MAASS SPEZIALSCHAR OF LEVEL N

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ABSTRACT. In this paper the image of the Saito-Kurokawa lift of level N with Dirichlet character is studied. We give a new characterization of this so called Maass Spezialschar of level N by symmetries involving Hecke operators related to $\Gamma_0(N)$. We finally obtain for all prime numbers p local Maass relations. This generalizes known results for level $N = 1$.

1. INTRODUCTION

In 2012 [Ib12], T. Ibukiyama gave a systematic treatment of Saito-Kurokawa lifts of level N with possible Dirichlet character. First results in the classical setting had been obtained by B. Ramakrishnan, M. Manickham, and T. Vasudewa [MRV93]. In this paper, we study the image of the lifting, the Maass Spezialschar of level N. We obtain a new characterization by symmetries, generalizing previous work on liftings for the full Siegel modular group of degree two [He10]. We refer to the original literature [Ma79I, Ma79II, Ma79III, Ku78] and [Za80] for the Saito-Kurokawa conjecture and the Maass Spezialschar. An excellent introduction is given in [EZ85]. See also Oda's general viewpoint of theta lifts [Od77].

Let $F \in M_k^2(\Gamma_0^2(N), \chi)$ be a Siegel modular form of Hecke type of integral weight k, degree 2 and level N with Dirichlet character χ. Here $\chi(-1) = (-1)^k$.

Let $\Delta_N(l)$ be the set of all integral matrices $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$ with determinant l, with $N|c$ and $(a, N) = 1$. We put $\chi(g) := \overline{\chi(a)}$ and $\Gamma_0(N) = \Delta_N(1)$. Let $|_k$ be the Petersson slash operator and $\widetilde{g}^{\uparrow\downarrow}$ be two dual embeddings of $\Delta_N(l)$ into the symplectic group $Sp_2(\mathbb{R})$. Then we have the following new characterisation of the Maass Spezialschar. The Siegel modular form $F \in M_k^2(\Gamma_0^2(N), \chi)$ is a lift if and only if for all $l \in \mathbb{N}$:

$$(1.1) \qquad \sum_{g \in \Gamma_0(N) \backslash \Delta_N(l)} \chi(g)^{-1} \left(F|_k \widetilde{g}^{\uparrow} \right) = \sum_{g \in \Gamma_0(N) \backslash \Delta_N(l)} \chi(g)^{-1} \left(F|_k \widetilde{g}^{\downarrow} \right). \qquad (*_l).$$

The level one case was previously proven [He10] by working out the relation of the Taylor expansion and properties of certain differential operators. In this paper we give a new and more simple proof by studying the Fourier-Jacobi expansion. This approach, involving well-known properties of the Hecke algebra $\mathcal{H}(\Gamma_0(N), \Delta_N)$ ([Mi06]) is more transparent and natural. Here Δ_N is the union of all $\Delta_N(l)$. The Hecke algebra is commutative, zero-divisor free and decomposes in local components. Hence

it is sufficient to check the symmetries only locally, which leads finally to the result that F is in the Maass Spezialschar iff $(*_p)$ is satisfied for all primes p. Of course the symmetries degenerate if $p|N$. For further generalisation, note that the following identity in the Hecke algebra $\mathcal{H}(\Gamma_0(N), \Delta_N)$ is crucial.

$$(1.2) \qquad T(m) \circ T(n) = \sum_{\substack{d|(m,n) \\ (d,N)=1}} d\, T(d,d)\, T\left(\frac{mn}{d^2}\right)$$

The element $T(l)$ degenerates if $(l, N) > 1$ (see Miyake [Mi06], Theorem 4.5.13 (i)).

The symmetries $(*_l)$ encode a new type of Maass relations for Saito-Kurokawa lifts of Hecke type. Let $\mathcal{X}$ denote the set of half-integral positive semi definite matrices $\left(\begin{smallmatrix} n & r/2 \\ r/2 & m \end{smallmatrix}\right)$. Let $\mathcal{X}^*$ be the subset, where the zero matrix is removed. We put $A(T) = 0$ if $T \notin \mathcal{X}$. Let $F \in M_k^2(\Gamma_0^2(N), \chi)$ with Fourier coefficients $A(T) = A(n, r, m)$. Then F is in the Maass space iff for all $T \in \mathcal{X}^*$ and $l \in \mathbb{N}$:

$$(1.3) \qquad \sum_{d|(n,r,l)} d^{k-1}\chi(d)\, A\left(\frac{nl}{d^2}, \frac{r}{d}, m\right) = \sum_{d|(l,r,m)} d^{k-1}\chi(d)\, A\left(n, \frac{r}{d}, \frac{ml}{d^2}\right).$$

Here (n, r, l) denotes the greatest common divisor. As a consequence we obtain the useful application that F is in the Maass Spezialschar iff for all $T \in \mathcal{X}^*$ and for all prime numbers p the following Maass p-relations are satisfied:

$$(1.4) \qquad A(pn, r, m) + p^{k-1}\chi(p)A\left(\frac{n}{p}, \frac{r}{p}, m\right)$$
$$= A(n, r, pm) + p^{k-1}\chi(p)A\left(n, \frac{r}{p}, \frac{m}{p}\right).$$

This gives a significant generalization to the known Maass p-relations for $N = 1$ (see the survey [FPRS13] for further background information). Note $\chi(p) = 0$ iff $p|N$. For $p|N$ we have $A(pn, r, m) = A(n, r, pm)$. In the literature (see [EZ85], [Ib12]) the equivalent Maass relations are stated as

$$(1.5) \qquad A(n, r, m) = \sum_{d|(n,r,m)} d^{k-1}\chi(d)\, A\left(\frac{nm}{d^2}, \frac{r}{d}, 1\right)$$

for all $T \in \mathcal{X}^*$.

Recently [HM15], together with Murase, we had been able to use a multiplicative version of the symmetry principle $(*_l)$ to give a characterization of holomorphic Borcherds lifts and a new proof of Bruiniers converse theorem for the discriminant kernel group. Borcherds proved that his lifts have certain special divisors and Bruinier proved that if a form has these special divisors, then the form is a lift. We refer to [Br16] for recent developments. It would be interesting to transfer some of the results of this paper to the theory of Borcherds lifts for congruence subgroups.

