cup\{S_\mathfrak{p} : \mathfrak{p}\in\mathcal{P}_\mathbf{u}\cap\mathcal{P}_H\}$. For every $\mathfrak{p}\in\mathcal{P}_H$ the set $K\cap S_\mathfrak{p}$ is finite as otherwise it would contain the unique limit point $\mathfrak{p}$ of the infinite set $S_\mathfrak{p}\cap E$ which is impossible. So it follows that the set $\cup\{S_\mathfrak{p} : \mathfrak{p}\in\mathcal{P}_H\}$ is H-finite. By 5.1.2.b each element of A belongs to at most finitely many horospheres, so the set $\mathcal{S}_H{=}\cup\{\mathfrak{p}\times S_\mathfrak{p} : \mathfrak{p}\in\mathcal{P}_H\}$ is H-finite too.

By the property 5.1.2.c, applied to $\mathcal{S}_H$, each $S_\mathfrak{p}$ is $H\cap\mathsf{St}_G\mathfrak{p}$-finite ($\mathfrak{p}\in\mathcal{P}_H$). It implies that the index of $H\cap\mathsf{St}_G\mathfrak{p}$ in $\mathsf{St}_G\mathfrak{p}$ is finite.

8.4. **Implication '**$\mathsf{a}\Rightarrow\mathsf{b}$**'.** We fix a subgroup H of G satisfying condition 'a' of Theorem B. Let Γ be a locally connected graph where G acts properly. Let O be a weakly quasiconvex H-orbit satisfying 'a' with the parameter q. Denote by A the vertex set Γ^0 and by $\widetilde{T}$ the union $A\sqcup\mathbf{\Lambda}G$.

Proposition 8.4.1. *For every system of horospheres S there exists a constant c such that $S_\mathfrak{p}\subset\mathsf{N}_cO$ for every $\mathfrak{p}\in\overline{O}\cap\mathsf{Par}$.*

Proof. Since the subgroup H is weakly α-quasiconvex in G there exists an α-isometric map $\varphi{:}H\to\Gamma$ with the distortion function α. Then by [GP09, Theorem C] the subgroup H is relatively hyperbolic with respect to the system $\{H\cap\mathsf{St}_G\mathfrak{p} : \mathfrak{p}\in\mathsf{Par}\}$. Thus this system of maximal parabolic subgroups of H contains at most finitely many H-conjugacy classes [Ge09, Main Theorem, a].

By Proposition 8.2.3 for every $\mathfrak{p}\in\overline{O}\cap\mathsf{Par}$ the set $P\cap H$ is infinite where $P{=}\mathsf{St}_G\mathfrak{p}$. Then by our assumption $|P{:}H\cap P|{<}\infty$. Since there are at most finitely many distinct H-conjugacy classes of such subgroups the set of all indices $|\mathsf{St}_G\mathfrak{p} : H\cap\mathsf{St}_G\mathfrak{p}|$ $(\mathfrak{p}\in\mathsf{Par})$ is bounded. The Proposition follows. □

Remark. In the above proof the fact that the subgroup H is relatively hyperbolic itself and can contain at most finitely many conjugacy classes of distinct parabolic subgroups was essential. In general there are examples of geometrically finite Kleinian groups containing finitely generated subgroups having infinitely many conjugacy classes of parabolic subgroups [KP91].

Definition. Let $O\subset A$ be an H-orbit of a point $v\in A$. The set

(8.4.2) $$F_v\leftrightharpoons\{x\in A : \mathsf{d}(x,O){=}\mathsf{d}(x,v)\}$$

is called Dirichlet set at v.

Remark. The set F_v is a discrete analog of the Dirichlet fundamental set for a discrete subgroup of the isometry group of the real hyperbolic space.

Proposition 8.4.3. *The set F_v is a v-star convex fundamental set for the action of H on A.*

Proof. For every point $x{\in}A$ there exists $w{=}h(v) \in O$ such that $d(x,w){=}d(x,O)$. So $h^{-1}x{\in}F_v$ and $A{=}\bigcup_{h\in H} hF_v$.

We have $v{\in}F_v$. To show that F_v is v-star convex we need to show that if $w{\in}F_v$ then for any $t{\in}[w,v]$ we have $t{\in}F_v$. Suppose not then there exists $u{\in}O\setminus v$ such that $d(t,u){<}d(t,v)$. Then $d(w,v){=}d(t,v)+d(t,w){>}d(t,u)+d(t,w){\geq}d(u,w)$ which is impossible as $w{\in}F_v$. □

The main step in proving the implication 'a⇒b' is the following.

