

ALGEBRAIC RATIONAL CELLS AND EQUIVARIANT INTERSECTION THEORY

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ABSTRACT. We provide a notion of algebraic rational cell with applications to intersection theory on singular varieties with torus action. Based on this notion, we study \mathbb{Q} -filtrable varieties: algebraic varieties where a torus acts with isolated fixed points, such that the associated Białynicki-Birula decomposition consists of algebraic rational cells. We show that the rational equivariant Chow group of any \mathbb{Q} -filtrable variety is freely generated by the classes of the cell closures. We apply this result to group embeddings, and more generally to spherical varieties.

1. INTRODUCTION AND MOTIVATION

Let \mathbb{k} be an algebraically closed field. The most commonly studied cell decompositions in algebraic geometry are those obtained by the method of Białynicki-Birula [B1]. If $\mathbb{G}_m \simeq \mathbb{k}^*$ acts on a smooth projective variety X with finitely many fixed points x_1, \dots, x_m , then $X = \bigsqcup X_i$, where

$$X_i = \{x \in X \mid \lim_{t \rightarrow 0} tx = x_i\}.$$

Moreover, the cells X_i are isomorphic to affine spaces. From this one concludes e.g. that the Chow groups of X are freely generated by the classes of the cell closures $\overline{X}_i \subseteq X$. This is quite notable, because the Chow groups of smooth varieties need not be finitely generated (consider e.g. a smooth projective curve of genus one). If $\mathbb{k} = \mathbb{C}$, then this decomposition implies that X has no singular cohomology in odd degrees, and that the cycle map $cl_X : A_*(X) \rightarrow H_*(X)$ is an isomorphism, to mention just a few interesting applications. The *BB-decomposition* makes sense even if X is singular, but the cells may no longer be so well-behaved.

In [G1] we study the BB-decompositions of possibly singular complex projective varieties, assuming that the cells are rationally smooth (i.e. *rational cells*). Recall that a complex algebraic variety X , of dimension n , is called *rationally smooth* if, for every $x \in X$, we have

$$H^m(X, X - \{x\}; \mathbb{Q}) = (0) \text{ if } m \neq 2n, \text{ and } H^{2n}(X, X - \{x\}; \mathbb{Q}) = \mathbb{Q}.$$

Such varieties satisfy Poincaré duality with rational coefficients. If X_i as above is a rational cell, then $\mathbb{P}(X_i) := (X_i \setminus \{x_i\})/\mathbb{G}_m$ is a rational cohomology complex projective space. Many important results on the equivariant

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cohomology of projective T -varieties admitting a BB-decomposition into rational cells are provided in [G1]; for instance, such varieties have no cohomology in odd degrees and their equivariant cohomology is freely generated by the classes of the cell closures.

The purpose of this paper is to provide analogues of such results in the context of intersection theory for schemes with an action of a torus T (i.e. T -schemes). For this, we introduce the notion of *algebraic rational cell*. Concisely, let X be an affine \mathbb{G}_m -variety with an attractive fixed point x . Then X is an algebraic rational cell if $\mathbb{P}(X) := [X \setminus \{0\}]/\mathbb{G}_m$ satisfies

$$A_*(\mathbb{P}(X))_{\mathbb{Q}} \simeq A_*(\mathbb{P}^{n-1})_{\mathbb{Q}},$$

where $n = \dim(X)$. The definition applies to actions of higher dimensional tori as well (Definition 3.1). Algebraic rational cells are modelled after (topological) rational cells [G1], although the resulting objects are not equivalent. In what follows, we show that algebraic rational cells are a good substitute for the notion of affine space in the study of Chow groups of singular varieties. This has applications to embedding theory (Section 5) and the geometry of spherical varieties (Section 6). In addition, some links between our present approach and that of [G1] are built (Theorems 5.5, 5.9, and 6.3). The techniques are mostly algebraic, and no essential use of the cycle map is made, except in Section 6.

Here is an outline of the paper. Section 2 briefly reviews equivariant Chow groups of T -schemes. We also recall and discuss the notion of equivariant multiplicities at nondegenerate fixed points. The section concludes with some inequalities relating Chow groups and fixed point loci. In Section 3 we study the intersection-theoretical properties of algebraic rational cells (Proposition 3.4, Theorem 3.5, Corollary 3.9). Next, in Section 4, we introduce the concept of (algebraically) \mathbb{Q} -filtrable spaces: projective T -varieties with isolated fixed points, such that the associated BB-decomposition is filtrable, and consists of algebraic rational cells (Definition 4.1). The key result is Theorem 4.4. It asserts that the rational equivariant Chow group of any \mathbb{Q} -filtrable variety is freely generated by the classes of the cell closures.

Having developed the theoretical framework for the study of \mathbb{Q} -filtrable varieties, we devote the last two sections to examples and applications. Let G be a connected reductive group. Recall that a normal G -variety X is called *spherical* if a Borel subgroup B of G has a dense orbit in X . Then it is known that G and B have finitely many orbits in X . It follows that X contains only finitely many fixed points of a maximal torus $T \subset B$, see e.g. [Ti]. These features make spherical varieties especially suitable for applying the techniques of this paper.

In Section 5 we apply our methods to a remarkable subclass of spherical varieties, namely, group embeddings. (We refer to that section for a definition of this key notion, and that of reductive monoids.) In this context, Theorem 5.5 states that reductive monoids which are algebraic rational cells

are characterized in the same way as rationally smooth monoids [R4]. The second half of Section 5 deals with projective group embeddings (i.e. projectivizations of reductive monoids). The outcome (Theorem 5.9) provides an extension of [G1, Theorem 7.4] to equivariant Chow groups.

Finally, in Section 6, we study complex spherical varieties. The purpose there is to compare the two notions of \mathbb{Q} -filtrable varieties, the algebraic one (Section 4) and the topological one [G1]. Roughly speaking, the main results of that section assert that *if* X is a spherical G -variety which is \mathbb{Q} -filtrable in the sense of [G1], then it is also \mathbb{Q} -filtrable in the sense of the present paper. Moreover, for such (possibly singular) X , the T -equivariant and non-equivariant cycle maps are isomorphisms. See Theorems 6.2 and 6.3 for precise statements.

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2. PRELIMINARIES

Conventions and notation. Throughout this paper, we work over an algebraically closed field \mathbb{k} of arbitrary characteristic (unless stated otherwise). All schemes and algebraic groups are assumed to be defined over \mathbb{k} . By a scheme we mean a separated scheme of finite type. A variety is a reduced scheme. Observe that varieties need not be irreducible. A subvariety is a closed subscheme which is a variety. A point on a scheme will always be a closed point.

We denote by T an algebraic torus. A scheme X provided with an algebraic action of T is called a T -scheme. For a T -scheme X , we denote by X^T the fixed point subscheme and by $i_T : X^T \rightarrow X$ the natural inclusion. If H is a closed subgroup of T , we similarly denote by $i_H : X^H \rightarrow X$ the inclusion of the fixed point subscheme. When comparing X^T and X^H we write $i_{T,H} : X^T \rightarrow X^H$ for the natural (T -equivariant) inclusion. For a scheme X , the dimension of the local ring of X at x is denoted $\dim_x X$. We denote by Δ the character group of T , and by S the symmetric algebra over \mathbb{Q} of the abelian group Δ . We denote by \mathcal{Q} the quotient field of S . Equivariant Chow groups are always considered with rational coefficients.

In this paper, torus actions are assumed to be *locally linear*, i.e. the schemes we consider are covered by invariant affine open subsets. This assumption is fulfilled e.g. by T -stable subschemes of normal T -schemes [Su].

2.1. The Bialynicki-Birula decomposition. The results in this subsection are due to Bialynicki-Birula [B1], [B2] (in the smooth case) and Konarski [Ko] (in the general case). For our purposes, it suffices to consider the case of torus actions with isolated fixed points.

Let T be an algebraic torus. Let X be a T -scheme with isolated fixed points. Then X^T is finite and we write $X^T = \{x_1, \dots, x_m\}$. Recall that a one-parameter subgroup $\lambda : \mathbb{G}_m \rightarrow T$ is called *generic* if $X^{\mathbb{G}_m} = X^T$, where \mathbb{G}_m acts on X via λ . Generic one-parameter subgroups always exist, due to local linearity of the action. Now fix a generic one-parameter subgroup λ of T . For each i , define the subset

$$X_+(x_i, \lambda) = \{x \in X \mid \lim_{t \rightarrow 0} \lambda(t) \cdot x = x_i\}.$$

Then $X_+(x_i, \lambda)$ is a locally closed T -invariant subscheme of X . The (disjoint) union of the $X_+(x_i, \lambda)$'s might not cover all of X , but when it does (e.g., when X is complete), the decomposition $\{X_+(x_i, \lambda)\}_{i=1}^m$ is called the Bialynicki-Birula decomposition, or *BB-decomposition*, of X associated to λ . Each $X_+(x_i, \lambda)$ is called a *cell* of the decomposition. Usually the *BB-decomposition* of a complete T -scheme is not a Whitney stratification; that is, it may happen that the closure of a cell is not a union of cells, even when the scheme is assumed to be smooth. For instance, see [B2, Example 1].