2. Modular Forms

For basic facts about elliptic modular forms and Hecke theory we refer to [Sh71, Mi06]. For Siegel modular forms especially of degree 2 we recommend [Fr83], [An87] and [EZ85] (also standard reference for Jacobi forms).

2.1. Preliminaries and Basic Notations.

Let R be a subring of the real numbers $\mathbb{R}$ and let N, k, n, r, m usually denote integers. Let χ be a Dirichlet character modulo N. We denote $e(Z) := \exp(\operatorname{trace}(Z))$ for every suitable matrix Z. The symplectic group $GSp^+(n, \mathbb{R})$ of positive similitudes of degree n acts on the Siegel upper half space $\mathbb{H}_n$. Further let F be a holomorphic function on $\mathbb{H}_n$ and let $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in GSp^+(n, R)$ and $Z \in \mathbb{H}_n$. Then

$$\gamma(Z) := (AZ + B)(CZ + D)^{-1}$$

$$F|_k \gamma \, (Z) := \det(CZ + D)^{-k} \, F(\gamma(Z))$$

$$\widetilde{\gamma} := \det(\gamma)^{\frac{1}{2n}}$$

$$g^\uparrow := \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$g^\downarrow := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & d \end{pmatrix}, \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$$

$$\Delta_N := \left\{ \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Q}) \cap \mathbb{Z}^{2,2} \mid (a, N) = 1, \ N|c, \ \det(\alpha) > 0 \right\}$$

$$Sp(n, R) := \{\gamma \in GSp^+(n, R) \mid \det(\gamma) = 1\}$$

$$\Gamma_0^{(n)}(N) := \{\gamma \in Sp(n, \mathbb{Z}) \mid C \equiv 0 \pmod{N}\}.$$

Let $\gamma \in \Gamma_0^{(n)}(N)$, we extend χ by $\chi(\gamma) := \chi(\det(D))$. We identify $GL_2(R)^+$ with $GSp^+(1, R)$ and $SL_2(R)$ with $Sp(1, R)$, and drop the index $n = 1$ for simplification. In the case $n = 2$ we also identify

$$Z = \begin{pmatrix} \tau_1 & z \\ z & \tau_2 \end{pmatrix} \text{ with } (\tau_1, z, \tau_2).$$

We further put

$$\mathcal{X} := \left\{ T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix} \mid n, r, m \in \mathbb{Z}, \ T \geq 0 \right\}.$$

Then $\mathcal{X}^* := \mathcal{X} - \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\mathcal{X}^+ := \{T \in \mathcal{X} \mid T > 0\}$. We identify T with (n, r, m). Note that for all T with $\det(T) = 0$ there exists a $U \in SL_2(\mathbb{Z})$ such that $T[U] :=$

$U^t T U = (l, 0, 0)$ with $l \in \mathbb{N}_0$. Let us further denote by $d|(n, m)$ or $d|(n, r, m)$ that d divides the gcd of the involved numbers. The condition $d|(0, 0, 0)$ is empty.

2.2. Modular forms of level N.

Definition. *Let k, N be natural numbers. Let χ a Dirichlet character modulo N. Let Γ be a congruence subgroup of $\Gamma_0^{(n)}(N)$. A holomorphic function F on $\mathbb{H}_n$ is denoted Siegel modular form of weight k, degree n and Dirichlet character χ with respect to Γ if for all $g \in \Gamma$ the functional equation*

$$(2.1) \qquad\qquad F|_k g = \chi(g)\ F$$

is satisfied. In the case $n = 1$ we additionally have to propose that F is regular at the cusps. The space of these forms is denoted by $M_k^n(\Gamma, \chi)$.

We refer briefly to the behavior of Saito-Kurokawa lifts at the cusps. The main focus of this paper is the characterization of lifts independent of their Fourier expansion, Although we consider the expansion at infinity to some extent.

Definition. *We denote by $S_k^n(\Gamma, \chi)$ the subspace of cusp forms. These are $F \in M_k^n(\Gamma, \chi)$ with $F|_k \gamma$ vanishing at all boundaries.*

See [Fr83, Mi06] and also [Ib12] for a more explicit version of the definition, guided by the Satake compactificaton. In a nutshell, let F be holomorphic on $\mathbb{H}_n$ satisfying the functional equation for all elements of Γ. Let $V(Y_0) := \{Y \in \mathbb{R}^{n,n} \,|\, Y \geq Y_0 > 0\}$ for Y_0 positive definite and Γ_χ kernel of χ on Γ. Then $F \in M_k^n(\Gamma, \chi)$ iff $F|\gamma$ is bounded on $V(Y_0)$ for all Y_0 and $\gamma \in \Gamma_\chi \backslash Sp(n, \mathbb{Z})$. This property is always satisfied for $n > 1$ (Koecher principle) and has only be checked for $n = 1$.

Further $F \in S_k^n(\Gamma, \chi)$ iff $\Phi(F|_k \gamma) = 0$ for all $\gamma \in \Gamma_\chi \backslash Sp(n, \mathbb{Z})$. Here Φ is the Siegel lowering operator. We refer to Freitag ([Fr83], chapter II, Satake compactification, see also section 3.1 and 3.2 [Ib12]).

Remark. Let $n = 2$ then it is sufficient to check to cuspidality for the representatives of

$$\gamma \in \Gamma_\chi \backslash Sp(2, \mathbb{Z}) / C_{2,1}(\mathbb{Z}).$$

Here $C_{2,1}(\mathbb{Z})$ is the subgroup of $Sp(2, \mathbb{Z})$ with last row given by $(0\,0\,0\,1)$.

