

THE DIVISIBILITY BY 2 OF RATIONAL POINTS ON ELLIPTIC CURVES

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ABSTRACT. We give a simple proof of the well-known divisibility by 2 condition for rational points on elliptic curves with rational 2-torsion. As an application of the explicit division by 2^n formulas obtained in Sec.2, we construct versal families of elliptic curves containing points of orders 4, 5, 6, and 8 from which we obtain an explicit description of elliptic curves over certain finite fields \mathbb{F}_q with a prescribed (small) group $E(\mathbb{F}_q)$. In the last two sections we study 3- and 5-torsion.

1. INTRODUCTION

Let E be an elliptic curve over a number field K . A famous *Mordell-Weil theorem* asserts that the (abelian) group $E(K)$ of K -points on E is finitely generated [3, 18, 21]. The first step in its proof (and actual finding a finite set that generates $E(K)$) is a *weak Mordell-Weil theorem* that asserts that the quotient $E(K)/2E(K)$ is a finite (abelian) group. This step is called 2-descent and its basic ingredient is a criterion for when a K -point on E is twice another K -point (under an additional assumption that all points of order 2 on E are defined over K). In this paper we give a new treatment of this criterion that seems to be less computational than previous ones ([10, Ch. 5, pp. 102–104], [4], [8, Th. 4.2 on pp. 85–87], [2, Lemma 7.6 on p. 67] [1, pp. 331–332]). This approach allows us to describe explicitly 2-power torsion on elliptic curves. In addition we obtain explicitly families of elliptic curves with various torsion subgroups over arbitrary fields of characteristic different from 2 (the problem of constructing elliptic curves with given torsion goes back to B.Levi [14]).

The paper is organized as follows. We work with elliptic curves E over an arbitrary field K with $\text{char}(K) \neq 2$. In Section 2 we discuss the criterion of divisibility by 2 and explicit formulas for the “half-points” in $E(K)$. Next we discuss a criterion of divisibility by any power of 2 in $E(K)$ (Section 3). In Section 4 we collect useful results about elliptic curves and their torsion. In Sections 5,6, and 7 we will use explicit formulas of Section 2 in order to construct *versal* families of elliptic curves E such that $E(K)$ contains a subgroup isomorphic to $\mathbb{Z}/2m\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ with $m = 2, 4, 3$, respectively. (In addition, in Section 5 we construct a *versal* family of elliptic curves E such that $E(K)$ contains a subgroup isomorphic to $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$.) Such families are parameterized by K -points of rational curves that are closely related to certain modular curves of genus zero (see [14, 9, 15, 16]); however, our approach remains

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quite elementary. In addition, in Sections 6 and 8 we construct *versal* families of elliptic curves E such that $E(K)$ contains a subgroup isomorphic to $\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/10\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, respectively. These two families are parameterized by K -points of curves that are closely related to certain modular curves of genus 1.

As an unexpected application, we describe explicitly (and without computations) elliptic curves E over small finite fields \mathbb{F}_q such that $E(\mathbb{F}_q)$ is isomorphic to a certain finite group (of small order). Using deep highly nontrivial results of B. Mazur [12] and of S. Kamienny and M. Kenku–F. Momose [5, 7], we describe explicitly elliptic curves E over the field \mathbb{Q} of rational numbers and over quadratic fields K such that the torsion subgroup $E(\mathbb{Q})_t$ of $E(\mathbb{Q})$ (resp. $E(K)_t$ of $E(K)$) is isomorphic to a certain finite group.

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2. DIVISION BY 2

Let K be a field of characteristic different from 2. Let

$$(1) \quad E : y^2 = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$$

be an elliptic curve over K , where $\alpha_1, \alpha_2, \alpha_3$ are *distinct* elements of K . This means that $E(K)$ contains all three points of order 2, namely, the points

$$(2) \quad W_1 = (\alpha_1, 0), W_2 = (\alpha_2, 0), W_3 = (\alpha_3, 0).$$

The following statement is pretty well known ([3, pp. 269–270], [10, Ch. 5, pp. 102–104], [4], [8, Th. 4.2 on pp. 85–87], [2, Lemma 7.6 on p. 67] [1, pp. 331–332], [21, pp. 212–214]; see also [22]).

Theorem 2.1. *Let $P = (x_0, y_0)$ be a K -point on E . Then P is divisible by 2 in $E(K)$ if and only if all three elements $x_0 - \alpha_i$ are squares in K .*

While the proof of the claim that the divisibility implies the squareness is straightforward, it seems that the known elementary proofs of the converse statement are more involved/computational. (Notice that there is another approach, which is based on Galois cohomology [17, Sect. X.1, pp. 313–315] and it works for hyperelliptic jacobians as well [13].)

We start with an elementary proof of the divisibility that seems to be less computational. (In additional, it will give us immediately explicit formulas for the coordinates of all four $\frac{1}{2}P$.)

Proof. So, let us assume that all three elements $x_0 - \alpha_i$ are squares in K , and let $Q = (x_1, y_1)$ be a point on E with $2Q = P$. Since $P \neq \infty$, we have $y_1 \neq 0$, and therefore the equation of the *tangent line* L to E at Q may be written in the form

$$L : y = lx + m.$$

(Here x_1, y_1, l, m are elements of an overfield of K .) In particular, $y_1 = lx_1 + m$. By the definition of Q and L , the point $-P = (x_0, -y_0)$ is the “third” common point of L and E ; in particular, $-y_0 = lx_0 + m$, i.e., $y_0 = -(lx_0 + m)$. Standard arguments (the restriction of the equation for E to L , see [18, pp. 25–27], [21, pp. 12–14], [1, p. 331]) tell us that the monic cubic polynomial

$$(x - \alpha_1)(x - \alpha_2)(x - \alpha_3) - (lx + m)^2$$

coincides with $(x - x_1)^2(x - x_0)$. This implies that

$$-(l\alpha_i + m)^2 = (\alpha_i - x_1)^2(\alpha_i - x_0) \text{ for all } i = 1, 2, 3.$$

Since $2Q = P \neq \infty$, none of $x_1 - \alpha_i$ vanishes. Recall that all $x_0 - \alpha_i$ are squares in K and they are obviously distinct. Consequently, the corresponding square roots [1, p. 331]

$$r_i := \frac{l\alpha_i + m}{x_1 - \alpha_i} = \sqrt{x_0 - \alpha_i}$$

are *distinct* elements of K . In other words, the transformation

$$z \mapsto \frac{lz + m}{-z + x_1}$$

of the projective line sends the three distinct K -points $\alpha_1, \alpha_2, \alpha_3$ to the three distinct K -points r_1, r_2, r_3 , respectively. This implies that our transformation is *not* constant, i.e., is an honest linear fractional transformation¹ and is defined over K . Since one of the “matrix entries”, -1 , is already a nonzero element of K , all other matrix entries l, m, x_1 also lie in K . Since $y_1 = lx_1 + m$, it also lies in K . So, $Q = (x_1, y_1)$ is a K -point of E . \square

Let us get explicit formulas for x_1, y_1, l, m in terms of r_1, r_2, r_3 . We have

$$\alpha_i = x_0 - r_i^2, \quad l\alpha_i + m = r_i(x_1 - \alpha_i),$$

and therefore

$$l(x_0 - r_i^2) + m = r_i[x_1 - (x_0 - r_i^2)] = r_i^3 + (x_1 - x_0)r_i,$$

which is equivalent to $r_i^3 + lr_i^2 + (x_1 - x_0)r_i - (lx_0 + m) = 0$, and this equality holds for all $i = 1, 2, 3$. This means that the monic cubic polynomial

$$h(t) = t^3 + lt^2 + (x_1 - x_0)t - (lx_0 + m)$$

coincides with $(t - r_1)(t - r_2)(t - r_3)$. Recall that $-(lx_0 + m) = y_0$ and get

$$(3) \quad r_1 r_2 r_3 = -y_0.$$

We also get

$$l = -(r_1 + r_2 + r_3), \quad x_1 - x_0 = r_1 r_2 + r_2 r_3 + r_3 r_1.$$

This implies that

$$(4) \quad x_1 = x_0 + (r_1 r_2 + r_2 r_3 + r_3 r_1).$$

Since $y_1 = lx_1 + m$ and $-y_0 = lx_0 + m$, we obtain that

$$m = -y_0 - lx_0 = -y_0 + (r_1 + r_2 + r_3)x_0,$$

and therefore

$$y_1 = -(r_1 + r_2 + r_3)[x_0 + (r_1 r_2 + r_2 r_3 + r_3 r_1)] + [-y_0 + (r_1 + r_2 + r_3)x_0],$$

i.e.,

$$(5) \quad y_1 = -y_0 - (r_1 + r_2 + r_3)(r_1 r_2 + r_2 r_3 + r_3 r_1).$$

Notice that there are precisely four points $Q \in E(K)$ with $2Q = P$,

$$(6) \quad Q = (x_0 + (r_1 r_2 + r_2 r_3 + r_3 r_1), -y_0 - (r_1 + r_2 + r_3)(r_1 r_2 + r_2 r_3 + r_3 r_1)),$$

¹Another way to see this is to assume the contrary. Then the *determinant* $lx_1 + m = 0$, i.e., $y_0 = 0$, and therefore $P = 2Q$ is the infinite point, which is not true.

each of which corresponds to one of the *four* choices of the three square roots $r_i = \sqrt{x_0 - \alpha_i} \in K$ ($i = 1, 2, 3$) with $r_1 r_2 r_3 = -y_0$. Using the latter equality, we may rewrite (5) as ²

$$(7) \quad y_1 = -(r_1 + r_2)(r_2 + r_3)(r_3 + r_1).$$

In addition,

$$(8) \quad x_1 = \alpha_i + (r_i + r_j)(r_i + r_k),$$

where i, j, k is any permutation of $1, 2, 3$. Indeed,

$$\begin{aligned} x_1 - \alpha_i &= (x_0 - \alpha_i) + r_1 r_2 + r_2 r_3 + r_3 r_1 = \\ &= r_i^2 + r_1 r_2 + r_2 r_3 + r_3 r_1 = (r_i + r_j)(r_i + r_k). \end{aligned}$$

The remaining four choices of the “signs” of r_1, r_2, r_3 bring us to the same values of abscissas and the opposite values of ordinates and give the results of division by 2 of the point $-P$.

Conversely, if we know $Q = (x_1, y_1)$, then we may recover the corresponding (r_1, r_2, r_3) . Namely, the equalities (8) and (7) imply that

$$\begin{aligned} r_j + r_k &= -\frac{y_1}{x_1 - \alpha_i}, \\ r_i &= \frac{-(r_j + r_k) + (r_i + r_j) + (r_i + r_k)}{2} \\ &= -\frac{y_1}{2} \cdot \left(-\frac{1}{x_1 - \alpha_i} + \frac{1}{x_1 - \alpha_j} + \frac{1}{x_1 - \alpha_k} \right) \end{aligned}$$

for any permutation i, j, k of $1, 2, 3$.

Example 2.2. Let us choose as $P = (x_0, y_0)$ the point $W_3 = (\alpha_3, 0)$ of order 2 on E . Then $r_3 = 0$, and we have two arbitrary independent choices of (nonzero) $r_1 = \sqrt{\alpha_3 - \alpha_1}$ and $r_2 = \sqrt{\alpha_3 - \alpha_2}$. Thus

$$Q = (\alpha_3 + r_1 r_2, -(r_1 + r_2)r_1 r_2) = (\alpha_3 + r_1 r_2, -r_1(\alpha_3 - \alpha_2) - r_2(\alpha_3 - \alpha_1))$$

is a point on E with $2Q = P$; in particular, Q is a point of order 4. The same is true for the (three remaining) points $-Q = (\alpha_3 + r_1 r_2, r_1(\alpha_3 - \alpha_2) + r_2(\alpha_3 - \alpha_1))$, $(\alpha_3 - r_1 r_2, -r_1(\alpha_3 - \alpha_2) + r_2(\alpha_3 - \alpha_1))$, and $(\alpha_3 - r_1 r_2, r_1(\alpha_3 - \alpha_2) - r_2(\alpha_3 - \alpha_1))$.

Recall that, in formula (6) for the coordinates of the points $\frac{1}{2}P$, one may arbitrarily choose the signs of r_1, r_2, r_3 under condition (3). Let Q be one of $\frac{1}{2}P$'s that corresponds to a certain choice of r_1, r_2, r_3 . The remaining three *halves* of P correspond to $(r_1, -r_2, -r_3)$, $(-r_1, r_2, -r_3)$, $(-r_1, -r_2, r_3)$. Let us denote these halves by $\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3$, respectively. For each $i = 1, 2, 3$, the difference $\mathcal{Q}_i - Q$ is a point of order 2 on E . Which one? The following assertion answers this question.

Theorem 2.3. *Let i, j, k be a permutation of $1, 2, 3$. Then*

- (i) *If $P = W_i$, then $\mathcal{Q}_i = -Q$.*
- (ii) *If $P \neq W_i$, then all three points $\mathcal{Q}_i, -Q, W_i$ are distinct.*
- (iii) *The points $\mathcal{Q}_i, -Q, W_i$ lie on the line*

$$y = (r_j + r_k)(x - \alpha_i).$$

- (iv) $\mathcal{Q}_i - Q = W_i$.

²This was brought to our attention by Robin Chapman.

