

The 2-braid group and Garside normal form

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Abstract

We investigate the relation between the Garside normal form for positive braids and the 2-braid group defined by Rouquier. Inspired by work of Brav and Thomas we show that the Garside normal form is encoded in the action of the 2-braid group on a certain categorified left cell module. This allows us to deduce the faithfulness of the 2-braid group in finite type. We also give a new proof of Paris' theorem that the canonical map from the generalized braid monoid to its braid group is injective in arbitrary type.

1 Introduction

The braid group is ubiquitous not only in knot theory, but also in topology, algebraic geometry and representation theory. Experience tells that the action of the Coxeter group (or its corresponding Hecke algebra) on the level of Grothendieck groups can often be upgraded to a braid group action on the underlying category. Examples of this phenomenon in representation theory can be seen in [Rou06], [BR12] and [CR08]. Rouquier suggests that not only the self-equivalences of the action are important, but also the morphisms between them possess some interesting structure. Thus he introduced in [Rou06] the 2-braid group as a concrete home to study these morphisms. The 2-braid group lives in the homotopy category of Soergel bimodules, but has many other incarnations (translation functors on Bernstein-Gelfand-Gelfand category \mathcal{O} , convolution functors, spherical twists, ...) as well.

The 2-braid group is a fundamental mathematical object. Its importance was underlined by its applications in categorified link invariants. Just as

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many knot invariants factor over the braid group, the 2-braid group can be used to construct triply-graded HOMFLY-PT homology (see [Kho07]).

The more natural a categorification is, the more structure of the underlying categorified mathematical object it reflects. Motivated by work of Brav and Thomas we looked for shadows of the Garside normal form of the positive braid monoid in the 2-braid group. It is remarkable that on the categorical level the Garside normal form only becomes apparent after acting on a certain categorified left cell module for the Hecke algebra (see Theorem 5.12). From this we can deduce the faithfulness of the 2-braid group in finite type (see Corollary 5.19) as conjectured by Rouquier in [Rou06]. Following Rouquier's philosophy a categorified braid group action should encode a lot of information about the category acted upon. For this purpose proving the faithfulness of the 2-braid group is a basic question. The faithfulness follows in type A from work by Khovanov and Seidel (see [KS02]) and in simply-laced, finite type from results by Brav and Thomas (see [BT11]). The shadows of the Garside normal form also enable us to give a new proof of Paris' theorem that the canonical map from the generalized braid monoid to its braid group is an injection in arbitrary type (see [Par02] and Corollary 5.20).

The following example illustrates that the categorified braid group action contains strictly more information than its decategorification. On the one hand, the categorified left cell module we construct admits a faithful braid group action (see Theorem 5.18). On the other hand, it gives a categorification of a twisted reduced Burau representation in type A_{n-1} (see Example 5.3) which is known not to be faithful for $n \geq 5$ (see [Big99]).

1.1 Structure of the paper

Sections 1.3 to 1.5 We introduce notation and recall important results about the Hecke algebra, cells with respect to the Kazhdan-Lusztig basis, Soergel bimodules, generalized braid groups and the 2-braid group.

Section 2 Using Soergel's Hom-formula we study the existence of degree 1 morphisms between indecomposable Soergel bimodules and rewrite the multiplication formula for the Kazhdan-Lusztig basis in our setting.

Section 3 We show that cell modules of the Hecke algebra can be categorified by mimicking the construction on the category of Soergel bimodules.

Section 4 We introduce the perverse filtration on the homotopy category of Soergel bimodules and recall some important results.

Section 5 After introducing the important notion of an anchor we prove our main results.

1.2 Acknowledgement

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1.3 The Hecke algebra and cells

Let (W, S) be a *Coxeter system*, i.e. W is a group together with a set of generators S admitting a particularly nice presentation:

$$W = \langle s \in S \mid \underbrace{sts\dots}_{m_{s,t} \text{ terms}} = \underbrace{tst\dots}_{m_{s,t} \text{ terms}}, s^2 = 1 \rangle$$

where $m_{s,t} \geq 2$ is the order of st for $s \neq t \in S$. For $w \in W$ denote the left (resp. right) descent set of w by $\mathcal{L}(w) = \{s \in S \mid sw < w\}$ (resp. $\mathcal{R}(w) = \{s \in S \mid ws < w\}$).

Let $\mathbf{H}_{(W,S)}$ be the corresponding *Hecke algebra* over $\mathbb{Z}[v, v^{-1}]$ which we will also denote by \mathbf{H} if there is no danger of confusion.

Denote by $\{H_w\}_{w \in W}$ the *standard* and by $\{\underline{H}_w\}_{w \in W}$ the *Kazhdan-Lusztig basis* in Soergel's normalization (see [Soe97]). Since we are not working with the usual normalization, let us state the relations for the standard basis:

$$\begin{aligned} H_s^2 &= (v^{-1} - v)H_s + 1 && \text{for all } s \in S, \\ \underbrace{H_s H_t H_s \dots}_{m_{s,t} \text{ terms}} &= \underbrace{H_t H_s H_t \dots}_{m_{s,t} \text{ terms}} && \text{for all } s \neq t \in S. \end{aligned}$$

Write $\underline{H}_x = \sum_{y < x} h_{y,x} H_y$ where $h_{y,x} \in \mathbb{Z}[v]$ are the *Kazhdan-Lusztig polynomials* up to simple renormalization (see [Soe97]). From the defining property of the Kazhdan-Lusztig basis, one sees immediately $h_{x,x} = 1$ and $h_{y,x} \in v\mathbb{Z}[v]$ for all $y < x$. Define $\mu(y, x)$ for $y \leq x \in W$ as the coefficient of v in $h_{y,x}$. We extend this definition of μ as follows. Set $\mu(x, y) := \mu(y, x)$ if $y < x$ and $\mu(x, y) = 0$ if x and y are incomparable in the Bruhat order.

There is a unique \mathbb{Z} -linear involution $\overline{(-)}$ on \mathbf{H} satisfying $\overline{v} = v^{-1}$ and $\overline{H_s} = H_s^{-1}$ for $s \in S$ and thus $\overline{H_x} = H_{x^{-1}}$. The Kazhdan-Lusztig basis element \underline{H}_x is the unique element in $H_x + \sum_{y < x} v\mathbb{Z}[v]H_y$ that is invariant under $\overline{(-)}$.

Moreover, there is a $\mathbb{Z}[v, v^{-1}]$ -linear anti-involution ι on \mathbf{H} satisfying $\iota(H_s) = H_s$ for $s \in S$ and thus $\iota(H_x) = H_{x^{-1}}$ and a \mathbb{Z} -linear anti-involution ω on \mathbf{H} satisfying $\omega(v) = v^{-1}$ and $\omega(\underline{H}_s) = \underline{H}_s$ for all $s \in S$.

Recall that a *trace* on \mathbf{H} is a $\mathbb{Z}[v, v^{-1}]$ -linear map $\varepsilon : \mathbf{H} \rightarrow \mathbb{Z}[v, v^{-1}]$ satisfying $\varepsilon(hh') = \varepsilon(h'h)$ for all $h, h' \in \mathbf{H}$. A calculation shows that the $\mathbb{Z}[v, v^{-1}]$ -linear map ε satisfying $\varepsilon(H_x) = \delta_{x,1}$, where $\delta_{x,1}$ is the Kronecker symbol, is a trace. This map is called the *standard trace*.

Using the standard trace and the \mathbb{Z} -linear anti-involution ω , we can now define the *standard pairing* $(-, -) : \mathbf{H} \times \mathbf{H} \rightarrow \mathbb{Z}[v, v^{-1}]$ via $(h, h') = \varepsilon(\omega(h)h')$ for all $h, h' \in \mathbf{H}$. Note that this pairing is $\mathbb{Z}[v, v^{-1}]$ -semilinear, i.e. $(v^{-1}h, h') = (h, vh') = v(h, h')$ for all $h, h' \in \mathbf{H}$. In addition, \underline{H}_s is self-biadjoint with respect to this pairing: $(\underline{H}_s h, h') = (h, \underline{H}_s h')$ and $(h \underline{H}_s, h') = (h, h' \underline{H}_s)$ for all $h, h' \in \mathbf{H}$.

Denote the *left* (resp. *right* or *two-sided*) *cell preorder* with respect to the Kazhdan-Lusztig basis by $\underset{L}{\leq}$ (resp. $\underset{R}{\leq}$ or $\underset{LR}{\leq}$). Recall that $\underset{L}{\leq}$ is generated by the relation $v \underset{L}{\leq} w$ for $v, w \in W$ if \underline{H}_v occurs with non-zero coefficient in $h \underline{H}_w$ for some $h \in \mathbf{H}$. Observe that this definition can be generalized to give a preorder on the indexing set of a basis of any based R -algebra.

For $w \in W$ let $\mathbf{H}(\underset{L}{\leq} w)$ (resp. $\mathbf{H}(\underset{L}{<} w)$) be the $\mathbb{Z}[v, v^{-1}]$ -span of all basis elements \underline{H}_v such that $v \underset{L}{\leq} w$ (resp. $v \underset{L}{<} w$). Use a similar notation for any left cell instead of w and for the other cell preorders. We will be interested in the best understood situation, namely a left cell inside the two-sided cell of all non-trivial elements with a unique reduced expression:

Let C be the set of all elements in $W \setminus \{id\}$ with a unique reduced expression and set $C_s := \{w \in C \mid ws < w\}$. Obviously the sets C_s for $s \in S$ form a partition of C .

Proposition 1.1 [Lus83, Proposition 3.8].

Assume (W, S) to be irreducible. Then:

- (i) C_s is a left cell in \mathbf{H} for all $s \in S$.
- (ii) C is a two-sided cell in \mathbf{H} .

To visualize the left cell C_s for $s \in S$ define an undirected graph Γ_s : The vertex set is given by C_s and the edge set contains an edge $\{x, y\}$ if $x^{-1}y$ lies in S . Define a map $\pi_s : C_s \rightarrow S$ by sending an element $w \in C_s$ to the unique element in its left descent set $\mathcal{L}(w)$.

Lemma 1.2 [Lus83, Proposition 3.8].

Assume (W, S) to be irreducible. For any $s \in S$:

- (i) Γ_s is a tree.
- (ii) π_s defines an isomorphism between Γ_s and the Coxeter graph of (W, S) if and only if the latter is a simply-laced tree.

(iii) For $x, y \in C_s$ the subset $\{x, y\}$ is an edge of Γ_s if and only if $\mathcal{L}(x) \neq \mathcal{L}(y)$ and $\mu(x, y) \neq 0$. In this case we have $\mu(x, y) = 1$.

1.4 Soergel bimodules

Let $\mathfrak{h} = \bigoplus_{s \in S} \mathbb{R}\alpha_s^\vee$ be the reflection representation of (W, S) and define the simple roots $\{\alpha_s \mid s \in S\} \subset \mathfrak{h}^*$ via:

$$\langle \alpha_s^\vee, \alpha_t \rangle = -2 \cos\left(\frac{\pi}{m_{s,t}}\right) \quad (1)$$

(This gives a *symmetric realization* in the sense of [EW13, Definition 3.1].)

Denote by $R = S(\mathfrak{h}^*)$ the symmetric algebra on \mathfrak{h}^* , viewed as a graded algebra with $\deg(\mathfrak{h}^*) = 2$. Since W acts on \mathfrak{h}^* via the contragredient representation ($s(\gamma) = \gamma - \langle \alpha_s^\vee, \gamma \rangle \alpha_s$ for all $\gamma \in \mathfrak{h}^*$), we can extend this to an action of W on R by degree-preserving algebra automorphisms.

Denote by $R\text{-grmod-}R$ the abelian, monoidal category of \mathbb{Z} -graded R -bimodules that are finitely generated as left and as right R -modules with degree-preserving bimodule homomorphisms as morphisms. Given a graded R -bimodule $M = \bigoplus_{i \in \mathbb{Z}} M^i$ we denote by $M(1)$ the R -bimodule with the grading shifted down by one: $M(1)^i = M^{i+1}$. For any two graded R -bimodules M and N denote by $\text{Hom}^\bullet(M, N) := \bigoplus_{n \in \mathbb{Z}} \text{Hom}_{R\text{-grmod-}R}(M, N(n))$ the bimodule homomorphisms from M to N of all degrees.

For $s \in S$ let $R^s \subseteq R$ be the subring of invariants under the action of s and define the R -bimodule $B_s = R \otimes_{R^s} R(1)$.

The *category of Bott-Samelson bimodules*, denoted by \mathcal{BS} , is the full additive, monoidal subcategory of $R\text{-grmod-}R$ generated by the B_s for $s \in S$ and their grading shifts. For an expression $\underline{w} = s_1 s_2 \cdots s_k$ with $s_i \in S$ for all $1 \leq i \leq k$ the graded R -bimodule $B_{\underline{w}} = B_{s_1} \otimes B_{s_2} \otimes \cdots \otimes B_{s_k}$ is called a *Bott-Samelson bimodule*. We usually omit all tensor products and thus $B_{\underline{w}}$ is written as $B_{s_1} B_{s_2} \cdots B_{s_k}$. Define the *category of Soergel bimodules* \mathcal{B} to be the Karoubi envelope of \mathcal{BS} . In other words, an indecomposable Soergel bimodule is a direct summand of a shifted Bott-Samelson bimodule and the morphisms between Soergel bimodules are degree-preserving. The following result is well known (see [EW13, Lemma 6.24]):

Lemma 1.3. *The category of Soergel bimodules \mathcal{B} is a Krull-Schmidt category with finite dimensional Hom-spaces.*

Recall that for an essentially small, additive, monoidal category \mathcal{C} , its split Grothendieck group $\mathfrak{K}^0(\mathcal{C})$ is an associative, unital ring with multiplication given by $[M][N] = [M \otimes N]$ for $M, N \in \mathcal{C}$ and the class of the monoidal

identity $[1_c]$ as unit. Applying this to the category of Soergel bimodules, we see that $\mathfrak{K}^0(\mathcal{B})$ is an associative, unital \mathbb{Z} -algebra. We can equip it with the structure of a $\mathbb{Z}[v, v^{-1}]$ algebra by defining $v[B] := [B(1)]$ for all $B \in \mathcal{B}$. Soergel proves that the category of Soergel bimodules gives a categorification of the Hecke algebra \mathbf{H} [Soe07, Theorem 1.10] and describes the graded rank of the homomorphism space between two Soergel bimodules (see [Soe07, Theorem 5.15]):

Theorem 1.4 (Soergel’s categorification Theorem).
There is a unique isomorphism of $\mathbb{Z}[v, v^{-1}]$ -algebras:

$$\begin{aligned} \varepsilon : \mathbf{H} &\xrightarrow{\cong} \mathfrak{K}^0(\mathcal{B}) \\ \underline{H}_s &\mapsto [B_s]. \end{aligned}$$

Theorem 1.5 (Soergel’s Hom-formula).