Definition 2.1. Let X be a T -scheme with finitely many fixed points. Let $\{X_+(x_i, \lambda)\}_{i=1}^m$ be the BB-decomposition associated to some generic one-parameter subgroup λ of T . The decomposition $\{X_+(x_i, \lambda)\}$ is said to be *filtrable* if there exists a finite increasing sequence $\Sigma_0 \subset \Sigma_1 \subset \dots \subset \Sigma_m$ of T -invariant closed subschemes of X such that:

- a) $\Sigma_0 = \emptyset$, $\Sigma_m = X$,
- b) $\Sigma_j \setminus \Sigma_{j-1}$ is a cell of the decomposition $\{X_+(x_i, \lambda)\}$, for each $j = 1, \dots, m$.

We will refer to Σ_j as the *j -th filtered piece* of X . In this context, it is common to say that X is *filtrable*. If, moreover, the cells $X_+(x_i, \lambda)$ are isomorphic to affine spaces \mathbb{A}^{n_i} , then X is called *T -cellular*.

Theorem 2.2 ([B1], [B2]). *Let X be a complete T -scheme with isolated fixed points, and let λ be a generic one-parameter subgroup. If X admits an ample T -linearized invertible sheaf, then the associated BB-decomposition $\{X_+(x_i, \lambda)\}$ is filtrable. Further, if X is smooth, then X is T -cellular. \square*

2.2. Review of equivariant Chow groups. Localization theorem.

Let X be a T -scheme of dimension n (not necessarily equidimensional). Let V be a finite dimensional T -module, and let $U \subset V$ be an invariant open subset such that a principal bundle quotient $U \rightarrow U/T$ exists. Then T acts freely on $X \times U$ and the quotient scheme $X_T := (X \times U)/T$ exists. Following Edidin and Graham [EG], we define the *i -th equivariant Chow group* $A_i^T(X)$ by $A_i^T(X) := A_{i+\dim U - \dim T}(X)$, if $V \setminus U$ has codimension more than $n - i$. Such a pair (V, U) always exist, and the definition is independent of the

choice of (V, U) , see [EG]. Finally, set $A_*^T(X) = \bigoplus_i A_*^T(X)$. If X is a T -scheme, and $Y \subset X$ is a T -stable closed subscheme, then Y defines a class $[Y]$ in $A_*^T(X)$. If X is smooth, then so is X_T , and $A_*^T(X)$ admits an intersection pairing; in this case, denote by $A_T^*(X)$ the corresponding ring graded by codimension. The equivariant Chow ring $A_T^*(pt)$ identifies to S , and $A_*^T(X)$ is a S -module, where Δ acts on $A_*^T(X)$ by homogeneous maps of degree -1 . This module structure is induced by pullback through the flat map $p_{X,T} : X_T \rightarrow U/G$. Restriction to a fiber of $p_{X,T}$ gives $i^* : A_*^T(X) \rightarrow A_*(X)$. If X is complete, we denote by $\int_X(\alpha) \in S$ the proper pushforward to a point of a class $\alpha \in A_*^T(X)$. See [EG] for details.

Next we state Brion's description [Br1] of the equivariant Chow groups in terms of invariant cycles. It also shows how to recover the usual Chow groups from equivariant ones.

Theorem 2.3. *Let X be a T -scheme. Then the S -module $A_*^T(X)$ is defined by generators $[Y]$ where Y is an invariant irreducible subvariety of X and relations $[\text{div}_Y(f)] - \chi[Y]$ where f is a rational function on Y which is an eigenvector of T of weight χ . Moreover, the map $A_*^T(X) \rightarrow A_*(X)$ vanishes on $\Delta A_*^T(X)$, and it induces an isomorphism*

$$A_*^T(X)/\Delta A_*^T(X) \rightarrow A_*(X).$$

□

The following is a slightly more general version of the localization theorem for equivariant Chow groups [Br1, Corollary 2.3.2]. For a proof, see e.g. [G3, Proposition 2.15].

Theorem 2.4. *Let X be a T -scheme, let $H \subset T$ be a closed subgroup, and let $i_H : X^H \rightarrow X$ be the inclusion of the fixed point subscheme. Then the induced morphism of equivariant Chow groups*

$$i_{H*} : A_*^T(X^H) \rightarrow A_*^T(X)$$

becomes an isomorphism after inverting finitely many characters of T that restrict non-trivially to H . □

Let X be a T -scheme. In many situations, Theorems 2.3 and 2.4 combined yield a relation between the dimensions of the \mathbb{Q} -vector spaces $A_*(X)$ and $A_*(X^T)$.

Lemma 2.5. *Let X be a T -scheme. If $A_*(X)$ is a finite-dimensional \mathbb{Q} -vector space, then the inequality $\dim_{\mathbb{Q}} A_*(X^T) \leq \dim_{\mathbb{Q}} A_*(X)$ holds. Furthermore, $\dim_{\mathbb{Q}} A_*(X^T) = \dim_{\mathbb{Q}} A_*(X)$ if and only if the S -module $A_*^T(X)$ is free.*

Proof. The degrees in $A_*^T(X)$ are at most the dimension of X , so by the graded Nakayama lemma [E, Exercise 4.6], the S -module $A_*^T(X)$ is finitely generated. The content of the corollary is now deduced from applying Lemma 2.6 and Remark 2.7 below to $M = A_*^T(X)$, taking into account

that $\dim_{S/\mathfrak{m}}(M/\mathfrak{m}M) = \dim_{\mathbb{Q}}(A_*(X))$ (Theorem 2.3), $\dim_{\mathbb{Q}}(M \otimes_S \mathcal{Q}) = \dim_{\mathbb{Q}}(A_*^T(X^T) \otimes \mathcal{Q})$ (Theorem 2.4), and observing that this corresponds to $\dim_{\mathbb{Q}} A_*(X^T)$, since $A_*^T(X^T) = A_*(X^T) \otimes_{\mathbb{Q}} S$. \square

Lemma 2.6. *Let S be a Noetherian positively graded ring such that S_0 is a field (e.g. $S = A_T^*(pt)$). Let \mathfrak{m} be the unique graded maximal ideal and suppose M is a non-zero finitely generated, graded, S -module. Suppose further that S is an integral domain. Then M is a free S -module if and only if*

$$\dim_{S/\mathfrak{m}}(M/\mathfrak{m}M) = \dim_{\mathbb{Q}}(M \otimes_S \mathcal{Q}),$$

where \mathcal{Q} is the quotient field of S .

Proof. If M is free, then clearly the equation above holds. Conversely, denote by n the common value of the two sides of the equation above. By the graded Nakayama lemma, M has a system $\{x_1, \dots, x_n\}$ of homogeneous generators. Now the elements $x_j \otimes 1$ generate the vector space $M \otimes_S \mathcal{Q}$ over \mathcal{Q} . But as by hypothesis this space is of dimension n over \mathcal{Q} , the elements $x_j \otimes 1$ are linearly independent over \mathcal{Q} . It follows that the x_j are linearly independent over S and so they form a basis of M . \square

Remark 2.7. The proof of Lemma 2.6 shows that that if M is a finitely generated, graded, S -module, then the inequality

$$\dim_{S/\mathfrak{m}}(M/\mathfrak{m}M) \geq \dim_{\mathbb{Q}}(M \otimes_S \mathcal{Q})$$

holds, as we can refine the generating set $\{x_j \otimes 1\}$ to get a basis of $M \otimes \mathcal{Q}$.

An important class of schemes to which Lemma 2.5 applies is the class of T -linear schemes. These are the equivariant analogues of the linear schemes considered by [Jann], [To], and [J]. T -linear schemes have been studied in [FMSS], [JK], [AP] and [G3]. Briefly, a T -linear scheme is a T -scheme that can be obtained by an inductive procedure starting with a finite dimensional T -representation, in such a way that the complement of a T -linear scheme equivariantly embedded in affine space is also a T -linear scheme, and any T -scheme which can be stratified as a finite disjoint union of T -linear schemes is a T -linear scheme. See [JK] or [G3] for details. It is known that if X is a T -linear scheme, then $A_*^T(X)_{\mathbb{Z}}$ is a finitely generated $S_{\mathbb{Z}}$ -module, and $A_*(X)_{\mathbb{Z}}$ is a finitely generated abelian group (see e.g. [G3, Lemma 2.7]). Below are the concrete examples we are interested in. For a proof of items (i)-(ii) see [JK, Proposition 3.6], for item (iii) see [G3, Theorem 2.5].

Theorem 2.8. *Let T be an algebraic torus. Then the following hold:*

- (i) *A T -cellular scheme is T -linear.*
- (ii) *Every T -scheme with finitely many T -orbits is T -linear. In particular, a toric variety with dense torus T is T -linear.*
- (iii) *Let B be a connected solvable linear algebraic group with maximal torus T . Let X be a B -scheme. If B acts on X with finitely many orbits, then X is T -linear. In particular, spherical varieties are T -linear. \square*

2.3. Nondegenerate fixed points and equivariant multiplicities. Let X be a T -scheme. A fixed point $x \in X$ is called *nondegenerate* if all weights of T in the tangent space $T_x X$ are non-zero. A fixed point in a nonsingular T -variety is nondegenerate if and only if it is isolated. To study possibly singular schemes, Brion developed a notion of *equivariant multiplicity* at nondegenerate fixed points [Br1]. The main features of this concept are outlined below.

