TOWARDS EFFECTIVE DETECTION OF THE BIFURCATION LOCUS OF REAL POLYNOMIAL MAPS

LUIS RENATO G. DIAS, SUSUMU TANABÉ, AND MIHAI TIBĂR

Abstract. We answer to a problem raised by recent work of Jelonek and Kurdyka: how can one detect by rational arcs the bifurcation locus of a polynomial map $\mathbb{R}^n \to \mathbb{R}^p$ in case $p > 1$. We describe an effective estimation of the “nontrivial” part of the bifurcation locus.

1. Introduction

The *bifurcation locus* of a polynomial map $f : \mathbb{R}^n \to \mathbb{R}^p$, $n \geq p$, is the smallest subset $B(f) \subset \mathbb{R}^p$ such that f is a locally trivial C^∞-fibration over $\mathbb{R}^p \setminus B(f)$. It is well known that $B(f)$ is the union of the set of critical values $f(\mathrm{Sing} f)$ and the set of *bifurcation values at infinity* $\mathcal{B}_\infty(f)$ (see Definition 2.1) which may be non-empty and disjoint from $f(\mathrm{Sing} f)$ even in very simple examples. Finding the bifurcation locus in the cases $p > 1$ or $p = 1$ and $n > 2$ is yet an unreached ideal. Nevertheless one can obtain approximations by supersets of $\mathcal{B}_\infty(f)$ from exploiting asymptotical regularity conditions [23], [19], [21], [9], [24], [16], [10], [6], [2], [13], [18], [15] etc.

Improving the effectivity of the detection of asymptotically non-regular values becomes an important issue, for instance it leads to applications in optimisation problems [11], [22]. Along this trend, Jelonek and Kurdyka [14] produced recently an algorithm for finding the set of *asymptotically critical values* $\mathcal{K}_\infty(f)$ in case $p = 1$. It is known that in this case $\mathcal{K}_\infty(f)$ is finite and includes $\mathcal{B}_\infty(f)$. A sharper estimation of $\mathcal{B}_\infty(f)$ has been found in the real setting [7] by approximating the set of *asymptotic ρ_a-nonregular values* of f. The later method provides a finite set of values $A(f)$ with the following property: $\mathcal{B}_\infty(f) \subset A(f) \subset \mathcal{K}_\infty(f)$.

In case $p > 1$ the bifurcation locus $\mathcal{B}_\infty(f)$ may be no more finite. Actually, by the Morse-Sard result proved by Kurdyka, Orro and Simon [16] for $\mathcal{K}_\infty(f)$, or by the one obtained in [6] for the sharper estimation $\mathcal{B}_\infty(f) \subset \mathcal{S}_0(f) \subset \mathcal{K}_\infty(f)$, one only knows that the sets $\mathcal{K}_\infty(f)$ and $\mathcal{S}_0(f)$ are contained in a 1-codimensional semi-algebraic subsets of $\mathbb{R}^p$.

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Our approach is based on the set $\mathcal{S}_\infty(f)$ of non-regular values at infinity with respect to the Euclidean distance function from any point as origin, and which includes $\mathcal{B}_\infty(f)$. Since the set of critical values $f(\mathrm{Sing} f)$ is the image of an algebraic set and the well-known estimation methods apply, we consider it as the "trivial" part of the job. The most difficult task is to apprehend the complements of $f(\mathrm{Sing} f)$ to the bifurcation locus $\mathcal{B}_\infty(f)$.

We shall detect here the "nontrivial" part $N\mathcal{S}_\infty(f)$ of the bifurcation locus at infinity (defined at §2.6) which, roughly speaking, contains the values of $\mathcal{S}_\infty(f)$ which are not comming from the branches at infinity of the singular locus $\mathrm{Sing} f$.

This note answers a question raised by the results [14] and [7], as of how can one detect the bifurcation locus by rational arcs in the case $p > 1$.

More precisely, given a polynomial map $f = (f_1, \ldots, f_p)\colon \mathbb{R}^n \to \mathbb{R}^p$, $\deg f_i \le d$, we find all the values of the "nontrivial" part $N\mathcal{S}_\infty(f)$ of $\mathcal{S}_\infty(f)$ and hence of nontrivial part $N\mathcal{B}_\infty(f)$ of the bifurcation locus $\mathcal{B}_\infty(f)$, as follows:

(1). We consider a set of rational paths: $(x(t), y(t)) = \left(\sum_{-ds \le i \le s} a_i t^i, \sum_{-ds \le j \le 0} b_j t^j\right) \subset \mathbb{R}^n \times \mathbb{R}^p$, where $s = [p(d-1)+1]^{n-p}[p(d-1)(n-p)+2]^{p-1}$.

This means a finite number of vectorial coefficients $a_i \in \mathbb{R}^n$, for $-ds \le i \le s$, and $b_j \in \mathbb{R}^p$, for $-ds \le j \le 0$.

(2). The coefficients are subject to several conditions, namely: $\|b_0\| = 1$, $\exists k > 0$, $a_k \ne 0 \in \mathbb{R}^n$, we ask the annulation of the coefficients of the terms with positive exponents in the expansion of $f(x(t))$ and the annulation of the coefficients of the terms with non-negative exponents in the expressions $x_i(t)\phi_j(x(t), y(t))$, for all $i, j \in \{1, \ldots, n\}$ (cf (13) for the definition).