2.3. Fourier and Fourier-Jacobi expansion.
Let Γ be a congruence subgroup of $Sp(2, \mathbb{Z})$ containing

$$\left\{ \begin{pmatrix} 1_2 & S \\ 0 & 1_2 \end{pmatrix} \,|\, S = S^t \in \mathbb{Z}^{2,2} \right\}.$$

Then $F \in M_k^2(\Gamma, \chi)$ has the Fourier expansion

$$(2.2) \qquad\qquad F(Z) \ = \ \sum_{T \in \mathcal{X}} A(T)\ e(TZ)$$

$$(2.3) \qquad\qquad\qquad = \ \sum_{(n,r,m) \in \mathcal{X}} A(n, r, m)\ e(n\tau_1 + rz + m\tau_2)$$

In the following we will also put $q_1 = e(\tau_1), \zeta = e(z)$ and $q_2 = e(\tau_2)$. Note that F is a cusp form then $A(T) = 0$ for all $T \in \mathcal{X}^+$. Note that the converse is not true. The Fourier-Jacobi expansion of F is given by

$$(2.4) \qquad F(\tau_1, z, \tau_2) = \sum_{m=0}^{\infty} F_m(\tau_1, z) \, q_2^m.$$

Then F_m is called the m-th Fourier Jacobi coefficient of F. It is a Jacobi form of weight k and index m. Note that F_m is a Jacobi cusp form, if F is a cusp form.

2.4. **Jacobi Group.** We consider the Jacobi group $G^J(R)$ as the semi-direct product of $GL_2^+(R)$ and the (additive written) Heisenberg group

$$H(R) = \{h = (\mu, \lambda; \kappa) \mid \mu, \lambda, \kappa \in R\}$$

(see [Ib12], Section 2). We consider $h^0 = (\lambda, \mu)$ as a row vector. Then

$$G^J(R) := \left\{ (g, h) \mid g \in GL_2^+(R), \, h \in H(R) \right\}.$$

The explicit group operation is given by:

$$(g_1, h_1)(g_2, h_2) = (g_1 g_2, \det(g_2)^{-1} \, (h_1^0 \, g_2, \kappa_1) + h_2).$$

We further define the following subgroups and monoids of $G^J(\mathbb{Z})$.

$$\begin{aligned}
\Gamma_0(N)^J &:= \left\{ (g, h) \in G^J(\mathbb{Z}) \mid g \in \Gamma_0(N) \right\} \\
\Delta_N^J &:= \left\{ (g, h) \in G^J(\mathbb{R}) \mid g \in \Delta_N \text{ and } h \in H(\mathbb{Z}) \right\}.
\end{aligned}$$

Let $\mathbb{H}^J := \mathbb{H} \times \mathbb{C}$. Let $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right) \in GSp^+(n, R)$. Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in GL_2^+(R)$ with $\det(g) = l$ and $h = (\mu, \lambda; \kappa) \in H(R)$. Let f be a complex valued function on $\mathbb{H}^J$ and F on $\mathbb{H}_n$. Let $k, m \in \mathbb{N}_0$.

$$\begin{aligned}
\widehat{f}(\tau_1, z, \tau_2) \quad &:= f(\tau_1, z) \, e(m\tau_2), \quad \text{for } (\tau_1, z, \tau_2) \in \mathbb{H}_2 \\[4pt]
\widehat{g} \quad &:= \begin{pmatrix} a & 0 & b & 0 \\ 0 & l & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\[4pt]
\widehat{h} \quad &:= \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & l & \mu & \kappa \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix} \\[4pt]
\widehat{G}^J(R) \quad &:= \left\{ \widehat{g}\,\widehat{h} \mid g \in GL_2^+(R), h \in H(R) \right\}.
\end{aligned}$$

Obviously the map $^\wedge$ is a group isomorphism between $G^J(R)$ and $\widehat{G}^J(R)$, where the semi-direct product structure can be recovered. Let $(g_1, h_1), (g_2, h_2) \in G^J(\mathbb{R})$. Then

$$\widehat{(g_1, h_1)}\widehat{(g_2, h_2)} = (\widehat{g_1}\widehat{g_2}) \left(\widehat{g_2}^{-1}\widehat{h_1}\widehat{g_2}\widehat{h_2} \right),$$

$$\widehat{g_2}^{-1}\widehat{h_1}\widehat{g_2} \in \widehat{H(\mathbb{R})}.$$

2.5. **Jacobi Forms of level** N. In this section we recall the definition of Jacobi forms of level N with Dirichlet character. Let $f : \mathbb{H} \times \mathbb{C} \longrightarrow \mathbb{C}$ and $k, m \in \mathbb{N}_0$. Let $g^J = (g, h) \in G^J(\mathbb{R})$ with $\det(g) = l$. Then we attach to f the function $\widetilde{f}$ defined by

$$(2.5) \qquad \left(\widehat{f}|_k\widehat{g^J}\right)(\tau_1, z, \tau_2) = \widetilde{f}(\tau_1, z) \, e(-ml\tau_2).$$

This leads to a canonical action of $G^J(\mathbb{R})$ on $\mathbb{H}^J$ and the definition of Jacobi forms. This avoids explicit calculations and displays the essential properties directly.

Definition. *Let Φ be a holomorphic function on $\mathbb{H}^J$. Let $k, m \in \mathbb{N}_0$. Let χ be a Dirichlet character modulo N. Let Γ^J be a congruence subgroup of $G^J(\mathbb{Z})$ with the same Heisenberg part. We denote by Φ a Jacobi form of weight k and index m with character χ with respect to Γ^J if Φ satisfies:*

   (1) *$\Phi|_{k,m}g^J = \chi(g)\Phi,$ for all $g^J = (g, h) \in \Gamma^J$*
   (ii) *For any $g \in GL_2^+(\mathbb{Q})$, $\Phi|_{k,m}g$ has the Fourier expansion*

$$\sum_{r,n\in\mathbb{Q}} c^g(n, r) \, q^n \, \zeta^r,$$

   where $c^g(n, r) = 0$ unless $4nm - r^2 \geq 0$.