Proposition 8.4.4. *The closure of the set F_v is disjoint from ∂O.*

Corollary. *The closure $\overline{F}_v$ of F_v in $\widetilde{T}$ is a compact fundamental set for the action of H on $\widetilde{T}\setminus\mathbf{\Lambda}H$.*

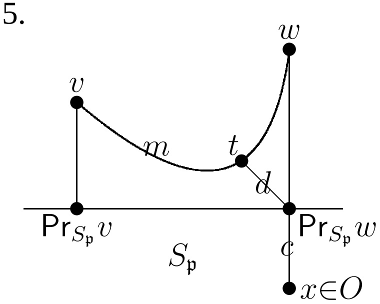
Proof of Corollary. Let $x\in\widetilde{T}\setminus\mathbf{\Lambda}H$, we need to show that there exists $h\in H\ :\ h(x)\in\overline{F}_v$. If $x\in A$ then it follows from 8.4.3.

Let $x\in\mathbf{\Lambda}G\setminus\mathbf{\Lambda}H$. Then there exists a sequence $(x_n)\subset A$ converging to x. Let $h_n\in H$ such that $h_n(x_n)\in F_v$. Suppose first that the set $\{h_n^{-1}(F_v)\}_n$ is infinite. By 8.4.4 $\overline{F}_v{\cap}\mathbf{\Lambda}H{=}\emptyset$ so up to passing to a subsequence we obtain $h_n^{-1}(y)\to x$ for every $y\in F_v$. Then $x\in\mathbf{\Lambda}H$ which is impossible. So the set $\{h_n^{-1}(F_v)\}_n$ is finite and up to a new subsequence we have $x_n\in h^{-1}F_v$ for a fixed $h\in H$. Thus $h(x)\in\overline{F}_v$. □

To prove 8.4.4 we need the following.

Lemma 8.4.5. *Let S be a system of horospheres for the action $G\curvearrowright\widetilde{T}$. There exists a constant D_S such that, for $\mathfrak{p}{\in}\overline{O}{\cap}\mathsf{Par}$, $\mathsf{diam}_\mathsf{d}\,\mathsf{Pr}_{S_\mathfrak{p}}F_v{\leqslant}D_S$.*

Proof. Let $w{\in}F_v$. Suppose that the distance between the projections of v and w is greater than the constant R of 8.2.5.



Then there exists d such that $\mathsf{d}(\mathsf{Pr}_{S_\mathfrak{p}}w,[w,v]){=}\mathsf{d}(\mathsf{Pr}_{S_\mathfrak{p}}w,t){\leqslant}d$ for some $t{\in}[w,v]$. Let $m{\leftrightharpoons}\mathsf{d}(t,v)$ and let c be the constant from Proposition 8.4.1. Then by Proposition 8.4.3 $t{\in}F_v$, so for every $x\in O$ we have
$m{\leqslant}d(t,x){\leqslant}d{+}c$, $\mathsf{d}(v,S_\mathfrak{p}){\leqslant}m{+}d{\leqslant}2d{+}c$; $\mathsf{d}(\mathsf{Pr}_{S_\mathfrak{p}}v,\mathsf{Pr}_{S_\mathfrak{p}}w){\leqslant}d(\mathsf{Pr}_{S_\mathfrak{p}}v,v){+}d(v,\mathsf{Pr}_{S_\mathfrak{p}}w))$
${\leqslant}2d{+}c{+}m{+}d{\leqslant}4d{+}2c{\leftrightharpoons}D_S$. □

Proof of 8.4.4. Suppose that the assertion is false, and let $\mathfrak{t}{\in}\overline{F_v}{\cap}\partial O$.

By compactness argument there exists an infinite α-geodesic γ starting at $v\in O$ and converging to $\mathfrak{t}$. It is the limit of a sequence of α-geodesics whose endpoints are v and $\mathfrak{t}_n{\in}O$ such that $\mathfrak{t}_n\to\mathfrak{t}$. So by the α-weak quasiconvexity we can assume that $\mathsf{Im}\gamma$ is contained in $\mathsf{N}_q O$.

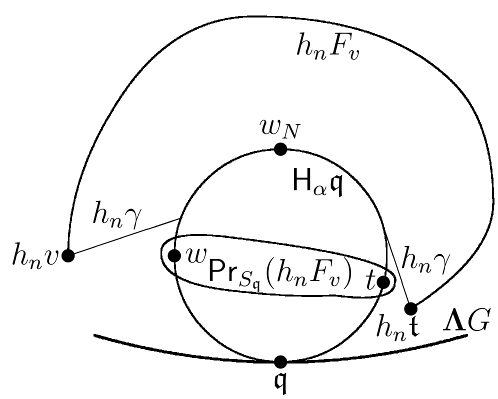
Choose $v_n{\in}\mathsf{Im}\gamma$ such that $\mathsf{d}(v_n,v){>}2n{+}q$. We claim that $\mathsf{d}(v_n,F_v){>}n$. Indeed let $o_n\in O$ be such that $d(o_n,v_n){\leqslant}q$. Take $v'_n\in\mathsf{Pr}_{F_v}v_n$. Then
$d(v'_n,v){\leqslant}d(v'_n,o_n){\leqslant}d(v'_n,v_n){+}d(v_n,o_n){\leqslant}d(v'_n,v_n)+q.$

Hence $2n+q{<}d(v,v_n){\leqslant}d(v,v'_n)+d(v'_n,v_n){\leqslant}2d(v'_n,v_n)+q$ and the claim follows.

