# CLASSIFICATION OF POINTED FUSION CATEGORIES OF DIMENSION 8 UP TO WEAK MORITA EQUIVALENCE 

ÁLVARO MUÑOZ AND BERNARDO URIBE


#### Abstract

In this paper we give a complete classification of pointed fusion categories over $\mathbb{C}$ of global dimension 8 . We first classify the equivalence classes of pointed fusion categories of dimension 8 , and then we proceed to determine which of these equivalence classes have equivalent categories of modules. This classificaction permits to classify the equivalence classes of braded tensor equivalences of twisted Drinfeld doubles of finite groups of order 8.


## Introduction

A fusion category ENO05 is a rigid semisimple $\mathbb{C}$-linear tensor category with only finitely many isomorphism classes of simple objects, such that the endomorphisms of the unit object is $\mathbb{C}$. It is moreover pointed if all its simple objects are invertible. Any pointed fusion category is equivalent to a fusion category $V e c t(H, \eta)$ of $H$-graded complex vector spaces with $H$ a finite group, together with an associativity constraint defined by a cocycle $\eta \in Z^{3}\left(H, \mathbb{C}^{*}\right)$. The skeletal tensor category $\mathcal{V}(H, \eta)$ associated to $\operatorname{Vect}(H, \eta)$ is a 2-group BL04 with only one object for each isomorphism class of simple objects in $\operatorname{Vect}(H, \eta)$, whose morphisms are only automorphisms and whose associator is defined by $\eta$. For any module category $\mathcal{M}$ over a fusion category $\mathcal{C}$ we may define the dual fusion category $\mathcal{C}_{\mathcal{M}}^{*}:=F u n_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$, and we say that two fusion categories $\mathcal{C}$ and $\mathcal{D}$ are weakly Morita equivalent if there exists an idecomposable module category $\mathcal{M}$ over $\mathcal{C}$ such that $\mathcal{C}_{\mathcal{M}}^{*}$ and $\mathcal{D}$ are tensor equivalent Müg03, Def. 4.2].

In Nai07] there were given necessary and sufficient conditions in terms of cocycles for two pointed fusion categories $\operatorname{Vect}(H, \eta)$ and $\operatorname{Vect}(\widehat{H}, \widehat{\eta})$ to be weakly Morita equivalent, and in [Uri15] a choice of appropriate coordinates permitted the second author to give a precise description of the groups $H, \widehat{H}$ and the cocycles $\eta, \widehat{\eta}$ for the pointed fusion categories to be weakly Morita equivalent. In this paper we follow the description done by the second author in Uri15] in order to classify the Morita equivalence classes of pointed fusion categories of global dimension 8

This work will be divided in two chapters. In the first chapter we will setup explicit basis for $H^{3}\left(H, \mathbb{C}^{*}\right)$ and we will calculate the space of orbits $H^{3}\left(H, \mathbb{C}^{*}\right) / A u t(H)$ for each of the five groups of order 8; this will determine the equivalence classes of pointed fusion categories of global dimension 8 . In the second chapter we will recall

[^0]the classification theorem of pointed fusion categories Uri15, Thm. 3.9] and we will use the Lyndon-Hochschild-Serre spectral sequence associated to group extensions to explicitly find the Morita equivalence classes of pointed fusion categories of global dimension 8. We will finish the paper with an application to the classification of braided tensor equivalences of twisted Drinfeld doubles of groups of order 8 .

Since this work is an application of the results obtained by the second author in Uri15], we will use the notation and the constructions done there. A more detailed version of the results presented in this paper appear in the Master thesis of the first author (Mn17].

## 1. Equivalence classes of pointed fusion categories of global DIMENSION 8

The pointed fusion categories $\operatorname{Vect}(H, \eta)$ and $\operatorname{Vect}(\widehat{H}, \widehat{\eta})$ are equivalent if and only if there is and isomorphism of groups $\phi: H \xlongequal{\cong} \widehat{H}$ such that $\left[\phi^{*} \widehat{\eta}\right]=[\eta]$ in $H^{3}\left(H, \mathbb{C}^{*}\right)$. Therefore the equivalence classes of pointed fusion categories of global dimension 8 is isomorphic to the union of the spaces of orbits $H^{3}\left(H, \mathbb{C}^{*}\right) / A u t(H)$ where $H$ runs over the groups $\mathbb{Z}_{2}^{3}, \mathbb{Z}_{4} \times \mathbb{Z}_{2}, \mathbb{Z}_{8}, D_{8}$ and $Q_{8}$. By the short exact sequence of coefficients $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{C} \rightarrow \mathbb{C}^{*} \rightarrow 0$ we now that for any finite group $H^{3}\left(H, \mathbb{C}^{*}\right) \cong H^{4}(H, \mathbb{Z})$; in what follows we will calculate $H^{4}(H, \mathbb{Z}) / A u t(H)$.
1.1. $(\mathbb{Z} / 2)^{3}$. We know that

$$
H^{4}\left((\mathbb{Z} / 2)^{3}, \mathbb{Z}\right)=\operatorname{ker}\left(S q^{1}: H^{4}\left((\mathbb{Z} / 2)^{3}, \mathbb{F}_{2}\right) \rightarrow H^{5}\left((\mathbb{Z} / 2)^{3}, \mathbb{F}_{2}\right)\right)
$$

where $\mathbb{F}_{2}$ is the field of 2 elements and $S q^{1}$ is the first Steenrod square operation in cohomology with $\mathbb{F}_{2}$ coefficients. Letting $H^{*}\left((\mathbb{Z} / 2)^{3}, \mathbb{F}_{2}\right)=\mathbb{F}_{2}[x, y, z]$ we obtain

$$
H^{4}\left((\mathbb{Z} / 2)^{3}, \mathbb{Z}\right) \cong \mathbb{Z} / 2\left\langle x^{4}, y^{4}, z^{4}, x^{2} y^{2}, x^{2} z^{2}, y^{2} z^{2}, x^{2} y z+x y^{2} z+x y z^{2}\right\rangle \cong(\mathbb{Z} / 2)^{7}
$$

The group of automorphisms of $(\mathbb{Z} / 2)^{3}$ is isomorphic to $G L\left(3, \mathbb{F}_{2}\right)$ and the 10 orbits of $H^{4}\left((\mathbb{Z} / 2)^{3}, \mathbb{Z}\right) / \operatorname{Aut}\left((\mathbb{Z} / 2)^{3}\right)$ are:

$$
\begin{gathered}
\text { orb }(0)=\{0\} \\
\operatorname{orb}\left(x^{4}\right)=\left\{x^{4}, y^{4}, z^{4}, x^{4}+y^{4}, x^{4}+z^{4}, y^{4}+z^{4}, x^{4}+y^{4}+z^{4}\right\} \\
\text { orb }\left(x^{2} y^{2}\right)= \\
\left\{x^{2} y^{2}, x^{2} y^{2}+y^{4}, x^{2} z^{2}+x^{4}, x^{2} y^{2}+y^{2} z^{2}, x^{4}+x^{2} y^{2}+x^{2} z^{2}+y^{2} z^{2}, x^{4}+x^{2} y^{2}+x^{2} z^{2},\right. \\
x^{2} z^{2}, x^{2} y^{2}+x^{4}, y^{2} z^{2}+y^{4}, x^{2} y^{2}+x^{2} z^{2}, y^{4}+x^{2} y^{2}+x^{2} z^{2}+y^{2} z^{2}, y^{4}+x^{2} y^{2}+y^{2} z^{2}, \\
y^{2} z^{2}, x^{2} z^{2}+z^{4}, y^{2} z^{2}+z^{4}, x^{2} z^{2}+y^{2} z^{2}, z^{4}+x^{2} y^{2}+x^{2} z^{2}+y^{2} z^{2}, z^{4}+x^{2} z^{2}+y^{2} z^{2}, \\
\left.x^{4}+y^{4}+x^{2} z^{2}+y^{2} z^{2}, z^{4}+y^{4}+x^{2} y^{2}+x^{2} z^{2}, x^{4}+z^{4}+x^{2} y^{2}+y^{2} z^{2}\right\} \\
\text { orb }\left(x^{4}+y^{2} z^{2}\right)= \\
\left\{x^{4}+y^{2} z^{2}, x^{4}+y^{4}+x^{2} y^{2}+x^{2} z^{2}, x^{4}+z^{4}+x^{2} z^{2}+y^{2} z^{2}, x^{4}+y^{4}+x^{2} z^{2}, x^{4}+z^{4}+x^{2} y^{2},\right. \\
y^{4}+x^{2} z^{2}, x^{4}+y^{4}+x^{2} y^{2}+y^{2} z^{2}, y^{4}+z^{4}+x^{2} y^{2}+y^{2} z^{2}, x^{4}+y^{4}+y^{2} z^{2}, y^{4}+z^{4}+x^{2} y^{2}, \\
z^{4}+x^{2} y^{2}, x^{4}+z^{4}+x^{2} y^{2}+x^{2} z^{2}, y^{4}+z^{4}+x^{2} z^{2}+y^{2} z^{2}, x^{4}+z^{4}+z^{2} y^{2}, y^{4}+z^{4}+x^{2} z^{2}, \\
x^{4}+y^{4}+z^{4}+x^{2} y^{2}, x^{4}+x^{2} y^{2}+y^{2} z^{2}, z^{4}+x^{2} y^{2}+y^{2} z^{2}, x^{4}+y^{4}+x^{2} y^{2}+x^{2} z^{2}+y^{2} z^{2}, \\
x^{4}+y^{4}+z^{4}+x^{2} z^{2}, x^{4}+x^{2} z^{2}+y^{2} z^{2}, y^{4}+x^{2} y^{2}+x^{2} z^{2}, x^{4}+z^{4}+x^{2} y^{2}+x^{2} z^{2}+y^{2} z^{2}, \\
x^{4}+y^{4}+z^{4}+y^{2} z^{2}, z^{4}+x^{2} y^{2}+x^{2} z^{2}, y^{4}+x^{2} z^{2}+y^{2} z^{2}, y^{4}+z^{4}+x^{2} y^{2}+x^{2} z^{2}+y^{2} z^{2}, \\
\left.x^{2} y^{2}+x^{2} z^{2}+y^{2} z^{2}\right\}
\end{gathered}
$$