Given any two Soergel bimodules B and B' , the morphism space $\mathrm{Hom}^\bullet(B, B')$ is free as a left (resp. right) R -module. Moreover, its graded rank is given by $(\varepsilon^{-1}[B], \varepsilon^{-1}[B'])$ where $(-, -)$ denotes the standard pairing on the Hecke algebra.

Using his Hom-formula, Soergel obtains a classification of the indecomposable Soergel bimodules in [Soe07, Theorem 6.14, (1) and (2)]:

Theorem 1.6. *Given any reduced expression \underline{w} of $w \in W$, the Bott-Samelson $B_{\underline{w}}$ contains up to isomorphism a unique indecomposable summand B_w which does not occur in $B_{\underline{y}}$ for any reduced expression \underline{y} of $y \in W$ with $l(y) < l(w)$. In addition, B_w does not depend up to isomorphism on the reduced expression \underline{w} . A complete set of representatives of the isomorphism classes of all indecomposable Soergel bimodules is given by:*

$$\{B_w(m) \mid w \in W \text{ and } m \in \mathbb{Z}\}.$$

Using Theorem 1.6 it follows that $\{[B_w] \mid w \in W\}$ is a $\mathbb{Z}[v, v^{-1}]$ basis of $\mathfrak{K}^0(\mathcal{B})$. It is a natural question what this basis is. Soergel explicitly constructs an inverse to ε , called the character map $ch : \mathfrak{K}^0(\mathcal{B}) \rightarrow \mathbf{H}$, and conjectures that the basis of the indecomposable Soergel bimodules $\{[B_w] \mid w \in W\}$ in $\mathfrak{K}^0(\mathcal{B})$ corresponds to the Kazhdan-Lusztig basis in \mathbf{H} . Elias and Williamson have recently proven Soergel’s conjecture for reflection faithful realizations over \mathbb{R} with linear independent sets of simple roots and co-roots satisfying a positivity condition. Libedinsky showed in [Lib08] that their results extend to the reflection representation:

Theorem 1.7 (Soergel’s conjecture).
For all $w \in W$ we have $\varepsilon(\underline{H}_w) = [B_w]$.

This has the following important consequence for us (see [EW14, top of page 15]; note that in [EW14] a different pairing is used):

Corollary 1.8. *For all $x, y \in W$ the homomorphism space $\text{Hom}^\bullet(B_x, B_y)$ is concentrated in non-negative degrees and $\dim \text{Hom}_{\mathcal{B}}(B_x, B_y) = \delta_{xy}$*

Proof. Soergel's Hom formula together with Soergel's conjecture imply that the graded rank of $\text{Hom}_{\mathcal{B}}^\bullet(B_x, B_y)$ is given by

$$\begin{aligned}
(\underline{H}_x, \underline{H}_y) &= \varepsilon(\underline{H}_{x^{-1}} \underline{H}_y) \\
&= \varepsilon\left(\left(\sum_{z^{-1} \leq x^{-1}} h_{z^{-1}, x^{-1}} H_{z^{-1}}\right) \left(\sum_{z \leq y} h_{z, y} H_z\right)\right) \\
&= \sum_{\substack{z \in W \text{ s.t.} \\ z^{-1} \leq x^{-1} \text{ and} \\ z \leq y}} h_{z^{-1}, x^{-1}} h_{z, y} \\
&= \sum_{\substack{z \in W \text{ s.t.} \\ z \leq x, y}} h_{z, x} h_{z, y} \in \begin{cases} v\mathbb{N}[v] & \text{if } x \neq y \\ 1 + v\mathbb{N}[v] & \text{if } x = y \end{cases} \quad (2)
\end{aligned}$$

where in the first step we applied $\omega(\underline{H}_x) = \underline{H}_{x^{-1}}$ (which follows from $\omega = \iota \circ \overline{(-)}$ and $\iota(\underline{H}_x) = \underline{H}_{x^{-1}}$), in the third step we used $\varepsilon(H_x H_y) = \delta_{x, y^{-1}}$ with $\delta_{x, y^{-1}}$ the Kronecker delta and in the last step we plugged in $h_{z^{-1}, x^{-1}} = h_{z, x}$ for all $z, x \in W$. Finally, Soergel's conjecture together with the definition of the Kazhdan-Lusztig polynomials implies that $h_{w', w} \in v\mathbb{N}[v]$ for $w' < w$ and $h_{w, w} = 1$ for all $w, w' \in W$. \square

1.5 Generalized braid groups and the 2-braid group

By dropping the condition $s^2 = 1$ for all $s \in S$ in the presentation of W , we get the presentation of the generalized braid group and the braid monoid corresponding to (W, S) :

$$\begin{aligned}
Br_{(W, S)} &= \langle s \in S \mid \underbrace{sts \dots}_{m_{s,t} \text{ terms}} = \underbrace{tst \dots}_{m_{s,t} \text{ terms}} \rangle \text{ in the category of groups;} \\
Br_{(W, S)}^+ &= \langle s \in S \mid \underbrace{sts \dots}_{m_{s,t} \text{ terms}} = \underbrace{tst \dots}_{m_{s,t} \text{ terms}} \rangle \text{ in the category of monoids.}
\end{aligned}$$

Note that there is a surjective monoid homomorphism $\psi : Br_{(W, S)}^+ \rightarrow W$ which is the identity on generators. The map ψ admits a set-theoretic section $\varphi : W \rightarrow Br_{(W, S)}^+$ which sends an element in W to the word in the braid monoid corresponding to one of its reduced expressions. (This is well defined as any two reduced expressions of an element in W can be related using only the braid relations.) The elements in its image are called *reduced braids*. Set $Br_{(W, S)}^{\text{red}} := \text{im}(\varphi)$. In this setting we have the *Garside normal form* for $Br_{(W, S)}^+$ as described in [Mic99, Section 4]:

Theorem 1.9 (Garside normal form). *For any positive braid $\sigma \in Br^+(W, S)$ there exists a unique sequence $(w_m, w_{m-1}, \dots, w_1)$ of reduced braids such that $\sigma = w_m w_{m-1} \dots w_1$ and for all $1 \leq i \leq m$ the reduced braid w_i is non-trivial and the unique maximal reduced braid that occurs as a right divisor of $w_m w_{m-1} \dots w_i$.*

The sequence $(w_m, w_{m-1}, \dots, w_1)$ is called the Garside normal form of σ . Each w_i for $1 \leq i \leq m$ is called a Garside factor.

The following two results will be important for us. The first one shows that being a Garside normal form can be checked locally (see [Mic99, Proposition 4.1]) and the second result relates a Garside normal form to the left and right descent sets of its factors in the Coxeter group (see [Mic99, Corollary 4.2]):

Proposition 1.10. *$(w_m, w_{m-1}, \dots, w_1)$ is a Garside normal form of the element $w_m w_{m-1} \dots w_1$ if and only if for all $1 \leq i < m$ the sequence (w_{i+1}, w_i) is a Garside normal form of $w_{i+1} w_i$.*

Lemma 1.11. *Let $x, y \in W$. $(\varphi(x), \varphi(y))$ is a Garside normal form of $\varphi(x)\varphi(y)$ in the positive braid monoid $Br^+_{(W,S)}$ if and only if $\mathcal{R}(x) \subseteq \mathcal{L}(y)$ in the Coxeter group.*

From the presentations given above, it is immediate that there is a canonical morphism of monoids $Br^+_{(W,S)} \rightarrow Br_{(W,S)}$ which is the identity on the generators in S . It is a natural question whether this morphism is injective, i.e. whether $Br^+_{(W,S)}$ can be identified with the submonoid of $Br_{(W,S)}$ generated by S . This is the case in all types. In finite type it is well known (see for example [Mic99, Corollary 3.2]) as $Br_{(W,S)}$ is the group of fractions of $Br^+_{(W,S)}$. It has been extended to arbitrary type in [Par02] using extensive calculations. In this paper we will give an alternative proof of this in arbitrary type.

The following result will be helpful to prove the faithfulness of the 2-braid group in finite type (see [BT11, Lemma 2.3] for a proof) and relies on the fact that in finite type the Garside normal form can be extended from $Br^+(W, S)$ to $Br(W, S)$:

Lemma 1.12. *Assume that (W, S) is a Coxeter system of finite type. A group homomorphism $\rho : Br_{(W,S)} \rightarrow G$ is injective if and only if the induced monoid homomorphism $\rho^+ : Br^+_{(W,S)} \hookrightarrow Br_{(W,S)} \rightarrow G$ is injective.*

Define the elementary Rouquier complexes corresponding to a simple reflection $s \in S$ as follows

$$F_s := (\quad \quad \quad 0 \longrightarrow B_s \longrightarrow R(1) \longrightarrow 0) \text{ with } a \otimes b \mapsto ab$$

$$E_s = F_{s^{-1}} := (0 \longrightarrow R(-1) \longrightarrow B_s \longrightarrow 0) \quad \text{with } 1 \mapsto \frac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s)$$

where in both complexes B_s sits in cohomological degree 0.

In [Rou06] Rouquier showed that in the bounded homotopy category of Soergel bimodules $K^b(\mathcal{B})$ these complexes are inverse to each other and satisfy the braid relations up to canonical isomorphism. We define *the 2-braid group*, denoted by $2-\mathcal{B}r$, as the full monoidal subcategory of $K^b(\mathcal{B})$ generated by $F_{s^{-1}}$ and F_s for $s \in S$. Note that the set of isomorphism classes of objects in $2-\mathcal{B}r$, denoted by $\text{Pic}(2-\mathcal{B}r)$, forms a group together with the binary operation induced by the tensor product. $\text{Pic}(2-\mathcal{B}r)$ is also called *the Picard group* of the monoidal category $2-\mathcal{B}r$. Rouquier formulates the following conjecture (see [Rou06, Conjecture 9.8]):

Conjecture 1.13 (Faithfulness of the 2-braid group).

The natural map $\text{Br}_{(W,S)} \longrightarrow \text{Pic}(2-\mathcal{B}r)$ is an isomorphism.

This conjecture follows in type A from work by Khovanov and Seidel (see [KS02]) and in simply-laced, finite type from results by Brav and Thomas (see [BT11]). In this paper we will prove the faithfulness of the 2-braid group in finite type (extending the previous results to non-simply laced finite type).

2 Consequences of the multiplication and the Hom-formula

Recall the left cells C_s for $s \in S$ and the two-sided cell C which were introduced at the end of Section 1.3. In this section we want to show that there is particularly nice choice of $s \in S$ for the left cell C_s , calculate all the Kazhdan-Lusztig polynomials for the elements in C_s and draw some conclusions using Soergel's Hom-formula.

First note that the graph Γ_s encodes all information necessary for the left cell module corresponding to C_s :

Lemma 2.1. *For $w \in C_s$ and $r \in S$ we have in $\mathbf{H}(\leq_C C_s)/\mathbf{H}(\leq_L C_s)$:*

$$\underline{H}_r \underline{H}_w = \begin{cases} (v + v^{-1}) \underline{H}_w & \text{if } r \in \mathcal{L}(w) \\ \sum_{\substack{y \in C_s \text{ s.t. } r \in \mathcal{L}(y) \\ \text{and } \{y,w\} \in E(\Gamma_s)}} \underline{H}_y & \text{if } r \notin \mathcal{L}(w) \end{cases}$$

Proof. The left handed multiplication formula from [KL79, Formula 2.3.a and 2.3.c]) reads in Soergel's normalization as follows:

$$\underline{H}_r \underline{H}_w = \begin{cases} (v + v^{-1}) \underline{H}_w & \text{if } rw < w, \\ \underline{H}_{rw} + \sum_{\substack{y < w \\ ry < y}} \mu(y, w) \underline{H}_y & \text{otherwise.} \end{cases} \quad (3)$$

Using [KL79, Claim 2.3.e] one can rewrite this as:

$$\underline{H}_r \underline{H}_w = \begin{cases} (v + v^{-1}) \underline{H}_w & \text{if } r \in \mathcal{L}(w), \\ \sum_{\substack{y \in W \\ r \in \mathcal{L}(y)}} \mu(y, w) \underline{H}_y & \text{if } r \notin \mathcal{L}(w). \end{cases} \quad (4)$$

The last formula together with Item 3 from Lemma 1.2 gives the claim. \square

Similarly, Eq. (3) implies (observe that in simply-laced type for any vertex of Γ_s the map π_s is injective on the set of its neighbours):

Lemma 2.2. *Assume (W, S) to be of simply-laced type. For the unique reduced expression $\underline{w} = s_1 s_2 \dots s_k$ of $w \in C_s$ we have in $\mathbf{H}(\leq C_s) / \mathbf{H}(\leq C_s)$:*

$$\underline{H}_w = \underline{H}_{\underline{w}} = \underline{H}_{s_1} \underline{H}_{s_2} \dots \underline{H}_{s_k}$$

Lemma 2.3. *Assume (W, S) to be of non-simply-laced type. Choose $s \in S$ among a pair $\{s, t\} \subseteq S$ with $m_{s,t} \geq 4$. Each element $w \in C_s$ can be uniquely written as $w = w_l \dots w_2 w_1$ for some $l \in \mathbb{N}$ where for each $1 \leq i \leq l$ odd $w_i = k_i \overline{(q_i, r_i)} = \dots q_i r_i$ (an alternating product of k_i terms) for some $k_i \in \mathbb{Z}$ lies in the standard parabolic subgroup generated by $\{q_i, r_i\} \subseteq S$ with $m_{q_i, r_i} \geq 4$ and for each $2 \leq j \leq l$ even w_j lies in a standard parabolic subgroup of simply-laced type. In $\mathbf{H}(\leq C_s) / \mathbf{H}(\leq C_s)$ we have*

$$\underline{H}_w = \underline{H}_{w_l} \dots \underline{H}_{w_2} \underline{H}_{w_1}$$

and the Lemma 2.2 may be applied to all \underline{H}_{w_i} with $2 \leq i \leq l$ even.

If in addition (W, S) is of finite type, then there is a unique pair $\{s, t\} \subseteq S$ with $m_{s,t} \geq 4$ and l in the form above is smaller or equal to 2.

From now on, we assume that if (W, S) is of non-simply laced type, then $s \in S$ is chosen among a pair $\{s, t\} \subseteq S$ with $m_{s,t} \geq 4$ (which is unique if W is finite).

Lemma 2.4. *For $x \neq y \in C_s$ there are only morphisms of degree one from B_x to B_y if either $\{x, y\}$ is an edge in Γ_s or $\mathcal{L}(x) = \mathcal{L}(y)$ and x and y are comparable in the Bruhat order.*

If x and y are connected by an edge in Γ_s , then a morphism of degree one from B_x to B_y is unique up to scalar.