Since N_qO is H-finite there exist $h_n{\in}H$ such that h_nv_n belong to a ball of a finite radius centered at v. By passing to a subsequence we can reduce the situation to the case when $h_nv_n{=}w$ independently on n. Moreover we can assume that the sequence $h_n\gamma$ converges in the Tikhonoff topology to an infinite α-geodesic $\beta:\mathbb{Z}\to A$ with $\beta(0){=}w$.

We have $d(w,h_nF_v){>}n$. By Proposition 8.4.3 h_nF_v is h_nv-star convex. Then by Proposition 2.4.1 we obtain that $\mathsf{diam}_{\overline{\boldsymbol{\delta}}_w}(h_nF_v)\to 0$ where $\overline{\boldsymbol{\delta}}_w$ is the shortcut metric with respect to a Floyd function forming with α an appropriate pair. Thus $\overline{\boldsymbol{\delta}}_w(h_nv,h_n\mathsf{t})\to 0$ and β is an α-horocycle based at a parabolic point $\mathsf{q}{\in}\mathsf{Par}$ (see 4.4.1). Then by 5.1.2 $\mathsf{Im}\beta$ is contained in the Q-finite set $S_\mathsf{q}{=}\mathsf{H}_\alpha\mathsf{q}$ where $Q{\leftrightharpoons}\mathsf{St}_G\mathsf{q}$. It follows that the intersection of O with the Q-finite set $\mathsf{N}_q\mathsf{H}_\alpha\mathsf{q}$ is infinite. By 8.2.1 the subgroup $H{\cap}Q$ is infinite and so $\mathsf{q}{\in}\overline{O}$.

By Lemma 8.4.5 the diameter of $\mathsf{Pr}_{S_\mathsf{q}}(h_nF_v){=}\mathsf{Pr}_{h_n^{-1}S_\mathsf{q}}F_v$ is bounded. Since the diameter of $(h_n\mathsf{Im}\gamma){\cap}\mathsf{Im}\beta$ tends to infinity there exists a point $w_N{\in}(h_n\mathsf{Im}\gamma){\cap}\mathsf{Im}\beta$ such that the number $N{=}d(w_N,\mathsf{Pr}_{S_\mathsf{q}}(h_nF_v))$ is arbitrarily large. To obtain a contradiction we choose N in few steps. The endpoints of the curve $h_n\gamma$ belong to $h_n\overline{F}_v$, so we have $\mathsf{Pr}_{S_\mathsf{q}}(\partial h_n\mathsf{Im}\gamma)\subset\mathsf{Pr}_{S_\mathsf{q}}(h_nF_v)$. Since $w_N{\in}S_\mathsf{q}$ we first assume that $N>R$ where R is the constant from Proposition 8.2.5. Then it implies that there exists $d{=}d(R)$ such that both α-distorted subpaths of $h_n\gamma$ joining w_N with its endpoints meet the d-neighborhood $U{=}\mathsf{N}_d(\mathsf{Pr}_{S_\mathsf{q}}(h_nF_v))$ of $\mathsf{Pr}_{S_\mathsf{q}}(h_nF_v)$ in A. On the picture below the points w and t belong to $\mathsf{Pr}_{S_\mathsf{q}}(h_nF_v)$ and are close to the subpaths of $h_n\gamma$.



Therefore $h_n\mathsf{Im}\gamma\subset\mathsf{H}_\alpha(U)$. By Proposition 4.1.2 there exists $s{=}s(d,\alpha)$ such that $h_n\mathsf{Im}\gamma\subset\mathsf{N}_s(\mathsf{Pr}_{S_\mathsf{q}}(h_nF_v))$. Assuming finally that $N>\max\{R,s\}$ we obtain $w_N{\in}h_n\mathsf{Im}\gamma\setminus N_s(\mathsf{Pr}_{S_\mathsf{q}}(h_nF_v))$ which is a contradiction. □

The condition 'b' of Theorem B does not depend on the choice of A. We formulate this independence as follows