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\(\operatorname{orb}\left(x^{4}+y^{4}+x^{2} y^{2}\right)=\)
    \(\left\{x^{4}+y^{4}+x^{2} y^{2}, x^{4}+y^{4}+z^{4}+x^{2} y^{2}+x^{2} z^{2}, x^{4}+z^{4}+x^{2} z^{2}, x^{4}+y^{4}+z^{4}+x^{2} y^{2}+y^{2} z^{2}\right.\),
    \(\left.y^{4}+z^{4}+y^{2} z^{2}, x^{4}+y^{4}+z^{4}+x^{2} z^{2}+y^{2} z^{2}, x^{4}+y^{4}+z^{4}+x^{2} y^{2}+x^{2} z^{2}+y^{2} z^{2}\right\}\)
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\(\operatorname{orb}\left(x^{4}+x^{2} y z+x y^{2} z+x y z^{2}\right)=\)
    \(\left\{x^{4}+x^{2} y z+x y^{2} z+x y z^{2}, x^{4}+y^{4}+z^{4}+x^{2} y z+x y^{2} z+x y z^{2}, x^{4}+z^{4}+x^{2} y z+x y^{2} z+x y z^{2}+y^{2} z^{2}\right.\),
    \(y^{4}+x^{2} y z+x y^{2} z+x y z^{2}, x^{4}+y^{4}+x^{2} y z+x y^{2} z+x y z^{2}+y^{2} z^{2}, y^{4}+x^{2} y z+x y^{2} z+x y z^{2}+y^{2} z^{2}\),
    \(z^{4}+x^{2} y z+x y^{2} z+x y z^{2}, z^{4}+x^{2} y z+x y^{2} z+x y z^{2}+y^{2} z^{2}, y^{4}+x^{2} y z+x y^{2} z+x y z^{2}+x^{2} y^{2}\),
    \(x^{4}+z^{4}+x^{2} y z+x y^{2} z+x y z^{2}+x^{2} y^{2}, y^{4}+z^{4}+x^{2} y z+x y^{2} z+x y z^{2}+x^{2} z^{2}\),
    \(y^{4}+z^{4}+x^{2} y z+x y^{2} z+x y z^{2}+x^{2} y^{2}, x^{4}+y^{4}+x^{2} y z+x y^{2} z+x y z^{2}+x^{2} z^{2}\),
    \(x^{4}+x^{2} y z+x y^{2} z+x y z^{2}+x^{2} y^{2}, z^{4}+x^{2} y z+x y^{2} z+x y z^{2}+x^{2} z^{2}, x^{4}+z^{4}+x^{2} y z+x y^{2} z+x y z^{2}+x^{2} y^{2}+y^{2} z^{2}\)
    \(x^{4}+x^{2} y z+x y^{2} z+x y z^{2}+x^{2} z^{2}, z^{4}+x^{2} y z+x y^{2} z+x y z^{2}+x^{2} y^{2}+x^{2} z^{2}\)
    \(y^{4}+z^{4}+x^{2} y z+x y^{2} z+x y z^{2}+x^{2} y^{2}+x^{2} z^{2}, y^{4}+x^{2} y z+x y^{2} z+x y z^{2}+x^{2} y^{2}+x^{2} z^{2}\)
    \(x^{2} y z+x y^{2} z+x y z^{2}+x^{2} y^{2}+x^{2} z^{2}, x^{4}+x^{2} y z+x y^{2} z+x y z^{2}+x^{2} y^{2}+y^{2} z^{2}\)
    \(x^{2} y z+x y^{2} z+x y z^{2}+x^{2} y^{2}+y^{2} z^{2}, y^{4}+x^{2} y z+x y^{2} z+x y z^{2}+x^{2} z^{2}+y^{2} z^{2}\),
    \(z^{4}+x^{2} y z+x y^{2} z+x y z^{2}+x^{2} y^{2}+y^{2} z^{2}, x^{4}+y^{4}+x y z+x y^{2} z+x y z^{2}+x^{2} z^{2}+y^{2} z^{2}\),
    \(\left.x y z+x y^{2} z+x y z^{2}+x^{2} z^{2}+y^{2} z^{2}, x^{4}+x y z+x y^{2} z+x y z^{2}+x^{2} z^{2}+y^{2} z^{2}\right\}\)
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        \(\operatorname{orb}\left(x^{4}+x^{2} y z+x y^{2} z+x y z^{2}+y^{2} z^{2}\right)=\)
            \(\left\{x^{4}+x^{2} y z+x y^{2} z+x y z^{2}+y^{2} z^{2}, x^{4}+z^{4}+x^{2} y z+x y^{2} z+x y z^{2}+x^{2} z^{2}+y^{2} z^{2}\right.\),
            \(y^{4}+x^{2} y z+x y^{2} z+x y z^{2}+x^{2} z^{2}, y^{4}+z^{4}+x^{2} y z+x y^{2} z+x y z^{2}+x^{2} z^{2}+y^{2} z^{2}\),
            \(z^{4}+x^{2} y z+x y^{2} z+x y z^{2}+x^{2} y^{2}, x^{4}+z^{4}+x^{2} y z+x y^{2} z+x y z^{2}+x^{2} y^{2}+x^{2} z^{2}\),
            \(x^{4}+y^{4}+x^{2} y z+x y^{2} z+x y z^{2}+x^{2} y^{2}+x^{2} z^{2}, x^{4}+y^{4}+x^{2} y z+x y^{2} z+x y z^{2}\),
            \(x^{4}+y^{4}+x^{2} y z+x y^{2} z+x y z^{2}+x^{2} y^{2}+y^{2} z^{2}, x^{4}+z^{4}+x^{2} y z+x y^{2} z+x y z^{2}\),
            \(y^{4}+z^{4}+x^{2} y z+x y^{2} z+x y z^{2}+x^{2} y^{2}+y^{2} z^{2}, y^{4}+z^{4}+x^{2} y z+x y^{2} z+x y z^{2}\),
            \(x^{4}+z^{4}+x^{2} y z+x y^{2} z+x y z^{2}+x^{2} z^{2}, x^{4}+y^{4}+z^{4}+x^{2} y z+x y^{2} z+x y z^{2}+x^{2} y^{2}\),
            \(x^{4}+y^{4}+x^{2} y z+x y^{2} z+x y z^{2}+x^{2} y^{2}, x^{4}+y^{4}+z^{4}+x^{2} y z+x y^{2} z+x y z^{2}+x^{2} z^{2}\),
            \(y^{4}+z^{4}+x^{2} y z+x y^{2} z+x y z^{2}+y^{2} z^{2}, x^{4}+y^{4}+z^{4}+x^{2} y z+x y^{2} z+x y z^{2}+y^{2} z^{2}\),
            \(x^{4}+y^{4}+z^{4}+x^{2} y z+x y^{2} z+x y z^{2}+x^{2} y^{2}+x^{2} z^{2}\),
            \(x^{4}+y^{4}+z^{4}+x^{2} y z+x y^{2} z+x y z^{2}+x^{2} y^{2}+y^{2} z^{2}\),
            \(\left.x^{4}+y^{4}+z^{4}+x^{2} y z+x y^{2} z+x y z^{2}+x^{2} z^{2}+y^{2} z^{2}\right\}\)
    orb $\left(x^{2} y z+x y^{2} z+x y z^{2}+x^{2} y^{2}+x^{2} z^{2}+y^{2} z^{2}\right)=$
$\left\{x^{2} y z+x y^{2} z+x y z^{2}+x^{2} y^{2}+x^{2} z^{2}+y^{2} z^{2}\right.$,
$x^{4}+z^{4}+x^{2} y z+x y^{2} z+x y z^{2}+x^{2} y^{2}+x^{2} z^{2}+y^{2} z^{2}, x^{4}+x^{2} y z+x y^{2} z+x y z^{2}+x^{2} y^{2}+x^{2} z^{2}+y^{2} z^{2}$,
$y^{4}+z^{4}+x^{2} y z+x y^{2} z+x y z^{2}+x^{2} y^{2}+x^{2} z^{2}+y^{2} z^{2}, z^{4}+x^{2} y z+x y^{2} z+x y z^{2}+x^{2} y^{2}+x^{2} z^{2}+y^{2} z^{2}$,
$\left.x^{4}+y^{4}+x^{2} y z+x y^{2} z+x y z^{2}+x^{2} y^{2}+x^{2} z^{2}+y^{2} z^{2}, y^{4}+x^{2} y z+x y^{2} z+x y z^{2}+x^{2} y^{2}+x^{2} z^{2}+y^{2} z^{2}\right\}$
$\operatorname{orb}\left(x^{2} y z+x y^{2} z+x y z^{2}\right)=$
$\left\{x^{2} y z+x y^{2} z+x y z^{2}\right.$,
$x^{2} y z+x y^{2} z+x y z^{2}+y^{2} z^{2}, x^{4}+x^{2} y z+x y^{2} z+x y z^{2}+x^{2} y^{2}+x^{2} z^{2}$,
$x^{2} y z+x y^{2} z+x y z^{2}+x^{2} y^{2}, y^{4}+x^{2} y z+x y^{2} z+x y z^{2}+x^{2} y^{2}+y^{2} z^{2}$,
$\left.x^{2} y z+x y^{2} z+x y z^{2}+x^{2} z^{2}, z^{4}+x^{2} y z+x y^{2} z+x y z^{2}+x^{2} z^{2}+y^{2} z^{2}\right\}$