Proof. Rewriting Eq. (2) we get for the graded rank of $\text{Hom}_{\mathcal{B}}^{\bullet}(B_x, B_y)$:

$$\begin{aligned} (\underline{H}_x, \underline{H}_y) &= \sum_{z \leq x, y} h_{z,x} h_{z,y} \\ &\in \mu(x, y)v + v^2\mathbb{N}[v] \end{aligned}$$

Thus there are morphisms of degree one from B_x to B_y if and only if $\mu(x, y)$ is non-zero. If this is the case, either the left descent sets of x and y disagree which by Lemma 1.2 (iii) is equivalent to $\{x, y\}$ being an edge in Γ_s or they agree which gives the second case of the statement as $\mu(-, -)$ is only non-zero for comparable elements.

The second part of the lemma is just a reformulation of Lemma 1.2 (iii). \square

Example 2.5. We want to give an example showing that the second case in Lemma 2.4 can actually occur, i.e. there may exist a degree 1 morphism between indecomposable Soergel bimodules corresponding to $x, y \in C_s$ such that $\mathcal{L}(x) = \mathcal{L}(y)$ and x and y are comparable in the Bruhat order. Consider an affine Coxeter group W of type \widetilde{A}_2 with generators $S = \{s, t, u\}$. A simple calculation shows $h_{s, stus} = v^3 + v$. Thus Soergel's Hom-formula implies that there is a degree 1 morphism from B_{stus} to B_s and B_s to B_{stus} .

For the rest of the section we will deal with the finite case in which we can say a little more.

Lemma 2.6. *Assume that (W, S) is a Coxeter group of finite type. Under the choices made above, all elements in C_s are rationally smooth. In other words, for $w \in C_s$ and $y \leq w$ we have: $h_{y,w} = v^{l(w)-l(y)}$.*

Proof. For all elements lying in the parabolic subgroup generated by $\{s, t\}$, the result is well known (see [Eli13, Claim 2.1]) as this is simply a dihedral group. Choose $w \in C_s$ that does not lie in this parabolic subgroup.

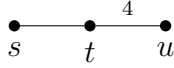
By induction, we may assume that we haven't proven the statement for all $w' \in C_s$ such that $w' < w$. Let $\pi_s(w) = r \in S$. Set $w' = rw < w$ and choose $y \leq w$. Note that r does not occur in rw . We obtain an inductive formula for the Kazhdan-Lusztig polynomials from the left-handed multiplication formula in Eq. (3) by expressing each Kazhdan-Lusztig basis element in terms of the standard basis and by comparing coefficients

$$\begin{aligned} h_{y,w} &= h_{ry, rw} + v^{c_y} h_{y, rw} - \sum_{\substack{y \leq z < rw \text{ s.t.} \\ rz < z}} \mu(z, rw) h_{y,z} \quad \text{where } c_y = \begin{cases} 1 & \text{if } ry > y, \\ -1 & \text{otherwise.} \end{cases} \\ &= \begin{cases} h_{ry, rw} & \text{if } ry < y, \\ v h_{y, rw} & \text{if } ry > y. \end{cases} \end{aligned}$$

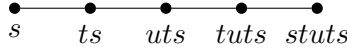
$$= v^{l(w)-l(y)}$$

where the sum $\sum \mu(z, rw)h_{y,z}$ vanishes as r does not occur in rw and thus there cannot be an element $z < rw$ with $r \in \mathcal{L}(z)$. In addition, we used $y \not\leq rw$ (resp. $ry \not\leq rw$) if $ry < y$ (resp. if $ry > y$) and thus $h_{y,rw} = 0$ (resp. $h_{ry,rw} = 0$) for basically the same reason. In the last step we simply plugged in the Kazhdan-Lusztig polynomials for $w' = rw$ which is by induction rationally smooth. \square

Remark 2.7. • Note that the last lemma is no longer true if s is not chosen among the unique pair $\{s, t\} \subseteq S$ with $m_{s,t} \geq 4$. Consider the example given by the following Coxeter graph:



For Γ_s we get:



A calculation yields the following Kazhdan-Lusztig polynomials

$$\begin{aligned} h_{us, stuts} &= v^3 + v \\ h_{u, stuts} &= v^4 + v^2 \\ h_{s, stuts} &= v^4 + v^2 \\ h_{e, stuts} &= v^5 + v^3 \end{aligned}$$

which show that $stuts$ is not rationally smooth.

- The last lemma holds in slightly more generality: It is true for all Coxeter groups whose Coxeter graph is a tree with at most one pair $\{s, t\} \subseteq S$ such that $m_{s,t} \geq 4$.

Lemma 2.8. *Assume that (W, S) is of finite type. Let $x, y \in C_s$. Choose $w \in W$ the unique maximal element such that $w < x$ and $w < y$. Then $\text{Hom}_{\mathbb{B}}^{\bullet}(B_x, B_y)$ is concentrated in degrees $\geq l(x) + l(y) - 2l(w)$ and is of dimension 1 in degree $l(x) + l(y) - 2l(w)$. Moreover, the morphisms are concentrated in even (resp. odd) degrees if $l(x) - l(y)$ is even (resp. odd). In particular, there are non-zero morphisms of degree one from B_x to B_y if and only if x and y are connected by an edge in Γ_s .*

Proof. First note that w lies in C_s and that it is the unique maximal subexpression shared by the unique reduced expressions of x and y . (In the simply laced case it is the first vertex the two paths from x to s and from y to s

share.) Then plugging the results from Lemma 2.6 into Eq. (2) we get for the graded rank of $\text{Hom}_{\mathcal{B}}^{\bullet}(B_x, B_y)$:

$$\begin{aligned} (\underline{H}_x, \underline{H}_y) &= \sum_{z \leq w} h_{z,x} h_{z,y} \\ &= \sum_{z \leq w} v^{l(x)+l(y)-2l(z)} \end{aligned}$$

The lowest degree term is $v^{l(x)+l(y)-2l(w)}$. Applying Soergel's Hom-formula from Theorem 1.5 yields the result. The graded rank shows that depending on the parity of $l(x)-l(y)$ all generators are in either even or odd degree. Due to our choice of grading on R , this implies that all morphisms are concentrated in even (resp. odd) degrees if $l(x)-l(y)$ is even (resp. odd). \square

Remark 2.9. It should be noted that for $x, y \in W$ the parity vanishing of $\text{Hom}_{\mathcal{B}}^{\bullet}(B_x, B_y)$ is a more general fact that holds in any Coxeter system.

3 Categorized left cell modules

The goal of this section is to construct a categorification of the left cell module corresponding to C_s by mimicking the definition of the left cell module on the categorical level. In order to do so, we will need some results that show that the Grothendieck group behaves well with respect to suitable quotients and subcategories.

Lemma 3.1. *Let \mathcal{C} be an essentially small Krull-Schmidt category and X a subclass of the indecomposable objects, closed under isomorphism. Let \mathfrak{J} be the 2-sided ideal of all morphisms factoring through a finite direct sum of objects in X . Then the following holds:*

- (i) \mathcal{C}/\mathfrak{J} is a Krull-Schmidt category.
- (ii) Let B be a set of representatives of all isoclasses of indecomposable objects in \mathcal{C} . A \mathbb{Z} -basis of $\mathfrak{K}^0(\mathcal{C}/\mathfrak{J})$ is given by $B \setminus X$.
- (iii) The functor $\mathcal{C} \rightarrow \mathcal{C}/\mathfrak{J}$ is additive and induces the following isomorphism of abelian groups on the level of Grothendieck groups:

$$\mathfrak{K}^0(\mathcal{C}) / \langle [M] \mid M \in X \rangle \xrightarrow{\sim} \mathfrak{K}^0(\mathcal{C}/\mathfrak{J})$$

Assume in addition that \mathcal{C} is graded, monoidal and that X is closed under grading shifts. Furthermore, assume that no indecomposable object Y is isomorphic to one of its grading shifts $Y(m)$ for $m \in \mathbb{Z}$. Let B be a set of

representatives of all isoclasses of indecomposable objects in \mathcal{C} up to grading shift. Assume that $B \cap X$ is the union of all two-sided cells belonging to a lower set in the two-sided cell preorder (see Section 1.3) of $\mathfrak{K}^0(\mathcal{C})$ with respect to the $\mathbb{Z}[v, v^{-1}]$ -basis B . Then the following holds:

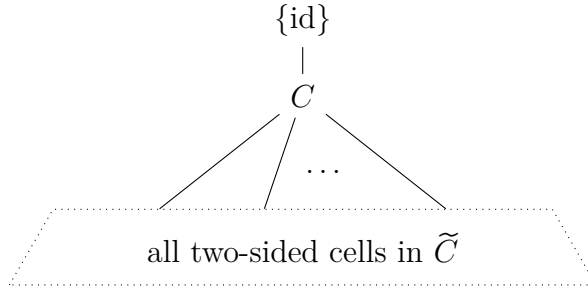
- (i) \mathcal{C}/\mathfrak{J} is a graded, monoidal category.
- (ii) A $\mathbb{Z}[v, v^{-1}]$ -basis of $\mathfrak{K}^0(\mathcal{C}/\mathfrak{J})$ is given by $B \setminus X$.
- (iii) The functor $\mathcal{C} \rightarrow \mathcal{C}/\mathfrak{J}$ is a strict monoidal functor and the isomorphism above is an isomorphism of $\mathbb{Z}[v, v^{-1}]$ -algebras.

Lemma 3.2. *Let \mathcal{C} be an essentially small (graded) Krull-Schmidt category such that no indecomposable object Y is isomorphic to one of its grading shifts $Y(m)$ for $m \in \mathbb{Z}$. Let B be a subset of the representatives of all isoclasses of indecomposable objects in \mathcal{C} (up to grading shift). Consider the full additive (graded) subcategory \mathcal{C}_B of \mathcal{C} generated by all objects isomorphic to an object in B , also denoted by $\langle M \mid M \in B \rangle_{\oplus}$ (resp. $\langle M \mid M \in B \rangle_{\oplus, (-)}$). Then the following holds:*

- (i) \mathcal{C}_B is a (graded) Krull-Schmidt category.
- (ii) B is a \mathbb{Z} - (resp. $\mathbb{Z}[v, v^{-1}]$ -) basis of $\mathfrak{K}^0(\mathcal{C}_B)$.
- (iii) The functor $\mathcal{C}_B \hookrightarrow \mathcal{C}$ induces the following isomorphism of \mathbb{Z} - (resp. $\mathbb{Z}[v, v^{-1}]$ -) modules on the level of Grothendieck-groups:

$$\mathfrak{K}^0(\mathcal{C}_B) \longrightarrow \langle [M] \mid M \in B \rangle \subset \mathfrak{K}^0(\mathcal{C})$$

Consider in W the set \tilde{C} of all elements that do not admit a unique reduced expression. Note that \tilde{C} is the union of all two-sided cells belonging to a lower set in the two-sided cell preorder of \mathbf{H} with respect to the Kazhdan-Lusztig basis. The Hasse diagram of the two-sided cell preorder in \mathbf{H} can be pictured as follows:



In order to see this, use the characterization of the two-sided cell preorder in terms of left and right descent sets. Let \mathfrak{J} be the two-sided ideal of all morphisms in \mathcal{B} factoring through a finite direct sum of objects in the class $\{M \in \mathcal{B} \mid M \cong B_w(k) \text{ for some } w \in \tilde{C} \text{ and } k \in \mathbb{Z}\}$. Lemma 3.1 shows that the quotient category \mathcal{B}/\mathfrak{J} , which will also be denoted by $\mathcal{B}/\langle B_w \mid w \in \tilde{C} \rangle_{\oplus, (-)}$, is a graded monoidal Krull-Schmidt category.

Inside this category we will study the full additive graded subcategory $\mathcal{C}_s = \langle B_w \mid w \in C_s \rangle_{\oplus, (-)}$. From Lemma 3.2 we see that \mathcal{C}_s is still a graded Krull-Schmidt category and that $\mathfrak{K}^0(\mathcal{C}_s)$ is a free $\mathbb{Z}[v, v^{-1}]$ -module with $\{[B_w] \mid w \in C_s\}$ as basis. In addition, $\mathfrak{K}^0(\mathcal{C}_s)$ sits inside $\mathfrak{K}^0(\mathcal{B}/\langle B_w \mid w \in \tilde{C} \rangle_{\oplus, (-)})$ which by Lemma 3.1 is isomorphic to $\mathfrak{K}^0(\mathcal{B})/\mathfrak{K}^0(\mathcal{B})(\underset{LR}{<} C)$ as a $\mathbb{Z}[v, v^{-1}]$ -algebra.

Note that for any element $y \in C_s$ and $x \in W$ such that $x \underset{L}{\leq} y$, the characterization of the left cell preorder in terms of left descent sets implies that x either lies in \tilde{C} or in C_s . Therefore the set $\{\underset{L}{\leq} C_s\}$ is contained in $\{\underset{LR}{\leq} C\}$. Due to $\mathbf{H}(\underset{L}{\leq} C_s) \subseteq \mathbf{H}(\underset{LR}{\leq} C)$, the following square can be completed:

$$\begin{array}{ccc} \mathbf{H}(\underset{L}{\leq} C_s) & \hookrightarrow & \mathbf{H} \\ \downarrow & & \downarrow \\ \mathbf{H}(\underset{L}{\leq} C_s)/\mathbf{H}(\underset{L}{\leq} C_s) & \hookrightarrow & \mathbf{H}/\mathbf{H}(\underset{LR}{\leq} C) \end{array}$$

which shows that the left cell module corresponding to C_s is isomorphic to the submodule spanned by $\{\underline{H}_w \mid w \in C_s\}$ inside $\mathbf{H}/\mathbf{H}(\underset{LR}{\leq} C)$.

After decategorifying all categorical constructions carry over to \mathbf{H} without difficulty:

- All ideals in $\mathfrak{K}^0(\mathcal{B})$ considered so far are generated by classes of indecomposable Soergel bimodules.
- Soergel's categorification theorem states that $\mathfrak{K}^0(\mathcal{B})$ and \mathbf{H} are isomorphic as $\mathbb{Z}[v, v^{-1}]$ -algebras.
- Soergel's conjecture implies that the $\mathbb{Z}[v, v^{-1}]$ -basis of $\mathfrak{K}^0(\mathcal{B})$ given by the classes of perverse indecomposable Soergel bimodules is matched with the Kazhdan-Lusztig basis of \mathbf{H} under this isomorphism.