Corollary. *Let a group G act on compacta X and $\widetilde{T}$ 3-discontinuously and 2-cocompactly such that the corresponding limit sets are proper subsets of them. Let $\varphi:X\to\widetilde{T}$ be a continuous equivariant map bijective on the limit sets. Then a subgroup $H{<}G$ acts cocompactly on $\widetilde{T}\setminus\boldsymbol{\Lambda}H$ if and only if H acts cocompactly on $X\setminus\boldsymbol{\Lambda}H$.* □

9. SUBGROUPS OF CONVERGENCE GROUPS WITH PROPER LIMIT SETS.

9.1. **Dynamical boundness of subgroups.** *Let G be a group acting 3-discontinuosly on a compactum $\widetilde{T}{=}T\sqcup A$ where $T{=}\mathbf{\Lambda}G$ and A is a non-empty, discrete and G-finite set (see Subsection 2.4).*

Remark. We do not assume in this Subsection that the action is 2-cocompact nor that G is finitely generated.

Definition. *Let G acts 3-discontinuously on a compactum T. A subgroup H of G is called dynamically bounded if every infinite set of elements $S{\subset}G$ contains an infinite subset S_0 such that $T\setminus \bigcup\limits_{s\in S_0} s(\mathbf{\Lambda}H)$ has a non-empty interior.*

We start by giving several equivalent reformulations of this notion.

Proposition 9.1.1. . *Let T be a metrisable compactum and G be a group acting 3-discontinuously on X. Then the following statements are equivalent:*

1) *H is dynamically bounded in G.*
2) *There exist finitely many proper closed subsets $F_1, ..., F_k$ of T such that*
$$\forall g{\in}G\ \exists\ i{\in}\{1, ..., k\}\ :\ g(\mathbf{\Lambda}H){\subset}F_i.$$
3) *In the space $\mathrm{Cl}(T)$ of closed subsets of a compactum T equipped with the Hausdorff topology one has*
$$T{\not\in}\overline{\{g(\mathbf{\Lambda}H)\ :\ g{\in}G\}}.$$

Corollary. *The dynamical boundness is a hereditary property with respect to subgroups, i.e. if a subgroup H of G is dynamically bounded then any subgroup of H is so.*

Proof of Corollary. It follows immediately e.g. from the condition 2). Indeed if $H_0 < H$ then $\mathbf{\Lambda}H_0{\subset}\mathbf{\Lambda}H$ and so the sets F_i existing for H work equally for H_0 $(i{=}1, ..., k)$. □

Proof of the Proposition. Let us prove the following implications : $2) \Rightarrow 1) \Rightarrow 3) \Rightarrow 2)$.

$2) \Rightarrow 1)$. Let $S{\in}G\setminus H$ an infinite set of pairwise distinct elements modulo H. Then there exists an infinite subset $S_0{\subset}S$ such that $\exists\ i{\in}\{1, ..., k\}\ \forall s{\in}S_0\ s(\mathbf{\Lambda}H){\subset}F_i$. The set $F_i'{=}T\setminus F_i$ is open so we are done.

$1) \Rightarrow 3)$. Suppose by contradiction that 3) is not true and there exists a sequence $(g_n){\subset}G$ such that $g_n(\mathbf{\Lambda}H) \to T$ in the Hausdorff topology. Then the same is true for any its subsequence contradicting the condition 1).

$3) \Rightarrow 2)$. We provide a topological proof. Recall first that every entourage $\mathbf{u}{\in}\mathrm{Ent}T$ defines a distance function $\Delta_{\mathbf{u}}$ on T which is the maximal one with the property $(x, y){\in}\mathbf{u}{\cap}\Theta^2T$ if and only if $\Delta_{\mathbf{u}}(x, y){\leqslant}1$ (see e.g. [GP10]). So for every $\mathbf{u}{\in}\mathrm{Ent}T$ we define the entourage $\mathbf{w}{=}\mathbf{u}^k$ $(k{\in}\mathbb{N})$ such that $(x, y){\in}\mathbf{w}$ if and only if $\Delta_{\mathbf{u}}(x, y){\leqslant}k$.

Let now $\mathbf{u}{\in}\mathrm{Ent}T$ be an entourage on T. For any subset $C{\subset}T$ its $\mathbf{u}$-neighborhood $C\mathbf{u}$ is the set $\{x{\in}T \mid \exists\, y{\in}C : (x, y){\in}\mathbf{u}\}$. By the condition 3) there exists $\mathbf{u}{\in}\mathrm{Ent}T$ such that $\forall g{\in}G\ g(\mathbf{\Lambda}H)\mathbf{u}{\neq}T$. In other words $\forall g{\in}G\ \exists\, p_g{\in}T : \forall\, y{\in}g(\mathbf{\Lambda}H)\ (p_g, y){\notin}\mathbf{u}$ (i.e. $p_g\mathbf{u} \cap g(\mathbf{\Lambda}H){=}\emptyset$). Take an entourage $\mathbf{v}{\in}\mathrm{Ent}T$ such that $\mathbf{v}^2{\subset}\mathbf{u}$ meaning that $(x, y){\in}\mathbf{v}$ and $(y, z){\in}\mathbf{v}$ implies $(x, z){\in}\mathbf{u}$. Since T is compact there exists a finite $\mathbf{v}$-net $\mathcal{P}{\subset}T$ such that $\forall x{\in}T\ \exists\ y{\in}\mathcal{P}\ :\ (x, y){\in}\mathbf{v}$. So for every $g \in G$