$$
\begin{aligned}
\operatorname{orb}\left(x^{4}+y^{4}+z^{4}+x^{2} y z\right. & \left.+x y^{2} z+x y z^{2}+x^{2} y^{2}+x^{2} z^{2}+y^{2} z^{2}\right) \\
& =\left\{x^{4}+y^{4}+z^{4}+x^{2} y z+x y^{2} z+x y z^{2}+x^{2} y^{2}+x^{2} z^{2}+y^{2} z^{2}\right\}
\end{aligned}
$$

1.2. $\mathbb{Z} / 4 \times \mathbb{Z} / 2$. By Kunneth's theorem we know that

$$
H^{4}(\mathbb{Z} / 4 \times \mathbb{Z} / 2, \mathbb{Z}) \cong \mathbb{Z}\left\langle v^{2}, u v, u^{2}\right\rangle /\left(4 v^{2}, 2 u v, 2 u^{2}\right) \cong \mathbb{Z} / 4 \oplus \mathbb{Z} / 2 \oplus \mathbb{Z} / 2
$$

where $H^{*}(\mathbb{Z} / 4, \mathbb{Z})=\mathbb{Z}[v] /(4 v)$ and $H^{*}(\mathbb{Z} / 2, \mathbb{Z})=\mathbb{Z}[u] /(4 u)$. The group of automorphisms of $\mathbb{Z} / 4 \times \mathbb{Z} / 2$ is the dihedral group $D_{8}$ generated by the automorphisms

$$
\begin{aligned}
\rho: \mathbb{Z} / 4 \times \mathbb{Z} / 2 & \longrightarrow \mathbb{Z} / 4 \times \mathbb{Z} / 2 & \sigma: \mathbb{Z} / 4 \times \mathbb{Z} / 2 & \longrightarrow \mathbb{Z} / 4 \times \mathbb{Z} / 2 \\
(1,0) & \longrightarrow(1,1) & (1,0) & \longrightarrow(1,1) \\
(0,1) & \longrightarrow(2,1) & (0,1) & \longrightarrow(0,1)
\end{aligned}
$$

with $\sigma \rho \sigma=\rho^{-1}$. The induced action in $H^{2}(\mathbb{Z} / 4 \times \mathbb{Z} / 2, \mathbb{Z})$ is given by the equations

$$
\rho^{*} u=u+v, \quad \rho^{*} v=2 u+v, \quad \sigma^{*} u=u, \quad \sigma^{*} v=2 u+v
$$

and therefore the 9 orbits are

$$
\begin{aligned}
& H^{4}(\mathbb{Z} / 4 \times \mathbb{Z} / 2, \mathbb{Z}) / A u t(\mathbb{Z} / 4 \times \mathbb{Z} / 2)= \\
& \quad\left\{\left\{u^{2}, u^{2}+v^{2}\right\},\left\{v^{2}\right\},\left\{u v, 2 u^{2}+u v+v^{2}, 2 u^{2}+u v, u v+v^{2}\right\},\left\{u^{2}+u v, 3 u^{2}+u v\right\}\right. \\
& \left.\quad\left\{3 u^{2}, 3 u^{2}+v^{2}\right\},\left\{u^{2}+u v+v^{2}, 3 u^{2}+u v+v^{2}\right\},\left\{2 u^{2}\right\},\left\{2 u^{2}+v^{2}\right\},\{0\}\right\}
\end{aligned}
$$

1.3. $\mathbb{Z} / 8$. The multiplicative group $\mathbb{Z} / 8^{*}$ of units of $\mathbb{Z} / 8$ is isomorphic to the automorphism group $\operatorname{Aut}(\mathbb{Z} / 8)$ and acts on $H^{*}(\mathbb{Z} / 8, \mathbb{Z})=\mathbb{Z}[s] /(8 s)$ by multiplication on the generator $s$. Since all the units of $\mathbb{Z} / 8$ square to 1 , the action of the units on $H^{4}(\mathbb{Z} / 8, \mathbb{Z})=\mathbb{Z}\left\langle s^{2}\right\rangle /\left(8 s^{2}\right)$ is trivial. Therefore

$$
H^{4}(\mathbb{Z} / 8, \mathbb{Z}) / A u t(\mathbb{Z} / 8)=\mathbb{Z}\left\langle s^{2}\right\rangle /\left(8 s^{2}\right)
$$

1.4. $D_{8}$. The dihedral group $D_{8}=\left\langle a, b \mid a^{4}=b^{2}=1, b a b=a^{-1}\right\rangle$ has for automorphisms a group isomorphic to $D_{8}$ generated by the automorphisms $\phi, \theta \in \operatorname{Aut}\left(D_{8}\right)$ with $\phi(a)=a, \phi(b)=b a, \theta(a)=a^{-1}, \theta(b)=b$, satisfying $\theta \phi \theta=\phi^{-1}$. The inner automorphisms are generated by $a d_{b}=\theta$ and $a d_{a}=\phi^{2}$, and since the inner automorphisms act trivially on the cohomology of the group, we only need to calculate the orbits that the action of $\phi^{*}$ induce on $H^{4}\left(D_{8}, \mathbb{Z}\right)$.

Consider the short exact sequence of groups

$$
\left\langle a^{2}\right\rangle=\mathbb{Z} / 2 \longrightarrow D_{8} \xrightarrow{\pi} \mathbb{Z} / 2 \times \mathbb{Z} / 2 \cong\langle a\rangle /{ }_{\left(a^{2}\right)} \times\langle\tau\rangle
$$

with $\pi(a)=(1,0)$ and $\pi(b)=(0,1)$, and define

$$
\begin{aligned}
\widehat{\phi}: \mathbb{Z} / 2 \times \mathbb{Z} / 2 & \longrightarrow \mathbb{Z} / 2 \times \mathbb{Z} / 2 \\
(1,0) & \longrightarrow(1,0) \\
(0,1) & \longrightarrow(1,1)
\end{aligned}
$$

Note that the isomorphism $\widehat{\phi}$ fits into the diagram

since $\phi$ fixes the center. We will fix a base for $H^{3}\left(D_{8}, \mathbb{C}^{*}\right)$ using the Lyndon-Hochschild-Serre spectral sequence associated to the short exact sequence, and since $\phi$ preserve the center, we can deduce the action of $\phi^{*}$ by looking at its action on the spectral sequence.

Let $H^{*}\left(\mathbb{Z} / 2 \times \mathbb{Z} / 2, \mathbb{F}_{2}\right)=\mathbb{F}_{2}[x, y]$ and note that the $k$-invariant of the extension of $D_{8}$ is $x y+x^{2}$ since on the restrition $\left.D_{8}\right|_{\mathbb{Z} / 2 \times\{0\}}$ we get the extension $0 \rightarrow \mathbb{Z} / 2 \rightarrow$ $\mathbb{Z} / 4 \rightarrow \mathbb{Z} / 2 \rightarrow 0$ whose $k$-invariant is $x^{2}$. The action of $\widehat{\phi}^{*}$ on $\mathbb{F}_{2}[x, y]$ is given by the equations $\widehat{\phi}^{*} x=x+y$ and $\widehat{\phi}^{*} y=y$.