Proof. First, assume that $P = W_i$. In this case, formulas (4) and (5) tell us that

$$Q = (\alpha_i + r_j r_k, -r_j r_k (r_j + r_k)),$$

which implies that

$$\mathcal{Q}_i = (\alpha_i + r_j r_k, r_j r_k (r_j + r_k)) = -Q$$

and

$$\mathcal{Q}_i - Q = -2Q = -P = P = W_i.$$

This proves (i) and a special case of (iv) when $P = W_i$. Now assume that $P \neq W_i$ and prove that the three points $\mathcal{Q}_i, -Q, W_i$ are *distinct*. Since none of \mathcal{Q}_i and $-Q$ is of order 2, none of them is W_i . On the other hand, if $\mathcal{Q}_i = -Q$, then

$$2Q = P = 2\mathcal{Q}_i = -2Q = -P,$$

and so P has order 2, say $P = W_j$. Applying (a) to j instead of i , we get $\mathcal{Q}_j = -Q$; but $\mathcal{Q}_i \neq \mathcal{Q}_j$ since $i \neq j$. Therefore $\mathcal{Q}_i, -Q, W_i$ are three *distinct* points. This proves (ii).

Let us prove (iii). Since

$$x_1 - \alpha_i = (r_i + r_j)(r_i + r_k), \quad y_1 = -(r_1 + r_2)(r_2 + r_3)(r_3 + r_1),$$

we have $y_1 = (r_j + r_k)(x_1 - \alpha_i)$. Further

$$x(-\mathcal{Q}_i) - \alpha_i = (r_i - r_j)(r_i - r_k),$$

$$y(-\mathcal{Q}_i) = (r_i - r_j)(-r_j - r_k)(-r_k + r_i) = (r_j + r_k)(x(-\mathcal{Q}_i) - \alpha_i).$$

Therefore $\mathcal{Q}_i, -Q, W_i$ lie on the line

$$y = (r_j + r_k)(x - \alpha_i).$$

We have already proven (iv) when $P = W_i$. So, let us assume that $P \neq W_i$. Now (iv) follows from (iii) combined with (i). \square

3. DIVISION BY 2^n

Using the formulas above that describe the division by 2 on E , one may easily deduce the following necessary and sufficient condition of divisibility by any power of 2. For an overfield L of K , we consider a sequence of points Q_μ in $E(L)$ such that $Q_0 = P$ and $2Q_{\mu+1} = Q_\mu$ for all $\mu = 0, 1, 2, \dots$. Let $r_1^{(\mu)}, r_2^{(\mu)}, r_3^{(\mu)}$ ($\mu = 0, 1, 2, \dots$) be arbitrary sequences of elements of L that satisfy the relations

$$(r_i^{(\mu)})^2 = x(Q_\mu) - \alpha_i.$$

Then for each permutation i, j, k of 1, 2, 3 we obtain, in light of the formula (8),

$$x(Q_{\mu+1}) - \alpha_i = (r_i^{(\mu)} + r_j^{(\mu)})(r_i^{(\mu)} + r_k^{(\mu)}),$$

which implies that

$$(r_i^{(\mu+1)})^2 = (r_i^{(\mu)} + r_j^{(\mu)})(r_i^{(\mu)} + r_k^{(\mu)}).$$

By changing the signs of $r_i^{(\mu)}, r_j^{(\mu)}, r_k^{(\mu)}$ in the product $(r_i^{(\mu)} + r_j^{(\mu)})(r_i^{(\mu)} + r_k^{(\mu)})$, we obtain all possible values of the abscissas of $Q_{(\mu+1)}$ with $2Q_{\mu+1} = Q_\mu$.

Suppose that $Q_\mu \in E(K)$. Then Q_μ is divisible by 2 in $E(K)$ if and only if one may choose $r_i^{(\mu)}, r_j^{(\mu)}, r_k^{(\mu)}$ in such a way that $(r_i^{(\mu)} + r_j^{(\mu)})(r_i^{(\mu)} + r_k^{(\mu)})$ are squares in K for all $i = 1, 2, 3$. We proved the following statement.

Theorem 3.1. *Let $P = (x_0, y_0) \in E(K)$. Let $r_1^{(\mu)}, r_2^{(\mu)}, r_3^{(\mu)}$ ($\mu = 0, 1, 2, \dots$) be sequences of elements of L that satisfy the relations*

$$(r_i^{(0)})^2 = r_i^2 = x_0 - \alpha_i, \quad (r_i^{(\mu+1)})^2 = (r_i^{(\mu)} + r_j^{(\mu)})(r_i^{(\mu)} + r_k^{(\mu)})$$

for all permutations i, j, k of $1, 2, 3$. Then P is divisible by 2^n in $E(K)$ if and only if all $x_0 - \alpha_i$ are squares in K , and, for each $\mu = 0, 1, \dots, n-1$, one may choose square roots $r_1^{(\mu)}, r_2^{(\mu)}, r_3^{(\mu)}$ in such a way that the products $(r_i^{(\mu)} + r_j^{(\mu)})(r_i^{(\mu)} + r_k^{(\mu)})$ are squares in K (and therefore all $r_i^{(\mu)}$ lie in K for $\mu = 0, 1, \dots, n-1$).

The knowledge of sequences $r_1^{(\mu)}, r_2^{(\mu)}, r_3^{(\mu)}$ allows us step by step to find the points $\frac{1}{2}P, \frac{1}{4}P, \frac{1}{8}P$ etc.

Example 3.2. Let $P = (x_0, y_0)$, let R be a point of E such that $4R = P$, and let $Q = 2R = (x_1, y_1)$. By formulas (4) and (7),

$$x_1 = x_0 + (r_1 r_2 + r_2 r_3 + r_3 r_1), \quad y_1 = -(r_1 + r_2)(r_2 + r_3)(r_3 + r_1),$$

where the square roots

$$r_i = \sqrt{x_0 - \alpha_i}, \quad i = 1, 2, 3,$$

are chosen in such a way that $r_1 r_2 r_3 = -y_0$. Further, let

$$r_i^{(1)} = \sqrt{(r_i + r_j)(r_i + r_k)}$$

be square roots that are chosen in such a way that

$$r_1^{(1)} r_2^{(1)} r_3^{(1)} = -y_1 = (r_1 + r_2)(r_2 + r_3)(r_3 + r_1).$$

In light of (4) and (7),

$$\begin{aligned} x(R) &= x_1 + r_1^{(1)} r_2^{(1)} + r_2^{(1)} r_3^{(1)} + r_3^{(1)} r_1^{(1)}, \\ y(R) &= -(r_1^{(1)} + r_2^{(1)})(r_2^{(1)} + r_3^{(1)})(r_3^{(1)} + r_1^{(1)}), \end{aligned}$$

which implies that

$$(9) \quad \begin{aligned} x(R) &= x_0 + (r_1 r_2 + r_2 r_3 + r_3 r_1) + (r_1^{(1)} r_2^{(1)} + r_2^{(1)} r_3^{(1)} + r_3^{(1)} r_1^{(1)}), \\ y(R) &= -(r_1^{(1)} + r_2^{(1)})(r_2^{(1)} + r_3^{(1)})(r_3^{(1)} + r_1^{(1)}). \end{aligned}$$

4. TORSION OF ELLIPTIC CURVES

In the sequel, we will freely use the following well-known elementary observation.

Let κ be a nonzero element of K . Then there is a canonical isomorphism of the elliptic curves

$$E : y^2 = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$$

and

$$E(\kappa) : y'^2 = \left(x' - \frac{\alpha_1}{\kappa^2}\right) \left(x' - \frac{\alpha_2}{\kappa^2}\right) \left(x' - \frac{\alpha_3}{\kappa^2}\right)$$

that is given by the change of variables

$$x' = \frac{x}{\kappa^2}, \quad y' = \frac{y}{\kappa^3}$$

and respects the group structure. Under this isomorphism the point $(\alpha_i, 0) \in E(K)$ goes to $(\alpha_i/\kappa^2, 0) \in E(\kappa)(K)$ for all $i = 1, 2, 3$. In addition, if $P = (0, y(P))$ lies in $E(K)$, then it goes (under this isomorphism) to $(0, y(P)/\kappa^3) \in E(\kappa)(K)$.

We will also use the following classical result of Hasse (Hasse bound) [21, Th. 4.2 on p. 97].

Theorem 4.1. *If q is a prime power, \mathbb{F}_q a q -element finite field and E is an elliptic curve over \mathbb{F}_q , then $E(\mathbb{F}_q)$ is a finite abelian group whose cardinality $|E(\mathbb{F}_q)|$ satisfies the inequalities*

$$(10) \quad q - 2\sqrt{q} + 1 \leq |E(\mathbb{F}_q)| \leq q + 2\sqrt{q} + 1.$$

Another result that we are going to use is the following immediate corollary of a celebrated theorem of B. Mazur ([12], [11, Th. 2.5.2 and p. 187]).

Theorem 4.2. *If E is an elliptic curve over \mathbb{Q} and the torsion subgroup $E(\mathbb{Q})_t$ of $E(\mathbb{Q})$ is not cyclic, then $E(\mathbb{Q})_t$ is isomorphic to $\mathbb{Z}/2m\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ with $m = 1, 2, 3$ or 4 . In particular, if $m = 3$ or 4 and $E(\mathbb{Q})$ contains a subgroup isomorphic to $\mathbb{Z}/2m\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, then $E(\mathbb{Q})_t$ is isomorphic to $\mathbb{Z}/2m\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.*

The next assertion follows readily from the list of possible torsion subgroups of elliptic curves over quadratic fields obtained by S. Kamienny [5] and M.A. Kenku - F. Momose [7] (see also [6, Th. 1]).

Theorem 4.3. *Let E be an elliptic curve over a quadratic field K . Assume that all points of order 2 on E are defined over K . Let $E(K)_t$ be the torsion subgroup of $E(K)$. Then $E(K)_t$ is isomorphic either to $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ or to $\mathbb{Z}/2m\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ with $1 \leq m \leq 6$.*

In particular, $E(K)_t$ enjoys the following properties.

- (1) *If $m = 5$ or 6 and $E(K)$ contains a subgroup isomorphic to $\mathbb{Z}/2m\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, then $E(K)_t$ is isomorphic to $\mathbb{Z}/2m\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.*
- (2) *If $E(K)$ contains a subgroup isomorphic to $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$, then $E(K)_t$ is isomorphic to $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$.*

5. RATIONAL POINTS OF ORDER 4

We are going to describe explicitly elliptic curves (1) that contain a K -point of order 4. In order to do that, we consider the elliptic curve

$$\mathcal{E}_{1,\lambda} : y^2 = (x + \lambda^2)(x + 1)x$$

over K . Here λ is an element of $K \setminus \{0, \pm 1\}$. In this case, we have

$$\alpha_1 = -\lambda^2, \alpha_2 = -1, \alpha_3 = 0.$$

Notice that

$$\mathcal{E}_{1,\lambda} = \mathcal{E}_{1,-\lambda}.$$

All three differences

$$\alpha_3 - \alpha_1 = \lambda^2, \alpha_3 - \alpha_2 = 1^2, \alpha_3 - \alpha_3 = 0^2$$

are squares in K . Dividing the order 2 point $W_3 = (0, 0) \in \mathcal{E}_{1,\lambda}(K)$ by 2, we get $r_3 = 0$ and the four choices

$$r_1 = \pm\lambda, r_2 = \pm 1.$$

Now Example 2.2 gives us four points Q with $2Q = W_3$, namely,

$$(\lambda, \mp(\lambda + 1)\lambda), (-\lambda, \pm(\lambda - 1)\lambda).$$

This implies that the group $\mathcal{E}_{1,\lambda}(K)$ contains the subgroup generated by any Q and W_1 , which is $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Remark 5.1. Our computations show that if Q is a K -point on $E_{1,\lambda}$, then

$$2Q = W_3 \text{ if and only if } x(Q) = \pm\lambda.$$

Both cases (signs) do occur.

Remark 5.2. There is another family of elliptic curves ([9, Table 3 on p. 217] (see also [15, Part 2], [11, Appendix E]))

$$\mathfrak{E}_{1,t} : y^2 + xy - \left(t^2 - \frac{1}{16}\right)y = x^3 - \left(t^2 - \frac{1}{16}\right)x^2$$

whose group of K -points contains a subgroup isomorphic to $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. If we put

$$y_1 := y + \frac{x - (t^2 - \frac{1}{16})}{2},$$

then the equation may be rewritten as

$$y_1^2 = x^3 - \left(t^2 - \frac{1}{16}\right)x^2 + \left[\frac{x - (t^2 - \frac{1}{16})}{2}\right]^2 = \left(x - t^2 + \frac{1}{16}\right) \left(x + \frac{t}{2} + \frac{1}{8}\right) \left(x - \frac{t}{2} + \frac{1}{8}\right).$$

If we put $x_1 := x - t^2 + 1/16$, then the equation becomes

$$y_1^2 = x_1 \left(x_1 + \left(t + \frac{1}{4}\right)^2\right) \left(x_1 + \left(t - \frac{1}{4}\right)^2\right),$$

which defines the elliptic curve $\mathcal{E}_{1,\lambda}(1/\kappa)$ with

$$\lambda = \frac{t - \frac{1}{4}}{t + \frac{1}{4}}, \quad \kappa = t + \frac{1}{4}.$$

In particular, $\mathfrak{E}_{1,t}$ is isomorphic to $\mathcal{E}_{1,\lambda}$.