We say Φ is a Jacobi cusp form if $c^g(n, r) = 0$ unless $4nm - r^2 > 0$ is satisfied. The space of Jacobi form is denoted by $J_{k,m}(\Gamma^J, \chi)$ and the subspace of cusp forms by $J_{k,m}^{cusp}(\Gamma^J, \chi)$

Remark. The property(ii) needs only be checked for $g \in SL_2(\mathbb{Z})$. Here the sum is running over $n \in h_g^{-1}\mathbb{Z}$, $r \in \mathbb{Z}$ with $c^g(n, r) = 0$, unless $4nm - r^2 \geq 0$ (and $>$ for being a cusp form).

Remark. Let F be a cusp form for a congruence subgroup on $\mathbb{H}_2$. Then F vanishes at every cusp. Equivalent the Fourier expansion at each cusp has only support (parametrization of Fourier coefficients) at positive definite half-integral matrices. Hence at each cusp the Fourier Jacobi coefficients are Jacobi cusp forms.

Next we recall the definition of the index shift operator $V_{l,\chi}$ (see [Ib12], section 3) and finally define the Saito-Kurokawa lift.

Definition. *For χ a Dirichlet character modulo N and an element of Δ_N:*

$$(2.6) \qquad\qquad\qquad \chi\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) := \overline{\chi(a)}.$$

Definition. *Let $k, N \in \mathbb{N}$ and let χ be a Dirichlet character modulo N. Let $\Phi \in J_{k,m}(\Gamma_0(N)^J, \chi)$, $m \in \mathbb{N}_0$. Then we define for all $l \in \mathbb{N}$ the index shift operator:*

$$V_{l,\chi} : J_{k,m}(\Gamma_0(N), \chi)^J \longrightarrow J_{k,ml}(\Gamma_0(N), \chi)^J$$

given by the explicit construction

$$V_{l,\chi}(\Phi) \;:=\; l^{k-1} \sum_{g \in \Gamma_0(N) \backslash \Delta_N(l)} \chi(g)^{-1} \, \Phi|_{k,m} g$$

$$= \; l^{k-1} \sum_{g \in \Gamma_0(N) \backslash \Delta_N(l)} \chi(a) \, (c\tau + d)^{-k} \, e^{-lm\frac{c\tau^2}{c\tau+d}} \, \Phi(g(\tau), \frac{lz}{c\tau + d})$$

$$= \; l^{k-1} V_{l,\chi}^{\circ}(\Phi).$$

Here $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $g(\tau) = \frac{a\tau+b}{c\tau+d}$.

Definition. *Let χ be a Dirichlet character modulo N. Let $\Phi \in J_{k,m}(\Gamma_0(N)^J, \chi)$. Then $\mathcal{L}_{N,\chi}(\Phi)$ is called the Saito-Kurokawa lift of Φ. It is defined by*

$$(2.7) \qquad \mathcal{L}_{N,\chi}(\Phi) \begin{pmatrix} \tau_1 & z \\ z & \tau_2 \end{pmatrix} := c(0) f_{k,\chi}(\tau_1) + \sum_{l=1}^{\infty} V_{l,\chi}(\Phi)(\tau_1, z) \, e(l\tau_2).$$

Here $c(0)$ is the constant term of Φ. For the definition of the Eisenstein series $f_{k,\chi}$ we refer to [Ib12], section 3.2.

Theorem 3.2 and Theorem 3.6 [Ib12] states that $\mathcal{L}$ is a linear injective map to $M_k(\Gamma_0^J(N), \chi)$. If Φ is a cusp form. Then $\mathcal{L}(\Phi)$ is a cusp form. The image of $\mathcal{L}$ is called Maass Spezialschar of level N.

3. Hecke Theory

References: Krieg [Kr90], Miyake [Mi06], Shimura [Sh71]. Let G be a group and Γ a subgroup. Two subgroups are commensurable if the intersection has finite index in each of the two subgroups. Let $\widetilde{\Gamma}$ be all elements $g \in \Gamma$ such that $g\Gamma g^{-1}$ is commensurable with the subgroup Γ itself.

Let $\Delta \subset G$ be a monoid and Ξ a set of commensurable subgroups Γ of G, such that $\Gamma \subset \Delta \subset \widetilde{\Gamma}$. Let R be a commutative ring with 1. Then we denote by

$$\mathcal{H}_R(\Gamma, \Delta) := \left\{ \sum_{\alpha \in \Delta} a_\alpha \, \Gamma \alpha \Gamma \mid a_\alpha \in R \text{ and } a_\alpha = 0 \text{ for almost all } \alpha \right\}$$

the free R-module generated double cosets. Let further $R[\Gamma \backslash \Delta]$ denote the free R-module generated by the $\Gamma \alpha$ cosets, where $\alpha \in \Delta$.

Next, let Δ act on a R-module M by $m \mapsto m^\alpha$. Let M^Γ be the submodule of Γ-invariant elements of M. Let $\Gamma \alpha \Gamma = \sqcup_i \Gamma \alpha_i \in R[\Gamma \backslash \Delta]$ be the disjoint coset decomposition. This identification leads to $\mathcal{H}_R(\Gamma, \Delta) = R[\Gamma \backslash \Delta]^\Gamma$. Note that $\mathcal{H} = \mathcal{H}_R(\Gamma, \Delta)$ acts on M^Γ via

$$m | \Gamma \alpha \Gamma := \sum_i m^{\alpha_i}.$$

Note that m^α is in general not invariant by Γ, but by $\alpha^{-1} \Gamma \alpha \cap \Gamma$. Let $\widetilde{M} := R[\Gamma \backslash \Delta]^\Gamma$. Then the action of $\mathcal{H}$ on $\widetilde{M}$ implies the following multiplication of double cosets. Let

$\Gamma \alpha \Gamma = \sqcup_i \Gamma \alpha_i$ and $\Gamma \beta \Gamma = \sqcup_i \Gamma \beta_i$. Then

$$(3.1) \qquad\qquad \Gamma \alpha \Gamma \circ \Gamma \beta \Gamma := \sum_\gamma \Gamma \gamma \Gamma,$$

where

$$c_\gamma = \sharp \left\{ (i,j) \,|\, \Gamma \alpha_i \beta_j = \Gamma \gamma \right\}.$$

3.1. Representations of Hecke algebras.

We make the assumption that $G = GL_2^+(\mathbb{R})$ and Γ a Fuchsian group with finite character χ. Let $\mathcal{H}$ be the Hecke algebra attached to the Hecke pair (Γ, Δ). We further assume that

(i) χ can be extended to a character of Δ and
(ii) that for $\alpha \in \Delta$ and $\gamma \in \Gamma$ with $\alpha \Gamma \alpha^{-1} \in \Gamma$:
$\chi(\alpha \gamma \alpha^{-1}) = \chi(\gamma)$.