Therefore we get:

$$\begin{array}{ccccc} \mathfrak{K}^0(\mathcal{C}_s) & \xrightarrow{\cong} & \langle [B_w] \mid w \in C_s \rangle & \hookrightarrow & \mathfrak{K}^0(\mathcal{B})/\mathfrak{K}^0(\mathcal{B})(\underset{LR}{\leq} C) \\ & & \downarrow \cong & & \downarrow \cong \\ \mathbf{H}(\underset{L}{\leq} C_s)/\mathbf{H}(\underset{L}{\leq} C_s) & \xrightarrow{\cong} & \langle \underline{H}_w \mid w \in C_s \rangle & \hookrightarrow & \mathbf{H}/\mathbf{H}(\underset{LR}{\leq} C) \end{array}$$

Note that for any Soergel bimodule $M \in \mathcal{B}$ we have an additive endofunctor on \mathcal{C}_s given by $M \otimes (-)$ because C_s is a left cell in $\mathfrak{K}^0(\mathcal{B})$ and we have killed all indecomposables indexed by some element $\underset{L}{\leq} C_s$ in $\mathcal{B} / \langle B_w \mid w \in \tilde{C} \rangle_{\oplus, (-)}$. This endofunctor descends to the $\mathbb{Z}[v, v^{-1}]$ -module endomorphism on $\mathfrak{K}^0(\mathcal{C}_s)$ given by left multiplication with $[X]$. Therefore we have shown:

Theorem 3.3. $\mathfrak{K}^0(\mathcal{C}_s)$ is isomorphic to the left cell module $\mathbf{H}(\underset{L}{\leq} C_s) / \mathbf{H}(\underset{L}{<} C_s)$ corresponding to C_s . This isomorphism matches the action of $\mathfrak{K}^0(\mathcal{B})$ on $\mathfrak{K}^0(\mathcal{C}_s)$ and the action of \mathbf{H} on $\mathbf{H}(\underset{L}{\leq} C_s) / \mathbf{H}(\underset{L}{<} C_s)$.

Next, we will define an action of $2\text{-}\mathcal{B}r$ on $K^b(\mathcal{C}_s)$. Since we killed all indecomposable Soergel bimodules indexed by elements in $\{\underset{L}{\leq} C_s\}$ (and their grading shifts) in \mathcal{C}_s , $K^b(\mathcal{C}_s)$ can be viewed as a module category over the monoidal category $K^b(\mathcal{B})$ (i.e. there exists a bifunctor $K^b(\mathcal{B}) \times K^b(\mathcal{C}_s) \rightarrow K^b(\mathcal{C}_s)$ induced by the tensor product of the corresponding complexes together with an associator and a left unitor satisfying the usual coherence conditions). Restricting this action of $K^b(\mathcal{B})$ on $K^b(\mathcal{C}_s)$ to $2\text{-}\mathcal{B}r$ yields our action. In particular, the action of $Br_{(W,S)}$ on $K^b(\mathcal{B})$ descends to $K^b(\mathcal{C}_s)$. More explicitly, for $\sigma \in Br_{(W,S)}$ we get an autoequivalence of $K^b(\mathcal{C}_s)$ via $F_\sigma \otimes (-)$. This yields a group homomorphism $Br_{(W,S)} \rightarrow Iso(Aut(K^b(\mathcal{C}_s)))$, where $Iso(Aut(K^b(\mathcal{C}_s)))$ are the isomorphism classes of autoequivalences on \mathcal{C}_s . We will show that this group homomorphism is faithful in finite type, even though its decategorification is not faithful in general (see Example 5.3).

4 The perverse Filtration

In this section we introduce the perverse t-structure on the homotopy category of Soergel bimodules as described by Elias and Williamson in [EW13, Remark 6.2]. The main idea is to play out the two gradings on a complex of Soergel bimodules, namely the cohomological and the internal grading, against each other and to consider linear complexes (see [MO05; Maz07; Maz10; Maz09]).

Let \mathcal{C} be the category of Soergel bimodules or a categorified left cell module \mathcal{C}_s for some $s \in S$ as introduced in Section 3. We try to use the following convention throughout: n denotes the cohomological degree, m the grading shift and k possible multiplicities.

Definition 4.1. Let $B \in \mathcal{C}$ be a Soergel bimodule. B is called *perverse* if B is a direct sum of copies of B_w for $w \in W$ without grading shifts. Fix a choice

of decomposition of B into indecomposable Soergel bimodules:

$$B = \bigoplus_{\substack{w \in W \\ m \in \mathbb{Z}}} B_w^{\oplus k_{w,m}}(m)$$

For $j \in \mathbb{Z}$ define the perverse filtration $\cdots \subseteq {}^p\tau_{\leq j-1}B \subseteq {}^p\tau_{\leq j}B \subseteq \cdots$ of B as follows:

$${}^p\tau_{\leq j}B := \bigoplus_{\substack{w \in W \\ m \geq -j}} B_w^{\oplus k_{w,m}}(m).$$

Set ${}^p\tau_{< j} := {}^p\tau_{\leq j-1}$ and define ${}^p\tau_{\geq j}B = B / {}^p\tau_{< j}B$. Similarly set ${}^p\tau_{> j} := {}^p\tau_{\geq j+1}$. Define the j -th perverse cohomology as:

$${}^p\mathcal{H}^j(B) := ({}^p\tau_{\leq j}(B) / {}^p\tau_{< j}(B))(j).$$

By definition the perverse filtration of a Soergel bimodule always splits and subquotients of the perverse filtration are isomorphic to shifted perverse Soergel bimodules. More explicitly ${}^p\tau_{\leq j}B / {}^p\tau_{< j}B$ is isomorphic to $C(-j)$ for some perverse Soergel bimodule $C \in \mathcal{C}$ and contains exactly all those indecomposable summands $B_w(-j)$ of B for $w \in W$. The perverse cohomology shifts the subquotients back in order for them to be perverse. Thus in this case ${}^p\mathcal{H}^j(B)$ exactly gives the perverse Soergel bimodule C .

Lemma 4.2. *Let $j \in \mathbb{Z}$. ${}^p\tau_{\leq j}(-)$ and ${}^p\tau_{\geq j}(-)$ give well-defined additive endofunctors on the category of Soergel bimodules. ${}^p\mathcal{H}^j(-)$ defines an additive functor from the category of Soergel bimodules to the full additive subcategory of perverse Soergel bimodules.*

Proof. This is an easy consequence of the fact that $\dim \operatorname{Hom}_{\mathcal{C}}(B_x, B_y) = \delta_{x,y}$ for $x, y \in W$ and that the morphisms of all degrees between two indecomposable Soergel bimodules are concentrated in non-negative degrees. \square

The following few results on homotopy minimal complexes hold in any Krull-Schmidt category. For concreteness, we will state them for the category of Soergel bimodules.

Definition 4.3. A complex $F \in C^b(\mathcal{C})$ is called *minimal* if F does not contain a contractible direct summand.

Since we use right superscripts to indicate homogeneous components of a graded module, we will use left superscripts to indicate the cohomological degree whenever we work with cochain complexes of graded modules.

It is easy to see that a complex of the form $\cdots \rightarrow 0 \rightarrow X \xrightarrow{\phi} Y \rightarrow 0 \rightarrow \cdots$ where ϕ is an isomorphism in \mathcal{C} is contractible. The following result is due to Bar-Natan (see [BN07]):

Lemma 4.4 (Gaussian elimination).

Given a complex $F \in C^b(\mathcal{C})$ which looks in cohomological degrees n and $n+1$ as follows:

$$\dots \longrightarrow {}^{n-1}F \xrightarrow{d^{n-1}} M \oplus B \xrightarrow{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}} M' \oplus B' \xrightarrow{d^{n+1}} {}^{n+2}F \longrightarrow \dots$$

where $\delta : B \rightarrow B'$ is an isomorphism, F is homotopy equivalent to a complex $F' \in C^b(\mathcal{C})$ which agrees with F outside the cohomological degrees n and $n+1$ and looks in these two degrees like:

$$\dots \longrightarrow {}^{n-1}F \xrightarrow{pr_M \circ d^{n-1}} M \xrightarrow{\alpha - \beta \circ \delta^{-1} \circ \gamma} M' \xrightarrow{d^{n+1} \circ \iota_{M'}} {}^{n+2}F \longrightarrow \dots$$

Moreover, F' is a direct summand of F . We will call the passage from F to F' a Gaussian elimination with respect to δ .

Given any complex $F \in C^b(\mathcal{C})$ one can successively eliminate contractible direct summands to obtain a direct summand $F_{\min} \overset{\oplus}{\subseteq} F$ such that F_{\min} is minimal. In particular, F is isomorphic to F_{\min} in $K^b(\mathcal{C})$ as F_{\min} is homotopy equivalent to F via the inclusion and projection.

Lemma 4.5. For a complex $F \in C^b(\mathcal{C})$ any two minimal complexes $F_1, F_2 \overset{\oplus}{\subseteq} F$ are isomorphic in $C^b(\mathcal{C})$.

Corollary 4.6. Homotopy equivalent minimal complexes in $C^b(\mathcal{C})$ are isomorphic in $C^b(\mathcal{C})$.

The filtration on \mathcal{C} from Definition 4.1 induces a split diagonal filtration on the full subcategory $C^b(\mathcal{C})_{\min}$ of minimal complexes in $C^b(\mathcal{C})$ as follows:

Definition 4.7. Let F be a minimal, bounded complex of Soergel bimodules. For $j \in \mathbb{Z}$ define the *perverse filtration* $\dots \subseteq {}^p\tau_{\leq j-1}F \subseteq {}^p\tau_{\leq j}F \subseteq \dots$ of F as follows:

$${}^n({}^p\tau_{\leq j}F) := {}^p\tau_{\leq j-n}({}^nF)$$

Define ${}^p\tau_{< j}$, ${}^p\tau_{\geq j}$ and ${}^p\tau_{> j}$ as above. Define the j -th perverse cohomology as:

$${}^p\mathcal{H}^j(F) := ({}^p\tau_{\leq j}(F) / {}^p\tau_{< j}(F))[j]$$

Note that ${}^p\tau_{\leq j}F$ is a well-defined subcomplex of F because Soergel's Hom-formula implies that the homomorphisms of all degrees from B_x to B_y for $x, y \in W$ are concentrated in non-negative degrees and that there can only be homomorphisms of degree 0 if $x = y$ and in that case every non-zero

homomorphism is invertible. Thus non-zero degree 0 homomorphisms cannot occur as components in minimal complexes and for every non-zero component $B_x(m_x) \rightarrow B_y(m_y)$ of the differential in F we have $m_y > m_x$. In addition, for any minimal complex F of Soergel bimodules as above, there is a level-wise split short exact sequence in $C^b(\mathcal{C})$ for all $j \in \mathbb{Z}$

$$0 \longrightarrow {}^p\tau_{\leq j}(F) \longrightarrow F \longrightarrow {}^p\tau_{> j}(F) \longrightarrow 0$$

which induces a distinguished triangle in the homotopy category $K^b(\mathcal{C})$.

The main idea of the next definition is to extend our previous definitions to $K^b(\mathcal{C})$ via the equivalence of categories induced by $C^b(\mathcal{C})_{\min} \hookrightarrow C^b(\mathcal{C})$ on the level of homotopy categories.

Definition 4.8. Let ${}^pK^b(\mathcal{C})^{\geq 0}$ be the full subcategory of $K^b(\mathcal{C})$ consisting of all complexes which are isomorphic to a minimal complex $F \in K^b(\mathcal{C})$ such that ${}^p\tau_{< 0}(F)$ vanishes.

Let ${}^pK^b(\mathcal{C})^{\leq 0}$ be the full subcategory of $K^b(\mathcal{C})$ consisting of all complexes which are isomorphic to a minimal complex $F \in K^b(\mathcal{C})$ such that ${}^p\tau_{> 0}(F)$ vanishes.

A complex $F \in K^b(\mathcal{C})$ is called *linear* or *perverse* if it lies in ${}^pK^b(\mathcal{C})^{\geq 0} \cap {}^pK^b(\mathcal{C})^{\leq 0}$. A *perverse shift* is a simultaneous shift of the form $(-k)[k]$ for some $k \in \mathbb{Z}$.

Proposition 4.9. *The following statements are equivalent for a complex $F \in K^b(\mathcal{C})$:*

- (i) $F \in {}^pK^b(\mathcal{C})^{\geq 0}$.
- (ii) Any minimal complex $F' \in C^b(\mathcal{C})$ which is isomorphic to F in $K^b(\mathcal{C})$ satisfies ${}^p\tau_{< 0}(F') = 0$.

Proof. ii) obviously implies i). Since homotopy equivalent minimal complexes in $C^b(\mathcal{C})$ are isomorphic in $C^b(\mathcal{C})$ by Corollary 4.6 and the Krull-Schmidt theorem holds in \mathcal{C} , we see that the converse also holds. See [Kra14, Proposition 3.2.1] for the proof that the Krull-Schmidt theorem holds in any Krull-Schmidt category. \square

Since this definition is a little technical, let us try to explain it a little: For a minimal complex in ${}^pK^b(\mathcal{C})^{\geq 0}$ an indecomposable Soergel bimodule occurring in cohomological degree n is of the form $B_w(m)$ for $m \leq n$. Loosely speaking, in ${}^pK^b(\mathcal{C})^{\geq 0}$ the possible shifts of the occurring indecomposables are bounded above by the cohomological degree in which the module occurs. For ${}^pK^b(\mathcal{C})^{\leq 0}$ replace “above” by “below” in the last slogan.

So, how should we visualize this definition? For a minimal complex one can keep track of the indecomposable bimodules occurring in the complex in the form of a table, where the columns denote the cohomological degree and the rows correspond to the grading shift. We will choose the following convention that the columns (resp. rows) are labelled by integers in increasing order from left to right (resp. from top to bottom). Thus we get a table of the following form

	-2	-1	0	1	2
-2					
-1					
0					
1					
2					

where an indecomposable Soergel bimodule $B_w(m)$ occurring in cohomological degree n appears in the column labelled with n and the row labelled with m . Therefore a minimal complex $F \in {}^p K^b(\mathcal{C})^{\geq 0}$ (resp. $F \in {}^p K^b(\mathcal{C})^{\leq 0}$) has only entries in cells on or above (resp. below) the marked grey diagonal. The entries in such a table for a minimal perverse complex are restricted to the grey diagonal.

Lemma 4.10. *(${}^p K^b(\mathcal{C})^{\leq 0}, {}^p K^b(\mathcal{C})^{\geq 0}$) gives a non-degenerate t-structure on $K^b(\mathcal{C})$, the perverse t-structure on the homotopy category of Soergel bimodules.*

Proof. By definition ${}^p K^b(\mathcal{C})^{\geq 0}$ and ${}^p K^b(\mathcal{C})^{\leq 0}$ are full subcategories. They are both proper because we have for example $F_s[1] \notin {}^p K^b(\mathcal{C})^{\geq 0}$ and $F_s[-1] \notin {}^p K^b(\mathcal{C})^{\leq 0}$.