there is $q_g\in\mathcal{P}\ :\ (p_g, q_g)\in\mathbf{v}$. It follows that $q_g\mathbf{v}\cap g(\mathbf{\Lambda}H)=\emptyset$ as otherwise $p_g\mathbf{u}\cap g(\mathbf{\Lambda}H)\neq\emptyset$. The set $F_q=(q_g\mathbf{v})'$ is the desired closed subset of T. □

Remark. In the above proof we need the metrisability of T only to prove the second implication as the choice of a sequence converging to an accumulation point in a topological space without countable basis is not possible in general.

Proposition 9.1.2. *If $H<G$ is dynamically quasiconvex then it is dynamically bounded.*

Proof. Let us fix an entourage $\mathbf{u}\in\mathrm{Ent}T$ such that T is not $\mathbf{u}^4$-small (i.e. $\mathrm{diam}_{\Delta_{\mathbf{u}^4}}(T)>1$). Then by compactness of T there exists a finite $\mathbf{u}$-net $\mathcal{P}$. So for any $x\in T\ \exists\ y\in\mathcal{P}$ such that $(x,y)\in\mathbf{u}$. Let $S\subset G$ be an infinite set of elements. Then there is an infinite subset $S_0\subset S$ such that $\exists y\in\mathcal{P}\ \forall s\in S_0\ y\in s(\mathbf{\Lambda}H)\mathbf{u}$. Since H is dynamically quasiconvex up to removing a finitely many elements we can assume that for all $s\in S_0$ we have $s(\mathbf{\Lambda}H)$ is $\mathbf{u}$-small. Therefore $\forall\ s\in S_0\ :$ $s(\mathbf{\Lambda}H)\subset U_y$ where U_y is an $\mathbf{u}^2$-small neighborhood of y.

Then there exists $z\in\mathcal{P}\setminus y$ having an $\mathbf{u}^2$-small neighborhood U_z such that $U_y\cap U_z=\emptyset$. Indeed otherwise every point of T would belong to an $\mathbf{u}^4$-small neighborhood of y which is impossible. So $T\setminus\bigcup_{s\in S_0}s(\mathbf{\Lambda}H)$ has a non-empty interior. □

The following Proposition shows that a dynamically bounded subgroup acting cocompactly outside the limit set on T do the same on $\widetilde{T}$.

Proposition 9.1.3. *Let G act 3-discontinuously on $\widetilde{T}=T\sqcup A$. Suppose H is a dynamically bounded subgroup of G acting cocompactly on $T\setminus\mathbf{\Lambda}H$. Then H acts cocompactly on $\widetilde{T}\setminus\mathbf{\Lambda}H$.*

By 9.1.2 every dynamically quasiconvex subgroup is dynamically bounded so we have.

Corollary. *Let T and G be as above. Let $H<G$ be a dynamically quasiconvex subgroup of G acting cocompactly on $T\setminus\mathbf{\Lambda}H$ then H acts cocompactly on $\widetilde{T}\setminus\mathbf{\Lambda}H$. In particular if H is a parabolic subgroup for the action of G on T then it is so for the action on $\widetilde{T}$.*

Remark. If one assumes in addition that the action $G\curvearrowright T$ is 2-cocompact then the latter fact also follows from [Ge09, Corollary, 7.2].

Proof of the Proposition. Suppose this is not true. Since $(T\setminus\mathbf{\Lambda}H)/H$ is compact there exists H-invariant subset W of A such that $|W/H|=\infty$ and all limit points of W are in $\mathbf{\Lambda}H$. The set A is G-finite, so we can assume that W is an orbit $Sa\ (a\in A)$ where S is an infinite set of elements of G representing distinct right cosets $H\backslash G$. Since H is dynamically bounded, S admits an infinite subset S_0 such that $C=T\setminus\bigcup_{g\in S_0}g^{-1}(\mathbf{\Lambda}H)$ has a non-empty interior. Choose $x\in C$ which admits a neighborhood $U_x\subset C$.