We denote by $\mathrm{Arc}_\infty(f)$ the algebraic subset of arcs obtained by this construction (steps (1) and (2) above), and by $\alpha_0(\mathrm{Arc}_\infty(f))$ the set of limits $\lim_{t\to\infty} f(x(t))$, i.e. the free coefficient in the expansion of $f(x(t)$ for $(x(t), y(t)) \in \mathrm{Arc}_\infty(f)$. Then our main result, Theorem 3.5, proves the inclusions:

$$N\mathcal{S}_\infty(f) \subset \alpha_0(\mathrm{Arc}_\infty(f)) \subset \mathcal{K}_\infty(f).$$

2. Regularity conditions at infinity and bifurcation loci

2.1. **Bifurcation locus.** Let $f = (f_1, \ldots, f_p)\colon \mathbb{R}^n \to \mathbb{R}^p$ be a polynomial map, $n \ge p$.

Definition 2.1. We say that $t_0 \in \mathbb{R}^p$ is a *typical value* of f if there exists a disk $D \subset \mathbb{R}^p$ centered at t_0 such that the restriction $f_|\colon f^{-1}(D) \to D$ is a locally trivial C^∞-fibration. Otherwise we say that t_0 is a *bifurcation value* (or atypical value). We denote by $B(f)$ the set of bifurcation values of f.

We say that f is *topologically trivial at infinity at* $t_0 \in \mathbb{R}^p$ if there exists a compact set $\mathcal{K} \subset \mathbb{R}^n$ and a disk $D \subset \mathbb{R}^p$ centered at t_0 such that the restriction $f_| : f^{-1}(D) \setminus \mathcal{K} \to D$ is a locally trivial C^∞-fibration. Otherwise we say that t_0 is a *bifurcation value at infinity of* f. We denote by $\mathcal{B}_\infty(f)$ the bifurcation locus at infinity of f.

2.2. **The rho-regularity.** Let $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ and let $\rho_a \colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $\rho_a(x) = (x_1 - a_1)^2 + \ldots + (x_n - a_n)^2$, be the Euclidian distance function to a. Let $f \colon \mathbb{R}^n \to \mathbb{R}^p$ be a polynomial map, where $n \geq p$.

Definition 2.2 (Milnor set at infinity and the ρ_a-nonregularity locus)**.** [7]
The critical set $\mathcal{M}_a(f)$ of the map $(f, \rho_a) \colon \mathbb{R}^n \to \mathbb{R}^{p+1}$ is called the *Milnor set of f* (with respect to the distance function). The following semi-algebraic set, cf [6, Theorem 5.7] and [7, Theorem 2.5]:

$$(1) \qquad \mathcal{S}_a(f) := \{t_0 \in \mathbb{R}^p \mid \exists \{x_j\}_{j\in\mathbb{N}} \subset \mathcal{M}_a(f), \lim_{j\to\infty} \|x_j\| = \infty \text{ and } \lim_{j\to\infty} f(x_j) = t_0\}$$

will be called the set of *asymptotic ρ_a-nonregular values*. If $t_0 \notin \mathcal{S}_a(f)$ we say that t_0 is *ρ_a-regular at infinity*. Let $\mathcal{S}_\infty(f) := \bigcap_{a\in\mathbb{R}^n} \mathcal{S}_a(f)$.

Lemma 2.3. *$\mathcal{S}_\infty(f)$ is a semi-algebraic set.*

Proof. Let $f \colon \mathbb{R}^n \to \mathbb{R}^p$ be a polynomial mapping and let us consider the following semi-algebraic set:

$$\mathcal{W} := \{(x, a) \in \mathbb{R}^n \times \mathbb{R}^n \mid x \in \mathcal{M}_a(f)\}.$$

By the definition of $\mathcal{S}_\infty(f)$, we have:

$$\mathcal{S}_\infty(f) := \{y \in \mathbb{R}^p \mid \forall a \in \mathbb{R}^n, \exists \{(x_k, a)\} \subset \mathcal{W} \text{ such that } f(x_k) \to y\},$$

which tells that $\mathcal{S}_\infty(f)$ can be writen by using first-order formulas. This means that $\mathcal{S}_\infty(f)$ is a semi-algebraic set, see for instance [4, pag.28-29] and [1, Prop. 2.2.4]. $\square$

It has been proved in [24], [6], [7] that one has the inclusion $\mathcal{B}_\infty(f) \subset \mathcal{S}_a(f)$, for any $a \in \mathbb{R}^n$, thus in particular:

$$(2) \qquad \mathcal{B}_\infty(f) \subset \mathcal{S}_\infty(f).$$

It was believed, cf [7, Conjecture 2.11], that (2) was an equality. We show here by an example that this is not the case, at least in the real setting.

2.3. **Example for $B_\infty(f) \neq \mathcal{S}_\infty(f)$.** We consider the two-variable real polynomial[1] constructed in [25], $f \colon \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = y(2x^2y^2 - 9xy + 12)$. We show that $\mathcal{S}_\infty(f) = \{0\}$ and $B_\infty(f) = \emptyset$.

It was already proved in [25] that f has no singular value, no bifurcation value and that $\mathcal{S}_0(f) \subset \{0\}$. We shall prove here that this inclusion is an equality. Moreover, we prove here that $\{0\} \subset \mathcal{S}_a(f)$ for any center $a \in \mathbb{R}^2$.