Theorem 5.3. *Let E be an elliptic curve over K . Then $E(K)$ contains a subgroup isomorphic to $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ if and only if there exists $\lambda \in K \setminus \{0, \pm 1\}$ such that E is isomorphic to $\mathcal{E}_{1,\lambda}$.*

Proof. We already know that $\mathcal{E}_{1,\lambda}(K)$ contains a subgroup isomorphic to $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Conversely, suppose that E is an elliptic curve over K such that $E(K)$ contains a subgroup isomorphic to $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Then $E(K)$ contains all three points of order 2, and therefore E can be represented in the form (1). It is also clear that at least one of the points (2) is divisible by 2 in $E(K)$. Suppose that W_3 is divisible by 2. We may assume that $\alpha_3 = 0$. By Theorem 2.1, both nonzero differences

$$-\alpha_1 = \alpha_3 - \alpha_1, \quad -\alpha_2 = \alpha_3 - \alpha_2$$

are squares in K ; in addition, they are *distinct* elements of K . Thus there are nonzero $a, b \in K$ such that $a \neq \pm b$ and $-\alpha_1 = a^2$, $-\alpha_2 = b^2$. Since $\alpha_3 = 0$, the equation for E is

$$E : y^2 = (x + a^2)(x + b^2)x.$$

If we put $\kappa = b$, then we obtain that E is isomorphic to

$$E(\kappa) : y'^2 = \left(x' + \frac{a^2}{b^2}\right)(x' + 1)x',$$

which is nothing else but $\mathcal{E}_{1,\lambda}$ with $\lambda = a/b$. □

Corollary 5.4. *Let E be an elliptic curve over \mathbb{F}_5 . The group $E(\mathbb{F}_5)$ is isomorphic to $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ if and only if E is isomorphic to the elliptic curve $y^2 = x^3 - x$.*

Proof. Suppose that $E(\mathbb{F}_5)$ is isomorphic to $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. By Theorem 5.3, E is isomorphic to

$$y^2 = (x + \lambda^2)(x + 1)x \quad \text{with } \lambda \in \mathbb{F}_5 \setminus \{0, 1, -1\}.$$

This implies that $\lambda = \pm 2, \lambda^2 = -1$, and so E is isomorphic to

$$\mathcal{E}_{1,2} : y^2 = (x - 1)(x + 1) = x^3 - x.$$

Now we need to check that $\mathcal{E}_{1,2}(\mathbb{F}_5) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. By Theorem 5.3, $E(\mathbb{F}_5)$ contains a subgroup isomorphic to $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$; in particular, 8 divides $|E(\mathbb{F}_5)|$. In order to finish the proof, it suffices to check that $|E(\mathbb{F}_5)| < 16$, but this inequality follows from the Hasse bound (10)

$$|E(\mathbb{F}_5)| \leq 5 + 2\sqrt{5} + 1 < 11.$$

□

Corollary 5.5. *Let E be an elliptic curve over \mathbb{F}_7 . The group $E(\mathbb{F}_7)$ is isomorphic to $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ if and only if E is isomorphic to the elliptic curve $y^2 = (x + 2)(x + 1)x$.*

Proof. Suppose that $E(\mathbb{F}_7)$ is isomorphic to $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. It follows from Theorem 5.3 that E is isomorphic to $y^2 = (x + \lambda^2)(x + 1)x$ with $\lambda \in \mathbb{F}_7 \setminus \{0, 1, -1\}$. This implies that $\lambda = \pm 2$ or ± 3 , and therefore $\lambda^2 = 4$ or 2 , i.e., E is isomorphic to one of the two elliptic curves

$$\mathcal{E}_{1,3} : y^2 = (x + 2)(x + 1)x, \quad \mathcal{E}_{1,2} : y^2 = (x + 4)(x + 1)x.$$

Since $1/4 = 2$ in \mathbb{F}_7 , the elliptic curve $\mathcal{E}_{1,3}$ coincides with $\mathcal{E}_{1,2}(2)$; in particular, $\mathcal{E}_{1,2}$ and $\mathcal{E}_{1,3}$ are isomorphic.

Now suppose that $E = \mathcal{E}_{1,2}$. We need to prove that $E(\mathbb{F}_7)$ is isomorphic to $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. By Theorem 5.3, $E(\mathbb{F}_7)$ contains a subgroup isomorphic to $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$; in particular, 8 divides $|E(\mathbb{F}_7)|$. In order to finish the proof, it suffices to check that $|E(\mathbb{F}_7)| < 16$, but this inequality follows from the Hasse bound (10)

$$|E(\mathbb{F}_7)| \leq 7 + 2\sqrt{7} + 1 < 14.$$

□

Theorem 5.6. *Suppose that K contains $\mathbf{i} = \sqrt{-1}$. Let a, b be nonzero elements of K such that $a \neq \pm b$, $a \neq \pm \mathbf{i}b$. Let us consider the elliptic curve*

$$E_{a,b} : y^2 = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$$

over K with $\alpha_1 = (a^2 - b^2)^2$, $\alpha_2 = (a^2 + b^2)^2$, $\alpha_3 = 0$. Then all points of order 2 on E are divisible by 2 in $E(K)$, i.e., $E(K)$ contains all twelve points of order 4. In particular, $E_{a,b}(K)$ contains a subgroup isomorphic to $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$.

Proof. Clearly, all α_i and $-\alpha_j$ are squares in K . In addition,

$$\alpha_2 - \alpha_1 = (a^2 + b^2)^2 - (a^2 - b^2)^2 = (2ab)^2, \quad \alpha_1 - \alpha_2 = (2\mathbf{i}ab)^2.$$

This implies that all $\alpha_i - \alpha_j$ are squares in K . It follows from Theorem 2.1 that all points $W_i = (\alpha_i, 0)$ of order 2 are divisible by 2 in $E(K)$, and therefore $E(K)$ contains all twelve (3×4) points of order 4. □

Keeping the notation and assumptions of Theorem 5.6, we describe explicitly all twelve points of order 4, using formula (6).

- (1) Dividing the point $W_2 = (\alpha_2, 0) = ((a^2 + b^2)^2, 0)$ by 2, we have $r_2 = 0$ and get four choices $r_1 = \pm 2ab$, $r_3 = \pm(a^2 + b^2)$. This gives us four points Q with $2Q = W_2$, namely, two points

$$\begin{aligned} & ((a^2 + b^2)^2 + 2ab(a^2 + b^2), \pm(a^2 + b^2 + 2ab)2ab(a^2 + b^2)) \\ & = ((a^2 + b^2)(a + b)^2, \pm 2ab(a^2 + b^2)(a + b)^2) \end{aligned}$$

and two points $((a^2 + b^2)(a - b)^2, \pm 2ab(a^2 + b^2)(a - b)^2)$.

- (2) Dividing the point $W_3 = (\alpha_3, 0) = (0, 0)$ by 2, we have $r_3 = 0$ and get four choices $r_1 = \pm i(a^2 - b^2)$, $r_2 = \pm i(a^2 + b^2)$. This gives us four points Q with $2Q = W_3$, namely, two points

$$\begin{aligned} & ((a^2 - b^2)(a^2 + b^2), \pm(i((a^2 - b^2) + i(a^2 + b^2))(a^2 - b^2)(a^2 + b^2))) \\ & = (a^4 - b^4, \pm 2ia^2(a^4 - b^4)) \end{aligned}$$

and two points $(b^4 - a^4, \pm 2ib^2(b^4 - a^4))$.

- (3) Dividing the point $W_1 = (\alpha_1, 0) = ((a^2 - b^2)^2, 0)$ by 2, we have $r_1 = 0$ and get four choices $r_2 = \pm 2iab$, $r_3 = \pm(a^2 - b^2)$. This gives us four points Q with $2Q = W_3$, namely, two points

$$\begin{aligned} & ((a^2 - b^2)^2 + 2iab(a^2 - b^2), \pm(2iab + (a^2 - b^2))2iab(a^2 - b^2)) \\ & = ((a^2 - b^2)(a + ib)^2, \pm 2iab(a^2 - b^2)(a + ib)^2) \end{aligned}$$

and two points $((a^2 - b^2)(a - ib)^2, \pm 2iab(a^2 - b^2)(a - ib)^2)$.

Remark 5.7. Let λ be an element of $K \setminus \{0, \pm 1, \pm\sqrt{-1}\}$. We write $\mathcal{E}_{2,\lambda}$ for the elliptic curve

$$\mathcal{E}_{2,\lambda} : y^2 = \left(x + \frac{(\lambda^2 - 1)^2}{(\lambda^2 + 1)^2} \right) (x + 1)x$$

over K . The elliptic curves $\mathcal{E}_{2,\lambda}$ and $E_{a,b}$ are isomorphic if $a = \lambda b$. Indeed, one has only to put $\kappa = a^2 + b^2$ and notice that $E_{a,b}(\kappa) = \mathcal{E}_{2,\lambda}$. It follows from Theorem 5.6 that $\mathcal{E}_{2,\lambda}(K)$ contains a subgroup isomorphic to $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$.

There is another family of elliptic curves with this property, namely,

$$y^2 = x(x - 1) \left(x - \frac{(u + u^{-1})^2}{4} \right)$$

([19], [15, pp. 451–453]; see also Remark 5.9).

Theorem 5.8. *Let E be an elliptic curve over K . Then $E(K)$ contains a subgroup isomorphic to $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ if and only if K contains $\sqrt{-1}$ and there exists $\lambda \in K \setminus \{0, \pm 1, \pm\sqrt{-1}\}$ such that E is isomorphic to $\mathcal{E}_{2,\lambda}$.*

Proof. Recall (Remark 5.7) that $\mathcal{E}_{2,\lambda}(K)$ contains a subgroup isomorphic to $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$.

Conversely, suppose that E is an elliptic curve over K and $E(K)$ contains a subgroup isomorphic to $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. By Theorem 5.3, there is $\delta \in K \setminus \{0, \pm 1\}$ such that E is isomorphic to

$$\mathcal{E}_{1,\delta} : y^2 = (x + \delta^2)(x + 1)x.$$

Hence we may assume that $\alpha_1 = -\delta^2, \alpha_2 = -1, \alpha_3 = 0$. It follows from Theorem 2.1 that all $\pm 1, \pm(\delta^2 - 1)$ are squares in K . (In particular, $\mathbf{i} = \sqrt{-1}$ lies in K .) So, there is $\gamma \in K$ with $\gamma^2 = 1 - \delta^2$. Clearly, $\gamma \neq 0, \pm 1$. We have

$$\delta^2 + \gamma^2 = 1.$$

The well-known parametrization of the “unit circle” (that goes back to Euler) tells us that there exists $\lambda \in K$ such that $\lambda^2 + 1 \neq 0$ and

$$\delta = \frac{\lambda^2 - 1}{\lambda^2 + 1}, \quad \gamma = \frac{2\lambda}{\lambda^2 + 1}.$$

Now one has only to plug in the formula for δ into the equation of $\mathcal{E}_{1,\delta}$ and get $\mathcal{E}_{2,\lambda}$. \square

Remark 5.9. Using a different parametrization of the unit circle in the proof of Theorem 5.8, we obtain the family of elliptic curves

$$E : y^2 = \left(x + \frac{(2\lambda)^2}{(\lambda^2 + 1)^2} \right) (x + 1)x$$

with the same property as the family $\mathcal{E}_{2,\lambda}$. Notice that, for each $\lambda \in K \setminus \{0, \pm 1\}$, the elliptic curve E is isomorphic to the elliptic curve

$$y^2 = x(x - 1)(x - (u + u^{-1})^2/4)$$

mentioned in Remark 5.7. Indeed, the latter differs from $E(\kappa)$, where $\kappa = 2\lambda\sqrt{-1}/(\lambda^2 + 1)$, only with the change of the parameter λ by u .

Corollary 5.10. *Let E be an elliptic curve over \mathbb{F}_q , where $q = 9, 13, 17$. The group $E(\mathbb{F}_q)$ is isomorphic to $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ if and only if E is isomorphic to one of elliptic curves $\mathcal{E}_{2,\lambda}$. If $q = 9$, then $E(\mathbb{F}_q)$ is isomorphic to $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ if and only if E is isomorphic to $y^2 = x^3 - x$.*

Proof. First, \mathbb{F}_q contains $\sqrt{-1}$. Suppose that $E(\mathbb{F}_q)$ is isomorphic to $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. It follows from Theorem 5.8 that E is isomorphic to $\mathcal{E}_{2,\lambda}$.

Conversely, suppose that E is isomorphic to one of these curves. We need to prove that $E(\mathbb{F}_q)$ is isomorphic to $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. By Theorem 5.8, $E(\mathbb{F}_q)$ contains a subgroup isomorphic to $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$; in particular, 16 divides $|E(\mathbb{F}_q)|$. In order to finish the proof, it suffices to check that $|E(\mathbb{F}_q)| < 32$, but this inequality follows from the Hasse bound (10)

$$|E(\mathbb{F}_q)| \leq q + 2\sqrt{q} + 1 \leq 17 + 2\sqrt{17} + 1 < 27.$$

Now assume that $q = 9$. Then λ is one of four $\pm(1 \pm \mathbf{i})$. For all such λ we have

$$\lambda^2 = \pm 2\mathbf{i} = \mp \mathbf{i}, \quad \frac{(\lambda^2 - 1)^2}{(\lambda^2 + 1)^2} = \frac{(1 \mp \mathbf{i})^2}{(-1 \mp \mathbf{i})^2} = \frac{\mp 2\mathbf{i}}{\pm 2\mathbf{i}} = -1.$$

Therefore the equation for $\mathcal{E}_{2,\lambda}$ is

$$y^2 = (x - 1)(x + 1)x = x^3 - x.$$

\square

Corollary 5.11. *Let E be an elliptic curve over \mathbb{F}_{29} . The group $E(\mathbb{F}_{29})$ is isomorphic to $\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ if and only if E is isomorphic to one of elliptic curves $\mathcal{E}_{2,\lambda}$.*

Proof. First, \mathbb{F}_{29} contains $\sqrt{-1}$. Suppose that $E(\mathbb{F}_{29})$ is isomorphic to $\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. Then $E(\mathbb{F}_{29})$ contains a subgroup isomorphic to $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. It follows from Theorem 5.8 that E is isomorphic to $\mathcal{E}_{2,\lambda}$.