For every $g\in S_0$ we have $g(x)\notin\mathbf{\Lambda}H$, so there exists $h_g\in H$ that $h_g(g(x))\in K$, where K is a compact fundamental set for the action $H\curvearrowright(T\setminus\mathbf{\Lambda}H)$. The set $S_1=\{\gamma_g\ :\ \gamma_g=h_g g,\ g\in S_0\}$ is infinite, so it admits a limit cross $(r,a)^\times=r\times T\cup T\times a$ where r and a are respectively repeller and attractor points [Ge09]. By our assumption we have $a\in\mathbf{\Lambda}H$.

We now claim that $r\neq x$. Suppose not. If first there exists $b\in\mathbf{\Lambda}H\setminus\{a\}$ then we can find $\gamma_g\in S_1$ close to $(r,a)^\times$ such that $\gamma_g^{-1}(b)\in U_x$. By the choice of U_x it is impossible. So we must have $\mathbf{\Lambda}H=\{a\}$. Then a is a parabolic point for the convergence action of $G\curvearrowright T$ [Bo99, Proposition 3.2], [Tu98, Theorem 3.A] (we note that the argument of these papers can be applied without assuming the metrisability of T). From the other hand we have $\gamma_g^{-1}(a)\notin U_x$ and $\gamma_g^{-1}(b)\in U_x$ for all $b\neq a$ and for all elements $\gamma_g\in S_1$ close to $(r,a)^\times$ (for which γ_g^{-1} is close to $(a,r)^\times$). It follows that a is a conical point for the action $G\curvearrowright T$ [Bo99]. This is a contradiction. We have proved $r\neq x$.

For any neighborhood $U_a \subset T$ of a we have $\gamma_g(x) \in U_a$ for some $\gamma_g \in S_1$. Since $\forall g \in S_0\ \gamma_g(x) \in K$ we obtain $K \cap U_a \neq \emptyset$. This is impossible as $a \in \mathbf{\Lambda} H$ and K is compact in $T \setminus \mathbf{\Lambda} H$. □

9.2. **Finite presentedness of dynamically bounded subgroups.** The property to act cocompactly outside the limit set for a subgroup of a RHG has several consequences which have been established in Sections 4 and 8. The following Proposition gives one more property of such subgroups.

Proposition 9.2.1. *Let G be a group acting 3-discontinuously and 2-cocompactly on a compactum $\widetilde{T}$. Suppose H is a subgroup of G acting cocompactly on $\widetilde{T} \setminus \mathbf{\Lambda} H$. If G is finitely presented then H also is.*

Proof. Let Γ be a connected graph on which G acts discontinuously and cocompactly. It is rather well-known that there exists a simply connected 2-dimensional CW-complex $C(\Gamma)$ such that $C(\Gamma)^1 = \Gamma$ and G acts cocompactly on $C(\Gamma)$. Since the action of G on Γ is not necessarily free we provide for the sake of completeness a short proof of it. Consider the Cayley graph $\Gamma_0 = \mathrm{Cay}(G, S)$ of G corresponding to the finite generating set S with finitely many defining relations. Let R_1 be a maximal subset of G-nonequivalent loops in Γ_0 corresponding to the elements of S. Since the action of G on both graphs Γ_0 and Γ is cocompact there is an equivariant finite-to-one quasi-isometry $\varphi : \Gamma_0 \to \Gamma$ which is injective everywhere outside the set of the preimages of the fixed points for the action $G \curvearrowright \Gamma$. Let $C(\Gamma_0)$ be the Cayley 2-dimensional simply connected CW-complex obtained by gluing 2-cells to the G-orbit of R_1. Denote by $R_2 \subset \Gamma_0^0$ a maximal subset of G-non-equivalent points on which φ is not injective. We now construct the 2-dimensional CW-complex $C(\Gamma)$ by attaching 2-cells to the G-orbits of the loops $\varphi(a) \in \Gamma^1$, where $a \in R_1$ or a is a path connecting a pair of points in R_2 mapped to the same point of Γ^0. The map φ extends continuously and equivariantly to a surjective map between the 2-skeletons $C^2(\Gamma_0) \to C^2(\Gamma)$. Every loop $\gamma \in \Gamma$ is a product $\prod_i \varphi(a_i)$ where each $\varphi(a_i)$ is trivial in $C(\Gamma)$. Therefore $C(\Gamma)$ is simply connected and satisfies the claim above.

We will now construct an H-invariant 2-dimensional simply connected CW-complex $\mathcal{E}$ such that $\mathcal{E}/H$ is compact. Let E be an H-finite and H-invariant subset of Γ^0. Set $\mathcal{E}^0 = E$. Join by an edge each pair of vertices of $\mathcal{E}^0$ situated within a distance at most C where C is the constant from Proposition 4.2.1. Denote by $\mathcal{E}^1$ the obtained graph. Let n be the maximal length of the boundary curves of the 2-cells of $C(\Gamma)$ corresponding to a finite set of generating relations of G. Attaching now a 2-cell to every closed curve of $\mathcal{E}^1$ of length at most n denote by $\mathcal{E}$ the obtained complex.