Theorem 2.9 ([Br1, Theorem 4.2]). *Let X be a T -scheme with an action of T , let $x \in X$ be a nondegenerate fixed point and let χ_1, \dots, χ_n be the weights of $T_x X$ (counted with multiplicity).*

(i) *There exists a unique S -linear map*

$$e_{x,X} : A_*^T(X) \longrightarrow \frac{1}{\chi_1 \cdots \chi_n} S$$

such that $e_{x,X}[x] = 1$ and that $e_{x,X}[Y] = 0$ for any T -invariant irreducible subvariety $Y \subset X$ which does not contain x .

(ii) *For any T -invariant irreducible subvariety $Y \subset X$, the rational function $e_{x,X}[Y]$ is homogeneous of degree $-\dim(Y)$ and it coincides with $e_{x,Y}[Y]$.*

(iii) *The point x is nonsingular in X if and only if $e_x[X] = \frac{1}{\chi_1 \cdots \chi_n}$. \square*

For any T -invariant irreducible subvariety $Y \subset X$, we set $e_{x,X}[Y] := e_x[Y]$, and we call $e_x[Y]$ the *equivariant multiplicity of Y at x* .

Proposition 2.10 ([Br1, Corollary 4.2]). *Let X be a T -scheme such that all fixed points in X are nondegenerate, and let $\alpha \in A_*^T(X)$. Then we have in $A_*^T(X) \otimes_S \mathcal{Q}$:*

$$\alpha = \sum_{x \in X^T} e_x(\alpha)[x].$$

\square

Next we consider a special class of nondegenerate fixed points. Let X be a T -variety. Call a fix point $x \in X$ *attractive* if all weights in the tangent space $T_x X$ are contained in some open half-space of $\Delta_{\mathbb{R}} = \Delta \otimes_{\mathbb{Z}} \mathbb{R}$, that is, some one-parameter subgroup of T acts on $T_x X$ with positive weights only. Below is a characterization.

Theorem 2.11 ([Br3, Proposition A2]). *Let X be a T -variety with a fixed point x . The following conditions are equivalent:*

(i) *x is attractive.*

(ii) *There exists a one-parameter subgroup $\lambda : \mathbb{G}_m \rightarrow T$ such that, for all y in a neighborhood of x , we have $\lim_{t \rightarrow 0} \lambda(t)y = x$.*

Moreover, if (i) or (ii) holds, then x admits a unique open affine T -stable neighborhood in X , denoted X_x , and X_x admits a closed equivariant embedding into $T_x X$. \square

Let X be a T -variety with an attractive fixed point x . Denote by χ_1, \dots, χ_n the weights of $T_x X$. Let Δ^* be the lattice of one-parameter subgroups of T , and let $\Delta_{\mathbb{R}}^*$ be the associated real vector space. Notice that the one-parameter subgroups λ satisfying Theorem 2.11 (ii) form the interior of a rational polyhedral cone $\sigma_x \subset \Delta_{\mathbb{R}}^*$, by setting

$$\sigma_x := \{\lambda \in \Delta_{\mathbb{R}}^* \mid \langle \lambda, \chi_i \rangle \geq 0 \text{ for } 1 \leq i \leq n\}.$$

It follows from Theorem 2.11 that X_x equals $X_+(x, \lambda)$ for any $\lambda \in \sigma_x^0$.

Proposition 2.12 ([Br1, Proposition 4.4]). *Notation being as above, the rational function $e_x[X]$, viewed as a rational function on $\Delta_{\mathbb{R}}^*$, is defined at λ and its value is the multiplicity of the algebra of regular functions on X_x graded via the action of λ . In particular, $e_x[X]$ is non-zero. \square*

2.4. Local study. Some inequalities relating Chow groups and fixed point loci. Let X be an affine T -variety with an attractive fixed point x . It follows from Proposition 2.12 that $X = X_+(x, \lambda)$ for any $\lambda \in \sigma_x^0$. Also, $\{x\}$ is the unique closed T -orbit in X , and X admits a closed T -equivariant embedding into $T_x X$. Observe that $\dim_x X = \dim X$, because x is contained in every irreducible component of X .

Choose $\lambda \in \sigma_x^0$. Then all the weights of the \mathbb{G}_m -action on $T_x X$ via λ are positive. Hence the geometric quotient

$$\mathbb{P}_{\lambda}(X) := (X \setminus \{x\})/\mathbb{G}_m$$

exists and is a projective variety. In fact, it is a closed subvariety of the weighted projective space $\mathbb{P}_{\lambda}(T_x X)$. On the other hand, by [Br3, Proposition A3], there exists a \mathbb{G}_m -module V and a finite equivariant surjective morphism $\pi : X \rightarrow V$ such that $\pi^{-1}(0) = \{x\}$ (as a set). This allows to estimate the size of the Chow groups of $\mathbb{P}_{\lambda}(X)$ in various cases.

Lemma 2.13. *In the situation above, π induces a surjection*

$$\pi_* : A_k(\mathbb{P}_{\lambda}(X)) \rightarrow A_k(\mathbb{P}_{\lambda}(V))$$

for all $k \geq 0$. Consequently, $A_k(\mathbb{P}_{\lambda}(X)) \neq 0$ if $0 \leq k \leq \dim(X)$, and $A_k(\mathbb{P}_{\lambda}(X)) = 0$ otherwise. \square

Remark 2.14. Clearly, $\pi_* : A_*(X) \rightarrow A_*(V)$ is also surjective. Observe that if X is equidimensional and d is the degree of π , then

$$e_x[X] = d \cdot e_0[V],$$

where $e_x[X]$ (resp. $e_0[V]$) is the \mathbb{G}_m -equivariant multiplicity of X at x (resp. of V at 0) [Br1, Proposition 4.3].

Now assume that $\dim_{\mathbb{Q}} A_*(\mathbb{P}_{\lambda}(X)) < \infty$. We record below a few elementary inequalities:

(1) $\dim A_*(\mathbb{P}_{\lambda}(X)) \geq \dim A_*(\mathbb{P}_{\lambda}(X)^T)$, by Lemma 2.5.

- (2) Notice that $\mathbb{P}_\lambda(X)^T = \bigsqcup_H \mathbb{P}_\lambda(X^H)$, where the union runs over all codimension-one subtori of T . In fact, by linearity of the action, there is only a finite collection of codimension-one subtori, say H_1, \dots, H_r , for which $X^{H_j} \neq X^T$. Thus

$$\dim_{\mathbb{Q}} A_*(\mathbb{P}_\lambda(X)^T) = \sum_{H_j} \dim_{\mathbb{Q}} A_*(\mathbb{P}_\lambda(X^{H_j})).$$

- (3) Let H_j be as in (2). We may also assume that x is an attractive fixed point of X^{H_j} , for the action of $\mathbb{G}_m \simeq T/H_j$. Hence, as in Lemma 2.13, there is a T -equivariant finite surjective map $\pi_j : X^{H_j} \rightarrow V_j$, where V_j is some T -module with a trivial action of H_j . Thus $\dim_{\mathbb{Q}} A_*(\mathbb{P}_\lambda(X^{H_j})) \geq \dim_{\mathbb{Q}} A_*(\mathbb{P}_\lambda(V_j))$, which in turn yields

$$\sum_{j=1}^r \dim_{\mathbb{Q}} A_*(\mathbb{P}_\lambda(X^{H_j})) \geq \sum_{j=1}^r \dim_{\mathbb{Q}} A_*(\mathbb{P}_\lambda(V_j)).$$

Equality holds if and only if the π_j 's induce isomorphisms on the Chow groups.

- (4) Since each $\mathbb{P}_\lambda(V_j)$ in (3) is a weighted projective space, we get

$$\sum_{j=1}^r \dim_{\mathbb{Q}} A_*(\mathbb{P}_\lambda(V_j)) = \sum_{j=1}^r \dim V_j = \sum_{j=1}^r \dim X^{H_j},$$

where the last equality stems from the fact that each π_j is finite and surjective.

- (5) Because x is an attractive fixed point, we have

$$\sum_{j=1}^r \dim X^{H_j} \geq \dim X,$$

by [Br3, Theorem 1.4]. The equality holds if and only if there is a T -module V and a T -equivariant finite surjective morphism $\pi : X \rightarrow V$ such that $\pi^{-1}(0) = \{x\}$. To show this, we follow closely the argument in [Br3, proof of Theorem 1.2]. First, assume that such a morphism π exists. Let H_1, \dots, H_r be subtori as in (2). Consider H_j and a point $y \in V^{H_j}$. Since H_j is connected, it acts trivially on the (finite) fiber $\pi^{-1}(y)$. This implies that the induced T -equivariant map $\pi_j : X^{H_j} \rightarrow V^{H_j}$ is finite and surjective. Hence, $\dim X = \dim V = \sum \dim V^{H_j} = \sum \dim X^{H_j}$. Conversely, if the equality $\dim X = \sum_j \dim X^{H_j}$ holds, then we can synchronize the maps π_j from (3) as follows. Given that X^{H_j} is T -stable and closed in X , we can extend π_j to an equivariant morphism $\pi'_j : X \rightarrow V_j$. Let V denote the product of all the V_j , and let $\pi : X \rightarrow V$ be the product morphism. Then $p(x) = 0$ and $V^T = \{0\}$, by construction. Notice that x , being an attractive fixed point, lies in the closure all the T -orbits in X . In particular, x is contained in all the irreducible components of $\pi^{-1}(0)$ (i.e. $\pi^{-1}(0)$ is connected). We

claim that the map π is finite. Indeed, $\{x\} = \pi^{-1}(0)$, for otherwise, $\pi^{-1}(0)$ would contain a T -stable curve upon which T acts through a non-trivial character [Br3, Proposition A.4]. But this is impossible, because π restricts to a finite morphism on each X^{H_j} . Finally, recall that $\dim V = \sum_j V_j = \sum_j X_j = \dim X$, by construction. Thus the map π is dominant, and hence surjective.