Conversely, suppose that E is isomorphic to one of these curves. We need to prove that $E(\mathbb{F}_{29})$ is isomorphic to $\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. By Theorem 5.8, $E(\mathbb{F}_{29})$ contains a subgroup isomorphic to $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$; in particular, 16 divides $|E(\mathbb{F}_{29})|$. The Hasse bound (10) tells us that

$$29 + 1 - 2\sqrt{29} \leq |E(\mathbb{F}_q)| \leq 29 + 1 + 2\sqrt{29}$$

and therefore

$$19 < |E(\mathbb{F}_{29})| < 41.$$

It follows that $|E(\mathbb{F}_{29})| = 32$; in particular, $E(\mathbb{F}_{29})$ is a finite 2-group. Clearly, $E(\mathbb{F}_{29})$ is isomorphic to a product of two cyclic 2-groups, each of which has order divisible by 4. It follows that $E(\mathbb{F}_{29})$ is isomorphic to $\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. \square

Theorem 5.12. *Let $K = \mathbb{Q}(\sqrt{-1})$ and E be an elliptic curve over $\mathbb{Q}(\sqrt{-1})$. Then the torsion subgroup $E(\mathbb{Q}(\sqrt{-1}))_t$ of $E(\mathbb{Q}(\sqrt{-1}))$ is isomorphic to $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ if and only if there exists $\lambda \in K \setminus \{0, \pm 1, \pm\sqrt{-1}\}$ such that E is isomorphic to $\mathcal{E}_{2,\lambda}$.*

Proof. By Theorem 4.3, if $E(\mathbb{Q}(\sqrt{-1}))$ contains a subgroup isomorphic to $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ then $E(\mathbb{Q}(\sqrt{-1}))_t$ is isomorphic to $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. Now the desired result follows from Theorem 5.3. \square

6. POINTS OF ORDER 8

Let us return to the curve $\mathcal{E}_{1,\lambda}$ and consider $Q \in \mathcal{E}_{1,\lambda}(K)$ with $2Q = W_3$. Let us try to divide Q by 2 in $E(K)$. By Remark 5.1, $x(Q) = \pm\lambda$. First, we assume that $x(Q) = \lambda$ (such a Q does exist).

Lemma 6.1. *Let Q be a point of $\mathcal{E}_{1,\lambda}(K)$ with $x(Q) = \lambda$. Then Q is divisible by 2 in $\mathcal{E}_{1,\lambda}(K)$ if and only if there exists $c \in K \setminus \{0, \pm 1, \pm 1 \pm \sqrt{2}, \pm\sqrt{-1}\}$ such that*

$$\lambda = \left[\frac{c - \frac{1}{c}}{2} \right]^2.$$

Proof. We have

$$\lambda - \alpha_1 = \lambda - (-\lambda^2) = \lambda + \lambda^2, \quad \lambda - \alpha_2 = \lambda - (-1) = \lambda + 1, \quad \lambda - \alpha_3 = \lambda - 0 = \lambda.$$

By Theorem 2.1, $Q \in 2\mathcal{E}_{1,\lambda}(K)$ if and only if all three $\lambda + \lambda^2, \lambda + 1, \lambda$ are squares in K . The latter means that both λ and $\lambda + 1$ are squares in K , i.e., there exist $a, b \in K$ such that $a^2 = \lambda + 1, \lambda = b^2$. This implies that the pair (a, b) is a K -point on the hyperbola

$$u^2 - v^2 = 1.$$

Recall that $\lambda \neq 0, \pm 1$. Using the well-known parametrization

$$u = \frac{t + \frac{1}{t}}{2}, \quad v = \frac{t - \frac{1}{t}}{2}$$

of the hyperbola, we obtain that both λ and $\lambda + 1$ are squares in K if and only if there exists a nonzero $c \in K$ such that

$$\lambda = \left[\frac{c - \frac{1}{c}}{2} \right]^2.$$

If this is the case, then

$$a = \pm \frac{c + \frac{1}{c}}{2}, \quad b = \pm \frac{c - \frac{1}{c}}{2}$$

and

$$\lambda + 1 = \left[\frac{c + \frac{1}{c}}{2} \right]^2.$$

Recall that $\lambda \neq 0, \pm 1$. This means that

$$\frac{c - \frac{1}{c}}{2} \neq 0, \pm 1, \pm\sqrt{-1}, \quad \text{i.e.,}$$

$$c \neq 0, \pm 1, \pm 1 \pm \sqrt{2}, \pm\sqrt{-1}.$$

□

Now let us assume that $x(Q) = -\lambda$ (such a Q does exist).

Lemma 6.2. *Let Q be a point of $\mathcal{E}_{1,\lambda}(K)$ with $x(Q) = -\lambda$. Then Q is divisible by 2 in $\mathcal{E}_{1,\lambda}(K)$ if and only if there exists $c \in K \setminus \{0, \pm 1, \pm 1 \pm \sqrt{2}, \pm\sqrt{-1}\}$ such that*

$$\lambda = - \left[\frac{c - \frac{1}{c}}{2} \right]^2.$$

Proof. Applying Lemma 6.1 to $-\lambda$ (instead of λ) and the curve $\mathcal{E}_{1,-\lambda} = \mathcal{E}_{1,\lambda}$, we obtain that $Q \in 2\mathcal{E}_{1,-\lambda}(K) = 2\mathcal{E}_{1,\lambda}(K)$ if and only if there exists

$$c \in K \setminus \{0, \pm 1, \pm 1 \pm \sqrt{2}, \pm\sqrt{-1}\}$$

such that

$$-\lambda = \left[\frac{c - \frac{1}{c}}{2} \right]^2.$$

□

Lemmas 6.1 and 6.2 give us the following statement.

Proposition 6.3. *The point $W_3 = (0, 0)$ is divisible by 4 in $\mathcal{E}_{1,\lambda}(K)$ if and only if there exists $c \in K$ such that $c \neq 0, \pm 1, \pm 1 \pm \sqrt{2}, \pm\sqrt{-1}$ and*

$$\lambda = \pm \left[\frac{c - \frac{1}{c}}{2} \right]^2, \quad \text{i.e.,} \quad \lambda^2 = \left[\frac{c - \frac{1}{c}}{2} \right]^4.$$

Proposition 6.4. *The following conditions are equivalent.*

- (i) *If $Q \in \mathcal{E}_{1,\lambda}(K)$ is any point with $2Q = W_3$, then Q lies in $2\mathcal{E}_{1,\lambda}(K)$.*
- (ii) *If R is any point of $\mathcal{E}_{1,\lambda}$ with $4R = W_3$, then R lies in $\mathcal{E}_{1,\lambda}(K)$.*
- (iii) *There exist $c, d \in K \setminus \{0, \pm 1, \pm 1 \pm \sqrt{2}, \pm\sqrt{-1}\}$ such that*

$$\lambda = \left[\frac{c - \frac{1}{c}}{2} \right]^2, \quad -\lambda = \left[\frac{d - \frac{1}{d}}{2} \right]^2.$$

If these equivalent conditions hold, then K contains $\sqrt{-1}$ and $\mathcal{E}_{1,\lambda}(K)$ contains all (twelve) points of order 4.

Proof. The equivalence of (i) and (ii) is obvious. It is also clear that (ii) implies that all points of order (dividing) 4 lie in $\mathcal{E}_{1,\lambda}(K)$.

Recall (Remark 5.1) that Q with $2Q = W_3$ are exactly the points of $\mathcal{E}_{1,\lambda}$ with $x(Q) = \pm\lambda$. Now the equivalence of (ii) and (iii) follows from Lemmas 6.1 and 6.2.

In order to finish the proof, we notice that $\lambda \neq 0$ and

$$-1 = \frac{-\lambda}{\lambda} = \left[\frac{\left[\frac{d-\frac{1}{d}}{2} \right]}{\left[\frac{c-\frac{1}{c}}{2} \right]} \right]^2.$$

□

Suppose that

$$\lambda = \left[\frac{c - \frac{1}{c}}{2} \right]^2 \quad \text{with } c \in K \setminus \{0, \pm 1, \pm 1 \pm \sqrt{2}, \pm \sqrt{-1}\}$$

and consider $Q = (\lambda, (\lambda + 1)\lambda) \in \mathcal{E}_{1,\lambda}(K)$ of order 4 with $2Q = W_3$. Let us find a point $R \in \mathcal{E}_{1,\lambda}(K)$ of order 8 with $2R = Q$. First, notice that

$$\begin{aligned} Q = (\lambda, (\lambda + 1)\lambda) &= \left(\left[\frac{c - \frac{1}{c}}{2} \right]^2, \left[\frac{c + \frac{1}{c}}{2} \right]^2 \cdot \left[\frac{c - \frac{1}{c}}{2} \right]^2 \right) \\ &= \left(\frac{(c^2 - 1)^2}{4c^2}, \frac{(c^4 - 1)^2}{4c^4} \right). \end{aligned}$$

We have

$$r_1 = \sqrt{\lambda + \lambda^2} = \sqrt{(\lambda + 1)\lambda}, \quad r_2 = \sqrt{\lambda + 1}, \quad r_3 = \sqrt{\lambda}; \quad r_1 r_2 r_3 = -(\lambda + 1)\lambda.$$

This means that

$$r_1 = \pm \frac{c - \frac{1}{c}}{2} \cdot \frac{c + \frac{1}{c}}{2}, \quad r_2 = \pm \frac{c + \frac{1}{c}}{2}, \quad r_3 = \pm \frac{c - \frac{1}{c}}{2},$$

and the signs should be chosen in such a way that the product $r_1 r_2 r_3$ coincides with

$$- \left[\frac{c - \frac{1}{c}}{2} \right]^2 \cdot \left[\frac{c + \frac{1}{c}}{2} \right]^2.$$

For example, we may take

$$r_1 = -\frac{c - \frac{1}{c}}{2} \cdot \frac{c + \frac{1}{c}}{2} = -\frac{c^2 - \frac{1}{c^2}}{4} = -\frac{c^4 - 1}{4c^2}, \quad r_2 = \frac{c + \frac{1}{c}}{2}, \quad r_3 = \frac{c - \frac{1}{c}}{2}$$

and get (since $r_2 + r_3 = c$ and $r_2 r_3 = (c^4 - 1)/4c^2$)

$$r_1 + r_2 + r_3 = -\frac{c^4 - 1}{4c^2} + c = \frac{-c^4 + 4c^3 + 1}{4c^2},$$

$$r_1 r_2 + r_2 r_3 + r_3 r_1 = c r_1 + r_2 r_3 = -\frac{c(c^4 - 1)}{4c^2} + \frac{c^4 - 1}{4c^2} = \frac{(1 - c)(c^4 - 1)}{4c^2}.$$

Now (4) and (7) tell us that the coordinates of the corresponding R with $2R = Q$ are as follows:

$$x(R) = x(Q) + r_1 r_2 + r_2 r_3 + r_3 r_1 = \frac{(c^2 - 1)^2}{4c^2} + \frac{(1 - c)(c^4 - 1)}{4c^2} = \frac{(1 - c)^3(c + 1)}{4c},$$

$$y(R) = -(r_1 + r_2)(r_2 + r_3)(r_1 + r_3) =$$

$$\begin{aligned}
& - \left(-\frac{c-\frac{1}{c}}{2} \cdot \frac{c+\frac{1}{c}}{2} + \frac{c+\frac{1}{c}}{2} \right) c \left(-\frac{c-\frac{1}{c}}{2} \cdot \frac{c+\frac{1}{c}}{2} + \frac{c-\frac{1}{c}}{2} \right) = \\
& - \left(1 - \frac{c-\frac{1}{c}}{2} \right) \cdot \frac{c+\frac{1}{c}}{2} \cdot c \cdot \left(1 - \frac{c+\frac{1}{c}}{2} \right) \frac{c-\frac{1}{c}}{2} = \\
& - \frac{c^2 - \frac{1}{c^2}}{16} \cdot \left(c - 2 - \frac{1}{c} \right) \left(c - 2 + \frac{1}{c} \right) c = - \frac{(c^2 - \frac{1}{c^2}) \left((c-2)^2 - \frac{1}{c^2} \right) c}{16}.
\end{aligned}$$

So, we get the K -point of order 8

$$R = \left(\frac{(1-c)^3(c+1)}{4c}, -\frac{(c^2 - \frac{1}{c^2}) \left((c-2)^2 - \frac{1}{c^2} \right) c}{16} \right)$$

on the elliptic curve

$$\mathcal{E}_{4,c} := \mathcal{E}_{1, \left(\pm \frac{c-\frac{1}{c}}{2} \right)^2} : y^2 = \left[x + \left(\frac{c-\frac{1}{c}}{2} \right)^4 \right] (x+1)x$$

for any $c \in K \setminus \{0, \pm 1, \pm 1 \pm \sqrt{2}, \pm \sqrt{-1}\}$. The group $\mathcal{E}_{4,c}(K)$ contains the subgroup generated by R and W_1 , which is isomorphic to $\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Theorem 6.5. *Let E be an elliptic curve over K . Then $E(K)$ contains a subgroup isomorphic to $\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ if and only if there exists $c \in K \setminus \{0, \pm 1, \pm 1 \pm \sqrt{2}, \pm \sqrt{-1}\}$ such that E is isomorphic to $\mathcal{E}_{4,c}$.*