Let $\mathsf{pr}_E : \Gamma^0 \to E$ denote a single valued branch of the multivalued map Pr_E obtained by choosing one element from the image of each vertex. The map pr_E extends to a continuous map $C(\Gamma)^1 \to \mathcal{E}^1$ which sends the edges of Γ to edges of $\mathcal{E}$ by 4.2.1. The projection of a path in Γ is a path in $\mathcal{E}^1$. The map pr_E is surjective so the graph $\mathcal{E}^1$ is connected.

Every 2-cell of $C(\Gamma)$ is bounded by a curve which is the product of curves of length at most n. By construction its projection to $\mathcal{E}$ is also a trivial loop with the same property. So the map pr_E extends to a map between the 2-skeletons $C(\Gamma)^2 \to \mathcal{E}^2$.

The complex $\mathcal{E}$ is simply connected. Indeed, let β be a simple loop in $\mathcal{E}^1$. Then it admits a preimage $\widetilde{\beta}$ in Γ which is either a loop; or a path connecting two points $v_i \in A$ such that $\mathsf{pr}_E(v_i) = v \in E$ $(i = 1, 2)$. In the first case since $C(\Gamma)$ is simply connected, the loop $\widetilde{\beta}$ is trivial and so β is trivial in $\mathcal{E}$. In the second case we have $v_i \in F_v$ for the set F_v introduced in 8.4.2. By Proposition 8.4.3, F_v is v-star geodesic and there exist two geodesics $l_i \subset F_v$ connecting v_i with v. The loop $\widetilde{\eta} = \widetilde{\beta} \cup l_1 \cup l_2$ is trivial in $C(\Gamma)$. We have $\mathsf{pr}_E(F_v) = \mathsf{pr}_E(l_i) = v$, so $\mathsf{pr}_E(\widetilde{\eta}) = \mathsf{pr}_E(\widetilde{\beta}) = \beta$

is as above a trivial loop in $\mathcal{E}$. So $\mathcal{E}$ is simply connected. Furthermore each relation in H corresponds to a 2-disk D in $\mathcal{E}$ such that $D = \mathsf{pr}_E(\widetilde{D})$ where $\widetilde{D}$ is a 2-disk in $C(\Gamma)$. By construction the projection $\mathsf{pr}_E : C(\Gamma)^1 \to \mathcal{E}^1$ is an isometric map. Therefore every relation in H follows from finitely many generating relations each of length at most n. Thus the subgroup H is finitely presented. □

Corollary. *Let a finitely presented group G act 3-discontinuously and 2-cocompactly on a compactum T. If H is a dynamically bounded subgroup of G acting cocompactly on $T \setminus \mathbf{\Lambda} H$ then H is finitely presented too.*

Proof. It follows immediately from Propositions 9.1.3 and 9.2.1.

Remarks. 1. In the above proof we could at once assume (w.l.o.g.) that $\Gamma{=}\Gamma_0$ is the Cayley graph. Indeed the proper quasi-isometry φ extends equivariantly to a homeomorphism of T keeping $\mathbf{\Lambda} H$ invariant [GP09, Lemma 2.5]. This gives an equivariant proper map $\Gamma_0 \sqcup T \to \Gamma \sqcup T$ preserving $\mathbf{\Lambda} H$. So the action of H on $(\Gamma_0 \sqcup T) \setminus \mathbf{\Lambda} H$ is cocompact too.

2. The above Proposition was inspired by [DG10, Theorem 1] establishing that the maximal parabolic subgroups of finitely presented relatively hyperbolic groups are finitely presented. This result follows from the above Corollary as maximal parabolic subgroups act cocompactly outside their limit points and are dynamically bounded (see Corollary of 9.1.3)

We finish the Section by a series of examples and questions.

Examples. 1) *An example of a subgroup acting cocompactly on the complement of its limit set and which is not a parabolic subgroup for any convergence action of the ambiant group*.

Let $G < \mathrm{Isom}\mathbb{H}^n$ be a uniform lattice. Let us fix two elements a and b of G having different fixed points (i.e. generating a non-elementary subgroup of G).