Set ${}^p K^b(\mathcal{C})^{\leq n} := {}^p K^b(\mathcal{C})^{\leq 0}[-n]$ and ${}^p K^b(\mathcal{C})^{\geq n} := {}^p K^b(\mathcal{C})^{\geq 0}[-n]$. Recall that an object $F \in K^b(\mathcal{C})$ lies in ${}^p K^b(\mathcal{C})^{\leq 0}$ if and only if for all $n \in \mathbb{Z}$ the shifts of the indecomposables occurring in cohomological degree n of a minimal complex isomorphic to F are bounded below by n . Therefore we see that a complex $F \in K^b(\mathcal{C})$ lies in ${}^p K^b(\mathcal{C})^{\leq 1}$ if and only if for all $n \in \mathbb{Z}$ the shifts of the indecomposables occurring in cohomological degree n of a minimal complex isomorphic to F are bounded below by $n-1$. This implies ${}^p K^b(\mathcal{C})^{\leq 0} \subseteq {}^p K^b(\mathcal{C})^{\leq 1}$.

Next, we want to show the Hom-vanishing condition. Let $X, Y \in K^b(\mathcal{C})$ be minimal complexes such that X lies in ${}^p K^b(\mathcal{C})^{\leq 0}$ and Y can be found in ${}^p K^b(\mathcal{C})^{\geq 1}$. Since $\mathrm{Hom}_{K^b(\mathcal{C})}(X, Y)$ is a quotient of $\mathrm{Hom}_{\mathcal{C}^b(\mathcal{C})}(X, Y)$ and a morphism f in $\mathrm{Hom}_{\mathcal{C}^b(\mathcal{C})}(X, Y)$ is a family of morphisms $\{f_n\}_{n \in \mathbb{Z}}$ such

that $f_n \in \text{Hom}_{\mathcal{C}}({}^n X, {}^n Y)$ and $d_n^Y \circ f_n = f_{n+1} \circ d_n^X$, it suffices to see that $\text{Hom}_{\mathcal{C}}({}^n X, {}^n Y)$ vanishes for all $n \in \mathbb{Z}$. This follows again from Soergel's Hom-formula (see Corollary 1.8) as for $n \in \mathbb{Z}$ the shifts of the indecomposables occurring in ${}^n X$ are bounded below by n and for the indecomposables in ${}^n Y$ they are bounded above by $n - 1$.

Since we have already noted the existence of the distinguished triangle of the required form prior to Definition 4.8, it remains to show the non-degeneracy of the perverse t-structure. This follows right away by considering a minimal complex and using the classification of the indecomposable Soergel bimodules from Theorem 1.6. \square

The following result implies that after applying F_s to a complex $F \in {}^p K^b(\mathcal{C})^{\geq 0}$, the negative perverse cohomology groups of FF_s remain zero (i.e. ${}^p \mathcal{H}^k(FF_s) = 0$ for all $k < 0$). It can be found in [EW14, Lemma 6.6].

Lemma 4.11. *F_s (resp. E_s) viewed as an additive endofunctor on $K^b(\mathcal{C})$ is left (resp. right) t-exact. In other words, if $F \in {}^p K^b(\mathcal{C})^{\geq 0}$ then $F_s F \in {}^p K^b(\mathcal{C})^{\geq 0}$.*

The left-handed multiplication formula from Eq. (3) from the proof of Lemma 2.1 immediately yields the following result:

Corollary 4.12. *Let $s \in S$ and $F \in {}^p K^b(\mathcal{C})^{\leq n}$ for some $n \in \mathbb{Z}$. Then $F_s F$ lies in ${}^p K^b(\mathcal{C})^{\leq n+1}$.*

5 Main results

For the rest of the section, assume that (W, S) is an irreducible Coxeter group of arbitrary type. Fix $s \in S$ as before. First we want to check that the action of $Br_{(W, S)}$ on $K^b(\mathcal{C}_s)$ behaves as expected:

Lemma 5.1. *In $K^b(\mathcal{C}_s)$ we have for $w \in C_s$ and $r \in S$:*

$$F_r(B_w) \cong \begin{cases} (B_w(-1) \rightarrow 0) & \text{if } \pi_s(w) = r, \\ \left(\bigoplus_{\substack{x \in C_s \text{ s.t.} \\ rx < x \text{ and} \\ \{x, w\} \in E(\Gamma_s)}} B_x \rightarrow B_w(1) \right) & \text{if } \pi_s(w) \neq r. \end{cases}$$

where in both complexes the term on the left sits in cohomological degree 0 and all maps between non-zero indecomposable Soergel bimodules are non-zero.

In particular, we have $F_r(B_w) \cong (0 \rightarrow B_w(1))$ if $\pi_s(w) \neq r$ and there is no edge between the vertices in $\pi_s^{-1}(r)$ and w in Γ_s .

Proof. By definition of F_r we have $F_r(B_w) = (B_r B_w \rightarrow B_w(1))$. Apply Lemma 2.1 to decompose $B_r B_w$ into indecomposable Soergel bimodules and note that all induced components of the differential are non-zero as $F_r(B_w)$ is indecomposable (F_r is an autoequivalence and B_w is indecomposable). In the case $\pi_s(w) = r$, apply a Gaussian elimination to the direct summand $(B_w(1) \xrightarrow{\cong} B_w(1))$ of $F_r(B_w)$ to obtain the result. \square

Comparing the results from Lemma 5.1 with the formula for $H_r \underline{H}_w$ obtained from Lemma 2.1 yields the following result:

Corollary 5.2. $K^b(\mathcal{C}_s)$ gives a categorification of $\mathbf{H}(\leq_L \mathcal{C}_s)/\mathbf{H}(\leq_L \mathcal{C}_s)$ viewed as $Br_{(W,S)}$ -module via the group homomorphism $\Psi : Br_{(W,S)} \rightarrow \mathbf{H}$, $r \mapsto H_r$. In other words, for $\sigma \in Br_{(W,S)}$ and $w \in \mathcal{C}_s$ the class $[F_\sigma(B_w)]$ coincides with $\Psi(\sigma) \cdot \underline{H}_w$ in $\mathbf{H}(\leq_L \mathcal{C}_s)/\mathbf{H}(\leq_L \mathcal{C}_s)$ under the isomorphism from Theorem 3.3.

Example 5.3. In type A_n the resulting representation of Br_{n+1} on the free $\mathbb{Z}[v, v^{-1}]$ -module $\mathfrak{R}^0(\mathcal{C}_s)$ gives a twisted, reduced Burau representation. Denote by σ_i the braid that passes the i -th strand under the $(i+1)$ -st strand and leaves the other strands untouched and let $S = \{s_i = (i, i+1) \mid 1 \leq i \leq n\}$ be the set of simple transpositions. By Lemma 1.2 the graph Γ_s is isomorphic to A_n for any $s \in S$. Denote by $[i]$ the unique element in \mathcal{C}_s with $\pi_s([i]) = s_i$.

As in [KS02], one may check that letting $\sigma_i \in Br_{n+1}$ act via the operator on $\mathfrak{R}^0(\mathcal{C}_s)$ induced by $F_{s_i}[-1](1)$ defines a representation isomorphic to the reduced Burau representation. More precisely, the action of σ_i in the $\mathbb{Z}[v, v^{-1}]$ -basis

$$(v[B_{[1]}], \dots, (-1)^{i-1}v^i[B_{[i]}], \dots, (-1)^n v^n[B_{[n]}])$$

of $\mathfrak{R}^0(\mathcal{C}_s)$ is given by the matrix:

$$\begin{pmatrix} 1 & & & & & & & & 0 \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & \boxed{\begin{matrix} 1 & 0 & 0 \\ v^{-2} & -v^{-2} & 1 \\ 0 & 0 & 1 \end{matrix}} & & & & & \\ & & & & 1 & & & & \\ 0 & & & & & & \ddots & & \\ & & & & & & & & 1 \end{pmatrix}$$

which we recognize as the transpose of the image of σ_i under the reduced Burau representation of Br_{n+1} (after substituting t for v^{-2}) as introduced in [KT08, chapter 3.3].

In [Big99] Bigelow shows that the Burau representation of Br_n is not faithful for $n \geq 5$. Thus it is remarkable that its categorification is faithful (see Theorem 5.18).

Definition 5.4. Let $F \in K^b(\mathcal{C}_s)$ be a perverse complex and $w \in W$. An indecomposable Soergel bimodule B_w is called an *anchor of F* if

$$\mathrm{Hom}_{K^b(\mathcal{C}_s)}(F, B_w(m)[-m])$$

is non-zero for some $m \in \mathbb{Z}$. We say that there is an anchor of F corresponding to $t \in S$ if $\mathrm{Hom}_{K^b(\mathcal{C}_s)}(F, \bigoplus_{x \in \pi_s^{-1}(t)} B_x(m)[-m])$ is non-zero for some $m \in \mathbb{Z}$.

We extend this important definition to non- perverse complexes as follows:

Definition 5.5. Let $0 \neq F \in K^b(\mathcal{C}_s)$ and $k \in \mathbb{Z}$ be maximal such that ${}^p\mathcal{H}^k(F)$ is non-zero. For $w \in W$ we say that B_w is an *anchor of F* if $\mathrm{Hom}_{K^b(\mathcal{C}_s)}(F, B_w(m-k)[-m])$ is non-zero for some $m \in \mathbb{Z}$.

The following result shows that being the anchor of a complex is equivalent to being the anchor of its highest non-zero perverse cohomology group.

Lemma 5.6. *Let $0 \neq F \in K^b(\mathcal{C}_s)$ and $k \in \mathbb{Z}$ be maximal such that ${}^p\mathcal{H}^k(F)$ is non-zero. Then the following statements are equivalent:*

- (i) B_w is an anchor of F .
- (ii) B_w is an anchor of ${}^p\mathcal{H}^k(F)$.

Proof. Consider the following distinguished triangle in $K^b(\mathcal{C}_s)$

$${}^p\tau_{\leq k-1}F \longrightarrow F \longrightarrow {}^p\tau_{\geq k}F \xrightarrow{[1]}$$

where the term on the right hand side is by assumption isomorphic to ${}^p\mathcal{H}^k(F)[-k]$. Applying $\mathrm{Hom}_{K^b(\mathcal{C}_s)}(-, B_w(m-k)[-m])$ and using the Hom-vanishing condition in the long exact sequence we obtain an isomorphism:

$$\mathrm{Hom}_{K^b(\mathcal{C}_s)}({}^p\mathcal{H}^k(F)[-k], B_w(m-k)[-m]) \cong \mathrm{Hom}_{K^b(\mathcal{C}_s)}(F, B_w(m-k)[-m])$$

□

Proposition 5.7. *Let $F \in K^b(\mathcal{C}_s)$ be a perverse complex. For $t \in S$ the following statements are equivalent:*

- (i) B_w is an anchor of F for some $w \in \pi_s^{-1}(t) \subseteq \mathcal{C}_s$.

(ii) There exists $m \in \mathbb{Z}$, $w \in \pi_s^{-1}(t)$ and a minimal complex F' isomorphic to F in $K^b(\mathcal{C}_s)$ such that $B_w(m)$ occurs as a direct summand in ${}^m F'$ and all components of the differential ending in this copy of $B_w(m)$ are 0.

(iii) ${}^p \mathcal{H}^1(F_t F) \neq 0$.

Proof. $i) \Rightarrow ii)$: Without loss of generality, we may assume that F itself is a minimal complex. Choose $m \in \mathbb{Z}$ such that $\text{Hom}_{K^b(\mathcal{C}_s)}(F, B_w(m)[-m])$ does not vanish. Decompose ${}^m F = B_w(m)^{\oplus k} \oplus C(m)$ for some perverse Soergel bimodule C in which B_w does not occur. Since F and $B_w(m)[-m]$ both are minimal perverse complexes, we have: $\text{Hom}_{K^b(\mathcal{C}_s)}(F, B_w(m)[-m]) = \text{Hom}_{C^b(\mathcal{C}_s)}(F, B_w(m)[-m])$. By assumption we thus have a map of chain complexes of the following form:

$$\begin{array}{ccccccc} \dots & \longrightarrow & {}^{m-1}F & \xrightarrow{d^{m-1}} & B_w(m)^{\oplus k} \oplus C(m) & \xrightarrow{d^{m+1}} & {}^{m+1}F & \longrightarrow & \dots \\ & & \downarrow & & \downarrow (f_1, f_2, \dots, f_k, 0) & & \downarrow & & \\ \dots & \longrightarrow & 0 & \longrightarrow & B_w(m) & \longrightarrow & 0 & \longrightarrow & \dots \end{array}$$

where $f_i = r_i \text{id}$ with $r_i \in \mathbb{R}$ for all $1 \leq i \leq k$ and at least one of them is non-zero. Up to automorphism of F we may assume that $r_1 = 1$ and $r_i = 0$ for all $2 \leq i \leq k$. Then the commutativity of the left square in the diagram above shows that all components of the differential ending in the first copy of $B_w(m)$ in ${}^m F$ vanish.

$ii) \Rightarrow iii)$: Let $F' \in K^b(\mathcal{C}_s)$ be a minimal complex as in ii). It suffices to show that ${}^p \mathcal{H}^1(F_t F')$ does not vanish. Let $m \in \mathbb{Z}$ and $w \in \pi_s^{-1}(t)$ be such that a copy of $B_w(m)$ occurs as summand in ${}^m F'$ with no non-zero incoming differential components. When tensoring with F_t this copy gives a summand $B_w(m-1)$ in ${}^m F_t F'$ which cannot be cancelled by any Gaussian eliminations by our assumptions (otherwise there would have to be a summand $B_w(m-1)$ in ${}^{m-1} F_t F'$ together with an isomorphism of the form $B_w(m-1) \xrightarrow{\iota} B_t({}^{m-1} F') \xrightarrow{1 \otimes d} B_t({}^m F') \xrightarrow{\text{pr}} B_w(m-1)$ which is impossible by assumption). This shows that ${}^p \mathcal{H}^1(F_t F')$ does not vanish.