For any fixed $a = (a_1, a_2) \in \mathbb{R}^2$, we have:

$$\mathcal{M}_a(f) = \{(x, y) \in \mathbb{R}^2 \mid y^2(4xy - 9)(y - a_2) = 6(x - a_1)(xy - 1)(xy - 2).\}$$

For $x = 0$ we eventually get solutions of the above equation but which have no influence on the set $\mathcal{S}_a(f)$. By removing these solutions from $\mathcal{M}_a(f)$, we pursue with the resulting

[1]We thank Y. Chen for suggesting us to test this example.

set, which we denote by $\mathcal{M}'_a(f)$. Thus, assuming that $x \neq 0$ and multiply the equation by x^3, we obtain:

$$\mathcal{M}'_a(f) = \{(x, y) \in \mathbb{R}^2 \mid x^2y^2(4xy - 9)(xy - xa_2) = 6x^3(x - a_1)(xy - 1)(xy - 2)\}. \tag{3}$$

We show that we can find solutions $(x_k, y_k)_{k\in\mathbb{N}}$ of the equality in (3) such that $\|(x_k, y_k)\| \to \infty$ and $f(x_k, y_k) \to 0$. Indeed, setting $z := xy$ our equation (3) becomes $z^2(4z - 9)(z - a_2x) = 6x^3(x - a_1)(z - 1)(z - 2)$. We then consider each side as a curve of variable z with x as parameter. We consider the graphs of these two curves and observe that for each sign of a_2 the two graphs intersect at least once for any fixed and large enough $|x|$ and that this happens at some value of z in the interval $]0, 1[$ (and in the interval $]1, 2[$ in case $a_2 = 0$, respectively). This shows that we can find solutions $(x_k, y_k) \in \mathcal{M}_a(f)$ with modulus tending to infinity and, since $z_k = x_ky_k$ is bounded and y_k tends to 0, we get that $f(x_k, y_k) \to 0$.

In conclusion, we have shown that $\mathcal{S}_\infty(f) = \{0\}$, which implies $\mathcal{B}_\infty(f) \neq \mathcal{S}_\infty(f)$.

2.4. Generic dimension of the nonsingular part of the Milnor set.

The following statement has been noticed in case $p = 1$ in [10] (see also [8, Lemma 2.2] or [7]). We outline the proof in case $p > 1$, some details of which will be used in §3.

Lemma 2.4. *Let $f = (f_1, \ldots, f_p)\colon \mathbb{R}^n \to \mathbb{R}^p$ be a polynomial map, where $n > p$ and $\deg f_i \leq d, \forall i$. There exists an open dense subset $\Omega_f \subset \mathbb{R}^n$ such that, for every $a \in \Omega_f$, the set $\mathcal{M}_a(f) \setminus \mathrm{Sing} f$ is either a smooth manifold of dimension p, or it is empty.*

Proof. We denote by $M_I[\mathrm{D}(f)(x)]$ (respectively $M_I[\mathrm{D}(f, \rho_a)(x)]$) the minor of the Jacobian matrix $\mathrm{D}(f)(x)$ (respectively $\mathrm{D}(f, \rho_a)(x)$) indexed by the multi-index I. We set

$$Z := \{(x, a) \in \mathbb{R}^n \times \mathbb{R}^n \mid x \in \mathcal{M}_a(f) \setminus \mathrm{Sing} f\}. \tag{4}$$

If $Z = \emptyset$, then $\mathcal{M}_a(f) \setminus \mathrm{Sing} f = \emptyset, \forall a \in \mathbb{R}^n$. From now on let us consider the case that $Z \neq \emptyset$. Let $(x_0, a_0) \in Z$. Since $\mathrm{Sing} f$ is closed, there is a neighborhood $U \subset \mathbb{R}^n$ of x_0 such that $U \cap \mathrm{Sing} f = \emptyset$. This means that there exists a multi-index $I = (i_1, \ldots, i_p)$ of size p, $1 \leq i_1 < \ldots < i_p \leq n$, such that $M_I[\mathrm{D}f(x)] \neq 0$, $\forall x \in U$.

Let $S_I := \{J = (j_1, \ldots, j_{p+1}) \mid I \subset J\}$ be the set of multi-indices of size $p + 1$ such that $1 \leq j_1 < \ldots < j_{p+1} \leq n$ and $i_1, \ldots, i_p \in \{j_1, \ldots, j_{p+1}\}$. There are $(n - p)$ multi-indices $J \in S_I$; we set

$$m_J(x, a) := M_J[\mathrm{D}(f, \rho_a)(x)], (x, a) \in U \times \mathbb{R}^n. \tag{5}$$

From the definitions of Z, U and the functions m_J, we have:

$$Z \cap (U \times \mathbb{R}^n) = \{(x, a) \in U \times \mathbb{R}^n \mid m_J(x, a) = 0; \forall J \in S_I\}. \tag{6}$$

Let $\varphi\colon U \times \mathbb{R}^n \to \mathbb{R}^{n-p}$ be the map consisting of the functions m_J for $J \in S_I$. Then $\varphi^{-1}(0) = Z \cap (U \times \mathbb{R}^n)$ and we notice that $\mathrm{D}\varphi(x, a)$ has rank $(n-p)$ at any $(x, a) \in U \times \mathbb{R}^n$. Indeed, let

$$\left(\frac{\partial \varphi}{\partial a_k}(x, a)\right)_{(n-p)\times(n-p)}, \, k \notin I, (x, a) \in U \times \mathbb{R}^n.$$

This is a minor of $\mathrm{D}\varphi(x,a)$ of size $(n-p)$. Interchanging if necessary the order of its lines, it is a diagonal matrix with all the entries on the diagonal equal to $-M_I[\mathrm{D}f(x)]$ and hence non-zero. This and (6) show that Z is a manifold of dimension $n+p$.