Let Ξ be the set of all subgroups of Γ of finite index. Let $k \in \mathbb{Z}$ be fixed. Let Γ_1 be any element of Ξ. Let $M_k(\Gamma_1, \chi)$ be the vector space of holomorphic functions on $\mathbb{H}$ (and the cusps) satisfying:

$$\begin{aligned}(f|_k \gamma)(\tau) & := j(\gamma, \tau)^{-k} f(\gamma(\tau)) \\ & = \chi(\gamma) f(\tau) \qquad \text{for all } \gamma \in \Gamma_1.\end{aligned}$$

Then Δ acts on the $\mathbb{Z}$-module

$$M := \bigcup_{\Gamma_1 \in \Xi} M_k(\Gamma_1, \chi)$$

by mapping $f \in M_k(\Gamma_1, \chi)$ to an element $f^\alpha \in M_k(\Gamma_1 \cap \alpha^{-1} \Gamma_1 \alpha, \chi)$:

$$f \mapsto f^\alpha := \overline{\chi(\alpha)} \, f|_k \alpha$$

(here we apply property (ii) from above). Let $\Gamma \alpha \Gamma = \sqcup_i \Gamma \alpha_i$. Then the operation of of the Hecke algebra $\mathcal{H}$ on M^Γ is given by

$$(3.2) \qquad\qquad f | \Gamma \alpha \Gamma := \sum_i f^{\alpha_i}.$$

This extends linearly to $h \in \mathcal{H}$ and called Hecke operators. We refer to Miyake [Mi06], Remark 2.8.1 and 2.8.2 for a short discussion on elements of the Hecke algebra and Hecke operators.

3.2. Structure of the Hecke Algebra $\mathcal{H}(\Gamma_0(N), \Delta_N)$.

Let χ be a Dirichlet character modulo N. We have extented (2.6) to Δ_N in such a way that (ii) is satisfied. The Hecke theory applies to $G = GL_2^+(\mathbb{R}), \Delta = \Delta_N, \Gamma = \Gamma_0(N)$ and $R = \mathbb{Z}$. Let $\mathcal{H} = \mathcal{H}(\Gamma_0(N), \Delta_N)$. Let $[a, d]$ be the diagonal matrix $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix} \right)$. Every double coset

$$\Gamma_0(N) \, \alpha \, \Gamma_0(N) = \Gamma_0(N) \, [a, d] \, \Gamma_0(N) =: T(a, b)$$

can be uniquely represented by a diagonal matrix $[a, d]$, where $(a, N) = 1$, $a|d$, and $ad = \det(\alpha)$. Further let

$$(3.3) \qquad T(l) = \sum_{\substack{ad=l,\, a|d, \\ (a,N)=1}} T(a, d) = \bigsqcup_{\substack{ad=l,\, (a,N)=1, \\ b \mod d}} \Gamma_0(N) \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix} \right) = \Gamma_0(N) \backslash \Delta_N(l).$$

Here we identified double cosets with elements in

$$\mathbb{Z}[\Gamma_0(N) \backslash \Delta_N]^{\Gamma_0(N)}.$$

Double cosets decompose in local components. Let $a_1 | d_1$ and $a_2 | d_2$. Then

$$(3.4) \qquad T(a_1 a_2, d_1 d_2) = T(a_1, d_1) \circ T(a_2, d_2) \text{ if } (d_1, d_2) = 1.$$

The Hecke algebra is commutative and decomposes as a restricted tensor product in local Hecke algebras $\mathcal{H}_p$ for all prime numbers p.

$$\mathcal{H} = \otimes_p \mathcal{H}_p,$$

where $\mathcal{H}_p$ is generated by $T(p)$ and $T(p, p)$ if $p \nmid N$ and $T(p)$ otherwise. Hence for every $h \in \mathcal{H}_p$ with $(p \nmid N)$ we have $h \in \mathbb{Z}[x, y]$, where

$$x = T(p) = \Gamma_0(N) \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix} \right) \bigsqcup_{b \pmod d} \Gamma_0(N) \left(\begin{smallmatrix} 1 & b \\ 0 & p \end{smallmatrix} \right)$$

$$y = T(p, p) = \Gamma_0(N) \left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix} \right).$$

Let $p|N$. Then $h \in \mathbb{Z}[T(p)]$, where

$$T(p) = \bigsqcup_{b \pmod d} \Gamma_0(N) \begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix} \quad \text{for } p|N.$$

We will transfer the result of [Mi06], Theorem 4.5.13 (1) to the theory of lifts.

Theorem 3.1. *Let m, n are natural numbers. Then we have the following identity in the Hecke algebra $\mathcal{H}(\Gamma_0(N), \Delta_N)$.*

$$(3.5) \qquad T(m) \circ T(n) = \sum_{\substack{d|(m,n) \\ (d,N)=1}} d \, T(d, d) \, T\left(\frac{mn}{d^2} \right).$$

To apply the general theory define Ξ to be the set of all subgroups of $\Gamma_0(N)$ and

$$M \quad := \bigcup_{\Gamma \in \Xi} M_k(\Gamma, \chi).$$

Then Δ_N operates on M by $f^\alpha := \overline{\chi(\alpha)} \, f|_k \alpha$. The Hecke algebra operates on $M^{\Gamma_0(N)}$. Actually it already operates on $M_k(\Gamma_0(N), \chi)$. We are mainly interested in

the operation of $T(l,l)$ and $T(l)$ on $M^{\Gamma_0(N)}$.