$iii) \Rightarrow i)$: The only indecomposable Soergel bimodules B_x occurring up to perverse shift in ${}^p \mathcal{H}^1(F_t F)$ satisfy $\pi_s(x) = t$. Any indecomposable Soergel bimodule in the lowest non-zero cohomological degree of ${}^p \mathcal{H}^1(F_t F)$ is an anchor. Since ${}^p \mathcal{H}^1(F_t F)$ does not vanish, we can find $m \in \mathbb{Z}$, $w \in \pi_s^{-1}(t)$ and a non-zero morphism from ${}^p \mathcal{H}^1(F_t F)$ to $B_w(m)[-m]$ in $K^b(\mathcal{C}_s)$. As ${}^p \mathcal{H}^k(F_t F)$ vanishes for all $k > 1$, this non-zero morphism shows that $\text{Hom}_{K^b(\mathcal{C}_s)}(F_t F, B_w(m-1)[-m])$ is not zero. From the fact that E_t is right

adjoint to F_t it follows that $\text{Hom}_{K^b(\mathcal{C}_s)}(F, E_t B_{t_w}(m-1)[-m])$ does not vanish. The dual version of Lemma 5.1 shows that $E_t B_w(m-1)[-m]$ is isomorphic to $B_w(m)[-m]$ in $K^b(\mathcal{C}_s)$ and thus the claim follows. \square

As a consequence of Lemma 5.6 and the equivalence iii) \Leftrightarrow i) in the last result, we get:

Corollary 5.8. *Let $0 \neq F \in K^b(\mathcal{C}_s)$ be a complex, $t \in S$ and $k \in \mathbb{Z}$ maximal such that ${}^p\mathcal{H}^k(F) \neq 0$. Then the following statements are equivalent:*

- (i) B_w is an anchor of F for some $w \in \pi_s^{-1}(t)$.
- (ii) Let $k' \in \mathbb{Z}$ be maximal such that ${}^p\mathcal{H}^{k'}(F_t F) \neq 0$. Then k' equals $k+1$ and all the indecomposable Soergel bimodules occurring in ${}^p\mathcal{H}^{k+1}(F_t F)$ are indexed by an element in $\pi_s^{-1}(t)$.

If this is the case and (W, S) is of finite type, there exist $m_w^1, m_w^2, \dots, m_w^{l_w} \in \mathbb{Z}$ for $l_w \geq 1$ and $w \in \pi_s^{-1}(t)$ such that

$${}^p\mathcal{H}^{k+1}(F_t F) \cong \bigoplus_{w \in \pi_s^{-1}(t)} \bigoplus_{i=1}^{l_w} B_w(m_w^i)[-m_w^i] \neq 0$$

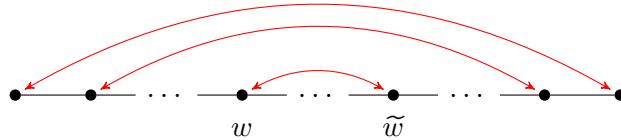
Proof. Only the second part of the corollary needs an explanation. For this note that by construction of Γ_s any two $x, y \in \mathcal{C}_s$ with $\pi_s(x) = t = \pi_s(y)$ are never neighboured in Γ_s and thus Lemma 2.8 shows that there do not exist any non-zero morphisms of degree one between B_x and B_y . It follows that ${}^p\mathcal{H}^{k+1}(F_t F)$ decomposes into a direct sum of perversely shifted copies of B_w for $w \in \pi_s^{-1}(t)$. \square

Finally we get:

Corollary 5.9. *Let $0 \neq F \in K^b(\mathcal{C}_s)$ be a complex. Let $k \in \mathbb{Z}$ (resp. $\tilde{k} \in \mathbb{Z}$) be maximal such that ${}^p\mathcal{H}^k(F) \neq 0$ (resp. ${}^p\mathcal{H}^{\tilde{k}}(F_t F) \neq 0$).*

If B_w is not an anchor of F for any $w \in \pi_s^{-1}(t)$, then $\tilde{k} = k$.

Let $q, r \in S$ with $m = m_{q,r} \geq 3$ and $w \in \pi_s^{-1}(r)$. For $1 \leq k \leq m$ denote by $\widehat{k}_r = \dots rqr$ the alternating word of length k in q and r having r in its right descent set. Similarly for \widehat{k}_q . The induced subgraph of Γ_s on $\pi_s^{-1}(\{q, r\})$ has a connected component Γ containing w which is a graph of type A_{m-1} . Let $w \mapsto \tilde{w} \in \mathcal{C}_s$ be the following graph automorphism of Γ :



Denote by $V(\Gamma)$ the vertex set of Γ . Note that all elements of $V(\Gamma)$ lie in the same left $\langle r, q \rangle$ -coset. Let $v \in C_s \cup \{1\}$ be the minimal element in this left $\langle r, q \rangle$ -coset in W . Let $t \in \{r, q\}$ be such that $tv > v$ and tv admits a unique reduced expression. Then $V(\Gamma)$ is in bijection with $\{k \in \mathbb{N} \mid 1 \leq k \leq m-1\}$ via $k \mapsto [k] := \widehat{k}_t v \in V$. Let w correspond to k . Then \tilde{w} corresponds to $m-k$. Denote by $d_1 = \min\{k-1, m-1-k\}$ and $d_2 = \max\{k-1, m-1-k\}$ the distances of w from the boundaries of Γ . Applying the graph automorphism swaps w and \tilde{w} and the values of d_1 and d_2 .

Lemma 5.10. *Then we have:*

$$F_{m-1_q}^- B_w \cong B_{\tilde{w}}(1)[-1]$$

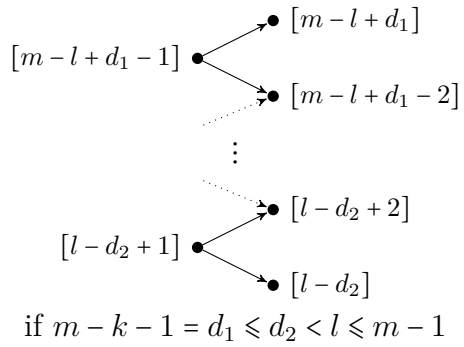
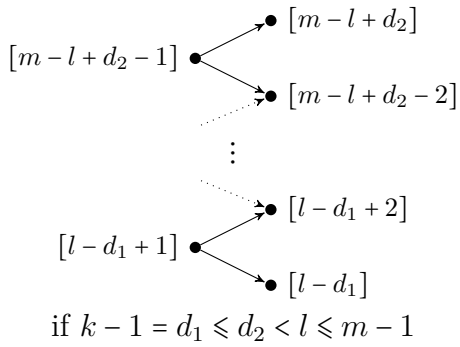
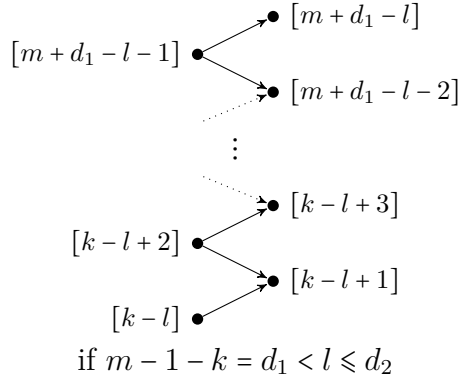
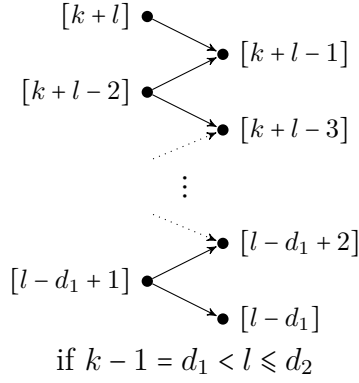
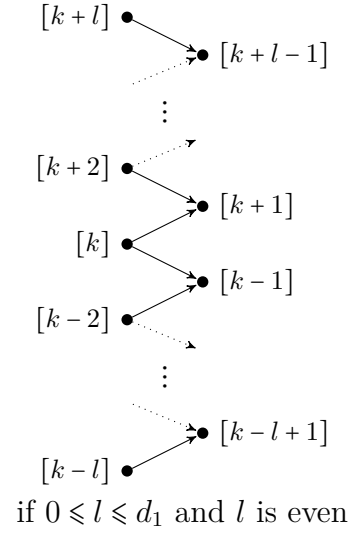
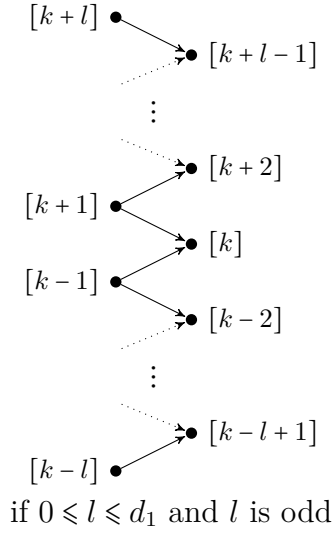
In particular, if m is odd, then $F_{m-1_q}^-$ turns an anchor corresponding to an element in $\pi_s^{-1}(r)$ into an anchor in $\pi_s^{-1}(q)$ and if m is even, then $F_{m-1_q}^-$ preserves setwise anchors corresponding to elements in $\pi_s^{-1}(r)$.

Proof. We will prove the statement by showing the following:

Claim. *In $K^b(\mathcal{C}_s)$ the complex $F_{l_q}^- B_w$ is isomorphic to a minimal complex corresponding to a graph in Table 1 where we denoted each indecomposable Soergel bimodule by its index. The graphs in Table 1 uniquely determine the isomorphism class of $F_{l_q}^- B_w$ in $K^b(\mathcal{C}_s)$ when placing the indecomposable Soergel bimodules in the left column of each graph without a grading shift in cohomological degree 0 and using that each arrow stands for a non-zero component of the differential given by the (up to scalar) unique morphism of degree one (see Lemma 2.4).*

Before proving the claim we want to make a few remarks. First note that as long as l satisfies $0 \leq l < d_1$ (resp. $l = d_1 \neq d_2$) the number of indecomposable Soergel bimodules occurring in the minimal complex increases by 2 (resp. 1) when l is increased by 1. For l in the range $d_1 < l < d_2$ (resp. $l = d_1 = d_2$) the number of indecomposable Soergel bimodules occurring in the minimal complex is constantly equal to $2d_1 + 2$ (resp. $2d_1 + 1 = m - 1$ in this case). As long as l satisfies $d_2 < l < m - 1$ (resp. $d_1 \neq d_2 = l$) the number of indecomposable Soergel bimodules occurring in the minimal complex decreases by 2 (resp. 1) when l is increased by 1. Thus for $l = m - 1$ there remains only one indecomposable Soergel bimodule in the right column of the graph (i.e. with grading shift (1) in the first cohomological degree of the corresponding minimal complex) (as $m - 1 - d_2 - 1 = d_1$) and it corresponds to \tilde{w} . Therefore the claim implies the statement of the lemma.

The reader should think of these graphs as describing a one-dimensional wave oscillating in a bounded region (i.e. the graph Γ) at discrete time values

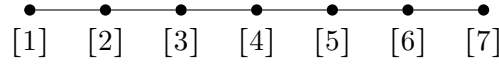
Table 1: Graphs describing $F_{l_q}^* B_w$

$t = 0, \dots, m-1$. At $t = 0$ the wave is stimulated in a point (i.e. the vertex B_w of Γ). From $t = n$ to $t = n+1$ all wave crests transform into wave troughs and vice versa. The evolution of the wave can be divided into three periods: First for $0 \leq t \leq d_1$ the wave front travels in both directions and thus the agitated region grows until the first wave front hits the nearest boundary where it gets reflected. Then for $d_1 < t \leq d_2$ the resulting wave packet propagates towards the other boundary where the wave front again gets reflected in such a way that the superposition of the resulting wave fronts leads to extinction. Finally for $d_2 < t \leq m-1$ the agitated region shrinks and the wave eventually vanishes. Each period corresponds to one row in the above table.

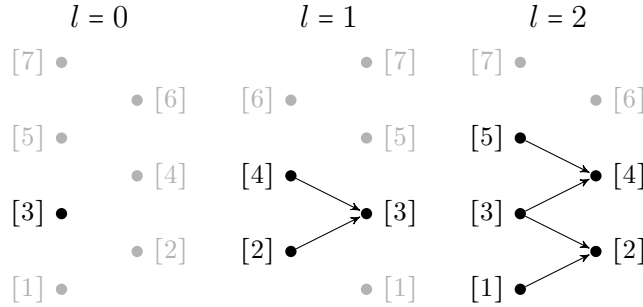
Finally it should be noted that for $0 < l < m-1$ all the sources (resp. sinks) in the minimal complex describing $F_{\widehat{\Gamma}_q} B_w$ are mapped under π_s to the unique element in $\mathcal{L}(\widehat{l}_q)$ (resp. $\{r, q\} \setminus \mathcal{L}(\widehat{l}_q) = \mathcal{L}(\widehat{l+1}_q)$).