Combining items (1) to (5), we obtain the chain of inequalities

$$\dim_{\mathbb{Q}} A_*(\mathbb{P}_{\lambda}(X)) \geq \sum_{j=1}^r \dim_{\mathbb{Q}} A_*(\mathbb{P}_{\lambda}(X^{H_j})) \geq \sum_{j=1}^r \dim_{\mathbb{Q}} A_*(\mathbb{P}_{\lambda}(V_j)) = \sum_{j=1}^r \dim X^{H_j} \geq \dim X.$$

Proposition 2.15. *Let X be an affine T -variety with an attractive fixed point x . Let H_1, \dots, H_r denote the finite list of all codimension one subtori satisfying $X^{H_j} \neq X^T$.*

(a) *The following are equivalent.*

(i) $\dim X = \sum_{j=1}^r \dim X^{H_j}$.

(ii) *There is a T -module V and a T -equivariant finite surjective morphism $\pi : X \rightarrow V$ such that $\pi(x) = 0$ and $V^T = \{0\}$. In particular, for all $1 \leq j \leq r$, the restriction of π to X^{H_j} , denoted π_j , induces a T -equivariant finite surjective morphism $\pi_j : X^{H_j} \rightarrow V_j$, where $V_j := V^{H_j}$.*

(b) *Let $\lambda \in \sigma_x^0$. If $\dim_{\mathbb{Q}} A_*(\mathbb{P}_{\lambda}(X)) = \dim X$, then conditions (i) and (ii) of (a) hold. Moreover, there is a chain of equalities*

$$\dim_{\mathbb{Q}} A_*(\mathbb{P}_{\lambda}(X)) = \sum_{j=1}^r \dim_{\mathbb{Q}} A_*(\mathbb{P}_{\lambda}(X^{H_j})) = \sum_{j=1}^r \dim_{\mathbb{Q}} A_*(\mathbb{P}_{\lambda}(V_j)) = \sum_{j=1}^r \dim X^{H_j} = \dim X,$$

and the maps π and π_j from (a) induce isomorphisms

$$\begin{aligned} \pi_* : A_k(\mathbb{P}_{\lambda}(X)) &\xrightarrow{\sim} A_k(\mathbb{P}_{\lambda}(V)), \\ \pi_{j*} : A_k(\mathbb{P}_{\lambda}(X^{H_j})) &\xrightarrow{\sim} A_k(\mathbb{P}_{\lambda}(V_j)), \end{aligned}$$

for all j and k .

Put in perspective, this result is our motivation for the material in the next section.

3. ALGEBRAIC RATIONAL CELLS

This section is devoted to the study of our main technical tool: algebraic rational cells. We thank M. Brion for leading us to the following definition.

Definition 3.1. Let X be an affine T -variety with an attractive fixed point x , and let $n = \dim X$. We say that (X, x) , or simply X , is an *algebraic rational cell* if and only if, for some $\lambda \in \sigma_x^0$, we have

$$A_k(\mathbb{P}_{\lambda}(X)) = \begin{cases} \mathbb{Q} & \text{if } 0 \leq k \leq n-1, \\ 0 & \text{otherwise.} \end{cases}$$

We abbreviate this condition by writing $A_*(\mathbb{P}_\lambda(X)) \simeq A_*(\mathbb{P}^{n-1})$.

Algebraic rational cells are such T -varieties for which Proposition 2.15 (b) holds. In principle, Definition 3.1 depends on a particular choice of $\lambda \in \sigma_x^0$. But, as we shall see next, it is independent of λ : if $A_*(\mathbb{P}_\lambda(X)) \simeq A_*(\mathbb{P}^{n-1})$ holds for some $\lambda \in \sigma_x^0$, then it holds for all $\lambda \in \sigma_x^0$.

Lemma 3.2. *Let X be an affine T -variety with an attractive fixed point x , and let $n = \dim X$. Then (X, x) is an algebraic rational cell if and only if*

$$A_k(X) = \begin{cases} \mathbb{Q} & \text{if } k = n \\ 0 & \text{if } k \neq n. \end{cases}$$

In particular, if (X, x) is an algebraic rational cell, then it is irreducible.

Proof. Let \mathbb{G}_m act on X via λ . Recall that we have a short exact sequence

$$0 \rightarrow A_*^{\mathbb{G}_m}(x) \rightarrow A_*^{\mathbb{G}_m}(X) \rightarrow A_*^{\mathbb{G}_m}(X \setminus \{x\}) \rightarrow 0,$$

which stems from the localization theorem (Theorem 2.4). As in Subsection 2.4, there exists a \mathbb{G}_m -equivariant finite surjective map $\pi : X \rightarrow \mathbb{A}^n$ such that $\pi^{-1}(0) = x$, and \mathbb{G}_m -acts on \mathbb{A}^n with positive weights only. This map induces the commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A_*^{\mathbb{G}_m}(x) & \xrightarrow{i_*} & A_*^{\mathbb{G}_m}(X) & \xrightarrow{j_*} & A_*^{\mathbb{G}_m}(X \setminus \{x\}) \longrightarrow 0 \\ & & \downarrow \pi_* & & \downarrow \pi_* & & \downarrow \pi_* \\ 0 & \longrightarrow & A_*^{\mathbb{G}_m}(0) & \xrightarrow{i_*} & A_*^{\mathbb{G}_m}(\mathbb{A}^n) & \xrightarrow{j_*} & A_*^{\mathbb{G}_m}(\mathbb{A}^n \setminus \{0\}) \longrightarrow 0. \end{array}$$

The left vertical map is clearly an isomorphism. We claim that the other two vertical maps are surjective. Indeed, since $\pi : X \rightarrow \mathbb{A}^n$ is finite and surjective, the induced map of mixed spaces $\pi : X_{\mathbb{G}_m} \rightarrow \mathbb{A}_{\mathbb{G}_m}^n$ inherits both properties, by descent [EG, Propositions 2 and 3]. Hence, $\pi_* : A_*^{\mathbb{G}_m}(X) \rightarrow A_*^{\mathbb{G}_m}(\mathbb{A}^n)$ is surjective. For the right vertical map, observe that $A_*^{\mathbb{G}_m}(X \setminus \{x\}) \simeq A_*(\mathbb{P}(X))$ by [EG, Theorem 3]. So this map represents $\pi_* : A_*(\mathbb{P}(X)) \rightarrow A_*(\mathbb{P}^{n-1})$, whose surjectivity is already known (Lemma 2.13). We conclude from the previous analysis that the right vertical map is an isomorphism if and only if so is the middle one. But the latter happens if and only if $A_*(X) \simeq A_*(\mathbb{A}^n) \simeq \mathbb{Q}$ (one direction is guaranteed by Theorem 2.3; for the other one use Lemma 2.5). This yields the first assertion of the lemma.

Finally, the second assertion follows from Lemma 3.3 below. \square

Lemma 3.3 ([EH, Proposition 1.6]). *Let X be a variety. Let X_1, \dots, X_m be the irreducible components of X . Then the classes $[X_i] \in A_*(X)_{\mathbb{Z}}$ generate a free abelian subgroup of rank m in $A_*(X)_{\mathbb{Z}}$. Here $A_*(X)_{\mathbb{Z}}$ denotes the integral Chow group of X .*

Proof. If X is equidimensional, then the result is well-known, see e.g. [F, Section 1.5]. If X is not equidimensional, then we argue as follows. Let

$n = \dim X$. For $0 \leq j \leq n$, let Y_j be the union of those X_i 's whose dimension is exactly j . By definition, Y_j is equidimensional. If $Y_j \neq \emptyset$, then $A_j(Y_j) = \mathbb{Z}^{c_j}$, where c_j is the number of X_i 's whose dimension is j . Moreover, if $j \neq k$, then $\dim Y_j \cap Y_k < \min\{j, k\}$. Now, for $0 \leq j \leq n$, define $Z_j := \bigcup_{i \neq j} Y_i$. Notice that $\dim Y_j \cap Z_j < j$. By [F, Example 1.8.1] we have an exact sequence

$$A_j(Y_j \cap Z_j) \rightarrow A_j(Y_j) \oplus A_j(Z_j) \rightarrow A_j(X) \rightarrow 0.$$

But $A_j(Y_j \cap Z_j) = 0$ because $\dim Y_j \cap Z_j < j$. Thus $A_j(Y_j) \oplus A_j(Z_j) \simeq A_j(X)$. So $\bigoplus_{j=0}^n A_j(Y_j) \subseteq A_*(X)$. But $\bigoplus_{j=0}^n A_j(Y_j) = \mathbb{Z}^m$, since $m = \sum_j c_j$. The proof is now complete. \square

Lemma 3.2 hints to a more general structural property of algebraic rational cells, with respect to the T -action.