The second page of the Lyndon-Hochschild-Serre spectral sequence

$$
E_{2}^{p, q}=H^{p}\left(\mathbb{Z} / 2 \times \mathbb{Z} / 2, H^{q}\left(\mathbb{Z} / 2, \mathbb{C}^{*}\right)\right)
$$

has for relevant terms

| 3 | $\left\langle z^{4}\right\rangle$ |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 2 | 0 | 0 |  |  |  |
| 1 | $\left\langle z^{2}\right\rangle$ | $\langle z x, z y\rangle$ | $\left\langle z x^{2}, z y^{2}, z x y\right\rangle$ |  |  |
| 0 | $\mathbb{C}^{*}$ | $\left\langle x^{2}, y^{2}\right\rangle$ | $\left\langle x^{2} y+x y^{2}\right\rangle$ | $\left\langle x^{4}, y^{4}, x^{2} y^{2}\right\rangle$ | $\left\langle x^{4} y+x^{3} y^{2}, x^{2} y^{3}+x y^{4}\right\rangle$ |
|  | 0 | 1 | 2 | 3 | 4 |

where we have taken in the fiber $H^{*}\left(\mathbb{Z} / 2, \mathbb{F}_{2}\right)=\mathbb{F}_{2}[z]$, and on $E_{2}^{*, 0}$ and $E_{2}^{0, *}$ we have taken the elements in $H^{*}(\mathbb{Z} / 2 \times \mathbb{Z} / 2, \mathbb{Z})$ and $H^{*}(\mathbb{Z} / 2, \mathbb{Z})$ respectively wich are trivial under the action of $S q^{1}$. Note that $\phi^{*}$ leaves $z$ fixed and on $x, y$ acts through the induced action of $\widehat{\phi}$.

The element $z^{4}$ survives the spectral sequence and the second differential $d_{2}$ : $H^{p}\left(\mathbb{Z} / 2 \times \mathbb{Z} / 2, \mathbb{F}_{2}\right) \rightarrow H^{p+2}\left(\mathbb{Z} / 2 \times \mathbb{Z} / 2, \mathbb{C}^{*}\right)$ is determined by the second differential $\bar{d}_{2}$ on the LHS spectral sequence with coefficients in $\mathbb{F}_{2}$. In this case $\bar{d}_{2}(z)=x y+x^{2}$ is the $k$-invariant and the second differential satisfies the equation

$$
d_{2}(z p(x, y))=S q^{1}\left(\left(x y+x^{2}\right) p(x, y)\right)
$$

where $p(x, y)$ is any polynomial in $x$ and $y$. The relevant terms of the third page of the spectral sequence are

and they all survive to the page at infinity.

The cohomology of the dihedral group is $H^{3}\left(D_{8}, \mathbb{C}^{*}\right)=\mathbb{Z} / 4 \oplus \mathbb{Z} / 2 \oplus \mathbb{Z} / 2$ and we may denote the generators

$$
\begin{equation*}
H^{3}\left(D_{8}, \mathbb{C}^{*}\right)=\langle\gamma\rangle \oplus\langle\alpha\rangle \oplus\langle\beta\rangle \tag{1.1}
\end{equation*}
$$

with $\beta=x^{4}, \alpha=y^{4}, E_{3}^{3,0} \cong\langle\gamma\rangle /(2 \gamma)$ and $E_{3}^{2,1} \cong\langle 2 \gamma\rangle /(4 \gamma)$. This choice of base agrees with the fact that when we restrict the spectral sequence to the subgroup $\langle a\rangle$ which fits in the short exact sequence $0 \rightarrow \mathbb{Z} / 2 \rightarrow \mathbb{Z} / 4 \rightarrow \mathbb{Z} / 2 \rightarrow 0$, the relevant surviving terms of the spectral sequence are only $E_{2}^{3,0}$ and $E_{3}^{2,1}$ (it is the same as making $y=0$ ).

Now, since $\phi^{*}$ leaves $z$ and $y$ fixed, and $\phi^{*} x=x+y$, we obtain that the 12 orbits of the action of $\phi^{*}$ are the following:

$$
\begin{aligned}
& H^{3}\left(D_{8}, \mathbb{C}^{*}\right) / \operatorname{Aut}\left(D_{8}\right)=\{\{0\},\{\gamma\},\{2 \gamma\},\{3 \gamma\},\{\alpha\},\{\beta, \alpha+\beta\},\{\gamma+\beta, \gamma+\alpha+\beta\} \\
&\{2 \gamma+\beta, 2 \gamma+\alpha+\beta\},\{3 \gamma+\beta, 3 \gamma+\alpha+\beta\},\{\gamma+\alpha\},\{2 \gamma+\alpha\},\{3 \gamma+\alpha\}\}
\end{aligned}
$$

1.5. $Q_{8}$. The group of quaternions $Q_{8}=\{ \pm 1, \pm i, \pm j, \pm k\}$ is a subgroup of $S U(2)$ and therefore has periodic cohomology with $H^{4}\left(Q_{8}, \mathbb{Z}\right)=\mathbb{Z} / 8$. The group of automorphisms $\operatorname{Aut}\left(Q_{8}\right)$ is isomorphic to the symetric group $\mathfrak{S}_{4}$.

The resolution

$$
\cdots \longrightarrow \mathbb{Z} Q_{8} \oplus \mathbb{Z} Q_{8} \xrightarrow{\delta_{1}} \mathbb{Z} Q_{8} \xrightarrow{\delta_{4}} \mathbb{Z} Q_{8} \xrightarrow{\delta_{3}} \mathbb{Z} Q_{8} \oplus \mathbb{Z} Q_{8} \xrightarrow{\delta_{2}} \mathbb{Z} Q_{8} \oplus \mathbb{Z} Q_{8} \xrightarrow{\delta_{1}} \mathbb{Z} Q_{8} \xrightarrow{\epsilon} \mathbb{Z}
$$

with differentials $\delta_{i}$ defined by the equations

$$
\begin{aligned}
& \delta_{1}\left(a_{1}, a_{2}\right)=a_{1}(i-1)+a_{2}(j-1) \\
& \delta_{2}\left(a_{1}, a_{2}\right)=\left(a_{1}(i+1)+a_{2}(i j+1),-a_{1}(j+1)+a_{2}(i-1)\right) \\
& \delta_{3}(a)=(a(i-1),-a(i j-1)) \\
& \delta_{4}(a)=a \sum_{\tau \in Q_{8}} \tau
\end{aligned}
$$

provides a periodic resolution of $\mathbb{Z}$ by free $\mathbb{Z} Q_{8}$-modules. Applying the functor $\operatorname{Hom}_{\mathbb{Z} Q_{8}}(\cdot, \mathbb{Z})$ we obtain a cochain complex that calculates the cohomology of $Q_{8}$ and on degree 4 we obtain $\operatorname{Hom}_{\mathbb{Z} Q_{8}}\left(\mathbb{Z} Q_{8}, \mathbb{Z}\right)$ which is invariant under the action of the automorphis group $\operatorname{Aut}\left(Q_{8}\right)$. Therefore $\operatorname{Aut}\left(Q_{8}\right)$ acts trivially on $H^{4}\left(Q_{8}, \mathbb{Z}\right)$ and we obtain 8 orbits

$$
H^{4}\left(Q_{8}, \mathbb{Z}\right) / A u t\left(Q_{8}\right)=H^{4}\left(Q_{8}, \mathbb{Z}\right)=\mathbb{Z}\langle t\rangle /(8 t)
$$

We conclude that are $10+9+8+12+8=47$ equivalence classes of pointed fusion categories of global dimension 8 .

## 2. Morita equivalence classes of pointed fusion categories of global DIMENSION 8

In this section we will explicitly determine which pointed fusion categories of global dimension 8 have equivalent categories of modules categories. We will use the notation and the results of Uri15] and we will recall the classification theorem Uri15, Thm. 3.9].

An skeletal indecomposable module category $\mathcal{M}=(A \backslash H, \mu)$ of $\mathcal{C}=\mathcal{V}(H, \eta)$ is determined by a transitive $H$-set $K:=A \backslash H$ with $A$ subgroup of $H$, and a cochain $\mu \in C^{2}\left(H, \operatorname{Map}\left(K, \mathbb{C}^{*}\right)\right)$ such that $\delta_{H} \mu=\pi^{*} \eta$ with $\pi^{*} \eta\left(k ; h_{1}, h_{2}, h_{3}\right)=$ $\eta\left(h_{1}, h_{2}, h_{3}\right)$ (see Uri15, §3.3]). The skeletal tensor category of the tensor category
$\mathcal{C}_{\mathcal{M}}^{*}=\operatorname{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ is equivalent to one of the form $\mathcal{V}(\widehat{H}, \widehat{\eta})$ whenever $A$ is normal and abelian in $H$ [Nai07] and if there exists a cochain $\gamma \in C^{1}\left(H, \operatorname{Map}\left(K, \mathbb{C}^{*}\right)\right)$ such that $\delta_{H} \gamma=\delta_{K} \mu$. In particular this implies that the cohomology class of $\eta$ belongs to the subgroup of $H^{3}\left(H, \mathbb{C}^{*}\right)$ defined by

$$
\Omega(H ; A):=\operatorname{ker}\left(\operatorname{ker}\left(H^{3}\left(H, \mathbb{C}^{*}\right) \rightarrow E_{\infty}^{0,3}\right) \rightarrow E_{\infty}^{1,2}\right)
$$

which fits into the short exact sequence Uri15, Cor. 3.2]

$$
0 \rightarrow E_{\infty}^{3,0} \rightarrow \Omega(G ; A) \rightarrow E_{\infty}^{2,1} \rightarrow 0
$$

where $E_{n}^{*, *}$ denotes the $n$-th page of the Lyndon-Hochschild-Serre spectral sequence associated to the group extension $1 \rightarrow A \rightarrow H \rightarrow K \rightarrow 1$.