Proof. We know that $\mathcal{E}_{4,c}(K)$ contains a subgroup isomorphic to $\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Conversely, suppose that $E(K)$ contains a subgroup isomorphic to $\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. This implies that $E(K)$ contains all three points of order 2, i.e., E may be represented in the form (1). Clearly, one of the points (2) is divisible by 4 in $E(K)$. We may assume that W_3 is divisible by 4. We may also assume that $\alpha_3 = 0$, i.e., $W_3 = (0, 0)$. Then we know that there exist distinct nonzero $a, b \in K$ such that $\alpha_1 = -a^2, \alpha_2 = -b^2$, i.e., the equation of E is

$$y^2 = (x+a^2)(x+b^2)x.$$

Replacing E by $E(b)$ and putting $\lambda = a/b$, we may assume that

$$E = \mathcal{E}_{1,\lambda} : y^2 = (x+\lambda^2)(x+1)x.$$

Since W_3 is divisible by 4 in $\mathcal{E}_{1,\lambda}(K)$, the desired result follows from Proposition 6.3. \square

Remark 6.6. There is another family of elliptic curves ([9, Table 3 on p. 217], [11, Appendix E])

$$y^2 + (1-a(t))xy - b(t)y = x^3 - b(t)x^2$$

with

$$a(t) = \frac{(2t+1)(8t^2+4t+1)}{2(4t+1)(8t^2-1)t}, \quad b(t) = \frac{(2t+1)(8t^2+4t+1)}{(8t^2-1)^2},$$

whose group of rational points contains a subgroup isomorphic to $\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Let us assume that t is an element of an arbitrary field K (with $\text{char}(K) \neq 2$) such that

$$t \neq 0, \quad 8t^2 - 1 \neq 0, \quad 4t + 1 \neq 0$$

and put

$$U(t) := (2t+1)(8t^2+4t+1), \quad A(t) = 2(4t+1)(8t^2-1)t \neq 0, \quad B(t) = (8t^2-1)^2 \neq 0,$$

$$a(t) = \frac{U(t)}{A(t)}, \quad b(t) = \frac{U(t)}{B(t)}.$$

Let us consider the cubic curve $\mathfrak{E}_{4,t}$ over K defined by the same equation

$$\mathfrak{E}_{4,t} : y^2 + (1 - a(t))xy - b(t)y = x^3 - b(t)x^2$$

as above. In light of Theorem 6.5, if $\mathfrak{E}_{4,t}$ is an elliptic curve over K , then $\mathfrak{E}_{4,t}$ is isomorphic to $\mathcal{E}_{4,c}$ for a certain $c \in K$. Let us find the corresponding λ (as a rational function of t). First, rewrite the equation for $\mathfrak{E}_{4,t}$ as

$$\left(y + \frac{(1 - a(t))x - b(t)}{2}\right)^2 = x^3 - b(t)x^2 + \left(\frac{(1 - a(t))x - b(t)}{2}\right)^2,$$

i.e.,

$$\left(y + \frac{(1 - a(t))x - b(t)}{2}\right)^2 = x^3 - \frac{U(t)}{B(t)} \cdot x^2 + \left(\frac{\left(1 - \frac{U(t)}{A(t)}\right)x - \frac{U(t)}{B(t)}}{2}\right)^2,$$

Second, multiplying the last equation by $(A(t)B(t))^6$ and introducing new variables

$$y_1 = (A(t)B(t))^3 \cdot \left(y + \frac{(1 - a(t))x - b(t)}{2}\right), \quad x_1 = (A(t)B(t))^2 \cdot x,$$

we obtain (with help of **magma**) the following equation for an isomorphic cubic curve $\tilde{\mathfrak{E}}_{4,t}$:

$$\begin{aligned} y_1^2 &= x_1^3 + \frac{-U(t)A(t)^2B(t) + ((U(t) - A(t))^2B(t)^2)}{4}x_1^2 \\ &+ \frac{(U(t) - A(t))U(t)A(t)^3B(t)^3}{2}x_1 + \frac{A(t)^6B(t)^4U(t)^2}{4} \\ &= (x_1 - \alpha_1)(x_1 - \alpha_2)(x_1 - \alpha_3), \end{aligned}$$

where

$$\begin{aligned} \alpha_1 &= -(-4194304t^{15} - 5242880t^{14} - 262144t^{13} + 2162688t^{12} + 753664t^{11} \\ &- 262144t^{10} - 172032t^9 - 2048t^8 + 14336t^7 + 2304t^6 - 320t^5 - 112t^4 - 8t^3), \end{aligned}$$

$$\begin{aligned} \alpha_2 &= -(4194304t^{16} + 4194304t^{15} - 1048576t^{14} - 2359296t^{13} - 327680t^{12} \\ &+ 491520t^{11} + 163840t^{10} - 40960t^9 - 25600t^8 + 1792t^6 + 192t^5 - 48t^4 - 8t^3), \end{aligned}$$

$$\begin{aligned} \alpha_3 &= -(-4194304t^{15} - 5242880t^{14} - 262144t^{13} + 2424832t^{12} + 1015808t^{11} \\ &- 294912t^{10} - 286720t^9 - 25600t^8 + 30720t^7 + 8960t^6 - 832t^5 \\ &- 720t^4 - 72t^3 + 16t^2 + 4t + 1/4). \end{aligned}$$

Using **magma**, we obtain that

$$\alpha_2 - \alpha_1 = -2^{22}t^4(t + 1/2)^4(t^2 - 1/8)^4, \quad \alpha_3 - \alpha_1 = -2^{18}(t + 1/4)^4(t^2 - 1/8)^4.$$

This implies that $\tilde{\mathfrak{E}}_{4,t}$ (and therefore $\mathfrak{E}_{4,t}$) is an elliptic curve over K (i.e., all three $\alpha_1, \alpha_2, \alpha_3$ are *distinct* elements of K) if and only if

$$t \neq 0, -\frac{1}{2}, -\frac{1}{4}, \pm \frac{1}{2\sqrt{2}}$$

and

$$\frac{\alpha_2 - \alpha_1}{\alpha_3 - \alpha_1} = \left(\frac{2t(t+1/2)}{t+1/4} \right)^4 \neq 1.$$

Assume that all these conditions hold. Then the change of variable $x_2 = x_1 + \alpha_1$ transforms $\tilde{\mathfrak{E}}_{3,t}$ to the elliptic curve

$$E : y_1^2 = x_2(x_2 - (\alpha_2 - \alpha_1))(x_2 - (\alpha_3 - \alpha_1)) =$$

$$x_2 (x_2 + 2^{22}t^4(t+1/2)^4(t^2-1/8)^4) (x_2 + 2^{18}(t+1/4)^4(t^2-1/8)^4).$$

If we put $\kappa = 2^9(t+1/4)^2(t^2-1/8)^2$, then

$$\kappa^2 = -(\alpha_3 - \alpha_1)$$

and E is isomorphic to the elliptic curve

$$E(\kappa) : y'^2 = x' \left(x' + \frac{\alpha_2 - \alpha_1}{\alpha_3 - \alpha_1} \right) (x' + 1) = x' \left(x' + \left(\frac{2t(t+1/2)}{t+1/4} \right)^4 \right) (x' + 1).$$

Notice that

$$\frac{2t(t+1/2)}{t+1/4} = \frac{2t(4t+2)}{(4t+1)} = \frac{4t(4t+2)}{2(4t+1)} = \frac{(4t+1)^2 - 1}{2(4t+1)} = \frac{(4t+1) - \frac{1}{(4t+1)}}{2},$$

and therefore $E(\kappa) = \mathcal{E}_{4,c}$ with $c = (4t+1)$. This implies that $\mathfrak{E}_{4,t}$ is isomorphic to $\mathcal{E}_{4,c}$ with $c = (4t+1)$.

Remark 6.7. Suppose that $K = \mathbb{F}_q$ with $q = 3, 5, 7$ or 9 . Then

$$\mathbb{F}_q \setminus \{0, 1, -1, \pm 1 \pm \sqrt{2}, \pm \sqrt{-1}\} = \emptyset.$$

Corollary 6.8. *Let E be an elliptic curve over \mathbb{F}_q , where $q = 11, 13, 17, 19$. The group $E(\mathbb{F}_q)$ is isomorphic to $\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ if and only if E is isomorphic to one of elliptic curves $\mathcal{E}_{4,c}$.*

Proof. Suppose that $E(\mathbb{F}_q)$ is isomorphic to $\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. It follows from Theorem 6.5 that E is isomorphic to one of the elliptic curves

$$\mathcal{E}_{4,c} : y^2 = \left[x + \left(\frac{c - \frac{1}{c}}{2} \right)^4 \right] (x+1)x$$

with $c \in K \setminus \{0, \pm 1, \pm \sqrt{-1}, \pm \sqrt{-1}\}$. Conversely, suppose that E is isomorphic to one of these curves. We need to prove that $E(\mathbb{F}_q)$ is isomorphic to $\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. By Theorem 6.5, $E(\mathbb{F}_q)$ contains a subgroup isomorphic to $\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$; in particular, 16 divides $|E(\mathbb{F}_q)|$. In order to finish the proof, it suffices to check that $|E(\mathbb{F}_q)| < 32$, but this inequality follows from the Hasse bound (10)

$$|E(\mathbb{F}_q)| \leq q + 2\sqrt{q} + 1 \leq 19 + 2\sqrt{19} + 1 < 29.$$

□

Corollary 6.9. *Let E be an elliptic curve over \mathbb{F}_{47} . The group $E(\mathbb{F}_{47})$ is isomorphic to $\mathbb{Z}/24\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ if and only if E is isomorphic to one of elliptic curves $\mathcal{E}_{4,c}$.*

Proof. Suppose that $E(\mathbb{F}_{47})$ is isomorphic to $\mathbb{Z}/24\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Then it contains a subgroup isomorphic to $\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. It follows from Theorem 6.5 that E is isomorphic to one of elliptic curves

$$\mathcal{E}_{4,c} : y^2 = \left[x + \left(\frac{c - \frac{1}{c}}{2} \right)^4 \right] (x + 1)x$$

with $c \in K \setminus \{0, \pm 1, \pm 1 \pm \sqrt{2}, \pm \sqrt{-1}\}$.

Conversely, suppose that E is isomorphic to one of these curves. We need to prove that $E(\mathbb{F}_{47})$ is isomorphic to $\mathbb{Z}/24\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. By Theorem 6.5, $E(\mathbb{F}_{47})$ contains a subgroup isomorphic to $\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$; in particular, 16 divides $|E(\mathbb{F}_{47})|$. The Hasse bound tells us that

$$47 + 1 - 2\sqrt{47} \leq |E(\mathbb{F}_{47})| \leq 47 + 1 + 2\sqrt{47}$$

and therefore $34 < |E(\mathbb{F}_{47})| < 62$. This implies that $|E(\mathbb{F}_{47})| = 48$; in particular, $E(\mathbb{F}_{47})$ contains a point of order 3. This implies that $E(\mathbb{F}_{47})$ contains a subgroup isomorphic to

$$(\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}/3\mathbb{Z} \cong \mathbb{Z}/24\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

Since this subgroup has the same order 48 as the whole group $E(\mathbb{F}_{47})$, we get the desired result. \square

Theorem 6.10. *Let $K = \mathbb{Q}$ and E be an elliptic curve over \mathbb{Q} . Then the torsion subgroup $E(\mathbb{Q})_t$ of $E(\mathbb{Q})$ is isomorphic to $\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ if and only if there exists $c \in \mathbb{Q} \setminus \{0, \pm 1\}$ such that E is isomorphic to $\mathcal{E}_{4,c}$.*

Proof. By Theorem 4.2 applied to $m = 4$, if $E(\mathbb{Q})$ contains a subgroup isomorphic to $\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ then $E(\mathbb{Q})_t$ is isomorphic to $\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Now the desired result follows from Theorem 6.5, since neither $\sqrt{2}$ nor $\sqrt{-1}$ lie in \mathbb{Q} . \square

Theorem 6.11. *Let E be an elliptic curve over K . Then $E(K)$ contains a subgroup isomorphic to $\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ if and only if K contains $\mathbf{i} = \sqrt{-1}$ and there exist*

$$c, d \in K \setminus \{0, \pm 1, \pm 1 \pm \sqrt{2}, \pm \sqrt{-1}\} \text{ such that } c - \frac{1}{c} = \mathbf{i} \left(d - \frac{1}{d} \right)$$

and E is isomorphic to $\mathcal{E}_{4,c}$.

Remark 6.12. The above equation defines an open dense set in the plane affine curve

$$(11) \quad \mathcal{M}_{8,4} : (c^2 - 1)d = \mathbf{i}(d^2 - 1)c.$$

It is immediate that the corresponding projective closure is a nonsingular cubic $\overline{\mathcal{M}}_{8,4}$ with a K -point, i.e., an elliptic curve. To obtain a Weierstrass normal form of $\overline{\mathcal{M}}_{8,4}$, we first slightly simplify equation(11) by the change of variables $d = s$, $\mathbf{i}c = t$ and get $s^2t + ts^2 + s - t = 0$. Then, using the birational transformation

$$s = \frac{\eta}{\xi + \xi^2}, \quad t = \frac{\eta}{1 + \xi},$$

we obtain $\eta^2 = \xi^3 - \xi$.³

³See [16, Example 1.4.2 on p. 88] for an explicit description of the (finite) set of all $\mathbb{Q}(\mathbf{i})$ -points on this elliptic curve; none of them corresponds to (c, d) that satisfy the conditions of Theorem 6.11.