Proposition 3.4. *Let X be an affine T -variety with an attractive fixed point x , and let $\lambda \in \sigma_x^0$. Let $n = \dim X$. Then the following conditions are equivalent.*

- (i) $A_*(\mathbb{P}_\lambda(X)) \simeq A_*(\mathbb{P}^{n-1})$.
- (ii) $A_*(X) \simeq A_*(\mathbb{A}^n)$.
- (iii) $A_*^T(X) \simeq A_*^T(pt) = S$.
- (iv) $A_*^T(\mathbb{P}_\lambda(X)) \simeq A_*(\mathbb{P}^{n-1}) \otimes_{\mathbb{Q}} S$

Proof. The equivalence (i) \Leftrightarrow (ii) follows from Lemma 3.2.

The equivalence (ii) \Leftrightarrow (iii) follows from Lemma 2.5.

The implication (iv) \Rightarrow (i) is deduced from Theorem 2.3 and Lemma 2.13.

Finally, we dispose of the direction (i) \Rightarrow (iv). Recall that (i) yields the existence of a T -equivariant finite surjective morphism $\pi : X \rightarrow V$, such that the induced map $\pi_* : A_*(\mathbb{P}_\lambda(X)) \rightarrow A_*(\mathbb{P}_\lambda(V))$ is an isomorphism (Proposition 2.15 (b)). By the graded Nakayama lemma, the corresponding map $\widetilde{\pi}_* : A_*^T(\mathbb{P}_\lambda(X)) \rightarrow A_*^T(\mathbb{P}_\lambda(V))$ is surjective. We claim that $\widetilde{\pi}_*$ is also injective (hence an isomorphism). Indeed, choose a basis z_1, \dots, z_n of $A_*(\mathbb{P}_\lambda(V))$. Now identify that basis with a basis of $A_*(\mathbb{P}_\lambda(X))$, via π_* , and lift it to a generating system of the S -module $A_*^T(\mathbb{P}_\lambda(X))$. This generating system is a basis, since its image under $\widetilde{\pi}_*$ is a basis of $A_*^T(\mathbb{P}_\lambda(V))$. \square

Next, we exhibit some additional features of algebraic rational cells. The result is an algebraic counterpart of [Br2, Theorem 18].

Theorem 3.5. *Let X be an irreducible affine T -variety with an attractive fixed point x . Then the following are equivalent.*

- (i) *For each subtorus $H \subset T$ of codimension one, (X^H, x) is an algebraic rational cell, and $\dim X = \sum_H \dim X^H$ (sum over all subtori of codimension one).*

(ii) For each subtorus $H \subset T$ of codimension one, (X^H, x) is an algebraic rational cell, and

$$e_x[X] = d \prod_H e_x[X^H],$$

where d is a positive rational number. If moreover each X^H is smooth, then d is an integer.

Furthermore, if (X, x) is an algebraic rational cell, then conditions (i) and (ii) hold.

Proof. Recall that there is only a finite collection of codimension one subtori, say H_1, \dots, H_r , for which $X^{H_j} \neq X^T$. The required equivalence is obtained arguing exactly as in [Br2, Theorem 18]. Indeed, if (i) holds, then there exists a T -equivariant finite surjective map $\pi : X \rightarrow V$, where V is a T -module (by Proposition 2.15 (a)). So $e_x[X] = de_0[V]$, where $d = \deg \pi$. But then $de_0[V] = d \prod_{H_j} e_0[V_j]$, because V is a T -module (Theorem 2.9 (iii)). In turn, the last expression identifies to $\frac{d}{\prod_j d_j} \prod_j e_x[X^{H_j}]$, where $d_j = \deg \pi_j$ and π_j is as in Proposition 2.15 (a). Condition (ii) is thus attained. Conversely, if (ii) holds, then the X^{H_j} 's are irreducible (Lemma 3.2). As X is irreducible by assumption, the equality $e_x[X] = d \prod_{H_j} e_x[X^{H_j}]$ yields $\dim X = \sum_{H_j} \dim X^{H_j}$ by Theorem 2.9 (ii).

Finally, if (X, x) is an algebraic rational cell, then condition (i) is deduced at once from Proposition 2.15 (b). \square

In general, it is not true that properties (i) or (ii) of Theorem 3.5 characterize algebraic rational cells. Here is an example, cf. [Br3, Remark 1.4].

Example 3.6. Let X be the hypersurface of \mathbb{A}^5 with equation $x^2 + yz + xtw = 0$. Note that X is irreducible, with singular locus $x = y = z = tw = 0$, a union of two lines meeting at the origin. Now consider the $\mathbb{G}_m \times \mathbb{G}_m$ -action on \mathbb{A}^5 given by $(u, v) \cdot (x, y, z, t, w) := (u^2v^2x, u^3vy, uv^3z, u^2t, v^2w)$. Then the origin of \mathbb{A}^5 is an attractive fixed point, X is T -stable of dimension four and X contains four closed irreducible T -stable curves, namely, the coordinate lines except for the x -axis. So X satisfies condition (i) of Theorem 3.5. Nevertheless, $(X, 0)$ is not an algebraic rational cell. To see this, consider the \mathbb{G}_m -action on \mathbb{A}^5 given by $u \cdot (x, y, z, t, w) := (x, uy, u^{-1}z, t, w)$. Then X is \mathbb{G}_m -stable and $X^{\mathbb{G}_m}$ is defined by $y = z = x^2 + xtw = 0$. Thus $X^{\mathbb{G}_m}$ is reducible at the origin. In fact $A_*(X^{\mathbb{G}_m}) = \mathbb{Q} \oplus \mathbb{Q}$ (since $X^{\mathbb{G}_m}$ consists of the union of two copies of \mathbb{A}^2). Thus $\dim_{\mathbb{Q}} A_*(X^{\mathbb{G}_m}) = 2$, and so $\dim_{\mathbb{Q}} A_*(X) \geq 2$, by Lemma 2.5. Therefore, in view of Lemma 3.2, $(X, 0)$ is not an algebraic rational cell.

Example 3.7 (Smooth rational cells). Let X be an affine T -variety with an attractive fixed point x . If X is smooth at x , then $X \simeq T_x X$, T -equivariantly. Thus $(X, x) \simeq (T_x X, 0)$ is an algebraic rational cell. This agrees with the fact that $\mathbb{P}_{\lambda}(T_x X)$ is a weighted projective space.

Remark 3.8. Let $\mathbb{k} = \mathbb{C}$. In general, there seems to be no immediate relation between algebraic rational cells and (topological) rational cells [G1]. Nevertheless, in some important cases, e.g. spherical varieties, strong connections exist, cf. Sections 5 and 6.

We conclude this section by computing equivariant multiplicities of algebraic rational cells. Recall that a primitive character χ of T is called *singular* if $X^{\ker(\chi)} \neq X^T$.

Corollary 3.9. *Let X be an irreducible T -variety with attractive fixed point x . Let X_x be the unique open affine T -stable neighborhood of x . If (X_x, x) is an algebraic rational cell, then the following hold:*

(i) $e_x[X]$ is the inverse of a polynomial. In fact,

$$e_x[X] = \frac{d}{\chi_1 \cdots \chi_r},$$

where the χ_i 's are singular characters, $r = \dim X$, and d is a positive rational number.

(ii) Additionally, if the number of closed irreducible T -stable curves through x is finite, say $\ell(x)$, then $\dim X = \ell(x)$. Furthermore, we may take for χ_1, \dots, χ_r the characters associated to these curves.

Proof. Replacing X by X_x we may assume that X is affine. Then (i) follows at once from Theorem 3.5 and its proof. As for (ii) simply use Theorem 3.5 and Corollary 3.9, to adapt the argument of [Br3, Corollary 1.4.2] and [G1, Corollary 5.6]. \square

In general, if X is an affine T -variety with attractive fixed point x , and $\ell(x)$ as above is finite, then $\dim_x X \leq \ell(x)$ [Br3, Corollary 1.4.2].

4. \mathbb{Q} -FILTRABLE VARIETIES AND EQUIVARIANT CHOW GROUPS

We aim at an inductive description of the equivariant Chow groups of filtrable T -varieties in the case when the cells are all algebraic rational cells. Our findings provide purely algebraic analogues of the topological results of [G1].

Definition 4.1. Let X be a T -variety. We say that X is \mathbb{Q} -filtrable if the following hold:

- (1) the fixed point set X^T is finite, and
- (2) there exists a generic one-parameter subgroup $\lambda : \mathbb{G}_m \rightarrow T$ for which the associated BB -decomposition of X is filtrable (Definition 2.1) and consists of T -invariant algebraic rational cells.

In particular, $X = \bigsqcup_j X_+(x_j, \lambda)$. Also, observe that the fixed points $x_j \in X^T$ need not be attractive in X , but they are so in their corresponding algebraic rational cells $X_+(x_j, \lambda)$. The following technical result will be of importance in the sequel.