Let us bring the last piece of notation. Denote the dual group $\mathbb{A}:=\operatorname{Hom}\left(A, \mathbb{C}^{*}\right)$ and consider cocycles $F \in Z^{2}(K, A)$ and $\widehat{F} \in Z^{2}(K, \mathbb{A})$. Denote by $G=A \rtimes_{F} K$ and $\widehat{G}=K \ltimes_{\widehat{F}} \mathbb{A}$ the groups defined by the multiplication laws

$$
\begin{aligned}
& \left(a_{1}, k_{1}\right)\left(a_{2}, k_{2}\right):=\left(a_{1}\left({ }^{k_{1}} a_{2}\right) F\left(k_{1}, k_{2}\right), k_{1} k_{2}\right) \\
& \left(k_{1}, \rho_{1}\right) \cdot\left(k_{2}, \rho_{2}\right):=\left(k_{1} k_{2},\left(\rho_{1}^{k_{2}}\right) \rho_{2} \widehat{F}\left(k_{1}, k_{2}\right)\right)
\end{aligned}
$$

respectively. The necessary and sufficient conditions for two pointed fusion categories to be Morita equivalent are the following (cf. [Nai07]):

Theorem 2.1. Uri15, Thm. 5.9] Let $H$ and $\widehat{H}$ be finite groups, $\eta \in Z^{3}\left(H, \mathbb{C}^{*}\right)$ and $\widehat{\eta} \in Z^{3}\left(\widehat{H}, \mathbb{C}^{*}\right)$. Then the tensor categories $\operatorname{Vect}(H, \eta)$ and $\operatorname{Vect}(\widehat{H}, \widehat{\eta})$ are weakly Morita equivalent if and only if the following conditions are satisfied:

- There exist isomorphisms of groups

$$
\phi: G=A \rtimes_{F} K \xlongequal{\cong} H \quad \widehat{\phi}: \widehat{G}=K \ltimes_{\widehat{F}} \mathbb{A} \cong \widehat{\rightrightarrows} \widehat{H}
$$

for some finite group $K$ acting on the abelian group $A$, with cocycles $F \in$ $Z^{2}(K, A)$ and $\widehat{F} \in Z^{2}(K, \mathbb{A})$.

- There exists $\epsilon: K^{3} \rightarrow \mathbb{C}^{*}$ such that $\widehat{F} \wedge F=\delta_{K} \epsilon$.
- The cohomology classes satisfy the equations $[\omega]=\left[\phi^{*} \eta\right]$ and $[\widehat{\omega}]=\left[\widehat{\phi^{*}} \widehat{\eta}\right]$ with

$$
\begin{aligned}
& \omega\left(\left(a_{1}, k_{1}\right),\left(a_{2}, k_{2}\right),\left(a_{3}, k_{3}\right)\right):=\widehat{F}\left(k_{1}, k_{2}\right)\left(a_{3}\right) \epsilon\left(k_{1}, k_{2}, k_{3}\right) \\
& \widehat{\omega}\left(\left(k_{1}, \rho_{1}\right),\left(k_{2}, \rho_{2}\right),\left(k_{3}, \rho_{3}\right)\right):=\epsilon\left(k_{1}, k_{2}, k_{3}\right) \rho_{1}\left(F\left(k_{2}, k_{3}\right)\right) .
\end{aligned}
$$

Note that $\mathbb{A}$ and $A$ are (non-canonically) isomorphic as $K$-modules, and therefore both $H$ and $\widehat{H}$ could be seen as extensions of $K$ by $A$. In order to calculate all possible Morita equivalences, we will analize the Morita equivalences that appear while fixing the group $K$ and the $K$-module $A$.

Let us recall the equivalence classes of normal an abelian subgroups of the groups of order 8 . We will say that two subgroups are equivalent if there is an automorphism of the group that maps one to the other. The following table contains the information on these subgroups:

| Isomorphic to | $D_{8}$ | $\mathbb{Z} / 4 \oplus \mathbb{Z} / 2$ | $Q_{8}$ | $\mathbb{Z} / 8$ |
| :---: | :---: | :---: | :---: | :---: |
| $\mathbb{Z} / 2$ | $\left\langle a^{2}\right\rangle$ | $\langle(2,0)\rangle$ | $\langle-1\rangle$ | $\langle 4\rangle$ |
| $\mathbb{Z} / 2$ |  | $\langle(0,1)\rangle,\langle(2,1)\rangle$ |  |  |
| $\mathbb{Z} / 2 \times \mathbb{Z} / 2$ | $\left\langle a^{2}, b a^{3}\right\rangle,\left\langle a^{2}, b\right\rangle$ | $\langle(2,0)\rangle \oplus\langle(0,1)\rangle$ |  |  |
| $\mathbb{Z} / 4$ | $\langle a\rangle$ | $\langle(1,0)\rangle,\langle(1,1)\rangle$ | $\langle i\rangle,\langle j\rangle,\langle k\rangle$ | $\langle 2\rangle,\langle 6\rangle$ |

In the group $(\mathbb{Z} / 2)^{3}$ all subgroups of order 2 are isomorphic to $\langle(0,0,1)\rangle$ and all subgroups of order 4 are isomorphic to $\langle(0,1,0),(0,0,1)\rangle$.

Now let us outline the procedure that we will follow. We will fix the groups $K$ and $A$, we will take the groups that are extensions of $K$ by $A$ and we will take explicit choices of subgroups from the table above that provide the extensions. Then we will calculate the relevant terms of the second page of the Lyndon-Hochschild-Serre spectral sequence, which are the same for all extensions of $K$ by $A$, and we will calculate the third page for each extension $0 \rightarrow A \rightarrow H \rightarrow K \rightarrow 1$. Then we will determine the cohomology class of the 2-cocycle $F$ that makes $H \cong A \rtimes_{F} K$ and we will calculate the cohomology classes in $\Omega(H ; A)$. With this information and Theorem 2.1 we will determine the Morita equivalence classes of pointed fusion categories for groups that are extensions of $K$ by $A$.
2.1. $K=\mathbb{Z} / 2$ and $A=\mathbb{Z} / 4$ with trivial action. The two possible extensions are $\mathbb{Z} / 8$ and $\mathbb{Z} / 4 \times \mathbb{Z} / 2$ with the following choices of subgroups:

$$
\begin{gathered}
1 \longrightarrow\langle(1,0)\rangle \longrightarrow \mathbb{Z} / 4 \times \mathbb{Z} / 2 \longrightarrow \mathbb{Z} / 2 \longrightarrow 1 \\
1 \longrightarrow\langle 2\rangle \longrightarrow \mathbb{Z} / 8 \longrightarrow \mathbb{Z} / 2 \longrightarrow 1 .
\end{gathered}
$$

For the group $\mathbb{Z} / 4 \times \mathbb{Z} / 2$ the relevant terms of the second page of the LHS spectral sequence are:
$\left.3 \left\lvert\, \begin{array}{ccccc}\mathbb{Z} / 4=\left\langle u^{2}\right\rangle & & & \\ 2 & 0 & 0 & \\ 1 \\ 0 & 0 & \mathbb{Z} / 4 & \mathbb{Z} / 2 & \mathbb{Z} / 2=\langle u v\rangle \\ \mathbb{Z} / 4 \\ \mathbb{C}^{*} & \mathbb{Z} / 2 & 0 & \mathbb{Z} / 2=\left\langle v^{2}\right\rangle & 0 \\ \hline 0 & 1 & 2 & 3 & 4\end{array}\right.\right]$

Since $H^{2}(\mathbb{Z} / 2, \mathbb{Z} / 4)=\mathbb{Z} / 2$ we have that $\mathbb{Z} / 8 \cong \mathbb{Z} / 2 \ltimes_{u v} \mathbb{Z} / 4$, and therefore the relevant terms of the third page of the LHS spectral sequence for $\mathbb{Z} / 8$ are:


We get the split short exact sequences

$$
\begin{aligned}
0 \rightarrow\left\langle v^{2}\right\rangle & \rightarrow \Omega(\mathbb{Z} / 4 \times \mathbb{Z} / 2 ;\langle(1,0)\rangle) \rightarrow\langle u v\rangle \rightarrow 0 \\
0 & \rightarrow \Omega(\mathbb{Z} / 8 ;\langle 4\rangle) \stackrel{\cong}{\rightrightarrows}\left\langle 4 s^{2}\right\rangle \rightarrow 0
\end{aligned}
$$

and we conclude that the only Morita equivalences that appear are:

$$
\operatorname{Vect}(\mathbb{Z} / 8,0) \simeq_{M} \operatorname{Vect}(\mathbb{Z} / 4 \times \mathbb{Z} / 2, u v) \simeq_{M} \operatorname{Vect}\left(\mathbb{Z} / 4 \times \mathbb{Z} / 2, u v+v^{2}\right)
$$

2.2. $K=\mathbb{Z} / 2$ and $A=\mathbb{Z} / 4$ with non-trivial action. In this case the possible extensions are:

$$
\begin{aligned}
& 1 \longrightarrow\langle a\rangle \longrightarrow D_{8} \longrightarrow \mathbb{Z} / 2 \longrightarrow 1 \\
& 1 \longrightarrow\langle i\rangle \longrightarrow Q_{8} \longrightarrow \mathbb{Z} / 2 \longrightarrow 1 .
\end{aligned}
$$

where $D_{8}$ is the trivial extension of $\mathbb{Z} / 2$ by the non-trivial module $\mathbb{Z} / 4$. For $D_{8}$ the relevant elements of the second page of the LHS spectral sequence are:

| 3 | $\mathbb{Z} / 4=\langle\gamma\rangle$ |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 2 | 0 | 0 |  |  |  |
| 1 | $\mathbb{Z} / 2$ | $\mathbb{Z} / 2$ | $\mathbb{Z} / 2=\langle\beta\rangle$ |  |  |
| 0 | $\mathbb{C}^{*}$ | $\mathbb{Z} / 2$ | 0 | $\mathbb{Z} / 2=\langle\alpha\rangle$ | 0 |
|  | 0 | 1 | 2 | 3 | 4 |

where the generators of the cohomology of $D_{8}$ were defined in (1.1). Note that the cohomology class $\alpha$ is the pullback of the generator of $H^{4}(\langle b\rangle, \mathbb{Z})$ and $\beta$ is the cohomology class in $H^{2}(\mathbb{Z} / 2, \mathbb{Z} / 4)$ that classifies the non-trivial extension determined by $Q_{8}$.

The second page of the LHS spectral sequence for the extension $Q_{8} \cong \mathbb{Z} / 2 \ltimes_{\beta} \mathbb{Z} / 4$ becomes:

where the second differential $d_{2}: E_{2}^{1,1} \cong E_{2}^{3,0}$ is an isomorphism since $H^{3}\left(Q_{8}, \mathbb{C}^{*}\right)=$ $\langle t\rangle /(8 t)$.

We get the split short exact sequences

$$
\begin{gathered}
0 \rightarrow\langle\alpha\rangle \rightarrow \Omega\left(D_{8} ;\langle a\rangle\right) \rightarrow\langle\beta\rangle \rightarrow 0 \\
0 \rightarrow \Omega\left(Q_{8} ;\langle i\rangle\right) \stackrel{\cong}{\rightrightarrows}\langle 4 t\rangle \rightarrow 0
\end{gathered}
$$

and we conclude that the only Morita equivalences that appear are:

$$
V e c t\left(Q_{8}, 0\right) \simeq_{M} \operatorname{Vect}\left(D_{8}, \beta\right) \simeq_{M} \operatorname{Vect}\left(D_{8}, \beta \oplus \alpha\right)
$$

2.3. $K=\mathbb{Z} / 4$ and $A=\mathbb{Z} / 2$. The two extensions are:

$$
\begin{gathered}
1 \longrightarrow\langle(0,1)\rangle \longrightarrow \mathbb{Z} / 4 \times \mathbb{Z} / 2 \rightarrow \mathbb{Z} / 4 \longrightarrow 1 \\
1 \longrightarrow\langle 4\rangle \longrightarrow \mathbb{Z} / 8 \longrightarrow \mathbb{Z} / 4 \longrightarrow 1
\end{gathered}
$$

and the relevant terms of the second page of the LHS spectral sequence for $\mathbb{Z} / 4 \times \mathbb{Z} / 2$ are:
3
2
$\left.1 \begin{array}{ccccc}\mathbb{Z} / 2=\left\langle v^{2}\right\rangle & & & \\ 0 & 0 & 0 & \\ 0 & \mathbb{Z} / 2 & \mathbb{Z} / 2 & \mathbb{Z} / 2=\langle u v\rangle & \\ \mathbb{C}^{*} & \mathbb{Z} / 4 & 0 & \mathbb{Z} / 4=\left\langle u^{2}\right\rangle & 0 \\ \hline 0 & 1 & 2 & 3 & 4\end{array}\right]$

We have again that $\mathbb{Z} / 8=\mathbb{Z} / 4 \rtimes_{u v} \mathbb{Z} / 2$, and the relevant terms of the third page of the LHS spectral sequence associated to the extension of $\mathbb{Z} / 8$ are:
3
2
1

0 | $\mathbb{Z} / 2=\left\langle s^{2}\right\rangle /\left(2 s^{2}\right)$ |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | 0 |  |  |
| $\mathbb{Z} / 2$ | 0 | $\mathbb{Z} / 2=\left\langle 2 s^{2}\right\rangle /\left(4 s^{2}\right)$ |  |  |
| $\mathbb{C}^{*}$ | $\mathbb{Z} / 4$ | 0 | $\mathbb{Z} / 2=\left\langle 4 s^{2}\right\rangle$ | 0 |
| 0 | 1 | 2 | 3 | 4 |

We obtain the short exact sequences

$$
\begin{gathered}
0 \rightarrow\left\langle v^{2}\right\rangle \rightarrow \Omega(\mathbb{Z} / 4 \times \mathbb{Z} / 2 ;\langle(0,1)\rangle) \rightarrow\langle u v\rangle \rightarrow 0 \\
0 \rightarrow\left\langle 4 s^{2}\right\rangle /\left(8 s^{2}\right) \rightarrow \Omega(\mathbb{Z} / 8 ;\langle 2\rangle) \rightarrow\left\langle 2 s^{2}\right\rangle /\left(4 s^{2}\right) \rightarrow 0
\end{gathered}
$$

where $\Omega(\mathbb{Z} / 8 ;\langle 2\rangle) \cong \mathbb{Z} / 4$. We conclude that the only Morita equivalences that appear are:

$$
\begin{gathered}
\operatorname{Vect}(\mathbb{Z} / 8,0) \simeq_{M} \operatorname{Vect}(\mathbb{Z} / 4 \times \mathbb{Z} / 2, u v) \\
\operatorname{Vect}\left(\mathbb{Z} / 8,4 s^{2}\right) \simeq_{M} \operatorname{Vect}\left(\mathbb{Z} / 4 \times \mathbb{Z} / 4, u v+u^{2}\right) \simeq_{M} \operatorname{Vect}\left(\mathbb{Z} / 4 \times \mathbb{Z} / 4, u v+3 u^{2}\right)
\end{gathered}
$$

2.4. $K=\mathbb{Z} / 2$ and $A=\mathbb{Z} / 2 \times \mathbb{Z} / 2$ with trivial action. The relevant extensions are:

$$
\begin{gathered}
1 \longrightarrow\langle(0,0,1),(0,1,0)\rangle \longrightarrow(\mathbb{Z} / 2)^{3} \rightarrow \mathbb{Z} / 2 \longrightarrow 1 \\
1 \longrightarrow\langle(2,0),(0,1)\rangle \longrightarrow \mathbb{Z} / 4 \times \mathbb{Z} / 2 \longrightarrow \mathbb{Z} / 2 \longrightarrow 1
\end{gathered}
$$

The relevant terms for the second page $E_{2}^{p, q}=H^{p}\left(\mathbb{Z} / 2, H^{q}\left(\mathbb{Z} / 2 \times \mathbb{Z} / 2, \mathbb{C}^{*}\right)\right.$ of the LHS spectral sequence are:

| 3 | $(\mathbb{Z} / 2)^{3}$ |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 2 | $\left\langle y^{2} z+y z^{2}\right\rangle$ | $\langle x y z\rangle$ |  |  |  |
| 1 | $\left\langle y^{2}, z^{2}\right\rangle$ | $\langle x y, x z\rangle$ | $\left\langle x^{2} y, x^{2} z\right\rangle$ |  |  |
| 0 | $\mathbb{C}^{*}$ | $\left\langle x^{2}\right\rangle$ | 0 | $\left\langle x^{4}\right\rangle$ | 0 |
|  | 0 | 1 | 2 | 3 | 4 |

where on the base we have $H^{*}\left(\mathbb{Z} / 2, \mathbb{F}_{2}\right) \cong \mathbb{F}_{2}[x]$ and on the fiber $H^{*}(\mathbb{Z} / 2 \times$ $\left.\mathbb{Z} / 2, \mathbb{F}_{2}\right) \cong \mathbb{F}_{2}[y, z]$. The cohomology classes that appear on the 0 -th row an the 0 -th column are the ones that are annihilated by the operation $S q^{1}$.