We next consider the projection $\tau\colon Z \to \mathbb{R}^n$, $\tau(x,a) = a$. Thus, $\tau^{-1}(a) = (\mathcal{M}_a(f) \setminus \mathrm{Sing} f) \times \{a\}$. By Sard's Theorem, we conclude that, for almost all $a \in \mathbb{R}^n$, $\tau^{-1}(a) = (\mathcal{M}_a(f) \setminus \mathrm{Sing} f) \times \{a\} \cong (\mathcal{M}_a(f) \setminus \mathrm{Sing} f)$ is either a smooth manifold of dimension p or an empty set. □

2.5. **The relation to the Malgrange-Rabier condition.**

Definition 2.5 ([21])**.** Let $f\colon \mathbb{R}^n \to \mathbb{R}^p$ be a polynomial map, $n \geq p$. Denote by $\mathrm{D}f(x)$ the Jacobian matrix of f at x. We consider

$$\begin{aligned} \mathcal{K}_\infty(f) \;\; := \;\; & \{t \in \mathbb{R}^p \mid \exists \{x_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^n, \lim_{j\to\infty} \|x_j\| = \infty, \\ & \lim_{j\to\infty} f(x_j) = t \text{ and } \lim_{j\to\infty} \|x_j\| \nu(\mathrm{D}f(x_j)) = 0\}, \end{aligned} \tag{7}$$

where

$$\nu(A) := \inf_{\|y\|=1} \|A^*(y)\|, \tag{8}$$

for a linear map A and its adjoint A^*.

We call the set $\mathcal{K}_\infty(f)$ of *asymptotic critical values of* f. If $t_0 \notin \mathcal{K}_\infty(f)$ we say that f verifies the Malgrange-Rabier condition at t_0.

We have the following relation between ρ_a-regularity and Malgrange-Rabier condition:

Theorem 2.6 ([7, Th. 2.8])**.** *Let* $f = (f_1, \ldots, f_p)\colon \mathbb{R}^n \to \mathbb{R}^p$ *be a polynomial map, where* $n > p$. *Let* $\phi :]0, \varepsilon[\to \mathcal{M}_a(f) \subset \mathbb{R}^n$ *be an analytic path such that* $\lim_{t\to 0} \|\phi(t)\| = \infty$ *and* $\lim_{t\to 0} f(\phi(t)) = \mathrm{c}$. *Then* $\lim_{t\to 0} \|\phi(t)\| \nu(\mathrm{D}f(\phi(t))) = 0$. *In particular* $\mathcal{S}_a(f) \subset \mathcal{K}_\infty(f)$ *for any* $a \in \mathbb{R}^n$, *and* $\mathcal{S}_\infty(f) \subset \mathcal{K}_\infty(f)$. □

Remark 2.7. See [6] and more precisely [7, Theorem 2.5] for a structure result and a fibration result on $\mathcal{S}_\infty(f)$. The inclusion $\mathcal{S}_\infty(f) \subset \mathcal{K}_\infty(f)$ may be strict (e.g. [20] and [7, Example 2.9]). The inclusion $B_\infty(f) \subset \mathcal{S}_\infty(f)$ may be strict, see the above Example §2.3. One may also have $\mathcal{S}_a(f) \neq \mathcal{S}_b(f)$ for some $a \neq b$, see [7, Example 2.10].

2.6. **The nontrivial bifurcation locus at infinity.** We have discussed up to now three types of bifurcation loci: $\mathcal{B}_\infty(f)$, $\mathcal{S}_\infty(f)$ and $\mathcal{K}_\infty(f)$. All of them may contain points of the critical locus $f(\mathrm{Sing} f)$. This locus can be estimated separately since it is the image by f of an algebraic set and the known estimation methods apply. What is more difficult to apprehend are the respective complements of $f(\mathrm{Sing} f)$. We define here the "nontrivial parts" of the bifurcation loci and next describe a procedure to estimate the one of $\mathcal{S}_\infty(f)$.

From the definitions of $\mathcal{M}_a(f)$ and $\mathcal{S}_a(f)$, we have the equality $\mathcal{S}_a(f) = J(f_{|\mathcal{M}_a(f)})$, where $J(f_{|\mathcal{M}_a(f)})$ is the *non-properness set* of $f_{|\mathcal{M}_a(f)}$. Jelonek defined this set in general:

Definition 2.8. ([12, Definition 3.3], [14]). Let $g : M \to N$ be a continuous map, where M, N are topological spaces. One says that g is *proper at the value* $t \in N$ if there exists an open neighbourhood $U \subset N$ of t such that the restriction $g_{|g^{-1}(U)} : g^{-1}(U) \to U$ is a proper map. We denote by $J(g)$ the set of points at which g is not proper.