The claim is proven by induction on l and explicit calculation. For $l = 0$ nothing has to be checked and the case $l = 1$ follows from Lemma 5.1. \square

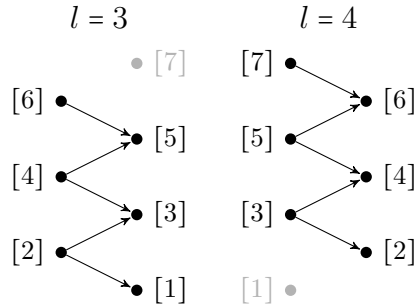
Example 5.11. We want to illustrate the claim of the last proof in the case $m_{q,r} = 8$. We get for $\Gamma = \pi_s^{-1}(\{q, r\})$



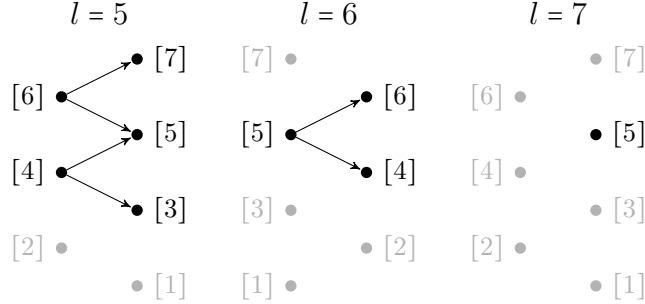
where $\pi_s^{-1}(r) = \{[1], [3], [5], [7]\}$ and $\pi_s^{-1}(q) = \{[2], [4], [6]\}$. For $k = 3$ we have the following graphs for $F_{\widehat{\Gamma}_q} B_{[k]}$ in the first phase:



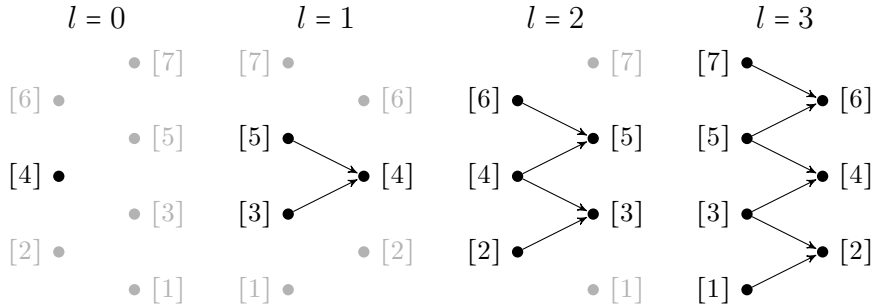
The second phase looks like:



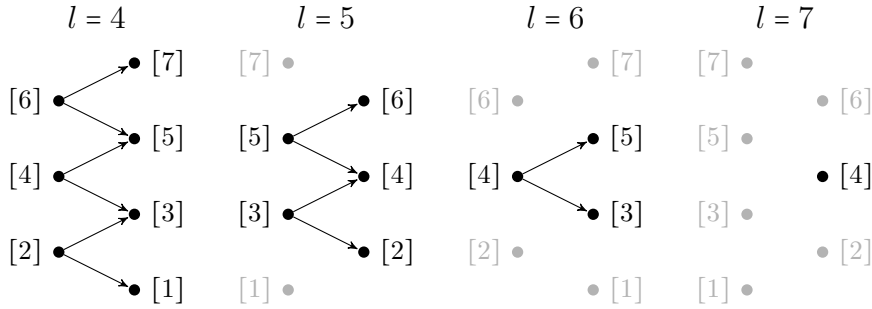
And in the last phase we get:



For $k = 4$ the following graphs describe $F_{\tilde{r}} B_{[k]}$ in the first phase:



The second phase completely collapses and in the third phase we have:



The main goal of this section is to prove the following theorem:

Theorem 5.12. *Let $\sigma = w_m w_{m-1} \dots w_1 \in Br_{(W,S)}^+$ be a non-trivial braid in Garside normal form. Set $B = \bigoplus_{w \in C_s} B_w$. Then the following holds:*

- (i) *If k is maximal such that ${}^p\mathcal{H}^k(F_\sigma(B))$ is non-zero, then $k = m$.*
- (ii) *Let C_s^σ be the set of Coxeter elements in C_s indexing an anchor in ${}^p\mathcal{H}^m(F_\sigma(B))$. Then π_s induces a surjection from C_s^σ onto $\mathcal{L}(w_m)$.*

If in addition (W, S) is a Coxeter group of simply-laced type whose Coxeter graph is a tree, π_s gives a bijection between C_s^σ and $\mathcal{L}(w_m)$.

Proof. To simplify notation, we will identify W with the set of reduced braids $\varphi(w) \subset Br_{(W,S)}^+$. We will state explicitly if a certain identity only holds in the Coxeter group.

We will prove by simultaneous induction on m and on $l(w_m)$ the following three statements from which the theorem follows:

- P) ${}^p\mathcal{H}^k(F_\sigma(B)) = 0$ for $k > m$.
- L) For all $t \in \mathcal{L}(w_m)$ there exists an element $w \in \pi_s^{-1}(t)$ such that the indecomposable Soergel bimodule B_w is an anchor of $F_\sigma(B)$.
- A) Any anchor B_w of $F_\sigma(B)$ satisfies $\pi_s(w) \in \mathcal{L}(w_m)$.

Note that under the additional assumptions that (W, S) is a Coxeter group of simply-laced type whose Coxeter graph is a tree, π_s is injective and thus the last statement of the theorem follows from L).

The proof strategy actually is as follows: We apply induction on m ; in the base case as well as in the inductive step we apply induction on $l(w_m)$. Since the inductive step for the induction on $l(w_m)$ does not depend on whether we are in the base case or the inductive step for the induction on m , we treat these cases together. Introduce the following notation for $x \in \{P, L, A\}$:

$X(n, l)$: Statement x) holds for all $\beta \in Br^+$ with Garside length $\leq n$ and length of the final Garside factor $\leq l$.

$X(n, \infty)$: $X(n, l)$ for all $l \in \mathbb{N}$.

In the case $m = 1$ and $w_m = t \in S$ all statements above follow immediately from Lemma 5.1. In other words: Lemma 5.1 $\Rightarrow P(1, 1), L(1, 1), A(1, 1)$

Suppose by induction that the statements above hold for any positive braid with fewer Garside factors than σ or with m Garside factors and shorter final (i.e. w_m) Garside factor than σ .

For simplicity of notation set $\beta = w_{m-1}w_{m-2} \dots w_1$. Since being a Garside normal form can be checked locally (see Proposition 1.10), β is in Garside normal form.

First, we prove P):

Lemma 5.13. $m > 1 + P(m - 1, \infty) \implies P(m, 1)$

Proof. Write $w_m = r \in S$. By induction on the number of Garside factors we know that ${}^p\mathcal{H}^k(F_\beta(B)) = 0$ for $k > m - 1$. Applying F_r , the left handed multiplication formula from Eq. (3) implies that the grading of any indecomposable Soergel bimodule gets shifted up at most by 1 (compare with Corollary 4.12). It follows that ${}^p\mathcal{H}^k(F_\sigma(B)) = 0$ for $k > m$. \square

Lemma 5.14. $l(w_m) > 1 + P(m, l(w_m) - 1) + A(m, l(w_m) - 1) \implies P(m, l(w_m))$

Proof. Write $w_m = tu$ with $t \in S$ and $l(u) = l(w_m) - 1 \geq 1$. By induction on the length of the final Garside factor, we know that the maximal k such that ${}^p\mathcal{H}^k(F_{u\beta}(B)) \neq 0$ satisfies $k \leq m$. Since $tu > u$ in the Coxeter group, it follows that $t \notin \mathcal{L}(u)$. Again by induction on the length of the final Garside factor, we can apply A) to conclude that no element in $\pi_s^{-1}(t)$ indexes an anchor of ${}^p\mathcal{H}^k(F_{u\beta}(B))$. Corollary 5.9 shows that ${}^p\mathcal{H}^{k+1}(F_\sigma(B)) \cong {}^p\mathcal{H}^1(F_t {}^p\mathcal{H}^k(F_{u\beta}(B)))$ vanishes. \square

Next, we will consider L) and A) and prove both $L(m, 1)$ and $A(m, 1)$ at the same time.

Lemma 5.15. $m > 1 + P(m - 1, \infty) + L(m - 1, \infty) \implies L(m, 1) + A(m, 1)$

Proof. Write $w_m = t \in S$. Lemma 1.11 shows that $\mathcal{R}(w_m) \subseteq \mathcal{L}(w_{m-1})$. Thus t lies in $\mathcal{L}(w_{m-1})$. By induction on the number of Garside factors, we may apply L) to see that there exists $w \in \pi_s^{-1}(t)$ such that B_w is an anchor of $F_\beta(B)$. Let $k \in \mathbb{Z}$ be maximal such that ${}^p\mathcal{H}^k(F_\beta(B))$ is non-zero. In addition, induction on the number of Garside factors applied to P) implies that $k \leq m - 1$. Corollary 5.8 shows that ${}^p\mathcal{H}^{k+1}(F_\sigma(B))$ is non-zero and that the indecomposable Soergel bimodules occurring in ${}^p\mathcal{H}^{k+1}(F_\sigma(B))$ are indexed by elements in $\pi_s^{-1}(t)$. Since t is the only element in $\mathcal{L}(w_m)$, L) and A) hold in this case. \square

Then we show the inductive step for A in the induction on $l(w_m)$.

Lemma 5.16. $l(w_m) > 1 + A(m, l(w_m) - 1) \implies A(m, l(w_m))$.

Proof. We show A) by showing its contrapositive. Fix $r \in S \setminus \mathcal{L}(w_m)$. Let $w \in \pi_s^{-1}(r)$ be arbitrary. We want to show that B_w is not an anchor of $F_\sigma(B)$. Write $w_m = tz$ with $t \in S$ and $l(z) = l(w_m) - 1 \geq 1$ in the Coxeter group. It follows that $t \neq r$. As before for $1 \leq k \leq m_{r,t}$ denote by ${}_t\widehat{k} = trt\dots$ (resp. $\widehat{k}_t = \dots trt$) the alternating word in r and t of length k having t in its left (resp. right) descent set. Similarly for ${}_r\widehat{k}$ and \widehat{k}_r .

Case 1: $m_{r,t} = 2$ (i.e. $rt = tr$). This implies that r does not lie in the left descent set of z . Let k be maximal such that ${}^p\mathcal{H}^k(F_{z\beta}(B))$ is not zero. By induction on the length of the final Garside factor, it follows that for all $v \in \pi_s^{-1}(\{r, t\})$ the indecomposable Soergel bimodule B_v is not an anchor of ${}^p\mathcal{H}^k(F_{z\beta}(B))$. Therefore

$$\mathrm{Hom}_{K^b(\mathcal{C}_s)}(F_{z\beta}(B), B_w(n - k)[-n])$$

vanishes for all $n \in \mathbb{Z}$. As there are no edges between $\pi_s^{-1}(t)$ and w in Γ_s Lemma 5.1 implies that $F_t B_w$ is isomorphic to $B_w(1)[-1]$. Applying the autoequivalence F_t to both terms of the Hom-space above implies that

$$\mathrm{Hom}_{K^b(\mathcal{C}_s)}(F_\sigma(B), B_w(n-k)[-n])$$

still vanishes for all $n \in \mathbb{Z}$. Using Corollary 5.9 for t shows that ${}^p\mathcal{H}^k(F_\sigma(B))$ is still the highest non-zero perverse cohomology group. Therefore we see that B_w is not an anchor of $F_\sigma(B)$.

In the following three cases we have $l < m_{r,t}$ due to $r \notin \mathcal{L}(w_m)$.

Case 2: $m_{r,t} \geq 3$, $w_m = {}_t\widehat{t}u$ with u minimal in its left $\langle r, t \rangle$ -coset, $1 \leq l < m_{r,t} - 1$ and $l(u) \geq 1$. Then r and t are not in $\mathcal{L}(u)$ and thus by induction on the length of the final Garside factor, for all $v \in \pi_s^{-1}(\{r, t\})$ the indecomposable Soergel bimodule B_v is not an anchor in $F_{u\beta}(B)$. Let k be maximal such that ${}^p\mathcal{H}^k(F_{u\beta}(B))$ is non-zero. Therefore

$$\mathrm{Hom}_{K^b(\mathcal{C}_s)}(F_{u\beta}(B), \bigoplus_{v \in \pi_s^{-1}(\{r, t\})} B_v(n-k)[-n])$$

vanishes for all $n \in \mathbb{Z}$. Applying the autoequivalence $F_{\widehat{t}}$ shows that

$$\mathrm{Hom}_{K^b(\mathcal{C}_s)}(F_\sigma(B), \bigoplus_{v \in \pi_s^{-1}(\{r, t\})} F_{\widehat{t}} B_v(n-k)[-n])$$

still vanishes for all $n \in \mathbb{Z}$. From the claim in the proof of Lemma 5.10 we know what each summand in

$$\bigoplus_{v \in \pi_s^{-1}(\{r, t\})} F_{\widehat{t}} B_v(n-k)[-n]$$

looks like and that we may choose $\tilde{w} \in \pi_s^{-1}(\{r, t\}) \subseteq C_s$ such that $B_w(1)[-1]$ occurs as summand in the first cohomological degree of $F_{\widehat{t}} B_{\tilde{w}}$. In particular there exist sets $I_t \subseteq \pi_s^{-1}(t)$ and $w \notin I_r \subseteq \pi_s^{-1}(r)$ such that $F_{\widehat{t}} B_{\tilde{w}}$ is isomorphic to a perverse two-term minimal complex of the following form:

$$\dots \longrightarrow 0 \longrightarrow \bigoplus_{v \in I_t} B_v \longrightarrow B_w(1) \oplus \bigoplus_{v \in I_r} B_v(1) \longrightarrow 0 \longrightarrow \dots$$

It follows that B_w cannot be an anchor of $F_\sigma(B)$ as any non-zero map from $F_\sigma(B)$ to $B_w(n-k)[-n]$ for some $n \in \mathbb{Z}$ would induce a non-zero map to $F_{\widehat{t}} B_{\tilde{w}}$ which is a direct summand of

$$\bigoplus_{v \in \pi_s^{-1}(\{r, t\})} F_{\widehat{t}} B_v(n-1-k)[-n+1]$$

by post-composition with the class of the following non-zero map in $K^b(\mathcal{C}_s)$:

$$\begin{array}{ccccccc} \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & B_w(n-k) & \longrightarrow & 0 & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow \left(\begin{array}{c} \text{id} \\ 0 \\ \vdots \\ 0 \end{array} \right) & & \downarrow & & \\ \dots & \longrightarrow & 0 & \longrightarrow & \bigoplus_{v \in I_t} B_v(n-1-k) & \longrightarrow & B_w(n-k) \oplus \bigoplus_{v \in I_r} B_v(n-k) & \longrightarrow & 0 & \longrightarrow & \dots \end{array}$$

where $B_w(n-k)$ sits in cohomological degree n . Indeed, first use repeatedly Corollary 5.9 to see that ${}^p\mathcal{H}^k(F_\sigma B)$ still is the highest non-zero perverse cohomology group of $F_\sigma B$. Then note that the composition of any representative of the non-zero map $F_\sigma(B) \rightarrow B_w(n-k)[-n]$ in $K^b(\mathcal{C}_s)$ with the morphism of cochain complexes pictured above is non-zero in $C^b(\mathcal{C}_s)$. Finally, use the following chain of isomorphisms

$$\begin{aligned} & \text{Hom}_{C^b(\mathcal{C}_s)}({}^p\mathcal{H}^k(F_\sigma B), \bigoplus_{v \in \pi_s^{-1}(\{r,t\})} F_{\widehat{t}} B_v(n-1)[-n+1]) \\ & \cong \text{Hom}_{K^b(\mathcal{C}_s)}({}^p\mathcal{H}^k(F_\sigma B), \bigoplus_{v \in \pi_s^{-1}(\{r,t\})} F_{\widehat{t}} B_v(n-1)[-n+1]) \\ & \cong \text{Hom}_{K^b(\mathcal{C}_s)}(F_\sigma(B), \bigoplus_{v \in \pi_s^{-1}(\{r,t\})} F_{\widehat{t}} B_v(n-1)[-n+1-k]) \end{aligned}$$

where we used the isomorphism from the proof of Lemma 5.6 in the second step and in the first step we assumed that ${}^p\mathcal{H}^k(F_\sigma B)$ is a minimal complex and thus by Soergel's conjecture all homotopies between minimal perverse complexes vanish. We conclude that B_w is not an anchor of $F_\sigma(B)$.