Lemma 4.2. *If (X, x) is an algebraic rational cell, then the equivariant multiplicity morphism $e_{X,x} : A_*^T(X) \rightarrow \mathcal{Q}$ is injective.*

Proof. By [Br1, Proposition 4.1] the map $i_* : A_*^T(x) \rightarrow A_*^T(X)$ is injective. Moreover, the image of i_* contains $\chi_1 \cdots \chi_n A_*^T(X)$, where χ_i are the T -weights of $T_x X$. Next, recall that e_x is defined as follows: given $\alpha \in A_*^T(X)$, we can form the product $\chi_1 \cdots \chi_n \alpha$. Thus, there exists $\beta \in S$ such that $i_*(\beta) = \chi_1 \cdots \chi_n \alpha$. Now let $e_x(\alpha) = \frac{\beta}{\chi_1 \cdots \chi_n}$. Since $A_*^T(X)$ is S -free (Proposition 3.4), it is clear from the construction that e_x is injective. \square

Let X be a \mathbb{Q} -filtrable T -variety. Then, by assumption, there is a closed algebraic rational cell $F = X_+(x_1, \lambda)$ (using the order of fixed points induced by the filtration, cf. Definition 2.1). Moreover $U = X \setminus F$ is also \mathbb{Q} -filtrable. We now proceed to describe $A_*^T(X)$ in terms of $A_*^T(F)$ and $A_*^T(U)$. Let $j_F : F \rightarrow X$ and $j_U : U \rightarrow X$ denote the inclusion maps.

Proposition 4.3. *Notation being as above, the maps $j_{F*} : A_*^T(F) \rightarrow A_*^T(X)$ and $j_U^* : A_*^T(X) \rightarrow A_*^T(U)$ fit into the exact sequence*

$$0 \rightarrow A_*^T(F) \rightarrow A_*^T(X) \rightarrow A_*^T(U) \rightarrow 0.$$

Proof. It is well-known that the sequence

$$A_*^T(F) \xrightarrow{j_{F*}} A_*^T(X) \xrightarrow{j_U^*} A_*^T(U) \longrightarrow 0$$

is exact. Thus it suffices to show that j_{F*} is injective. But this follows easily from the factorization $e_{x,F} = e_{x,X} \circ j_{F*}$. Indeed, since $e_{x,F}$ is injective (Lemma 4.2), so is j_{F*} . \square

Arguing by induction on the length of the filtration leads to the following.

Theorem 4.4. *Let X be a \mathbb{Q} -filtrable T -variety. Then the T -equivariant Chow group of X is a free S -module of rank $|X^T|$. In fact, it is freely generated by the classes of the closures of the cells $X_+(x_i, \lambda)$. Consequently, $A_*(X)$ is also freely generated by the classes of the cell closures $\overline{X_+(x_i, \lambda)}$.* \square

If X is a \mathbb{Q} -filtrable variety, then each filtered piece Σ_i is also \mathbb{Q} -filtrable, and so Theorem 4.4 applies at each step of the filtration. Our approach, based on equivariant multiplicities, is more flexible than the general approach which compares (equivariant) Chow groups with (equivariant) homology via the (equivariant) cycle map. This flexibility will be illustrated in the next sections.

5. APPLICATIONS TO EMBEDDING THEORY

We now furnish our theory with its first set of examples: \mathbb{Q} -filtrable embeddings of reductive groups. We show that the notion of algebraic rational cell is well adapted to the study of group embeddings.

Further notation. We denote by G a connected reductive linear algebraic group with Borel subgroup B and maximal torus $T \subset B$. We denote by W the Weyl group of (G, T) .

An affine algebraic monoid M is called *reductive* if it is irreducible, normal, and its unit group is a reductive algebraic group. See [R2] for details. Let M be a reductive monoid with zero and unit group G . We denote by $E(M)$ the idempotent set of M , that is, $E(M) = \{e \in M \mid e^2 = e\}$. Likewise, we denote by $E(\overline{T})$ the idempotent set of the associated affine torus embedding \overline{T} . One defines a partial order on $E(\overline{T})$ by declaring $f \leq e$ if and only if $fe = f$. Denote by $\Lambda \subset E(\overline{T})$, the cross section lattice of M . The Renner monoid $\mathcal{R} \subset M$ is a finite monoid whose group of units is W and contains $E(\overline{T})$ as idempotent set. Any $x \in \mathcal{R}$ can be written as $x = fu$, where $f \in E(\overline{T})$ and $u \in W$. Given $e \in E(\overline{T})$, we write $C_W(e)$ for the centralizer of e in W . Denote by \mathcal{R}_k the set of elements of rank k in \mathcal{R} , that is, $\mathcal{R}_k = \{x \in \mathcal{R} \mid \dim Tx = k\}$. Analogously, one has $\Lambda_k \subset \Lambda$ and $E_k \subset E(\overline{T})$. For $e \in E(M)$, set $M_e := \overline{\{g \in G \mid ge = eg = e\}}$. Then M_e is an irreducible, normal reductive monoid with e as its zero element [Br4].

5.1. Group embeddings. A normal irreducible variety X is called an *embedding* of G , or a *group embedding*, if X is a $G \times G$ -variety containing an open orbit isomorphic to G . Due to the Bruhat decomposition, group embeddings are spherical $G \times G$ -varieties. Substantial information about the topology of a group embedding is obtained by restricting one's attention to the induced action of $T \times T$. When $G = B = T$, we get back the notion of toric varieties. Group embeddings are classified as follows.

(I) *Affine case:* Let M be a reductive monoid with unit group G . Then $G \times G$ acts naturally on M via $(g, h) \cdot x = gxh^{-1}$. The orbit of the identity is $G \times G / \Delta(G) \simeq G$. Thus M is an affine embedding of G . Remarkably, by a result of Rittatore [Ri], reductive monoids are exactly the affine embeddings of reductive groups.

(II) *Projective case:* Let M be a reductive monoid with zero and unit group G . Then there exists a central one-parameter subgroup $\epsilon : \mathbb{G}_m^* \rightarrow T$, with image Z , such that $\lim_{t \rightarrow 0} \epsilon(t) = 0$. Moreover, the quotient space

$$\mathbb{P}_\epsilon(M) := (M \setminus \{0\})/Z$$

is a normal projective variety on which $G \times G$ acts via $(g, h) \cdot [x] = [gxh^{-1}]$. Hence, $\mathbb{P}_\epsilon(M)$ is a normal projective embedding of the quotient group G/Z . These varieties were introduced by Renner in his study of algebraic monoids ([R3], [R4]). Notably, normal projective embeddings of connected reductive groups are exactly the projectivizations of normal algebraic monoids [R1].

5.2. Algebraic monoids and algebraic rational cells.

Lemma 5.1. *Let $\varphi : L \rightarrow M$ be a finite surjective morphism of normal, reductive monoids. Then φ is the quotient map by the finite group $\ker(\varphi|_{G_L})$, where G_L is the unit group of L .*

Proof. Let $\mu = \ker(\varphi|_{G_L})$. Because μ is a finite and normal subgroup of the connected reductive group G_L , it is central. Hence $\mu \subset T_L$ (for the center of G_L is the intersection of all its maximal tori). It follows that the induced map $\tilde{\varphi} : L/\mu \rightarrow M$ is bijective and birational. But M is normal, so $\tilde{\varphi}$ is an isomorphism. \square

The following is crucial for our purposes.

Lemma 5.2 ([Gr, Lemma 1]). *Let G be a connected linear algebraic group. Let X be a G -variety. Then the action of G on $A_*(X)$ is trivial.* \square

Corollary 5.3. *Let $\varphi : L \rightarrow M$ be a finite dominant morphism of normal algebraic monoids. Then φ induces an isomorphism of (rational) Chow groups, namely, $A_*(L) \simeq A_*(M)$.*

Proof. By Lemma 5.1 and [F], Example 1.7.6, we have $(A_*(L))^\mu \simeq A_*(M)$. Now, since the action of μ on $A_*(L)$ comes induced from the action of G_L on $A_*(L)$ we have $(A_*(L))^{G_L} \subset (A_*(L))^\mu$. But, by Lemma 5.2, we have $(A_*(L))^{G_L} = A_*(L)$. We conclude that $(A_*(L))^\mu = A_*(L)$. \square

Let M and N be reductive monoids. Following Renner [R3], we write $M \sim_0 N$ if there is a reductive monoid L and finite dominant morphisms $L \rightarrow M$ and $L \rightarrow N$ of algebraic monoids. One checks that this gives rise to an equivalence relation. The following basic result, a consequence of Corollary 5.3, states that rational Chow groups are an invariant of the equivalence classes.

Corollary 5.4. *Let M and N be reductive monoids. If $M \sim_0 N$, then $A_*(M) \simeq A_*(N)$.* \square

Now let M be a reductive monoid with zero and unit group G . Recall that $T \times T$ acts on M via $(s, t) \cdot x = txs^{-1}$ and 0 is the unique attractive fixed point for this action (see e.g. [Br4, Lemma 1.1.1]). The number of closed irreducible $T \times T$ -invariant curves in M is finite (all of them passing through 0), and it equals $|\mathcal{R}_1|$. Indeed, each closed $T \times T$ -curve of M can be written as $\overline{T x T}$, where $x \in \mathcal{R}_1$, for they correspond to the $T \times T$ -fixed points of $\mathbb{P}_\epsilon(M)$, see [G2, Theorem 3.1]. It follows that $\dim M \leq |\mathcal{R}_1|$. Similarly, \overline{T} is an affine T -variety with 0 as its unique attractive fixed point and with finitely many T -stable curves. The number of these curves equals $|E_1|$ and so $\dim T \leq |E_1|$. Next we provide combinatorial criteria for showing when M is an algebraic rational cell (for the $T \times T$ -action). This adds to the list of equivalences from [R3] and [R4].