Proof of Theorem 6.11. We have already seen that $\mathcal{E}_{4,c}(K)$ contains an order 8 point R with $4R = W_3$. It follows from Proposition 6.4 that $\mathcal{E}_{4,c}(K)$ contains all points of order 4. In particular, it contains an order 4 point \mathcal{Q} with $2\mathcal{Q} = W_1$. Clearly, R and \mathcal{Q} generate a subgroup isomorphic to $\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$.

Conversely, suppose that $E(K)$ contains a subgroup isomorphic to $\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. This implies that $E(K)$ contains all twelve points of order 4. In particular, E may be represented in the form (1). Clearly, one of the points of order 2 is divisible by 4 in $E(K)$. We may assume that W_3 is divisible by 4. The same arguments as in the proof of Theorem 6.5 allow us to assume that

$$E = \mathcal{E}_{1,\lambda} : y^2 = (x + \lambda^2)(x + 1)x.$$

Since W_3 is divisible by 4 in $\mathcal{E}_{1,\lambda}(K)$ and all points of order dividing 4 lie in $\mathcal{E}_{1,\lambda}(K)$, every point R of $\mathcal{E}_{1,\lambda}$ with $4R = W_3$ also lies in $\mathcal{E}_{1,\lambda}(K)$. It follows from Proposition 6.3 that K contains $\mathbf{i} = \sqrt{-1}$ and there exist

$$c, d \in K \setminus \{0, 1, -1, \pm 1 \pm \sqrt{2}, \pm \sqrt{-1}\}$$

such that

$$\lambda = \left[\frac{c - \frac{1}{c}}{2} \right]^2, \quad -\lambda = \left[\frac{d - \frac{1}{d}}{2} \right]^2.$$

This implies that

$$c - \frac{1}{c} = \pm \mathbf{i} \left(d - \frac{1}{d} \right).$$

Replacing if necessary d by $-d$, we obtain the desired

$$c - \frac{1}{c} = \mathbf{i} \left(d - \frac{1}{d} \right).$$

□

7. POINTS OF ORDER 3

The following assertion gives a simple description of points of order 3 on elliptic curves.

Proposition 7.1. *A point $P = (x_0, y_0) \in E(K)$ has order 3 if and only if one can choose three square roots $r_i = \sqrt{x_0 - \alpha_i}$ in such a way that*

$$r_1 r_2 + r_2 r_3 + r_3 r_1 = 0.$$

Proof. Indeed, let P be a point of order 3. Then $2(-P) = P$. Hence, all $x_0 - \alpha_i$ are squares in K . By (4),

$$x(-P) = x_0 + (r_1 r_2 + r_2 r_3 + r_3 r_1)$$

for a suitable choice of r_1, r_2, r_3 . Since $x(-P) = x(P) = x_0$, we get $r_1 r_2 + r_2 r_3 + r_3 r_1 = 0$.

Conversely, suppose that there exists a triple of square roots $r_i = \sqrt{x_0 - \alpha_i}$ such that $r_1 r_2 + r_2 r_3 + r_3 r_1 = 0$. Since $P \in E(K)$,

$$(r_1 r_2 r_3)^2 = (x_0 - \alpha_1)(x_0 - \alpha_2)(x_0 - \alpha_3) = y_0^2,$$

i.e., $r_1 r_2 r_3 = \pm y_0$. Replacing r_1, r_2, r_3 by $-r_1, -r_2, -r_3$ if necessary, we may assume that $r_1 r_2 r_3 = -y_0$. Then there exists a point $Q = (x(Q), y(Q)) \in E(K)$ such that

$2Q = P$, and $x_1 = x(Q), y_1 = y(Q)$ are expressed in terms of r_1, r_2, r_3 as in (6). Therefore

$$\begin{aligned} x(Q) &= x_0 + (r_1r_2 + r_2r_3 + r_3r_1) = x_0, \\ y(Q) &= -y_0 - (r_1 + r_2 + r_3)(r_1r_2 + r_2r_3 + r_3r_1) = -y_0, \end{aligned}$$

i.e., $Q = -P, 2(-P) = P$, and so P has order 3. \square

Theorem 7.2. *Let a_1, a_2, a_3 be elements of K such that all a_1^2, a_2^2, a_3^2 are distinct. Let us consider the elliptic curve*

$$E = E_{a_1, a_2, a_3} : y^2 = (x + a_1^2)(x + a_2^2)(x + a_3^2)$$

over K . Let $P = (0, a_1a_2a_3)$. Then P enjoys the following properties.

- (i) P is divisible by 2 in $E(K)$. More precisely, there are four points $Q \in E(K)$ with $2Q = P$, namely,

$$\begin{aligned} &(a_2a_3 - a_1a_2 - a_3a_1, (a_1 - a_2)(a_2 + a_3)(a_3 - a_1)), \\ &(a_3a_1 - a_1a_2 - a_2a_3, (a_1 - a_2)(a_2 - a_3)(a_3 + a_1)), \\ &(a_1a_2 - a_2a_3 - a_3a_1, (a_1 + a_2)(a_2 - a_3)(a_3 - a_1)), \\ &(a_1a_2 + a_2a_3 + a_3a_1, (a_1 + a_2)(a_2 + a_3)(a_3 + a_1)). \end{aligned}$$

- (ii) The following conditions are equivalent.

- (1) P has order 3.
- (2) None of a_i vanishes, i.e., $\pm a_1, \pm a_2, \pm a_3$ are six distinct elements of K , and one of the following four equalities holds:

$$\begin{aligned} a_2a_3 &= a_1a_2 + a_3a_1, & a_3a_1 &= a_1a_2 + a_2a_3, \\ a_1a_2 &= a_2a_3 + a_3a_1, & a_1a_2 + a_2a_3 + a_3a_1 &= 0. \end{aligned}$$

- (iii) Suppose that equivalent conditions (i)–(ii) hold. Then one of four points Q coincides with $-Q$ and has order 3, while the three other points are of order 6. In addition, $E(K)$ contains a subgroup isomorphic to $\mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Remark 7.3. Clearly, $E_{a_1, a_2, a_3} = E_{\pm a_1, \pm a_2, \pm a_3}$.

Proof of Theorem 7.2. We have

$$\alpha_1 = -a_1^2, \quad \alpha_2 = -a_2^2, \quad \alpha_3 = -a_3^2.$$

Let us try to divide P by 2 in $E(K)$. We have

$$r_1 = \pm a_1, \quad r_2 = \pm a_2, \quad r_3 = \pm a_3.$$

Since all r_i lie in K , the point $P = (0, a_1a_2a_3)$ is divisible by 2 in $E(K)$. Let Q be a point on E with $2Q = P$. By (4) and (7),

$$x(Q) = r_1r_2 + r_2r_3 + r_3r_1, \quad y(Q) = -(r_1 + r_2)(r_2 + r_3)(r_3 + r_1)$$

with $r_1r_2r_3 = -a_1a_2a_3$. Plugging in $r_i = \pm a_i$ into the formulas for $x(Q)$ and $y(Q)$, we get explicit formulas for points Q as in the statement of the theorem. This proves (i).

Let us prove (ii). Suppose that P has order 3. Since P is not of order 2, we have $0 = x(P) \neq \alpha_i$ for all $i = 1, 2, 3$. Since

$$\{\alpha_1, \alpha_2, \alpha_3\} = \{-a_1^2, -a_2^2, -a_3^2\},$$

none of a_i vanishes. It follows from Proposition 7.1 that one may choose the signs for r_i in such a way that $r_1r_2 + r_2r_3 + r_3r_1 = 0$. Plugging in $r_i = \pm a_i$ into this formula, we get four relations between a_1, a_2, a_3 as in (ii)(2).

Now suppose that one of the relations as in (ii)(2) holds. This means that one may choose the signs of $r_i = \pm a_i$ in such a way that $r_1 r_2 + r_2 r_3 + r_3 r_1 = 0$. It follows from Proposition 7.1 that P has order 3. This proves (ii).

Let us prove (iii). Since P has order 3, $2(-P) = P$, i.e., $-P$ is one of the four Q 's. Suppose that Q is a point of E with $2Q = P$, $Q \neq -P$. Clearly, the order of Q is either 3 or 6. Assume that Q has order 3. Then $P = 2Q = -Q$ and therefore $Q = -P$, which is not the case. Hence Q has order 6. Then $3Q$ has order 2, i.e., coincides with $W_i = (-a_i^2, 0)$ for some $i \in \{1, 2, 3\}$. Pick $j \in \{1, 2, 3\} \setminus \{i\}$ and consider the point $W_j = (-a_j^2, 0) \neq W_i$. Then the subgroup of $E(K)$ generated by Q and W_j is isomorphic to $\mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. This proves (iii). \square

Remark 7.4. In Theorem 7.2 we do *not* assume that $\text{char}(K) \neq 3$!

Corollary 7.5. *Let a_1, a_2, a_3 be elements of K such that a_1^2, a_2^2, a_3^2 are distinct. Then the following conditions are equivalent.*

- (i) *The point $P = (0, a_1 a_2 a_3) \in E_{a_1, a_2, a_3}(K)$ has order 3.*
- (ii) *None of a_i vanishes, and one may choose signs for*

$$a = \pm a_1, \quad b = \pm a_2, \quad c = \pm a_3$$

in such a way that $c = ab/(a + b)$.

If these conditions hold, then

$$E_{a_1, a_2, a_3} = E_{\lambda, b} : y^2 = (x^2 + (\lambda b)^2) (x + b^2) \left(x + \left(\frac{\lambda}{\lambda + 1} b \right)^2 \right),$$

where $\lambda = a/b \in K \setminus \{0, \pm 1, -2, -\frac{1}{2}\}$.

Proof. Suppose that condition (ii) of the corollary holds, i.e., none of a_i vanishes, and one may choose signs for

$$a = \pm a_1, \quad b = \pm a_2, \quad c = \pm a_3$$

in such a way that $c = ab/(a + b)$. Then none of a, b, c vanishes and $ab = ac + bc$. By Theorem 7.2(ii), $\mathcal{P} = (0, abc)$ is a point of order 3 on the elliptic curve

$$E_{\lambda, b} = E_{a_1, a_2, a_3}.$$

Since $abc = \pm a_1 a_2 a_3$, either $\mathcal{P} = P$ or $\mathcal{P} = -P$. In both cases P has order 3.

Notice that $\pm a_1, \pm a_2, \pm a_3$ are six distinct elements of K . This means that $\pm a, \pm b, \pm c$ are also six distinct elements of K . If we put $\lambda = a/b$, then

$$\pm \lambda b, \quad \pm b, \quad \pm \frac{\lambda + 1}{\lambda} b$$

are six distinct elements of K . This means (in light of the inequalities $a \neq 0, b \neq 0$) that

$$\lambda \neq 0, \pm 1, -2, -\frac{1}{2}.$$

Suppose P has order 3. By Theorem 7.2(ii), none of a_i vanishes and one of the following four equalities holds:

$$\begin{aligned} a_2 a_3 &= a_1 a_2 + a_3 a_1, & a_3 a_1 &= a_1 a_2 + a_2 a_3, \\ a_1 a_2 &= a_2 a_3 + a_3 a_1, & a_1 a_2 + a_2 a_3 + a_3 a_1 &= 0. \end{aligned}$$

Here are the corresponding choices of a, b, c with $c = ab/(a + b)$:

$$\begin{aligned} a = a_1, b = -a_2, c = a_3; \quad a = a_1, b = -a_2, c = a_3; \\ a = a_1, b = a_2, c = a_3; \quad a = a_1, b = a_2, c = -a_3. \end{aligned}$$

In order to finish the proof, we just need to notice that $a = \lambda b$ and

$$c = \frac{ab}{a + b} = \frac{\lambda b \cdot b}{\lambda b + b} = \frac{\lambda}{\lambda + 1} b.$$

□

Theorem 7.6. *Let E be an elliptic curve over K . Then $E(K)$ contains a subgroup isomorphic to $\mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ if and only if there exists $\lambda \in K \setminus \{0, \pm 1, -2, -\frac{1}{2}\}$ such that E is isomorphic to*

$$\mathcal{E}_{3,\lambda} : y^2 = (x^2 + \lambda^2)(x + 1) \left(x + \left(\frac{\lambda}{\lambda + 1} \right)^2 \right).$$

Proof of Theorem 7.6. Let $\lambda \in K \setminus \{0, \pm 1, -2, -1/2\}$ and put $a_1 = \lambda, a_2 = 1, a_3 = \lambda/(\lambda + 1)$. Then all a_i do not vanish, a_1^2, a_2^2, a_3^2 are three distinct elements of K , $a_1 a_2 = a_2 a_3 + a_3 a_1$, and $\mathcal{E}_{3,\lambda} = E_{a_1, a_2, a_3}$. It follows from Theorem 7.2 that $\mathcal{E}_{3,\lambda}$ contains a subgroup isomorphic to $\mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Conversely, suppose that E is an elliptic curve over K such that $E(K)$ contains a subgroup isomorphic to $\mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. It follows that all three points of order 2 lie in $E(K)$, and therefore E can be represented in the form (1). It is also clear that $E(K)$ contains a point of order 3. Let us choose a point $P = (x(P), y(P)) \in E(K)$ of order 3. We may assume that $x(P) = 0$. We have $P = 2(-P)$, and therefore P is divisible by 2 in $E(K)$. By Theorem 2.1, all $x(P) - \alpha_i = -\alpha_i$ are squares in K . This implies that there exist elements $a_1, a_2, a_3 \in K$ such that $\alpha_i = -a_i^2$. Clearly, all three a_1^2, a_2^2, a_3^2 are distinct. Since P lies on E ,

$$y(P)^2 = (x(P) + a_1^2)(x(P) + a_2^2)(x(P) + a_3^2) = a_1^2 a_2^2 a_3^2 = (a_1 a_2 a_3)^2,$$

and therefore $y(P) = \pm a_1 a_2 a_3$. Replacing P by $-P$ if necessary, we may assume that $y(P) = a_1 a_2 a_3$, i.e., $P = (0, a_1 a_2 a_3)$ is a K -point of order 3 on

$$E = E_{a_1, a_2, a_3} : y^2 = (x + a_1)^2 (x + a_2)^2 (x + a_3)^2.$$

It follows from Corollary 7.5 that there exist *nonzero* $b \in K$ and $\lambda \in K \setminus \{0, \pm 1, -2, -1/2\}$ such that

$$E = E_{a_1, a_2, a_3} = E_{\lambda, b} : y^2 = (x + (\lambda b)^2)(x + b^2) \left(x + \left[\frac{\lambda}{\lambda + 1} b \right]^2 \right).$$