Case 3: $m_{r,t} \geq 3$, $w_m = \widehat{t}$ with $1 < l \leq m_{r,t} - 1$. For some $q \in \{r, t\}$ we have $\widehat{t} = \widehat{q}$. Let $q' \in \{r, t\} \setminus \{q\}$. Let k be maximal such that ${}^p\mathcal{H}^k(F_{q\beta}(B))$ is non-zero. From the case $w_m = q$ and $m \geq 1$ we know that all indecomposable Soergel bimodules occurring in ${}^p\mathcal{H}^k(F_{q\beta}(B))$ are indexed by elements in $\pi_s^{-1}(q)$.

By Lemma 5.10 there exists $\widetilde{w} \in \pi_s^{-1}(\{r, t\})$ such that $F_{\widehat{t}(\overline{m_{r,t}-1})} B_{\widetilde{w}}$ is isomorphic to $B_w(1)[-1]$. Note that $F_{\widehat{q}(\overline{m_{r,t}-l})} B_{\widetilde{w}}$ has sources (resp. sinks) indexed by elements in $\pi_s^{-1}(q)$ (resp. $\pi_s^{-1}(q')$) and since each source has non-zero outgoing components of the differential we get (for degree reasons):

$$\text{Hom}_{C^b(\mathcal{C}_s)}({}^p\mathcal{H}^k(F_{q\beta}(B)), F_{\widehat{q}(\overline{m_{r,t}-l})} B_{\widetilde{w}}(n-1-k)[-n+1+k]) = 0 \quad \forall n \in \mathbb{Z}$$

Now use the following chain of isomorphisms

$$\begin{aligned} & \text{Hom}_{C^b(\mathcal{C}_s)}({}^p\mathcal{H}^k(F_{q\beta}(B)), F_{\widehat{q}(\overline{m_{r,t}-l})} B_{\widetilde{w}}(n-1-k)[-n+1+k]) \\ & \cong \text{Hom}_{K^b(\mathcal{C}_s)}({}^p\mathcal{H}^k(F_{q\beta}(B)), F_{\widehat{q}(\overline{m_{r,t}-l})} B_{\widetilde{w}}(n-1-k)[-n+1+k]) \\ & \cong \text{Hom}_{K^b(\mathcal{C}_s)}(F_{q\beta}(B), F_{\widehat{q}(\overline{m_{r,t}-l})} B_{\widetilde{w}}(n-1-k)[-n+1]) \\ & \cong \text{Hom}_{K^b(\mathcal{C}_s)}(F_\sigma(B), F_{\widehat{t}(\overline{m_{r,t}-1})} B_{\widetilde{w}}(n-1-k)[-n+1]) \end{aligned}$$

$$\cong \mathrm{Hom}_{K^b(\mathcal{C}_s)}(F_\sigma(B), B_w(n-k)[-n])$$

where for the first step one should note that all homotopies between minimal perverse complexes are 0, in the second step the isomorphism from the proof of Lemma 5.6 is used, in the third step the autoequivalence $F_{\widehat{t^{-1}}}$ is applied and finally the choice of \tilde{w} comes into play.

This shows that B_w cannot be an anchor of $F_\sigma(B)$ as induction applied to A) and Corollary 5.9 yield that ${}^p\mathcal{H}^k(F_\sigma(B))$ is still the highest non-zero perverse cohomology group.

Case 4: $m_{r,t} \geq 3$, $w_m = \widehat{(\overline{m_{r,t} - 1})}u$ with u minimal in its left (r, t) -coset and $l(u) \geq 1$. Then r and t are not in $\mathcal{L}(u)$ and thus by induction on the length of the final Garside factor, for all $v \in \pi_s^{-1}(\{r, t\})$ the indecomposable Soergel bimodule B_v is not an anchor in $F_{u\beta}(B)$. By Lemma 5.10 there exists $\tilde{w} \in \pi_s^{-1}(\{r, t\})$ such that $F_{\widehat{(\overline{m_{r,t} - 1})}}B_{\tilde{w}}$ is isomorphic to $B_w(1)[-1]$. Since

$$\mathrm{Hom}_{K^b(\mathcal{C}_s)}(F_{u\beta}(B), B_{\tilde{w}}(n-k)[-n])$$

vanishes for all $n \in \mathbb{Z}$, we may apply the autoequivalence $F_{\widehat{(\overline{m_{r,t} - 1})}}$ to deduce that

$$\mathrm{Hom}_{K^b(\mathcal{C}_s)}(F_\sigma(B), B_w(n-k)[-n])$$

is zero for all $n \in \mathbb{Z}$. Induction on the length of the Garside factor together with Corollary 5.9 imply that ${}^p\mathcal{H}^k(F_\sigma(B))$ is still the highest non-zero perverse cohomology group of $F_\sigma(B)$. Therefore B_w is not an anchor of $F_\sigma(B)$. \square

Finally, we prove L):

Lemma 5.17. $l(w_m) > 1 + L(m, l(w_m) - 1) + Am, l(w_m) \implies L(m, l(w_m))$

Proof. Fix $r \in \mathcal{L}(w_m)$ and write $w_m = rtz$ with $t \in S$, $l(z) = l(w_m) - 2$ and z possibly trivial. Observe that $r \neq t$. We want to show that B_w is an anchor in $F_\sigma(B)$ for some $w \in \pi_s^{-1}(r)$.

Case 1: $m_{r,t} = 2$ (i.e. $rt = tr$). By induction on the length of the final Garside factor, B_w is an anchor of $F_{rz\beta}(B)$ for some $w \in \pi_s^{-1}(r)$. Let k be maximal such that ${}^p\mathcal{H}^k(F_{rz\beta}(B))$ is non-zero. By definition of an anchor, there exists some $n \in \mathbb{Z}$ such that

$$\mathrm{Hom}_{K^b(\mathcal{C}_s)}(F_{rz\beta}(B), B_w(n-k)[-n])$$

does not vanish. Applying the autoequivalence F_t and using that $F_t B_w$ is isomorphic to $B_w(1)[-1]$ by Lemma 5.1 as there are no edges between $\pi_s^{-1}(t)$ and w in Γ_s , it follows that

$$\mathrm{Hom}_{K^b(\mathcal{C}_s)}(F_\sigma(B), B_w(n+1-k)[-n-1])$$

is non-zero. By induction on the length of the Garside factor applied to A, we know that B_t is not an anchor of $F_{rz\beta}(B)$. Therefore, Corollary 5.9 shows that a new non-zero highest perverse cohomology group is not created when applying F_t and thus B_w is an anchor of $F_\sigma(B)$.

Case 2: $m_{r,t} \geq 3$, $w_m = {}_r\widehat{lu}$ with u minimal in its left $\langle r, t \rangle$ -coset, possibly trivial and $2 \leq l \leq m_{r,t} - 1$. Let $q \in \{r, t\}$ be such that ${}_r\widehat{l} = \widehat{l}_q$ and $q' \in \{r, t\} \setminus \{q\}$. By induction on the length of the final Garside factor, there exists some $\widetilde{w} \in \pi_s^{-1}(x)$ such that $B_{\widetilde{w}}$ is an anchor of $F_{qu\beta}(B)$. Let k be maximal such that ${}^p\mathcal{H}^k(F_{qu\beta}(B))$ is non-zero. By definition of an anchor, there exists some $n \in \mathbb{Z}$ such that

$$\mathrm{Hom}_{K^b(\mathcal{C}_s)}(F_{qu\beta}(B), B_{\widetilde{w}}(n-k)[-n])$$

does not vanish. Applying the autoequivalence $F_{(\overline{l-1})_{q'}}$ shows:

$$\mathrm{Hom}_{K^b(\mathcal{C}_s)}(F_\sigma(B), F_{(\overline{l-1})_{q'}} B_{\widetilde{w}}(n-k)[-n]) \neq 0$$

The claim in the proof of Lemma 5.10 shows that there exist sets $I_t \subseteq \pi_s^{-1}(t)$ and $I_r \subseteq \pi_s^{-1}(r)$ such that $F_{(\overline{l-1})_{q'}} B_{\widetilde{w}}$ is isomorphic to a perverse two-term minimal complex of the following form:

$$\dots \longrightarrow 0 \longrightarrow \bigoplus_{v \in I_r} B_v \longrightarrow \bigoplus_{v \in I_t} B_v(1) \longrightarrow 0 \longrightarrow \dots$$

where $\bigoplus_{v \in I_r} B_v$ sits in cohomological degree 0. Since u is minimal in its left $\langle r, t \rangle$ -coset and $l < m_{r,t}$, it follows that $t \notin \mathcal{L}(w_m)$. By A we know that there is no anchor of $F_\sigma(B)$ corresponding to t . Therefore a non-zero map from $F_\sigma(B)$ to $F_{(\overline{l-1})_{q'}} B_{\widetilde{w}}$ must have a non-zero component to a summand B_w for $w \in I_r$ occurring in cohomological degree 0. Thus post-composition with the non-zero map

$$\begin{array}{ccccccc} \dots & \longrightarrow & 0 & \longrightarrow & B_w(n-k) \oplus \bigoplus_{v \in I_r \setminus \{w\}} B_v(n-k) & \longrightarrow & \bigoplus_{v \in I_t} B_v(n+1-k) \longrightarrow 0 \longrightarrow \dots \\ & & \downarrow & & \downarrow (\mathrm{id}, 0, \dots, 0) & & \downarrow & & \downarrow \\ \dots & \longrightarrow & 0 & \longrightarrow & B_w(n-k) & \longrightarrow & 0 & \longrightarrow & 0 \longrightarrow \dots \end{array}$$

in $K^b(\mathcal{C}_s)$ shows that $\mathrm{Hom}_{K^b(\mathcal{C}_s)}(F_\sigma(B), B_w(n-k)[-n])$ does not vanish. As before, we conclude that B_w is an anchor of $F_\sigma(B)$.

Case 3: $m_{r,t} \geq 3$, $w_m = {}_r(\widehat{m_{r,t}})u$ with u minimal in its left $\langle r, t \rangle$ -coset and possibly trivial. Let $q \in \{r, t\}$ be such that ${}_r(\widehat{m_{r,t}}) = (\widehat{m_{r,t}})_q$ and $q' \in \{r, t\} \setminus \{q\}$. We have $w_m = {}_r(\widehat{m_{r,t}})_q u = {}_t(\widehat{m_{r,t}})_{q'} u$. By induction on the length of the final Garside factor, we know that there exists $\widetilde{w} \in \pi_s^{-1}(q')$ such that $B_{\widetilde{w}}$ is an anchor of $F_{q'u\beta}(B)$. Let k be maximal such that ${}^p\mathcal{H}^k(F_{q'u\beta}(B))$ is non-zero. By definition of an anchor, there exists some $n \in \mathbb{Z}$ such that

$$\mathrm{Hom}_{K^b(\mathcal{C}_s)}(F_{q'u\beta}(B), B_{\widetilde{w}}(n-k)[-n])$$

does not vanish. Since $F_{(\overline{m_{r,t}-1})_q} B_{\tilde{w}}$ is isomorphic to $B_w(1)[-1]$ for some $w \in \pi_s^{-1}(r)$ by Lemma 5.10 (due to $(\overline{m_{r,t}-1})_q = {}_t(\overline{m_{r,t}-1})$) we can apply the autoequivalence $F_{(\overline{m_{r,t}-1})_q}$ to get a non-zero map from $F_\sigma(B)$ to $B_w(n+1-k)[-n-1]$. Induction applied to A together with Corollary 5.9 show that ${}^p\mathcal{H}^k(F_\sigma(B))$ is still the highest non-zero perverse cohomology group of $F_\sigma(B)$. Therefore B_w is an anchor of $F_\sigma(B)$. \square

This concludes the proof of the main theorem. \square

From this we will easily deduce the faithfulness of the action in finite type using Lemma 1.12:

Theorem 5.18. *Assume that (W, S) is a Coxeter system of finite type. The action of $Br_{(W,S)}$ on $K^b(\mathcal{C}_s)$ is faithful.*

Proof. Let $\rho : Br_{(W,S)} \rightarrow Iso(Aut(K^b(\mathcal{C}_s)))$ be the group homomorphism corresponding to the action of $Br_{(W,S)}$ on $K^b(\mathcal{C}_s)$ where $Iso(Aut(K^b(\mathcal{C}_s)))$ is the group of isomorphism classes of autoequivalences on $K^b(\mathcal{C}_s)$. Due to Lemma 1.12 it is enough to show the injectivity for its restriction ρ^+ to the braid monoid $Br^+(W, S)$. We will show that for any positive braid $\sigma \in Br^+(W, S)$ its Garside normal form and thus σ itself can fully be recovered from the action of σ and its subwords on $K^b(\mathcal{C}_s)$. Set $B := \bigoplus_{w \in \mathcal{C}_s} B_w$.

Consider $F_\sigma(B)$ and its highest non-zero perverse cohomology group ${}^p\mathcal{H}^m(F_\sigma(B))$. Theorem 5.12 implies that the number of Garside factors of σ is m . Let $\sigma = w_m w_{m-1} \dots w_1$ be the Garside normal form of σ . To simplify notation, set $\beta = w_{m-1} \dots w_1$.

To determine a reduced expression of w_m we will proceed as follows: Let B_w be an anchor in ${}^p\mathcal{H}^m(F_\sigma(B))$. Theorem 5.12 shows that $\pi_s(w) = s_1$ lies in $\mathcal{L}(w_m)$. Write $w_m = s_1 u$ with $l(u) < l(w_m)$. Since the Rouquier complexes satisfy up to isomorphism the braid relations, we know: $E_{s_1} F_\sigma \cong F_{u\beta}$. Now consider the action of $F_{u\beta}$ on B . If ${}^p\mathcal{H}^m(F_{u\beta}(B))$ is zero, then $u = 1$, $u\beta$ has a Garside normal form with $m-1$ Garside factors and $w_m = s_1$. Otherwise we repeat the argument above to find an element in the left descent set of u . After a finite number of steps we have reconstructed a reduced expression $w_m = s_1 s_2 \dots s_k$.

By repeating the whole process we will eventually find all Garside factors of σ and we know their order. Thus we have determined σ itself. \square

The following result is an immediate consequence as the faithful action from Theorem 5.18 factors over the 2-braid group.

Corollary 5.19 (Faithfulness of the 2-braid group in finite type).

Let (W, S) be a Coxeter group of finite type. For two distinct braids $\sigma \neq \beta \in$

$Br(W, S)$ the corresponding Rouquier complexes F_σ and F_β in the 2-braid group are non-isomorphic.

Since in arbitrary type we can still differentiate the images of different positive braids in $Br(W, S)$, we get the following known result (the main result of [Par02, Theorem 1.1]):

Corollary 5.20 (Injectivity of the canonical map).

Let (W, S) be a Coxeter group of arbitrary type. The canonical morphism $Br^+(W, S) \rightarrow Br(W, S)$ is injective.

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