Theorem 5.5. *Let M be a reductive monoid with zero and unit group G . Then the following are equivalent.*

- (a) $M \sim_0 \prod_i M_{n_i}(\mathbb{k})$.
- (b) If T is a maximal torus of G , then $\dim T = |E_1|$.
- (c) $\overline{T} \sim_0 \mathbb{A}^n$.

(d) $(\overline{T}, 0)$ is an algebraic rational cell.

(e) $(M, 0)$ is an algebraic rational cell.

(f) $\dim M = |\mathcal{R}_1|$.

Proof. The equivalence of (a), (b) and (c) is proven in [R3, Theorem 2.1] (no use of rational smoothness is made there). The implication (c) \Rightarrow (d) follows from Corollary 5.4 and Proposition 3.4. On the other hand, condition (d) implies (b) because of Corollary 3.9 and the fact that $|E_1(\overline{T})|$ is the number of T -invariant curves of \overline{T} passing through 0. Hence conditions (a), (b), (c) and (d) are all equivalent.

Certainly (a) implies (e), by Corollary 5.4 and Proposition 3.4. In turn, (e) yields (f) due to Corollary 3.9 and the fact that the number of closed irreducible $T \times T$ -curves in M equals $|\mathcal{R}_1|$. So to conclude the proof it suffices to show that (f) implies (b). For this we argue as follows.

Assume (f) and recall that each closed $T \times T$ -curve in M can be written as $\overline{T}x\overline{T}$, with $x \in \mathcal{R}_1$. Moreover, if we write $x = ew$, with $e \in E_1$ and $w \in W$, then $T \times T$ acts on $\overline{T}x\overline{T}$ through the character $(\lambda_e, \lambda_e(\text{int}(w)))$, where $\lambda_e : T \rightarrow eT \simeq \mathbb{k}^*$ is the character sending t to et .

Now, for each $x = ew \in \mathcal{R}_1$, we can find a finite $T \times T$ -equivariant surjective map $\pi_x : \overline{T}x\overline{T} \rightarrow \mathbb{k}_x$. Here, $T \times T$ -acts on $\mathbb{k}_x \simeq \mathbb{k}$ via $(\lambda_e, \lambda_e(\text{int}(w)))$. Since $\overline{T}x\overline{T}$ is $T \times T$ -invariant and closed in M , we can extend π_x to a $T \times T$ -equivariant morphism $\pi_x : M \rightarrow \mathbb{k}_x$. Synchronizing efforts via the product map, we obtain a $T \times T$ -equivariant map

$$\pi : M \rightarrow V = \prod_{x \in \mathcal{R}_1} \mathbb{k}_x, \quad m \mapsto (\pi_x(m))_{x \in \mathcal{R}_1}.$$

By construction, π is finite (cf. proof of item (5) in page 9), and given that $\dim M = |\mathcal{R}_1|$, it is also surjective.

Let $\Delta T \subset T \times T$ be the diagonal torus. We know that the fixed point set $M^{\Delta T}$ equals \overline{T} (see the proof of [R2, Theorem 5.5]). Let us look at the restriction map

$$\pi : \overline{T} \rightarrow V^{\Delta T}.$$

We claim that $\dim V^{\Delta T} = |E_1(\overline{T})|$. Indeed, it is clear that for $e \in E_1(\overline{T})$, the $T \times T$ -invariant curve $\mathbb{k}_e \subset V$ is fixed by ΔT , since $tet^{-1} = e$ (recall that \overline{T} is commutative). Hence

$$\prod_{e \in E_1} \mathbb{k}_e \subset V^{\Delta T}.$$

Thus,

$$|E_1(\overline{T})| = \dim \prod_{e \in E_1(\overline{T})} \mathbb{k}_e \leq \dim V^{\Delta(T)} = \dim T.$$

But, in general, $\dim T \leq |E_1(\overline{T})|$. Hence $\dim T = |E_1(\overline{T})|$. As this is condition (b), the proof is now complete. \square

Remark 5.6. Let M be a reductive monoid with zero. Theorem 5.5 gives a converse to Corollary 3.9: $(M, 0)$ is an algebraic rational cell if and only if $\dim M = |\mathcal{R}_1|$.

Theorem 5.5 and [R4, Theorem 2.4] immediately give the following. Notice that the cycle map is not needed in the proof.

Corollary 5.7. *Let $\mathbb{k} = \mathbb{C}$. Let M be a reductive monoid with zero, and let \overline{T} be the associated affine toric variety. Then M (resp. \overline{T}) is rationally smooth if and only if M (resp. \overline{T}) is an algebraic rational cell. \square*

5.3. \mathbb{Q} -filtrable projective group embeddings. We start by recalling [R3, Definition 2.2].

Definition 5.8. A reductive monoid M with zero element is called *quasi-smooth* if, for any minimal non-zero idempotent $e \in E(M)$, M_e satisfies the conditions of Theorem 5.5.

In other words, M is quasismooth if and only if M_e is an algebraic rational cell, for any minimal non-zero idempotent $e \in E(M)$.

Now consider the projective group embedding $\mathbb{P}_\epsilon(M) = (M \setminus \{0\})/Z$ (as in Section 5.1). When $\mathbb{k} = \mathbb{C}$, it is worth noting that M is quasismooth if and only if $\mathbb{P}_\epsilon(M)$ is rationally smooth [R4, Theorem 2.5].

Next is the second main result of this section. It is an extension of [G1, Theorem 7.4] to equivariant Chow groups.

Theorem 5.9. *Let M be a reductive monoid with zero. If M is quasismooth, then the projective group embedding $\mathbb{P}_\epsilon(M)$ is \mathbb{Q} -filtrable (as in Section 4).*

Proof. The strategy is to adapt the proof of [G1, Theorem 7.4] in light of Proposition 3.4 and Theorem 5.5. Recall that, by [R3, Theorem 3.4], $\mathbb{P}_\epsilon(M)$ comes equipped with a BB-decomposition

$$\mathbb{P}_\epsilon(M) = \bigsqcup_{r \in \mathcal{R}_1} C_r,$$

where \mathcal{R}_1 identifies to $\mathbb{P}_\epsilon(M)^{T \times T}$. (In fact these cells are $B \times B$ -invariant, where B is a Borel subgroup of G .) Given that $\mathbb{P}_\epsilon(M)$ is normal, projective, and \mathcal{R}_1 is finite, this BB-decomposition is filtrable (Theorem 2.2). So we just need to show that these cells are algebraic rational cells. Furthermore, since the C_r are affine $T \times T$ -varieties with an attractive fixed point $[r]$, Proposition 3.4 reduces the proof to showing that $A_*(C_r) \simeq \mathbb{Q}$.

Bearing this in mind, we delve a bit further into the structure of these cells. By [R3, Lemma 4.6 and Theorem 4.7], each C_r is isomorphic to

$$U_1 \times C_r^* \times U_2,$$

where the U_i are affine spaces. Moreover, writing $r \in \mathcal{R}_1$ as $r = ew$, with $e \in E_1(\overline{T})$ and $w \in W$, yields $C_r^* = C_e^*w$. Hence, by the Kunneth formula (which holds, because the U_i 's are affine spaces), we are further reduced to showing that $A_*(C_e^*) \simeq \mathbb{Q}$, for $e \in E_1(\overline{T})$.

Now we call the reader's attention to [R3, Theorem 5.1]. It states that if M is quasismooth, then

$$C_e^* = f_e M(e)/Z,$$

for some unique $f_e \in E(\overline{T})$, where $M(e) = M_e Z$, and M_e is reductive monoid with e as its zero. By hypothesis, we know that M_e is an algebraic rational cell, that is, $A_*(M_e) \simeq \mathbb{Q}$. Since $M(e)/Z$ is a reductive monoid with $[e]$ as its zero, and $M_e \sim_0 M(e)/Z$, Corollary 5.4 yields $A_*(M(e)/Z) \simeq \mathbb{Q}$. Now, by [Br4, Lemma 1.1.1], one can find a one-parameter subgroup $\lambda : \mathbb{G}_m \rightarrow T$, with image S , such that $\lambda(0) = f$ and

$$f_e M(e)/Z = (M(e)/Z)^S.$$

That is, $f_e M(e)/Z$ is the S -fixed point set of $M(e)/Z$. But now we invoke Lemma 2.5 to get

$$\dim_{\mathbb{Q}} A_*((M(e)/Z)^S) \leq \dim_{\mathbb{Q}} A_*(M(e)/Z) = 1.$$

Hence, *a fortiori*

$$\dim_{\mathbb{Q}} A_*((M(e)/Z)^S) = \dim_{\mathbb{Q}} A_*(f_e M(e)/Z) = \dim_{\mathbb{Q}} A_*(C_e^*) = 1.$$

This shows that $A_*(C_e^*) = \mathbb{Q}$, concluding the argument. \square

It is well-known that for projective simplicial toric varieties (equivalently, rationally smooth projective toric varieties) the equivariant cycle map is an isomorphism over \mathbb{Q} . Below we extend this result to all rationally smooth projective group embeddings.