In our setting $f\colon \mathbb{R}^n \to \mathbb{R}^p$, let us define the *nontrivial ρ-bifurcation set at infinity* $N\mathcal{S}_\infty(f) := \bigcap_{a\in\mathbb{R}^n} N\mathcal{S}_a(f)$, where:

$$N\mathcal{S}_a(f) := \{t \in \mathbb{R}^p \mid \exists\{x_j\}_{j\in\mathbb{N}} \subset \mathcal{M}_a(f) \setminus \mathrm{Sing} f, \lim_{j\to\infty} \|x_j\| = \infty, \text{ and } \lim_{j\to\infty} f(x_j) = t\}$$

and note that $\mathcal{S}_\infty(f) = N\mathcal{S}_\infty(f) \cup J(f_{|\mathrm{Sing} f})$ and that $N\mathcal{S}_\infty(f)$ is a *closed set* since each set $N\mathcal{S}_a(f)$ is closed, which fact follows from the arguments of [6, Theorem 5.7(a)].

Similarly, we introduce the following notation for the *nontrivial bifurcation set at infinity* which is the object of our main result, Theorem 3.5:

$$N\mathcal{B}_\infty(f) := \mathcal{B}_\infty(f) \setminus J(f_{|\mathrm{Sing} f}). \tag{9}$$

By the above definitions and by Theorem 2.6, we immediately get:

Proposition 2.9.

$$N\mathcal{B}_\infty(f) \subset N\mathcal{S}_\infty(f) \subset \mathcal{K}_\infty(f).$$

□

Remark 2.10. If f has a compact singular set $\mathrm{Sing} f$ or, more generally, if $J(f_{|\mathrm{Sing} f}) = \emptyset$, then $N\mathcal{S}_\infty(f) = \mathcal{S}_\infty(f)$, and $N\mathcal{B}_\infty(f) = \mathcal{B}_\infty(f)$. However these equalities mai fail whenever $J(f_{|\mathrm{Sing} f}) \neq \emptyset$.

In this matter, let us point out here that the proofs of [7, Proposition 3.1, Theorem 3.4] run actually for the set $N\mathcal{S}_\infty(f)$; one therefore needs to replace $\mathcal{S}_\infty(f)$ by $N\mathcal{S}_\infty(f)$ in the statements of those results.

3. Detection of bifurcation values at infinity by parametrized curves

3.1. Effective Curve Selection Lemma at infinity via the Milnor set.

If $t_0 \in N\mathcal{S}_\infty(f)$ then $t_0 \in N\mathcal{S}_a(f)$ for any $a \in \mathbb{R}^n$ and in particular for $a \in \Omega_f$, where Ω_f is as in Lemma 2.4.

Theorem 3.1. *Let $f = (f_1, \ldots, f_p)\colon \mathbb{R}^n \to \mathbb{R}^p$ be a polynomial mapping such that* $\deg f_i \leq d, \forall i = 1, \ldots, p$, *and $n > p$. Let $t_0 \in N\mathcal{S}_a(f)$ for some $a \in \Omega_f$. Then there exists an analytic path:*

$$x(t) = \sum_{-\infty\leq i\leq s} a_i t^i, \tag{10}$$

with

$$s \leq [p(d-1)+1]^{n-p}[p(d-1)(n-p)+2]^{p-1}$$

and such that:

(a) $x(t) \in \mathcal{M}_a(f) \setminus \mathrm{Sing} f$, for any $t \geq R$, for some large enough $R \in \mathbb{R}_+$;
(b) $\|x(t)\| \to \infty$, *as* $t \to \infty$;

(c) $f(x(t)) \to t_0$, *as* $t \to \infty$.

Proof. The case $p = 1$ is [7, Theorem 3.4]. We assume in the following that $p > 1$.

From Lemma 2.4 we have that $\mathcal{M}_a(f) \setminus \mathrm{Sing} f$ is a smooth semi-algebraic set of dimension p since non-empty by our hypothesis on t_0. From the proof of Lemma 2.4 the set $\mathcal{M}_a(f) \setminus \mathrm{Sing} f$ is locally a complete intersection defined by $(n-p)$ equations, each of which is of degree at most $p(d-1)+1$. So let us denote by $g_1, \ldots, g_{n-p}$ these functions.

We use coordinates $(x_1, \ldots, x_n)$ for the affine space $\mathbb{R}^n$ and coordinates $[x_0 : x_1 : \ldots : x_n]$ for the projective space $\mathbb{P}^n$. We identify the affine space $\mathbb{R}^n$ with the chart $\{x_0 \neq 0\}$ of $\mathbb{P}^n$. Let $\mathbb{X} = \overline{\mathrm{graph} f}$ be the closure of the graph of f in $\mathbb{P}^n \times \mathbb{R}^p$ and let $\mathbb{X}^\infty$ the intersection of $\mathbb{X}$ with the hyperplane at infinity $\{x_0 = 0\}$. Let $i : \mathbb{R}^n \to \mathbb{X} \subset \mathbb{P}^n \times \mathbb{R}^p$, $x \mapsto (x, f(x))$ be the graph embedding. Consider the closure in $\mathbb{X}$ of the image $i(\mathcal{M}_a(f) \setminus \mathrm{Sing} f)$ and denote it (abusively) by $\overline{\mathcal{M}_a(f) \setminus \mathrm{Sing} f}$.