$$f \;\mapsto\; T(l,l)(f) = f^{[l,l]} = \overline{\chi(l)}\, l^{-k}\, f$$

$$f \;\mapsto\; T(l)(f) = \sum_{\substack{ad=l,\,(a,N)=1 \\ b \pmod d}} \overline{\chi(a)}\, f|_k \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}.$$

We have the two Hecke algebras $\mathcal{H} := \mathcal{H}(\Gamma_0(N)), \Delta_N)$ and $\mathcal{H}^J := \mathcal{H}(\widehat{\Gamma}_0^J(N)), \widehat{\Delta}_N^J)$. We are mainly interested in the image of the embedding

$$\iota : \mathcal{H} \hookrightarrow \mathcal{H}^J, \qquad \Gamma_0(N)\alpha\Gamma_0(N) \mapsto \widehat{\Gamma}_0(N)^J\, \widehat{\alpha}\, \widehat{\Gamma}_0(N)^J.$$

This map respects the coset decomposition

$$\bigsqcup \Gamma_0(N)\, \alpha_i \mapsto \bigsqcup \widehat{\Gamma}_0(N)^J\, \widehat{\alpha}_i.$$

Note that this property is implicitly used in the definition of the operator $V_{N,\chi}(l)$ in [EZ85], [Ib12] (see also [He99], section 3). Let

$$(3.6) \qquad\qquad M^J := \bigcup_{m=0}^{\infty} \bigcup_{\Gamma \in \Xi} \widehat{J}_{k,m}\left(\Gamma \ltimes H(\mathbb{Z}), \chi\right).$$

Here Ξ denotes the set of all congruence subgroup of $\Gamma_0(N)$. Let $\alpha \in \Delta_N(l)$, then $M^J \longrightarrow M^J, \Phi \mapsto \Phi^{\widehat{\alpha}}$, where $\Phi \in J_{k,m}(\Gamma_0^J(N), \chi)$. Then

$$\Phi|\Gamma_0^J(N)\,\alpha\,\Gamma_0^J(N) \;:=\; \sum_i \Phi^{\alpha_i} \in J_{k,ml}(\Gamma_0^J(N), \chi)$$

$$= \;\sum_i \overline{\chi(\alpha_i)}\; \Phi|_{k,m}\alpha_i.$$

We frequently switch between Φ and $\widehat{\Phi}$ and consider α as element of Δ_N, Δ_N^J, and $\widehat{\Delta}_N$ accordingly. We make all the obvious identifications if clear from the context. Note that cusp forms map to cusp forms. Finally we perform the translation of the formula (3.1 into the Hecke-Jacobi algebra. Note that a priori it was not clear that this is possible, since the general Hecke-Jacobi algebra is not abelian and has zero divisors [He99]. Let $V^0(m)$ correspond to $T(m)$ and $V^0(d,d)$ if $(d,N)=1$ as elements of $\mathcal{H}^J$. Then we obtain inside the Hecke algebra $\mathcal{H}(\Gamma_0^J(N), \Delta_N)$ the important algebraic identity

$$(3.7) \qquad\qquad V^0(m) \circ V^0(n) = \sum_{\substack{d|(m,n) \\ (d,N)=1}} d\, V^0(d,d)\, V^0\left(\frac{mn}{d^2}\right).$$

4. Main Results

The Maass Spezialschar of level N is given by

$$(4.1) \qquad\qquad M_k^{\mathrm{Spez}}(\Gamma_0^2(N), \chi) := \left\{ \mathcal{L}_{N,\chi}(\Phi) \mid \Phi \in J_{k,1}(\Gamma_0^J(N), \chi) \right\}.$$

The subspace of cusp form we denote by $S_k^{\mathrm{Spez}}(\Gamma_0^2(N), \chi)$. A Siegel modular form $FM_k^2(\Gamma_0(N)^J, \chi)$ is in the Maass Spezialschar iff all the Fourier coefficients of F satisfy the general Maass relations

$$(4.2) \qquad A(n, r, m) = \sum_{d | (n,r,m)} d^{k-1} \chi(d) \, A\left(\frac{nm}{d^2}, \frac{r}{d}, 1\right).$$

See also [Ib12]section 3.4 and the observations at the end of the proof of Theorem 4.1. Our argument is the following. All Fourier coefficients $A(T)$, $T \in \mathcal{X}^*$ are determined by the first Fourier-Jacobi coefficient of F. This is a Jacobi form of weight k and index 1 of level N and the relations reflect exactly the definition of $\mathcal{L}_{N,\chi}$.

In this section we prove that F is a Saito-Kurokawa lift iff F satisfies symmetries $(*_l)$ for all $l \in \mathbb{N}$. We state two applications. First, it is sufficient to check $(*_p)$ for prime numbers and second we obtain symmetric Maass relations (of course equivalent to (4.2). Combined we obtain local Maass p-relations generalizing the known level $N = 1$ case, discovered first by Pitale, Schmidt and the author [FPRS13].

4.1. Maass Spezialschar and Symmetries.

Saito-Kurokawa lifts, elements in the Maass Spezialschar, can be characterized by symmetries. Note that these symmetries $(*_l)$ for all $l \in \mathbb{N}$ make it possible to study Saito-Kurokawa lifts by properties of the Hecke algebra $\mathcal{H}(\Gamma_0(N), \Delta_N)$ originally constructed to study elliptic modular forms.