The second differential of the LHS spectral sequence of the extension of $\mathbb{Z} / 4 \times \mathbb{Z} / 2$, whose $k$-invariant is the class $x^{2}$, maps

$$
y^{2} z+y z^{2} \mapsto x^{2} y, \quad x z \mapsto x^{4}, \quad x y \mapsto 0, \quad x y z \mapsto x^{3} y
$$

Therefore the relevant terms of the third page of the LHS spectral sequence for $\mathbb{Z} / 4 \times \mathbb{Z} / 2$ are (using the base defined in $\$ 1.2$ ):


We get the split short exact sequences

$$
\begin{gathered}
0 \rightarrow\left\langle x^{4}\right\rangle \rightarrow \Omega\left((\mathbb{Z} / 2)^{3} ;\langle(0,0,1),(0,1,0)\rangle\right) \rightarrow\left\langle x^{2} z^{2}, x^{2} y^{2}\right\rangle \rightarrow 0 \\
\quad 0 \rightarrow \Omega(\mathbb{Z} / 4 \times \mathbb{Z} / 2 ;\langle(2,0),(0,1)\rangle) \stackrel{\cong}{\leftrightarrows}\left\langle 2 u^{2}\right\rangle \rightarrow 0
\end{gathered}
$$

where all the non-trivial classes in $\left\langle x^{2} z^{2}, x^{2} y^{2}\right\rangle$ define an extension isomorphic to $\mathbb{Z} / 4 \times \mathbb{Z} / 2$, and $\mathbb{Z} / 4 \times \mathbb{Z} / 2 \cong \mathbb{Z} / 2 \ltimes_{x^{2} y^{2}}(\mathbb{Z} / 2 \times \mathbb{Z} / 2)$. We conclude that the only Morita equivalence that appear is:

$$
V e c t(\mathbb{Z} / 4 \times \mathbb{Z} / 2,0) \simeq_{M} \operatorname{Vect}\left((\mathbb{Z} / 2)^{3}, x^{2} z^{2}\right)
$$

2.5. $K=\mathbb{Z} / 2$ and $A=\mathbb{Z} / 2 \times \mathbb{Z} / 2$ with non-trivial action. Let us take the action of $\mathbb{Z} / 2$ that flips the coordinates in $\mathbb{Z} / 2 \times \mathbb{Z} / 2$, therefore the only extension is equivalent to the following one:

$$
0 \longrightarrow\left\langle b, b a^{2}\right\rangle \longrightarrow D_{8} \longrightarrow \mathbb{Z} / 2 \longrightarrow 1
$$

Since there is up to isomorphism only one extension, this choice of $K$ and $A$ does not produce any Morita equivalence between non-equivalent pointed fusion categories.
2.6. $K=\mathbb{Z} / 2 \times \mathbb{Z} / 2$ and $A=\mathbb{Z} / 2$. In this last case the relevant extensions are:

$$
\begin{gathered}
1 \longrightarrow\langle(0,0,1)\rangle \longrightarrow(\mathbb{Z} / 2)^{3} \longrightarrow \mathbb{Z} / 2 \times \mathbb{Z} / 2 \longrightarrow 1 \\
1 \longrightarrow\langle(2,0)\rangle \longrightarrow \mathbb{Z} / 4 \times \mathbb{Z} / 2 \longrightarrow \mathbb{Z} / 2 \times \mathbb{Z} / 2 \longrightarrow 1 \\
1 \longrightarrow\left\langle a^{2}\right\rangle \longrightarrow D_{8} \longrightarrow \mathbb{Z} / 2 \times \mathbb{Z} / 2 \longrightarrow 1 \\
1 \longrightarrow\langle-1\rangle \longrightarrow Q_{8} \longrightarrow \mathbb{Z} / 2 \times \mathbb{Z} / 2 \longrightarrow 1 .
\end{gathered}
$$

The relevant terms of the second page $E_{2}^{p, q}=H^{p}\left(\mathbb{Z} / 2 \times \mathbb{Z} / 2, H^{q}\left(\mathbb{Z} / 2, \mathbb{C}^{*}\right)\right)$ of the LHS spectral sequence are:

| 3 | $\left\langle z^{4}\right\rangle$ |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 2 | 0 | 0 | 0 |  |  |
| 1 | $\left\langle z^{2}\right\rangle$ | $\langle z x, z y\rangle$ | $\left\langle z x^{2}, z y^{2}, z x y\right\rangle$ |  |  |
| 0 | $\mathbb{C}^{*}$ | $\left\langle x^{2}, y^{2}\right\rangle$ | $\left\langle x^{2} y+x y^{2}\right\rangle$ | $\left\langle x^{4}, y^{4}, x^{2} y^{2}\right\rangle$ | $\left\langle x^{4} y+x^{3} y^{2}, x^{2} y^{3}+x y^{4}\right\rangle$ |
|  | 0 | 1 | 2 | 3 | 4 |

where we have assumed that $H^{*}\left(\mathbb{Z} / 2 \times \mathbb{Z} / 2, \mathbb{F}_{2}\right)=\mathbb{F}_{2}[x, y]$ for the base, $H^{*}\left(\mathbb{Z} / 2, \mathbb{F}_{2}\right)=$ $\mathbb{F}_{2}[z]$ for the fiber, and the elements on the 0 -th row and the 0 -th column are the classes that get annihilated by the operation $S q^{1}$.

The calculation of the second differential between the first row and the 0 -th row

$$
d_{2}^{G}: H^{p}\left((\mathbb{Z} / 2)^{2} ; \mathbb{F}_{2}\right) \otimes H^{1}\left(\mathbb{Z} / 2 ; \mathbb{F}_{2}\right) \rightarrow H^{p+2}\left((\mathbb{Z} / 2)^{2}, \mathbb{C}^{*}\right)
$$

depends on the $k$-invariant of the extension $G$. In $₫ 1.4$ we explain the case for $D_{8}$ on which the differential was

$$
d_{2}^{D_{8}}(z p(x, y))=S q^{1}\left(\left(x y+x^{2}\right) p(x, y)\right)
$$

and following the same argument we get that the second differentials for the other two groups are

$$
\begin{aligned}
& d_{2}^{\mathbb{Z} / 4 \times \mathbb{Z} / 2}(z p(x, y))=S q^{1}\left(\left(x^{2}\right) p(x, y)\right) \\
& d_{2}^{Q_{8}}(z p(x, y))=S q^{1}\left(\left(x^{2}+x y+x^{2}\right) p(x, y)\right)
\end{aligned}
$$

2.6.1. For the group $(\mathbb{Z} / 2)^{3}$ we obtain a split extension

$$
0 \rightarrow\left\langle x^{4}, y^{4}, x^{2} y^{2}\right\rangle \rightarrow \Omega\left((\mathbb{Z} / 2)^{3} ;\langle(0,0,1)\rangle\right) \rightarrow\left\langle z x^{2}, z y^{2}, z x y\right\rangle \rightarrow 0
$$

where the classes $x^{2}, x^{2}+y^{2}, y^{2}$ in $H^{2}\left((\mathbb{Z} / 2)^{2}, \mathbb{F}_{2}\right)$ define an extension isomorphic to $\mathbb{Z} / 4 \times \mathbb{Z} / 2$, the classes $x y, x y+x^{2}, x y+y^{2}$ define an extension isomorphic to $D_{8}$ and the class $x^{2}+x y+y^{2}$ define an extension isomorphic to $Q_{8}$. Therefore it is enough to analize the cases determined by the classes $x^{2}, x y+x^{2}$ and $x^{2}+x y+y^{2}$.
2.6.2. For the group $\mathbb{Z} / 4 \times \mathbb{Z} / 2$ we have that it is isomorphic to the group $\mathbb{Z} / 2 \ltimes_{x^{2}}$ $(\mathbb{Z} / 2 \times \mathbb{Z} / 2)$ and the relevant terms of the third page of the LHS are:


Following the notations of $\$ 1.2$ we know that $E_{3}^{3,0}$ is generated by $v^{2}$ and $E_{3}^{2,1}$ is generated by $u v$ and $2 u^{2}$, where $u v$ corresponds to the class $y^{2} z$ and $2 u^{2}$ corresponds to the class $x^{2} z$. Therefore we obtain

$$
0 \rightarrow\left\langle v^{2}\right\rangle \rightarrow \Omega(\mathbb{Z} / 4 \times \mathbb{Z} / 2 ;\langle(2,0)\rangle) \rightarrow\left\langle u v, 2 u^{2}\right\rangle \rightarrow 0
$$

and since all the classes in $E_{3}^{2,1}$ induce extensions isomorphic to $\mathbb{Z} / 4 \times \mathbb{Z} / 2$, we conclude that the only Morita equivalences that we obtain in this case are:

$$
\begin{aligned}
\operatorname{Vect}(\mathbb{Z} / 4 \times \mathbb{Z} / 2,0) & \simeq_{M} \operatorname{Vect}\left((\mathbb{Z} / 2)^{3}, x^{2} z^{2}\right) \\
\operatorname{Vect}\left(\mathbb{Z} / 4 \times \mathbb{Z} / 2, v^{2}\right) & \simeq_{M} \operatorname{Vect}\left((\mathbb{Z} / 2)^{3}, x^{2} z^{2}+y^{4}\right) .
\end{aligned}
$$