Let then $w := (\underline{x}, t_0) \in \overline{\mathcal{M}_a(f) \setminus \mathrm{Sing} f} \cap \mathbb{X}^\infty$. We shall work in some affine chart $U \simeq \mathbb{R}^n$ at infinity of $\mathbb{P}^n$ assuming (without loss of generality) that the point $\underline{x}$ is the origin. We may then use an "effective curve selection lemma" to show that there is a curve $\Gamma \subset \mathcal{M}_a(f) \setminus \mathrm{Sing} f$ such that $w \in \overline{\Gamma}$ and that this curve has a one-sided bounded parametrization. To do so, we combine Milnor's basic construction in [17] with the idea of Jelonek and Kurdyka given in [14, Lemma 6.4].

Namely we consider small enough spheres centered at $w \in U$ of equation $\rho_w = \beta$ and a function $h_l := x_0 l$, for some linear function l in the local coordinates. One can then prove like in [14, Lemma 6.4] (where an apparently more particular situation was considered, but the proof works as well) that, for a general such linear function l, the set of critical points of the map $(\rho_w, h_l) : U \cap \overline{\mathcal{M}_a(f) \setminus \mathrm{Sing} f} \to \mathbb{R}_+ \times \mathbb{R}$ is an analytic curve and its branches are the singular points of the restrictions of the quadratic function h_l to the levels $\{\rho_w = \beta\} \cap \mathcal{M}_a(f) \setminus \mathrm{Sing} f$. It is shown in [14, Lemmas 6.5 and 6.6] that these singular points are all Morse for a generic choice of l, and that there is at least one Morse point on each level, for small enough $\beta > 0$.

Let us then consider a branch of this analytic curve as our $x(t)$. By its definition, this curve is a solution of the following system of equations: $g_1 = 0, \ldots, g_{n-p} = 0$ and $\mathrm{d}g_1 \wedge \cdots \wedge \mathrm{d}g_{n-p} \wedge \mathrm{d}\rho_w \wedge \mathrm{d}h_l = 0$, the first of which are of degree at most $p(d-1)+1$ and the last one means the annulation of $p-1$ minors of degree at most $p(d-1)(n-p)+2$. Thus our algebraic set of solutions has degree δ verifying the inequality:

$$\delta \leq [p(d-1)+1]^{n-p}[p(d-1)(n-p)+2]^{p-1}.$$

Finally, by using the effective Curve Selection Lemma of Jelonek and Kurdyka [14, Lemma 3.1 and Lemma 3.2] which says that there exists a parametrization of our curve $x(t)$ bounded by the degree δ of the curve, we get exactly an expansion like (10). This finishes the proof of our theorem. □

3.2. **Finite length expansion for curves detecting asymptotically critical values.** We need a preliminary result which follows by applying [14, Lemma 3.3] to each function h_i in the following statement:

Lemma 3.2. *Let $h = (h_1, \dots, h_m)\colon \mathbb{R}^k \to \mathbb{R}^m$ be a polynomial map and $\deg h_i \le \tilde{d}, \forall i$. Let $x(t) = \sum_{-\infty \le i \le s} a_i t^i$, where $t \in \mathbb{R}$, $a_i \in \mathbb{R}^k$, $s > 0$ and that $\|x(t)\| \to \infty$ and $h(x(t)) \to b$. Then, for any $D \le -\tilde{d}s + s$, the truncated curve*

$$\tilde{x}(t) = \sum_{D \le i \le s} a_i t^i,$$

verifies $\|\tilde{x}(t)\| \to \infty$ and $h(\tilde{x}(t)) \to b$. □

If we try to replace $x(t)$ given in (10) by a truncated path, we may go out of the set $\mathcal{M}_a(f) \setminus \mathrm{Sing} f$. Bearing in mind the inclusion $\mathcal{S}_a(f) \subset \mathcal{K}_\infty(f)$ of Theorem 2.6, instead of searching in vain a truncated expansion inside the Milnor set, we may show that there exists a truncation which verifies the Malgrange-Rabier condition (7). The proof of the following result employs the technique of [6, Theorem 3.2] and [5, Theorem 2.4.8], where we have used the t-regularity to find a geometric interpretation for $\mathcal{K}_\infty(f)$.

Proposition 3.3. *Let $f = (f_1, \dots, f_p)\colon \mathbb{R}^n \to \mathbb{R}^p$ be a polynomial map such that $n > p$ and that $\deg f_i \le d, \forall i$. Let*

$$x(t) = (x_1(t), \dots, x_n(t)) = \sum_{-\infty \le i \le s} a_i t^i,$$

where $t \in \mathbb{R}$, $a_i \in \mathbb{R}^n$, $s > 0$ and such that:

(a) $\|x(t)\| \to \infty$, *as* $t \to \infty$;
(b) $f(x(t)) \to b$, *as* $t \to \infty$;
(c) $\|x(t)\|\nu(\mathrm{D}f(x(t))) \to 0$, *as* $t \to \infty$.