Theorem 4.1. *Let k and N be positive integers. Suppose χ is a Dirichlet character modulo N satisfying $\chi(-1) = (-1)^k$. Let $F \in M_k(\Gamma_0^{(2)}(N), \chi)$ be a Siegel modular form of weight k, degree 2, and level N with Dirichlet character χ. Then F is a Saito-Kurowaka lift if and only if F satisfies for all $l \in \mathbb{N}$ the symmetry relation $(*_l)$ given by*

$$(4.3) \qquad \sum_{g \in \Gamma_0(N) \backslash \Delta_N(l)} \chi(g)^{-1} \left(F|_k \widetilde{g}^{\uparrow}\right) = \sum_{g \in \Gamma_0(N) \backslash \Delta_N(l)} \chi(g)^{-1} \left(F|_k \widetilde{g}^{\downarrow}\right). \qquad (*_l).$$

Proof. Note that $(*_l)$ is well-defined, since

$$\chi(\gamma g) = \chi(g) = \chi(a)^{-1}; \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta_N(l) \text{ and } \gamma \in \Gamma_0(N).$$

First we show that $(*_l)$ implies that the l-th Fourier-Jacobi (FJ) coefficients F_l of F satisfy $F_l = V_{l,\chi}(F_1)$. This implies that for $F \in S_k(\Gamma_0^{(2)}(N), \chi)$ all FJ coefficients are obtained by $V_{l,\chi}(F_1)$, where $F_1 \in J_{k,1}^{\mathrm{cusp}}(\Gamma_0^J(N), \chi)$. Hence $F = \mathcal{L}_{n,\chi}(F_1)$. For the general case we refer to the end of this proof. Let

$$A := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} \sqrt{l} & 0 \\ 0 & \sqrt{l}^{-1} \end{pmatrix}, \text{ then } A \begin{pmatrix} \tau_1 & z \\ z & \tau_2 \end{pmatrix} = \left(\tau_1, \sqrt{l}z, l\tau_2\right).$$

We deform $(*_l)$ on both sides by $|_k A$. This breaks the symmetry of $(*_l)$. Nevertheless the projective matrices $\widetilde{g}^{\uparrow\downarrow}$ become integral and the iff part of the Theorem still

remains. Let $(*_l^A)$ be given by

$$(4.4) \qquad \sum_{g \in \Gamma_0(N) \backslash \Delta_N(l)} \chi(g)^{-1} \left(F|_k \widetilde{g}^\uparrow A \right) = \sum_{g \in \Gamma_0(N) \backslash \Delta_N(l)} \chi(g)^{-1} \left(F|_k \widetilde{g}^\downarrow A \right). \qquad (*_l^A).$$

We calculate the left side of $(*_l^A)$. Note that

$$\widetilde{g}^\uparrow A = \sqrt{l}^{-1} \left(g \times \begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix} \right),$$

which implies that $\left(F|_k \widetilde{g}^\uparrow A \right) = l^k F|_k \widehat{g}$. This action is compatible with the FJ expansion of F:

$$F(\tau_1, z, \tau_2) = \sum_{m=1}^\infty F_m(\tau_1, z) \, q_2^m \text{ with } q_2 = e(\tau_2).$$

Finally we obtain for the l-th FJ coefficient of the left side of $(*_l^A)$ the expression

$$l^k V_{l,\chi}^0(F_1).$$

Next we determine the l-th FJ-coefficient of the right side of $(*_l^A)$. We fix for $\Gamma_0(N) \backslash \Delta_N(l)$ the special representation system (3.3) and obtain:

$$\sum_{\substack{a,d \in \mathbb{N}; \ ad=l \\ b \pmod d}} \left(\frac{d}{l} \right)^{-k} \chi(a) F\left(\tau_1, az, a^2\tau_2 + \frac{b}{d} \right).$$

The l-th of this expression is equal to

$$\sum_{\substack{a,d \in \mathbb{N}; \ ad=l}} \left(\frac{d}{l} \right)^{-k} \chi(a) F_{\frac{l}{a^2}}(\tau_1, az) \left(\sum_{b \pmod d} e\left(\frac{l}{a^2} \frac{b}{d} \right) \right),$$

which simplifies to $l \, F_l$ (only the term $d = l$ contributes).

Conversely, assuming that $V_{l,\chi}(F_1) = F_l$ for all implies $(*_l)$ for all $l \in \mathbb{N}$ by applying a pure algebraic relation in a corresponding Hecke algebra. We start in comparing the m-th Fourier Jacobi coefficients of both sides of $(*_l^A)$, where $m = l_1 l_2$ and $l_2 = l$. We obtain for the left side:

$$l_2^k \sum_{\substack{a,d \in \mathbb{N}; \ ad=l \\ b \pmod d}} \chi(a) F_{l_1}\left(\frac{a\tau_1 + b}{d}, az, l_2\tau_2 \right)$$

$$= l_2^k \, V_{l_2,\chi}^0(F_{l_1}) = l_2^k \, l_1^{k-1} \left(V_{l_2,\chi}^0 \circ V_{l_1,\chi}^0 \right)(F_1).$$

For the right side we obtain:

$$\sum_{a,d \in \mathbb{N};\; ad=l_2} \left(\frac{d}{l_2}\right)^{-k} \chi(a)\, F_{\frac{m}{a^2}}(\tau_1, az) \left(\sum_{b \pmod{d}} e\left(\frac{m}{a^2}\frac{b}{d}\right)\right)$$

$$= \sum_{ad=l_2,\; a|l_1} d\, \left(\frac{d}{l_2}\right)^{-k} \chi(a)\, F_{\frac{m}{a^2}}(\tau_1, az)$$

$$= l_2 \sum_{a|(l_1,l_2)} a^{k-1}\, \chi(a)\, F_{\frac{m}{a^2}}(\tau_1, az)$$

$$= l_2\, (l_1 l_2)^{k-1} \sum_{a|(l_1,l_2)} a^{1-k}\, \chi(a)\, V^0_{\frac{l_1 l_2}{a^2}\chi}(F_1)(\tau_1, az).$$

The operator $V^0(a,a)$ is defined by

$$V^0(a,a)(F)(\tau_1, z, \tau_2) \;:=\; \chi(a)\, F|_k \widehat{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}}(\tau_1, z, \tau_2)$$

$$= \chi(a)\, a^{-k} F(\tau_1, az, a^2\tau_2),$$

which leads to an action on Jacobi forms. Hence the right side is equal to

(4.5)
$$l_2\, (l_1 l_2)^{k-1} \sum_{\substack{a|(l_1,l_2) \\ (a,N)=1}} a\left(V^0(a,a) \circ V^0_{\frac{l_1 l_2}{a^2}\chi}\right)(F_1).$$

Comparing the left and right side, we are left with showing the following identity inside the Hecke algebra $\mathcal{H}(\Gamma^J_0(N), \Delta^J_N)$:

(4.6)
$$V^0(m) \circ V^0(n) = \sum_{\substack{d|(m,n) \\ (d,N)=1}} d\, V^0(d,d)\, V^0\left(\frac{mn}{d^2}\right).$$

This is pure algebraic and independent of the involved Jacobi forms and Fourier Jacobi expansions. This formula has been obtained in section 3 on Hecke theory.