Corollary 5.10. *Let $\mathbb{k} = \mathbb{C}$. If M is a quasismooth monoid with zero, then the equivariant cycle map*

$$cl_{\mathbb{P}_\epsilon(M)}^T : A_*^T(\mathbb{P}_\epsilon(M)) \rightarrow H_*^T(\mathbb{P}_\epsilon(M))$$

is an isomorphism of free S -modules. Moreover, the usual cycle map

$$cl_{\mathbb{P}_\epsilon(M)} : A_*(\mathbb{P}_\epsilon(M)) \rightarrow H_*(\mathbb{P}_\epsilon(M))$$

is an isomorphism of \mathbb{Q} -vector spaces.

Proof. By [G1, Theorem 7.4] $\mathbb{P}_\epsilon(M)$ has no homology in odd degrees, and each cell is rationally smooth, so $H_{*,c}(C_r) \simeq \mathbb{Q}$ and $H_{*,c}^T(C_r) \simeq S$. Now Theorem 5.9 implies that the cycle maps $cl_{C_r} : A_*(C_r) \rightarrow H_{*,c}(C_r)$ and $cl_{C_r}^T : A_*^T(C_r) \rightarrow H_{*,c}^T(C_r)$ are isomorphisms. Arguing by induction on the length of the filtration concludes the proof. \square

In [R3] and [R4], Renner has computed the H -polynomial of a quasismooth monoid. This polynomial counts the number of algebraic rational cells (of each dimension) that appear in the BB-decomposition of $\mathbb{P}_\epsilon(M)$. In particular, when $\mathbb{P}_\epsilon(M)$ is simple (i.e. it contains a unique $G \times G$ -orbit), [R3] shows that the number of cells of dimension k equals the number of cells of dimension $n - k$, where $n = \dim \mathbb{P}_\epsilon(M)$. By Theorem 4.4, this yields

$$\dim_{\mathbb{Q}} A_k(\mathbb{P}_\epsilon(M)) = \dim_{\mathbb{Q}} A_{n-k}(\mathbb{P}_\epsilon(M)).$$

Over the complex numbers, this is equivalent to the fact that $\mathbb{P}_\epsilon(M)$ satisfies Poincaré duality for rational singular cohomology.

6. CONNECTIONS WITH TOPOLOGY: SPHERICAL VARIETIES

In this section we work over the complex numbers. The aim is to relate the results of this paper with those of [G1] in the case of spherical varieties.

The following result is a particular case of [To, Theorem 3].

Theorem 6.1. *Let Γ be connected solvable linear algebraic group. For any Γ -variety Y with a finite number of orbits, the natural map*

$$A_i(Y) \otimes \mathbb{Q} \longrightarrow W_{-2i}H_{2i}^{BM}(Y, \mathbb{Q}),$$

from the Chow groups into the smallest subspace of Borel-Moore homology with respect to the weight filtration is an isomorphism. \square

Let G be a connected reductive linear algebraic group with Borel subgroup B and maximal torus $T \subset B$. Recall that given a one-parameter subgroup $\lambda : \mathbb{C}^* \rightarrow T$, we can define

$$G(\lambda) = \{g \in G \mid \lambda(t)g\lambda(t)^{-1} \text{ has a limit as } t \rightarrow 0\}.$$

It is well-known that $G(\lambda)$ is a parabolic subgroup of G with unipotent radical $R_u G(\lambda) = \{g \in G \mid \lim_{t \rightarrow 0} \lambda(t)g\lambda(t)^{-1} = 1\}$. Moreover, the centralizer $C_G(\lambda)$ of the image of λ is connected, and the product morphism $R_u G(\lambda) \times C_G(\lambda) \rightarrow G(\lambda)$ is an isomorphism of varieties. Also, the parabolic subgroups $G(\lambda)$ and $G(-\lambda)$ are opposite. Finally, $G(\lambda) = B$ if and only if λ lies in the interior of the Weyl chamber associated with B . See e.g. [Sp, Theorem 13.4.2].

Next we show that algebraic rational cells are naturally found on rationally smooth spherical varieties.

Theorem 6.2. *Let X be a G -spherical variety with an attractive T -fixed point x . Let X_x be the unique open affine T -stable neighborhood of x . If X is rationally smooth at x , then (X_x, x) is an algebraic rational cell.*

Proof. Because x is attractive, we may choose λ such that $X_x = X_+(x, \lambda)$ and $G(\lambda) = B$. Since X is rationally smooth at x , so is the open subset X_x . Moreover, X_x is rationally smooth everywhere, and so X_x is a rational cell [G1, Definition 3.4]. By Theorem 6.1 we have

$$A_i(X_x) \simeq W_{-2i}H_{2i}^{BM}(X_x, \mathbb{Q}) \simeq H_c^{2i}(X_x, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } i = \dim_{\mathbb{C}} X_x \\ 0 & \text{otherwise,} \end{cases}$$

where the last two identifications follow from the fact that X_x is a cone over a rational cohomology sphere [G1, Corollary 3.11]. \square

Let X be a G -spherical variety. Recall that X^T is finite. For convenience, we use the following nomenclature. We say that

- (a) X has an *algebraic* \mathbb{Q} -filtration, if it satisfies Definition 4.1 for some generic one-parameter subgroup λ of T .
- (b) X has a *topological* \mathbb{Q} -filtration, if there exists a generic one-parameter subgroup $\lambda : \mathbb{C}^* \rightarrow T$ for which the associated BB-decomposition of X is filtrable, and consists of *rational cells* [G1].

Theorem 6.3. *Let X be a G -spherical variety. If X has a topological \mathbb{Q} -filtration, then this filtration is also an algebraic \mathbb{Q} -filtration.*

Proof. Let $X^T = \{x_1, \dots, x_m\}$. By assumption, there exists a generic one-parameter subgroup such that $X^\lambda = X^T$, and the cells $X_j := X_+(x_j, \lambda)$ are rational cells. Consider the parabolic subgroup $G(\lambda)$. We claim that the cells X_j are invariant under $G(\lambda)$. Indeed, $G(\lambda) = R_u(\lambda) \times C_G(\lambda)$, and $C_G(\lambda)$, being connected, fixes each $x_j \in X^\lambda$. Now let $x \in X_j$ and write $g \in G(\lambda)$ as $g = uh$, with $u \in R_u(\lambda)$ and $h \in C_G(\lambda)$. Then

$$\lambda(t)g \cdot x = \lambda(t)uh\lambda(t)^{-1}\lambda(t) \cdot x = \lambda(t)u\lambda(t)^{-1}h\lambda(t) \cdot x.$$

Taking limits at 0 gives the claim. Because X is spherical, it contains only finitely many orbits of any Borel subgroup of G . Therefore, a Borel subgroup of $G(\lambda)$ has finitely many orbits in X_j . Applying Theorem 6.1 to each X_j yields

$$A_i(X_j) \simeq W_{-2i}H_{2i}^{BM}(X_j, \mathbb{Q}) \simeq H_c^{2i}(X_j, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } i = \dim_{\mathbb{C}} X_j \\ 0 & \text{otherwise,} \end{cases}$$

noting that X_j is a cone over a rational cohomology sphere [G1, Corollary 3.11]. Therefore, by Lemma 3.2, the cells X_j are algebraic rational cells. This concludes the proof. \square

Remark 6.4. Let X be a G -spherical variety, and let λ be a generic one-parameter subgroup. Then the argument above shows that the cells $X_+(x_j, \lambda)$ are T' -linear varieties, where $T' \subset G(\lambda)$ is a maximal torus of G .

Arguing by induction on the length of the filtration, using the fact that a T -variety with a topological \mathbb{Q} -filtration has no (co)homology in odd degrees, gives immediately the following.

Corollary 6.5. *Let X be a spherical G -variety with a topological \mathbb{Q} -filtration, say $\emptyset = \Sigma_0 \subset \Sigma_1 \subset \dots \subset \Sigma_m = X$. Then, for every j , both cycle maps, $cl_{\Sigma_j} : A_*(\Sigma_j) \rightarrow H_*(\Sigma_j)$ and $cl_{\Sigma_j}^T : A_*^T(\Sigma_j) \rightarrow H_*^T(\Sigma_j)$ are isomorphisms. \square*

We should remark that Theorem 6.3 provides another proof of Theorem 5.9. However, in the case of group embeddings, the approach taken in Section 5 is more intrinsic, for it uses the rich structure of the Chow groups and the fine combinatorial structure of algebraic monoids. Notice that the results of Section 5 are independent of Theorem 6.1. This shows how the notion of algebraic rational cells is well adapted to embedding theory, and opens the way for further work in this direction. For instance, the results of this paper, together with those of [G3], yield some characterizations of

Poincaré duality for the equivariant operational Chow *rings* of projective group embeddings, and related spherical varieties [G5]. This will appear elsewhere.

Finally, observe that, when looking for concrete examples, topological \mathbb{Q} -filtrations are slightly more approachable, for they are built using the classical topology of a complex variety, and could be obtained e.g. via Hamiltonian actions. Our Theorem 6.3 guarantees that the topological knowledge thus acquired gets transformed into algebraic information about the Chow groups. This provides examples of singular spherical varieties for which the cycle map is an isomorphism (e.g. rationally smooth group embeddings). It is worth noting, however, that the study of algebraically \mathbb{Q} -filtrable varieties can be carried out intrinsically, via equivariant intersection theory.

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