Then the truncated expansion

$$\tilde{x}(t) = \sum_{-ds \le i \le s} a_i t^i,$$

verifies the following conditions:

(i) $\|\tilde{x}(t)\| \to \infty$, *as* $t \to \infty$;
(ii) $f(\tilde{x}(t)) \to b$, *as* $t \to \infty$;
(iii) $\|\tilde{x}(t)\|\nu(\mathrm{D}f(\tilde{x}(t))) \to 0$, *as* $t \to \infty$.

Proof. We treat here the case $p > 1$. See Remark 3.4 for the case $p = 1$.

By the definition of ν (Definition 2.5 and (8)), condition (c) means:

$$\|x(t)\| \left(\inf_{\|y\|=1} \|\mathrm{D}f(x(t))^*(y)\| \right) \to 0, \text{ as } t \to \infty, \tag{11}$$

where $\mathrm{D}f(x(t))^*$ denotes the adjoint of $\mathrm{D}f(x(t))$.

Since ν is a semi-algebraic mapping (see e.g [16, Proposition 2.4]), the Curve Selection Lemma and (11) imply that there there exists an analytic path (see also the proofs of [6, Theorem 3.2] and [3, Proposition 2.4] for this argument):

$$y(t) = \sum_{-\infty \le i \le 0} b_j t^j = (y_1(t), \ldots, y_p(t)), b_j \in \mathbb{R}^p,$$

such that $\|y(t)\| = 1, \forall t \gg 0$, and that:

$$\|x(t)\| \left\| y_1(t)\frac{\partial f_1}{\partial x}(x(t)) + \cdots + y_p(t)\frac{\partial f_p}{\partial x}(x(t)) \right\| \to 0, \text{ as } t \to \infty, \tag{12}$$

where $\frac{\partial f_i}{\partial x}(x(t)) := \left(\frac{\partial f_i}{\partial x_1}(x(t)), \ldots, \frac{\partial f_i}{\partial x_n}(x(t)) \right)$ for $i = 1, \ldots, p$.

For any fixed $j \in \{1, \cdots, n\}$ we set $\phi_j \colon \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$,

$$\phi_j(x, y) := \left(y_1 \frac{\partial f_1}{\partial x_j}(x) + \cdots + y_p \frac{\partial f_p}{\partial x_j}(x) \right). \tag{13}$$

It then follows that $\deg \phi_j \le d$ and that our path:

$$(x(t), y(t)) := \left(\sum_{-\infty \le i \le s} a_i t^i, \sum_{-\infty \le i \le 0} b_j t^j \right)$$

verifies the conditions:

(1) $\|x(t)\| \to \infty$ as $t \to \infty$, and $\|y(t)\| = 1$;
(2) $x_i(t)\phi_j(x(t), y(t)) \to 0$ as $t \to \infty$, for any $i, j \in \{1, \ldots, n\}$.

Applying Lemma 3.2 to the mapping $(x_i\phi_j)_{i,j=1}^n$, we get that, for any $D \le -(d+1)s+s = -ds$, the truncated path:

$$(\tilde{x}(t), \tilde{y}(t)) := \left(\sum_{D \le i \le s} a_i t^i, \sum_{D \le i \le 0} b_j t^j \right)$$

verifies the conditions:

(1') $\|\tilde{x}(t)\| \to \infty$ and $\|\tilde{y}(t)\| \to 1$ as $t \to \infty$;
(2') $\tilde{x}_i(t)\phi_j(\tilde{x}(t), \tilde{y}(t)) \to 0$, as $t \to \infty$, for any $i, j \in \{1, 2, \ldots, n\}$.

These imply:

$$\|\tilde{x}(t)\| \left\| \tilde{y}_1(t)\frac{\partial f_1}{\partial x}(\tilde{x}(t)) + \cdots + \tilde{y}_p(t)\frac{\partial f_p}{\partial x}(\tilde{x}(t)) \right\| \to 0 \text{ as } t \to \infty, \tag{14}$$

and, since $\|\tilde{y}(t)\| \to 1$, we obtain:

$$\|\tilde{x}(t)\| \frac{1}{\|\tilde{y}(t)\|} \left\| \tilde{y}_1(t)\frac{\partial f_1}{\partial x}(\tilde{x}(t)) + \cdots + \tilde{y}_p(t)\frac{\partial f_p}{\partial x}(\tilde{x}(t)) \right\| \to 0, \text{ as } t \to \infty. \tag{15}$$

The later implies that $\|\tilde{x}(t)\|\nu(\mathrm{D}f(\tilde{x}(t))) \to 0$, as $t \to \infty$, which shows (iii).

Next, (i) follows by (1'), and (ii) follows from Lemma 3.2 for $h := f$, since $-ds < -ds+s$. $\square$

REMARK 3.4. In case $p = 1$, in the proof of Proposition 3.3 we may consider $\phi_j : \mathbb{R}^n \to \mathbb{R}, \phi_j(x, y) = \frac{\partial f}{\partial x_j}(x)$ since in this case $y = 1$. Then $\deg \phi_j \le d-1$ and by applying Lemma 3.2 as above to the mapping $(x_i\phi_j)_{i,j=1}^n$ we get that, for any $D \le -ds + s$, the truncation $\tilde{\tilde{x}}(t) = \sum_{D \le i \le s} a_i t^i$ satisfies (i), (ii) and (iii).

In the definition of $\mathrm{Arc}(f)$, the lower bound is $-ds + s$ instead of $-ds$. Since the value of the degree s from Theorem 3.1 is d^{n-1} in case $p = 1$, we recover the result in [7].