But $E_{\lambda, b}$ is isomorphic to

$$E_{\lambda, b}(b) : y'^2 = (x' + \lambda^2)(x' + 1) \left(x' + \left[\frac{\lambda}{\lambda + 1} \right]^2 \right)$$

while the latter coincides with $\mathcal{E}_{3,\lambda}$. □

Remark 7.7. There is a family of elliptic curves over \mathbb{Q} [9, Table 3 on p. 217] (see also [11, Appendix E]),

$$\mathfrak{E}_{3,t} : y^2 + (1 - a(t))xy - b(t)y = x^3 - b(t)x^2,$$

with

$$a(t) = \frac{10 - 2t}{t^2 - 9}, \quad b(t) = \frac{-2(t-1)^2(t-5)}{(t^2 - 9)^2}$$

(with $t \in \mathbb{Q} \setminus \{1, 5, \pm 3, 9\}$), whose group of rational points contains a subgroup isomorphic to $\mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. (The point $(0, 0)$ of $\mathfrak{E}_{3,t}$ has order 6, *ibid.*) Let us assume that $t \neq \pm 3$ is an element of an arbitrary field K (with $\text{char}(K) \neq 2$) and consider the cubic curve $\mathfrak{E}_{3,t}$ over K defined by the same equation as above.

In light of Theorem 7.6, if $\mathfrak{E}_{3,t}$ is an elliptic curve over K , then $\mathfrak{E}_{3,t}$ is isomorphic to $\mathcal{E}_{3,\lambda}$ for a certain $\lambda \in K$. Let us find the corresponding λ (as a rational function of t). First, rewrite the equation for $\mathcal{E}_{3,\lambda}$ as

$$\left(y + \frac{(1 - a(t)x - b(t))}{2}\right)^2 = x^3 - b(t)x^2 + \left(\frac{(1 - a(t)x - b(t))}{2}\right)^2.$$

Second, multiplying the last equation by $(t^2 - 9)^6$ and introducing new variables

$$y_1 = (t^2 - 9)^3 \cdot \left(y + \frac{(1 - a(t)x - b(t))}{2}\right), \quad x_1 = (t^2 - 9)^2 \cdot x,$$

we obtain (with help of **magma**) the equation for an isomorphic cubic curve

$$\tilde{\mathfrak{E}}_{3,t} : y_1^2 = (x_1 - \alpha_1)(x_1 - \alpha_2)(x_1 - \alpha_3),$$

where

$$\begin{aligned} \alpha_1 &= -(2t^3 - 10t^2 - 18t + 90) = -2(t-5)(t-3)(t+3), \\ \alpha_2 &= -(2t^3 - 10t^2 + 14t - 6) = -2(t-3)(t-1)^2, \\ \alpha_3 &= -\left(\frac{1}{4}t^4 - t^3 - \frac{5}{2}t^2 + 7t - \frac{15}{4}\right) = -\frac{1}{4}(t-5)(t+3)(t-1)^2. \end{aligned}$$

We have

$$\alpha_1 - \alpha_2 = -2^5(t-3), \quad \alpha_2 - \alpha_3 = \frac{1}{4} \cdot (t-1)^3(t-9), \quad \alpha_3 - \alpha_1 = -\frac{1}{4} \cdot (t-5)^3(t+3).$$

This implies that $\tilde{\mathfrak{E}}_{3,t}$ (and therefore $\mathfrak{E}_{3,t}$) is an elliptic curve over K if and only if

$$t \in K \setminus \{1, \pm 3, 5, 9\}.$$

Further we assume that this condition holds and therefore $\tilde{\mathfrak{E}}_{3,t}$ and $\mathfrak{E}_{3,t}$ are elliptic curves over K . Clearly, all three points of order 2 on $\tilde{\mathfrak{E}}_{3,t}$ are defined over K and the K -point

$$Q = (x_1(Q), y_1(Q)) = (0, -(t-5)(t-3)(t+3)(t-1)^2)$$

lies on $\tilde{\mathfrak{E}}_{3,t}$. We prove that Q has order 6. Let us consider the point $P = 2Q \in E(K)$ with coordinates $x_1(P), y_1(P) \in K$. (Since $y_1(P) \neq 0$, $P \neq \infty$.) According to formulas of Section 1, there exists a unique triple r_1, r_2, r_3 of distinct elements of K such that

$$(r_1 + r_2)(r_2 + r_3)(r_3 + r_1) = -y_1(Q) = (t-5)(t-3)(t+3)(t-1)^2$$

and for all $i = 1, 2, 3$

$$x_1(P) - \alpha_i = r_i^2,$$

$$0 \neq -\alpha_i = x_1(Q) - \alpha_i = (r_i + r_j)(r_i + r_k),$$

where (i, j, k) is a permutation of $(1, 2, 3)$. This implies that

$$r_1 + r_2 = \frac{(t-5)(t-3)(t+3)(t-1)^2}{-\alpha_3} = \frac{(t-5)(t-3)(t+3)(t-1)^2}{\frac{1}{4}(t-5)(t+3)(t-1)^2} = 4(t-3),$$

$$r_2 + r_3 = \frac{(t-5)(t-3)(t+3)(t-1)^2}{-a_1} = \frac{(t-5)(t-3)(t+3)(t-1)^2}{2(t-5)(t-3)(t+3)} = \frac{1}{2} \cdot (t-1)^2,$$

$$r_3 + r_1 = \frac{(t-5)(t-3)(t+3)(t-1)^2}{-a_2} = \frac{(t-5)(t-3)(t+3)(t-1)^2}{2(t-3)(t-1)^2} = \frac{1}{2} \cdot (t-5)(t+3).$$

Hence

$$r_1 + r_2 = 4(t-3), \quad r_2 + r_3 = \frac{(t-1)^2}{2}, \quad r_3 + r_1 = \frac{(t+3)(t-5)}{2},$$

and therefore

$$r_1 + r_2 + r_3 = \frac{1}{2} \cdot ((r_1 + r_2) + (r_2 + r_3) + (r_3 + r_1)) = \frac{1}{2} \cdot (t^2 + 2t - 19),$$

which, in turn, implies that

$$r_1 = 2t - 10 = 2(t-5), \quad r_2 = 2t - 2 = 2(t-1), \quad r_3 = \frac{1}{2} \cdot (t-1)(t-5) = \frac{1}{8}r_1r_2.$$

One may easily check that

$$c(t) := -2t^3 + 14t^2 - 22t + 10 = r_i^2 + \alpha_i \text{ for all } i = 1, 2, 3.$$

This implies that

$$x_1(P) = c(t), \quad c(t) - \alpha_i = r_i^2 \text{ for all } i = 1, 2, 3$$

and $\tilde{\mathfrak{E}}_{3,t}$ is isomorphic to the elliptic curve

$$E_{r_1, r_2, r_3} : y_1^2 = (x_2 + r_1^2)(x_2 + r_2^2)(x_3 + r_3^2)$$

with $x_2 = x_1 - c(t)$. In addition,

$$y_1(P) = -r_1r_2r_3 = -2(t-1)^2(t-5).$$

We have

$$r_1r_2 = 8r_3, \quad r_2 - r_1 = 8.$$

This implies that $(r_2 - r_1)r_3 = r_1r_2$, which means that

$$(-r_1)r_2 + r_2r_3 + (-r_1)r_3 = 0.$$

It follows from Proposition 7.1 that P has order 3 in $\tilde{\mathfrak{E}}_{3,t}(K)$. (In particular, all $r_i \neq 0$.) Since $2Q = P$, the order of Q in $\tilde{\mathfrak{E}}_{3,t}$ is 6.

Notice that

$$-r_3 = \frac{(-r_1)r_2}{(-r_1) + r_2}$$

and

$$E_{r_1, r_2, r_3} = E_{-r_1, r_2, -r_3}.$$

It follows from Corollary 7.5 and the end of the proof of Theorem 7.6 that E_{r_1, r_2, r_3} is isomorphic to $\mathcal{E}_{3,\lambda}$ with

$$\lambda = \frac{-r_1}{r_2} = \frac{-(2t-10)}{2t-2} = -\frac{t-5}{t-1}.$$

This implies that $\mathfrak{E}_{3,t}$ is isomorphic to $\mathcal{E}_{3,\lambda}$ with $\lambda = -(t-5)/(t-1)$.

Corollary 7.8. *Let E be an elliptic curve over \mathbb{F}_q , where $q = 7, 9, 11, 13$. The group $E(\mathbb{F}_q)$ is isomorphic to $\mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ if and only if E is isomorphic to one of elliptic curves $\mathcal{E}_{3,\lambda}$.*

Proof. Suppose that $E(\mathbb{F}_q)$ is isomorphic to $\mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. By Theorem 7.6, E is isomorphic to one of elliptic curves $\mathcal{E}_{3,\lambda}$.

Conversely, suppose that E is isomorphic to one of these curves. We need to prove that $E(\mathbb{F}_q)$ is isomorphic to $\mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. By Theorem 7.6, $E(\mathbb{F}_q)$ contains a subgroup isomorphic to $\mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$; in particular, 12 divides $|E(\mathbb{F}_q)|$. In order to finish the proof, it suffices to check that $|E(\mathbb{F}_q)| < 24$, but this inequality follows from the Hasse bound (10)

$$|E(\mathbb{F}_q)| \leq q + 2\sqrt{q} + 1 \leq 13 + 2\sqrt{13} + 1 < 22.$$

□

Corollary 7.9. *Let E be an elliptic curve over \mathbb{F}_{23} . The group $E(\mathbb{F}_{23})$ is isomorphic to $\mathbb{Z}/12\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ if and only if E is isomorphic to one of elliptic curves $\mathcal{E}_{3,\lambda}$.*

Proof. Suppose that $E(\mathbb{F}_{23})$ is isomorphic to $\mathbb{Z}/12\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Then it contains a subgroup isomorphic to $\mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. It follows from Theorem 7.6 that E is isomorphic to one of elliptic curves $\mathcal{E}_{3,\lambda}$.

Conversely, suppose that E is isomorphic to one of these curves. We need to prove that $E(\mathbb{F}_{23})$ is isomorphic to $\mathbb{Z}/12\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. By Theorem 7.6, $E(\mathbb{F}_{23})$ contains a subgroup isomorphic to $\mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$; in particular, 12 divides $|E(\mathbb{F}_{23})|$. The Hasse bound (10) tells us that

$$23 + 1 - 2\sqrt{23} \leq |E(\mathbb{F}_{23})| \leq 23 + 1 + 2\sqrt{23}$$

and therefore $14 < |E(\mathbb{F}_{23})| < 34$. It follows that $|E(\mathbb{F}_{23})| = 24$; in particular the 2-primary component $E(\mathbb{F}_{23})(2)$ of $E(\mathbb{F}_{23})$ has order 8. On the other hand, $E(\mathbb{F}_{23})(2)$ is isomorphic to a product of two cyclic groups, each of which has even order. This implies that $E(\mathbb{F}_{23})(2)$ is isomorphic to $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Taking into account that $E(\mathbb{F}_{23})$ contains a point of order 3, we conclude that it contains a subgroup isomorphic to

$$(\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}/3\mathbb{Z} \cong \mathbb{Z}/12\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}.$$

This subgroup has the same order 24 as the whole group $E(\mathbb{F}_{23})$, which ends the proof. □

Theorem 7.10. *Let $K = \mathbb{Q}$ and E an elliptic curve over \mathbb{Q} . Then the torsion subgroup $E(\mathbb{Q})_t$ of $E(\mathbb{Q})$ is isomorphic to $\mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ if and only if there exists $\lambda \in \mathbb{Q} \setminus \{0, \pm 1, -2, -\frac{1}{2}\}$ such that E is isomorphic to $\mathcal{E}_{3,\lambda}$.*

Proof. By Theorem 4.2 applied to $m = 3$, if $E(\mathbb{Q})$ contains a subgroup isomorphic to $\mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ then $E(\mathbb{Q})_t$ is isomorphic to $\mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Now the desired result follows from Theorem 7.6. □

8. POINTS OF ORDER 5

The following assertion gives a description of points of order 5 on elliptic curves.