For a sufficiently big n_0 the subgroup $H{=} < b, a^{n_0} b a^{-n_0} >$ is free (Schottky) and quasiconvex in G. Since G contains no parabolics it follows that the limit set of H is a proper Cantor subset of $\mathbb{S}^{n-1}$. The group H acts geometrically finitely (without parabolic elements) on $\mathbb{S}^{n-1} \setminus \mathbf{\Lambda} H$, and $(\mathbb{S}^{n-1} \setminus \mathbf{\Lambda} H)/H$ is a compact $(n{-}1)$-manifold homeomorphic to the connected sum $(\mathbb{S}^{n-2} \times \mathbb{S}^1) \# (\mathbb{S}^{n-2} \times \mathbb{S}^1)$. We have $gbg^{-1} {\in} H \cap gHg^{-1}$ where $g{=}a^{n_0} {\notin} H$. Thus the subgroup $H {\cap} gHg^{-1}$ is infinite. It is well known (e.g. follows from our Proposition 5.1.3) that H cannot be parabolic for any geometrically finite action of G. □

2) *An example of a dynamically bounded subgroup which is not dynamically quasiconvex.*

Take a 3-dimensional uniform arithmetic lattice $G < \mathrm{Isom}\mathbb{H}^3$ such that $\mathbb{H}^3/G$ fibers over the circle. Let H be a normal finitely generated subgroup of G of infinite index which is the group of the fiber manifold. It acts non-geometrically finitely on $\mathbb{H}^n$ $(n \geq 3)$. The group G can be embedded into another arithmetic lattice $G_0 < \mathrm{Isom}\mathbb{H}^n$ $(n > 3)$. Since G is (dynamically) quasiconvex in G_0, by Proposition 9.1.2 it is also dynamically bounded. Then by the hereditary property (see Corollary of 9.1.1) H is a dynamically bounded subgroup of G_0.

3) *An example of a dynamically bounded subgroup of a relatively hyperbolic finitely presented group which is not finitely presented itself.*

It is proved in [KPV08] that any arithmetic non-uniform lattice $G_0 < \mathrm{Isom}\mathbb{H}^{\mathrm{n}}$ $(\mathrm{n} \geq 6)$ contains a geometrically finite subgroup $G < \mathrm{Isom}\mathbb{H}^4$ which contains a normal finitely generated but infinitely presented subgroup F. The subgroup G is dynamically quasiconvex in G_0, and so is dynamically bounded. As in Example 2 by the hereditary property F is dynamically bounded too. This example shows that Proposition 9.2.1 is not true for dynamically bounded subgroups without assuming the cocompactness of the action outside the limit set.

Here are several questions which seem to be intriguing and open.

Questions. 1) Suppose that the action $G \curvearrowright \widetilde{T}$ is 3-discontinuous and 2-cocompact. Let $H < G$ be a finitely generated subgroup acting cocompactly on $T \setminus \mathbf{\Lambda} H$. Is it true that H acts cocompactly on $\widetilde{T} \setminus \mathbf{\Lambda} H$?

2) Suppose that the action $G \curvearrowright \widetilde{T}$ is 3-discontinuous and 2-cocompact. Can G contain a finitely generated subgroup H such that $\mathbf{\Lambda} H \subsetneqq \mathbf{\Lambda} G$ and H is not dynamically bounded ?

In particular can a discrete finitely generated subgroup $H < \mathrm{Isom}\mathbb{H}^n$ of a geometrically finite group G such that $\mathbf{\Lambda} H \subsetneqq \mathbf{\Lambda} G$ be not dynamically bounded ?

Comments. If the answer to the second question is "no" then by 9.1.3 the answer to the first question is "yes". Then by 4.2.2 the subgroup H acts cocompactly on $\widetilde{T} \setminus \mathbf{\Lambda} H$ so H is quasiconvex in G. This would in particular imply that a geometrically infinite finitely generated group in $\mathrm{Isom}\mathbb{H}^n$ acting cocompactly on the non-empty set $\mathbb{S}^{n-1} \setminus \mathbf{\Lambda} H$ (*totally degenerate* Kleinian group) cannot appear as a subgroup of a lattice in $\mathrm{Isom}\mathbb{H}^n$. This fact is true in dimension $n{=}3$ and follows from so called *covering* theorem due to W. Thurston [Mo84, Proposition 7.1]. From the other hand totally degenerate Kleinian groups exist in dimension 3 and appear on the boundary of the Teuchmüller spaces of surfaces [Be70]. It is not known whether they exist in higher dimensions.

We also note that every lattice G in $\mathrm{Isom}\mathbb{H}^n$ contains an infinitely generated subgroup H which is not dynamically bounded. We thank Misha Kapovich for indicating to us that there always exists an infinitely generated subgroup of G whose limit set is a proper subset of $\mathbb{S}^{n-1}$ containing deep points (see Introduction). This follows from the fact that the images of $\mathbf{\Lambda} H$ under a sequence of $(g_n) \subset G \setminus H$ whose repeller point is a conical limit point of G, belonging to $\mathbf{\Lambda} H$, are dense in $\mathbb{S}^{n-1}$ [Ka00, Section 8.5].

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