3.3. **Arc space and the main result.** We may now apply to a polynomial map $f = (f_1, \ldots, f_p) : \mathbb{R}^n \to \mathbb{R}^p$, $\deg f_i \le d$, a similar procedure as the one described by Jelonek and Kurdyka [14] in case $p = 1$. Thus, in case $p > 1$, we consider the following space of arcs associated to f:

$$\mathrm{Arc}(f) := \left\{ (x(t), y(t)) = \left(\sum_{-ds \le i \le s} a_i t^i, \sum_{-ds \le j \le 0} b_j t^j \right),\ (a_i, b_i) \in \mathbb{R}^n \times \mathbb{R}^p \right\}, \tag{16}$$

where $s := [p(d-1)+1]^{n-p}[p(d-1)(n-p)+2]^{p-1}$, as in Theorem 3.1. Then $\mathrm{Arc}(f)$ is a vector space of finite dimension.

Referring to the notations in (16), we define, in a similar manner as [14, Definition 6.10], the *asymptotic variety of arcs* $\mathrm{Arc}_\infty(f) \subset \mathrm{Arc}(f)$, as the algebraic subset of the rational arcs $(x(t), y(t)) \in \mathrm{Arc}(f)$ verifying the following conditions:

(a') $\exists k > 0$ such that $a_k \neq 0 \in \mathbb{R}^n$, and $\|b_0\| = 1$.
(b') $\mathrm{ord}_t f(x(t)) \le 0$.
(c') $\mathrm{ord}_t\,(x_i(t)\phi_j(x(t), y(t))) < 0$, for any $i, j \in \{1, \ldots, n\}$, where ϕ_j is defined at (13) in the proof of Proposition 3.3.

Let us then set $\alpha_0 : \mathrm{Arc}_\infty(f) \to \mathbb{R}^p$, $\alpha_0(\xi(t)) := \lim_{t\to\infty} f(x(t))$, where $\xi(t) = (x(t), y(t))$.

In view of the above results, we may now give an estimation of the nontrivial ρ-bifurcation set at infinity $N\mathcal{S}_\infty(f)$, thus of the nontrivial bifurcation locus $N\mathcal{B}_\infty(f)$, cf Proposition 2.9:

Theorem 3.5. *$N\mathcal{S}_\infty(f) \subset \alpha_0(\mathrm{Arc}_\infty(f)) \subset \mathcal{K}_\infty(f)$.*

Proof. If $\alpha \in N\mathcal{S}_\infty(f)$ then $\alpha \in N\mathcal{S}_a(f)$ for any fixed $a \in \Omega_f$. By Theorem 3.1, there exists a path

$$x(t) = \sum_{-\infty \le i \le s} a_i t^i \in \mathcal{M}_a(f) \setminus \mathrm{Sing} f,$$

such that $\lim_{t\to\infty} f(x(t)) = \alpha$. It follows from Theorem 2.6 that $x(t)$ verifies the conditions (a)–(c) of Proposition 3.3. Moreover, the truncation $\tilde{x}$ defined in the same Proposition 3.3 verifies the properties (i)–(iii). Since conditions (i)–(iii) are equivalent to conditions (a')–(c'), we conclude that the first inclusion holds.

The second inclusion $\alpha_0(\mathrm{Arc}_\infty(f)) \subset \mathcal{K}_\infty(f)$ is a direct consequence of the definitions of $\mathrm{Arc}_\infty(f)$ and $\mathcal{K}_\infty(f)$ since properties (a'), (b') and (c') characterize the values $\alpha_0 \in \mathcal{K}_\infty(f)$ as shown in the proof of Proposition 3.3. This completes our proof. □

Let us remark that the first inclusion can be strict, as shown by the next example:

EXAMPLE 3.6 ([7, Example 2.10]). Let $f\colon \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = y(x^2y^2 + 3xy + 3)$. We have $N\mathcal{S}_\infty(f) = \emptyset$, $0 \in \alpha_0(\mathrm{Arc}_\infty(f))$ and $0 \in \mathcal{K}_\infty(f)$.

In trying to prove the equality in place of the second inclusion in Theorem 3.5 one notices that the inverse inclusion depends on the possibility of truncating paths which detect some value $\alpha_0 \in \mathcal{K}_\infty(f)$ at the order provided by Theorem 3.1. But our Theorem 3.1 is based on paths in the Milnor set $\mathcal{M}_a(f) \setminus \mathrm{Sing} f$, which provide in principle lower degrees than working with the Malgrange-Rabier condition (7), and we know that the later is not equivalent to ρ-regularity (cf §2). Else, for the same reason, it would be difficult to obtain examples to disprove the inverse inclusion.

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Faculdade de Matemática, Universidade Federal de Uberlândia, Av. João Naves de Ávila 2121, 1F-153 - CEP: 38408-100, Uberlândia, Brazil.
E-mail address: `lrgdias@famat.ufu.br`

Mathématiques, Université de Galatasaray, 34357 Istanbul, Turquie.
E-mail address: `tanabesusumu@hotmail.com`

Univ. Lille, CNRS, UMR 8524 - Laboratoire Paul Painlevé, F-59000 Lille, France
E-mail address: `tibar@math.univ-lille1.fr`