Finally we consider the case when F is not necessarily a cusp form. Let $A(n,r,m)$ be the Fourier coefficients of F. Then $(*_l)$ implies that

(4.7)
$$\sum_{d|(n,l)} d^{k-1} \chi(d)\, A\left(\frac{nl}{d^2}, 0, 0\right) = \sum_{d|\,l} d^{k-1} \chi(d)\, A(n, 0, 0).$$

Let $a(l) := A(l, 0, 0)$. Then we obtain

$$a(l) = \left(\sum_{d|\,l} d^{k-1} \chi(d)\right) a(1).$$

All possible $a(0)$ such that

$$(4.8) \qquad f(\tau) = \sum_{n=0}^{\infty} a(n) \; q^n \; \in \; M_k(\Gamma_0, \chi)$$

are classified in [Ib12]. Hence F is a Saito-Kurokawa lift in the sense of Ibukiyama.

<div align="right">□</div>

4.2. **Applications.**

Corollary 4.2. *Let* $F \in M_k^2(\Gamma_0^{(2)}(N), \chi)$ *with Fourier expansion*

$$F(\tau_1, z, \tau_2) = \sum_{T=(n,r,m) \in \mathcal{X}} A(n, r, m) \; q_1^n \; \zeta^r \; q_2^m.$$

Then the following properties are equal.

(i) *F is a Saito-Kurokawa lift (also called Maass lift)*
(ii) *All the Fourier coefficients of F satisfy:*

$$A(n, r, m) = \sum_{d | (n,r,m)} d^{k-1} \chi(d) \; A\left(\frac{nm}{d^2}, \frac{r}{d}, 1\right).$$

(iii) *All the Fourier coefficients of F satisfy for all $l \in \mathbb{N}$:*

$$\sum_{d|(n,r,l)} d^{k-1} \chi(d) \; A\left(\frac{nl}{d^2}, \frac{r}{d}, m\right) = \sum_{d|(l,r,m)} d^{k-1} \chi(d) \; A\left(n, \frac{r}{d}, \frac{ml}{d^2}\right).$$

Proof. The Maass lift (called also Saito-Kurokawa lift, see [Ib12] Introduction) and the relations of Fourier coefficients is given in [Ib12], section 3.4. For the readers convenience we recall the equivalence of (i) and (ii). Let F be a Maass lift then (ii) is satisfied (Proposition 3.8, [Ib12]). If (ii) is satisfied then F is uniquely determined by the first Fourier-Jacobi coefficient and all the other Fourier-Jacobi coefficients are the expected lifts in the setting of Jacobi forms.
(iii) implies (i) by putting $m = 1$ in formula (iii). Next we show that (i) implies (iii). We have already proven that F is a Maass lift if and only if

$$\sum_{g \in \Gamma_0(N) \backslash \Delta_N(l)} \chi(g)^{-1} \left(F|_k \widetilde{g}^{\uparrow}\right) = \sum_{g \in \Gamma_0(N) \backslash \Delta_N(l)} \chi(g)^{-1} \left(F|_k \widetilde{g}^{\downarrow}\right). \qquad (*_l)$$

for all $l \in \mathbb{N}$. We fix for $\Gamma \backslash \Delta_N(l)$ the special representative system

$$(4.9) \qquad \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \; \Big| \; a, b \in \mathbb{N}; \; ad = l; \; (a, N) = 1; \; b = 0, 1, \ldots, d-1 \right\}.$$

Note that $\chi(g)^{-1} = \chi(a)$. Then the left side of the equation $(*_l)$ is equal to

$$\sum_{\substack{a,b \in \mathbb{N}; \; ad=l \\ b \pmod d}} l^{\frac{k}{2}} d^{-k} \chi(a) \, F\left(\frac{a\tau_1 + b}{d}, l^{-\frac{1}{2}} az, \tau_2\right)$$

We consider the left side of $(*_l)$. For convenience we map $z \mapsto \sqrt{l}\, z$ and keep in mind that $\sum_{b \pmod d} e(n\frac{b}{d}) = d$ if $d|n$ and 0 otherwise. We obtain

$$\sum_{n,r,m} l^{\frac{k}{2}} \sum_{\substack{a,d\in\mathbb{N} \\ ad=l,\, d|n}} d^{-k+1} \chi(a)\, A(n,r,m)\, q_1^{\frac{an}{d}} \zeta^{ra} q_2^m.$$

This is equal to

$$l^{1-\frac{k}{2}} \sum_{n,r,m} \sum_{a|(n,r,l)} a^{k-1} \chi(a)\, A\left(\frac{nl}{a^2}, \frac{r}{a}, m\right) q_1^n \zeta^r q_2^m.$$

Since the left side of the relation $(*_l)$ is symmetric to the right side this leads to the proof. $\qquad\square$

Actually one has to check the relations in (iii) only for l prime numbers. This follows from the observation

Corollary 4.3. *Let F be a Siegel modular form of level N. Then F is a Saito-Kurowaka lift if and only if F satisfies the symmetry relation $(*_l)$ for all prime numbers l.*

Proof. This follows from the results of section 3.2. $\qquad\square$

Putting this together leads to

Corollary 4.4. *(Maass p-relations)*
Let F be a Siegel modular form of level N. Then F is a Saito-Kurokawa lift iff the Fourier coefficients of F satisfy

$$(4.10) \quad A(np,r,m) + p^{k-1}\chi(p)\, A\left(\frac{n}{p}, \frac{r}{p}, m\right) = A(n,r,pm) + p^{k-1}\chi(p)\, A\left(n, \frac{r}{p}, \frac{m}{p}\right).$$

for all prime numbers p. Note that for $p|N$ the relations degenerate to

$$A(np,r,m) = A(n,r,pm).$$

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