2.6.3. The calculation of the relevant terms of the third page of the LHS spectral sequence for the group $D_{8} \cong \mathbb{Z} / 2 \ltimes_{x y+x^{2}}(\mathbb{Z} / 2 \times \mathbb{Z} / 2)$ was done in $\oint 1.4$ and therefore we obtain the short exact sequence

$$
0 \rightarrow\langle\alpha, \beta\rangle \rightarrow \Omega\left(D_{8} ;\left\langle a^{2}\right\rangle\right) \rightarrow\langle 2 \gamma\rangle \rightarrow 0
$$

Since $E_{3}^{2,1}$ is generated by the class $z\left(x y+y^{2}\right)$ and $x y+x^{2}$ is the $k$-invariant for $D_{8}$, we only get the following Morita equivalences:

$$
\begin{gathered}
V e c t\left(D_{8}, 0\right) \simeq_{M} \operatorname{Vect}\left((\mathbb{Z} / 2)^{3}, x^{2} y z+x y^{2} z+x y z^{2}+x^{2} z^{2}\right) \\
V e c t\left(D_{8}, \alpha\right) \simeq_{M} \operatorname{Vect}\left((\mathbb{Z} / 2)^{3}, y^{4}+x^{2} y z+x y^{2} z+x y z^{2}+x^{2} z^{2}\right) \\
\operatorname{Vect}\left(D_{8}, \beta\right) \simeq_{M} \operatorname{Vect}\left((\mathbb{Z} / 2)^{3}, x^{4}+x^{2} y z+x y^{2} z+x y z^{2}+x^{2} z^{2}\right)
\end{gathered}
$$

2.6.4. The relevant terms of the third page for the LHS spectral sequence for the group $Q_{8} \cong \mathbb{Z} / 2 \ltimes_{x^{2}+x y+y^{2}}(\mathbb{Z} / 2 \times \mathbb{Z} / 2)$ are:


Since $H^{3}\left(Q_{8}, \mathbb{C}^{*}\right) \cong \mathbb{Z}\langle t\rangle /(8 t)$ we know that $E_{3}^{0,3} \cong \mathbb{Z}\langle t\rangle /(2 t), E_{3}^{2,1} \cong \mathbb{Z}\langle 2 t\rangle /(4 t)$ and $E_{3}^{3,0} \cong \mathbb{Z}\langle 4 t\rangle /(8 t)$. Therefore we get that $\Omega\left(Q_{8} ;\langle-1\rangle\right)=\mathbb{Z}\langle 2 t\rangle /(8 t)$ which fits into the short exact sequence:

$$
0 \rightarrow \mathbb{Z}\langle 4 t\rangle /(8 t) \rightarrow \Omega\left(Q_{8} ;\langle-1\rangle\right) \rightarrow \mathbb{Z}\langle 2 t\rangle /(4 t) \rightarrow 0
$$

Since the $E_{3}^{2,1}$ is generated by the class $z\left(x^{2}+x y+y^{2}\right)$ and $x^{2}+x y+y^{2}$ is the $k$-invariant of $Q_{8}$, we only get the following Morita equivalences:

$$
\begin{gathered}
\operatorname{Vect}\left(Q_{8}, 0\right) \simeq_{M} \operatorname{Vect}\left((\mathbb{Z} / 2)^{3}, x^{2} y z+x y^{2} z+x y z^{2}+x^{2} z^{2}+y^{2} z^{2}\right) \\
\operatorname{Vect}\left(Q_{8}, 4 t\right) \simeq_{M} \operatorname{Vect}\left((\mathbb{Z} / 2)^{3}, x^{2} y z+x y^{2} z+x y z^{2}+x^{2} z^{2}+y^{2} z^{2}+x^{2} y^{2}\right)
\end{gathered}
$$

2.7. We conclude that the only weak Morita equivalences between pointed fusion categories of global dimension 8 are the ones that appear on each row of the following table:

| $(\mathbb{Z} / 2)^{3}$ | $\mathbb{Z} / 4 \times \mathbb{Z} / 2$ | $\mathbb{Z} / 8$ | $D_{8}$ | $Q_{8}$ |
| :---: | :---: | :---: | :---: | :---: |
| $\operatorname{orb}\left(x^{2} y^{2}\right)$ | $\{0\}$ |  |  |  |
| $\operatorname{orb}\left(x^{4}+y^{2} z^{2}\right)$ | $\left\{v^{2}\right\}$ |  |  |  |
| $\operatorname{orb}\left(x^{2} y z+x y^{2} z+x y z^{2}\right)$ |  |  | $\{0\}$ |  |
| $\operatorname{orb}\left(x^{4}+x^{2} y z+x y^{2} z+x y z^{2}\right)$ |  | $\{\alpha+\beta, \beta\}$ | $\{0\}$ |  |
| $\operatorname{orb}\left(x^{4}+x^{2} y z+x y^{2} z+x y z^{2}+y^{2} z^{2}\right)$ |  | $\{\alpha\}$ |  |  |
| $\operatorname{orb}\left(x^{2} y z+x y^{2} z+x y z^{2}+x^{2} y^{2}+x^{2} z^{2}+y^{2} z^{2}\right)$ |  |  | $\{4 t\}$ |  |
|  | $\operatorname{orb}(u v)$ | $\{0\}$ |  |  |
|  | $\operatorname{orb}\left(u v+u^{2}\right)$ | $\left\{4 s^{2}\right\}$. |  |  |

Therefore there are only $47-9=36$ Morita equivalence classes of pointed fusion categories of global dimension 8 .
2.8. Twisted Drinfeld double. The center of a fusion category is again a fusion category and it is moreover braided. In ENO11, Thm. 3.1] it is shown that two tensor categories are weakly Morita equivalent if and only if their centers are braided equivalent. In particular, if $\mathcal{M}$ is an indecomposable module category over $\mathcal{C}$, there is a canonical equivalence of braided tensor categories $\mathcal{Z}(\mathcal{C}) \simeq \mathcal{Z}\left(\mathcal{C}_{\mathcal{M}}^{*}\right)$ Ost03, Prop. 2.2]. The center $\mathcal{Z}(\operatorname{Vect}(H, \eta))$ of the tensor category $\operatorname{Vect}(H, \eta)$ contains the information necessary for constructing the braided quasi-Hopf algebra that is known as the twisted Drinfeld double $D^{\eta}(H)$ of the group $H$ twisted by $\eta$ (see DPR90, §3.2]), and moreover the centers of two pointed fusion categories are braided equivalent if and only if the associated twisted Drinfeld doubles are isomorphic as braided quasi-Hopf algebras.

Therefore the twisted Drinfeld doubles $D^{\eta}(H)$ and $D^{\widehat{\eta}}(\widehat{H})$ are isomorphic as braided quasi-Hopf algebras if and only if the pointed fusion categories $\operatorname{Vect}(H, \eta)$ and $V e c t(\widehat{H}, \widehat{\eta})$ are weakly Morita equivalent.

We conclude that there are 38 isomorphism classes of twisted Drinfeld doubles of groups of order 8,18 of them corresponding to twisted Drinfeld doubles which are commutative as algebras and 20 corresponding to twisted Drinfeld doubles which are non-commutative as algebras. The commutative twisted Drinfeld doubles are the ones constructed from the groups $\mathbb{Z} / 8, \mathbb{Z} / 4 \times \mathbb{Z} / 2$ with any associator and from the group $(\mathbb{Z} / 2)^{3}$ with associator any cohomology class not containing the element $S q^{1}(x y z)=x^{2} y z+x y^{2} z+x y z^{2}$. The non-commutative twisted Drinfeld doubles are the ones constructed from the groups $D_{8}, Q_{8}$ with any associator and from the group $(\mathbb{Z} / 2)^{3}$ with associator any cohomology class containing the element $S q^{1}(x y z)=x^{2} y z+x y^{2} z+x y z^{2}$.

The isomorphism classes of twisted Drinfeld doubles of groups of order 8 was carried out with complete different methods in MN01] whenever the algebra structure
was commutative and in GMN07] whenever the algebra structure was not commutative. Our results are compatible with the ones of the previous two references and independent of them.

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[^1]:    Departamento de Matemáticas y Estadística, Universidad del Norte, Km. 5 Vía Antigua a Puerto Colombia, Barranquilla, Colombia.

    E-mail address: munozfa@uninorte.edu.co
    E-mail address: bjongbloed@uninorte.edu.co