Proposition 8.1. *Let $P = (x_0, y_0) \in E(K)$. The point P has order 5 if and only if, for any permutation i, j, k of 1, 2, 3, one can choose square roots $r_i = \sqrt{x_0 - \alpha_i}$ and $r_i^{(1)} = \sqrt{(r_i + r_j)(r_i + r_k)}$ in such a way that*

$$(12) \quad \begin{aligned} (r_1 r_2 + r_2 r_3 + r_3 r_1) + (r_1^{(1)} r_2^{(1)} + r_2^{(1)} r_3^{(1)} + r_3^{(1)} r_1^{(1)}) &= 0, \\ r_1 r_2 + r_2 r_3 + r_3 r_1 &\neq 0. \end{aligned}$$

Remark 8.2. Notice that if we drop the condition $r_1 r_2 r_3 = -y_0$ in formulas (4) and (7), then we get 8 points Q such that $2Q = \pm P$. Similarly, if we drop the conditions $r_1 r_2 r_3 = -y_0$, $r_1^{(1)} r_2^{(1)} r_3^{(1)} = (r_1 + r_2)(r_2 + r_3)(r_3 + r_1)$ in the formulas (9), then we obtain all points R for which $4R = \pm P$.

Proof of Proposition 8.1. Suppose that P has order 5. Then $-P$ is a 1/4th of P . Therefore there exist r_i and $r_j^{(1)}$ such that

$$x(-P) = x(P) + (r_1 r_2 + r_2 r_3 + r_3 r_1) + (r_1^{(1)} r_2^{(1)} + r_2^{(1)} r_3^{(1)} + r_3^{(1)} r_1^{(1)}).$$

Since $x(P) = x(-P)$, we have

$$(r_1 r_2 + r_2 r_3 + r_3 r_1) + (r_1^{(1)} r_2^{(1)} + r_2^{(1)} r_3^{(1)} + r_3^{(1)} r_1^{(1)}) = 0.$$

On the other hand, if $r_1 r_2 + r_2 r_3 + r_3 r_1$, then the corresponding Q (with $2Q = P$) satisfies

$$x(Q) = x(P) + (r_1 r_2 + r_2 r_3 + r_3 r_1) = x(P)$$

and therefore $Q = P$ or $-P$. Since $2Q = P$, either $P = 2P$ or $Q = -P = -2Q$ has order 5. Clearly, $P \neq 2P$. If $Q = -2Q$ then Q has order dividing 3, which is not true, because its order is 5. The contradiction obtained proves that $r_1 r_2 + r_2 r_3 + r_3 r_1 \neq 0$.

Conversely, suppose there exist square roots

$$r_i = \sqrt{x_0 - \alpha_i} \quad \text{and} \quad r_i^{(1)} = \sqrt{(r_i + r_j)(r_i + r_k)}$$

that satisfy (12). Replacing if necessary all r_i by $-r_i$, we may and will assume that $r_1 r_2 r_3 = -y(P)$. Let $Q = (x(Q), y(Q))$ be the corresponding half of P with $x(Q) = x(P) + (r_1 r_2 + r_2 r_3 + r_3 r_1)$. Since $r_1 r_2 + r_2 r_3 + r_3 r_1 \neq 0$, we have $x(Q) \neq x(P)$; in particular, $Q \neq -P$. Replacing if necessary all $r_i^{(1)}$ by $r_i^{(1)}$, we may and will assume that

$$r_1^{(1)} r_2^{(1)} r_3^{(1)} = (r_1 + r_2)(r_2 + r_3)(r_3 + r_1) = -y(Q).$$

Let $R = (x(R), y(R))$ be the corresponding half of Q . Then $4R = 2(2R) = 2Q = P$ and

$$x(R) = x(P) + (r_1 r_2 + r_2 r_3 + r_3 r_1) + (r_1^{(1)} r_2^{(1)} + r_2^{(1)} r_3^{(1)} + r_3^{(1)} r_1^{(1)}) = x(P).$$

This means that either $R = P$ or $R = -P$. If $R = P$, then $R = 4R$ and R has order 3. This implies that both $Q = 2R$ and $P = 4R$ also have order 3. It follows that $P = -Q$, which is not the case. Therefore $R = -P$. This means that $R = -4R$, i.e., R has order 5 and therefore $P = -R$ also has order 5. \square

In what follows we will use the following identities in the polynomial ring $\mathbb{Z}[t_1, t_2, t_3]$ that could be checked either directly or by using **magma**.

$$(13) \quad \begin{aligned} & (-t_1^2 + t_2^2 + t_3^2)(t_1^2 - t_2^2 + t_3^2) + (t_1^2 - t_2^2 + t_3^2)(t_1^2 + t_2^2 - t_3^2) \\ & + (t_1^2 + t_2^2 - t_3^2)(-t_1^2 + t_2^2 + t_3^2) = \\ & -(t_1 + t_2 + t_3)(-t_1 + t_2 + t_3)(t_1 - t_2 + t_3)(t_1 + t_2 - t_3), \end{aligned}$$

$$\begin{aligned}
(14) \quad & (-t_1^2 + t_2^2 + t_3^2)(t_1^2 - t_2^2 + t_3^2) + (t_1^2 - t_2^2 + t_3^2)(t_1^2 + t_2^2 - t_3^2) \\
& + (t_1^2 + t_2^2 - t_3^2)(-t_1^2 + t_2^2 + t_3^2) + 4t_1^2t_2t_3 + 4t_1t_2^2t_3 + 4t_1t_2t_3^2 \\
& = t_1^4 + t_2^4 + t_3^4 - 2t_1^2t_2^2 - 2t_2^2t_3^2 - 2t_1^2t_3^2 - 4t_1^2t_2t_3 - 4t_1t_2^2t_3 - 4t_1t_2t_3^2 \\
& = (t_1 + t_2 + t_3)(t_1^3 + t_2^3 + t_3^3 - t_1^2t_2 - t_1t_2^2 - t_2^2t_3 - t_2t_3^2 - t_1^2t_3 - t_1t_3^2 - 2t_1t_2t_3).
\end{aligned}$$

Theorem 8.3. *Let a_1, a_2, a_3 be elements of K such that $\pm a_1, \pm a_2, \pm a_3$ are six distinct elements of K and none of three elements*

$$\beta_1 = -a_1^2 + a_2^2 + a_3^2, \beta_2 = a_1^2 - a_2^2 + a_3^2, \beta_3 = a_1^2 + a_2^2 - a_3^2$$

vanishes. Then the following conditions hold.

- (i) *None of a_i vanishes and $\beta_1^2, \beta_2^2, \beta_3^2$ are three distinct elements of K .*
- (ii) *Let us consider an elliptic curve*

$$E_{5;a_1,a_2,a_3} : y^2 = \left(x + \frac{\beta_1^2}{4}\right) \left(x + \frac{\beta_2^2}{4}\right) \left(x + \frac{\beta_3^2}{4}\right)$$

with $P = (0, -\beta_1\beta_2\beta_3/8) \in E_{5;a_1,a_2,a_3}(K)$.

Then P enjoys the following properties.

- (1) *$P \in 2E_{5;a_1,a_2,a_3}(K)$.*
- (2) *Assume that*

$$\begin{aligned}
(15) \quad & a_1^3 + a_2^3 + a_3^3 - a_1^2a_2 - a_1a_2^2 - a_2^2a_3 - a_2a_3^2 - a_1^2a_3 - a_1a_3^2 - 2a_1a_2a_3 = 0, \\
& (a_1 + a_2 + a_3)(a_1 - a_2 - a_3)(a_1 + a_2 - a_3)(a_1 - a_2 + a_3) \neq 0.
\end{aligned}$$

Then P has order 5. In addition, $E_{5;a_1,a_2,a_3}(K)$ contains a subgroup isomorphic to $\mathbb{Z}/10\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Proof. (i) Since $a_i \neq -a_i$, none of a_i vanishes. Let $i, j \in \{1, 2, 3\}$ be two distinct indices and $k \in \{1, 2, 3\}$ be the third one. Then

$$\beta_i - \beta_j = a_j^2 - a_i^2 \neq 0, \quad \beta_i + \beta_j = 2a_k^2 \neq 0.$$

This implies that $\beta_i^2 \neq \beta_j^2$.

(ii) Keeping our notation, we obtain that

$$r_1 = \pm \frac{\beta_1}{2} = \pm \frac{-a_1^2 + a_2^2 + a_3^2}{2}, r_2 = \pm \frac{\beta_2}{2} = \pm \frac{a_1^2 - a_2^2 + a_3^2}{2}, r_3 = \pm \frac{\beta_3}{2} = \pm \frac{a_1^2 + a_2^2 - a_3^2}{2},$$

$$r_i^{(1)} = \pm \sqrt{(r_i + r_j)(r_i + r_k)}$$

where i, j, k is any permutation of $1, 2, 3$. Thanks to Proposition 8.1, it suffices to check that one may choose the square roots r_i and $r_i^{(1)}$ in such a way that $r_1r_2 + r_2r_3 + r_3r_1 \neq 0$ and

$$(16) \quad (r_1r_2 + r_2r_3 + r_3r_1) + (r_1^{(1)}r_2^{(1)} + r_2^{(1)}r_3^{(1)} + r_3^{(1)}r_1^{(1)}) = 0.$$

Let us put

$$r_i = \frac{\beta_i}{2} = \frac{-a_i^2 + a_j^2 + a_k^2}{2}.$$

We have

$$r_1 + r_2 = a_3^2, \quad r_1 + r_3 = a_2^2, \quad r_2 + r_3 = a_1^2.$$

It follows that

$$(r_1^{(1)})^2 = a_2^2 a_3^2, \quad (r_2^{(1)})^2 = a_1^2 a_3^2, \quad (r_3^{(1)})^2 = a_1^2 a_2^2.$$

Let us put

$$r_1^{(1)} = a_2 a_3, \quad r_2^{(1)} = a_1 a_3, \quad r_3^{(1)} = a_1 a_2.$$

Then the condition (16) may be rewritten as follows.

$$\begin{aligned} & (-a_1^2 + a_2^2 + a_3^2)(a_1^2 - a_2^2 + a_3^2) + (a_1^2 - a_2^2 + a_3^2)(a_1^2 + a_2^2 - a_3^2) \\ & + (a_1^2 + a_2^2 - a_3^2)(-a_1^2 + a_2^2 + a_3^2) + 4a_1^2 a_2 a_3 + 4a_1 a_2^2 a_3 + 4a_1 a_2 a_3^2 = 0. \end{aligned}$$

In light of (14), the condition (16) may be rewritten as

$$(a_1 + a_2 + a_3)(a_1^3 + a_2^3 + a_3^3 - a_1^2 a_2 - a_1 a_2^2 - a_2^2 a_3 - a_2 a_3^2 - a_1^2 a_3 - a_1 a_3^2 - 2a_1 a_2 a_3) = 0.$$

The latter equality follows readily from the assumption (15) of Theorem. By Proposition 8.1, it suffices to check that $r_1 r_2 + r_2 r_3 + r_3 r_1 \neq 0$. In other words, we need to prove that

$$(17) \quad \begin{aligned} & (-a_1^2 + a_2^2 + a_3^2)(a_1^2 - a_2^2 + a_3^2) + (a_1^2 - a_2^2 + a_3^2)(a_1^2 + a_2^2 - a_3^2) \\ & + (a_1^2 + a_2^2 - a_3^2)(-a_1^2 + a_2^2 + a_3^2) \neq 0. \end{aligned}$$

In light of (13), this inequality is equivalent to

$$(a_1 + a_2 + a_3)(a_1 - a_2 - a_3)(a_1 + a_2 - a_3)(a_1 - a_2 + a_3) \neq 0.$$

But the latter inequality holds, by the assumption (15) of the theorem. Hence, P has order 5. Clearly, P and all points of order 2 generate a subgroup that is isomorphic to $\mathbb{Z}/10\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. \square

Theorem 8.4. *Let E be an elliptic curve over K . Then the following conditions are equivalent.*

- (i) $E(K)$ contains a subgroup isomorphic to $\mathbb{Z}/10\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.
- (ii) There exists a triple $\{a_1, a_2, a_3\} \subset K$ that satisfies all the conditions of Theorem 8.3, including (15), and such that E is isomorphic to $E_{5;a_1,a_2,a_3}$.

Proof. (i) follows from (ii), thanks to Theorem 8.3.

Suppose (i) holds. In order to prove (ii) it suffices to check that E is isomorphic to a certain $E_{5;a_1,a_2,a_3}$ over K . We may assume that E is defined by an equation of the form (1). Suppose that $P = (0, y(P)) \in E(K)$ has order 5. Then $P = 4(-P)$ is divisible by 4 in $E(K)$. This implies the existence of square roots $r_i = \sqrt{-\alpha_i} \in K$ and $r_i^{(1)} = \sqrt{(r_i + r_j)(r_i + r_k)} \in K$ in such a way that

$$\begin{aligned} x(-P) &= x(P) + (r_1 r_2 + r_2 r_3 + r_3 r_1) + (r_1^{(1)} r_2^{(1)} + r_2^{(1)} r_3^{(1)} + r_3^{(1)} r_1^{(1)}), \\ r_1^{(1)} r_2^{(1)} r_3^{(1)} &= (r_1 + r_2)(r_2 + r_3)(r_3 + r_1). \end{aligned}$$

Since $x(-P) = x(P) = 0$,

$$(18) \quad (r_1 r_2 + r_2 r_3 + r_3 r_1) + (r_1^{(1)} r_2^{(1)} + r_2^{(1)} r_3^{(1)} + r_3^{(1)} r_1^{(1)}) = 0.$$

Since the order of P is not 3,

$$(19) \quad r_1 r_2 + r_2 r_3 + r_3 r_1 \neq 0.$$

Recall that none of $r_i + r_j$ vanishes. Let us choose square roots

$$b_1 = \sqrt{r_2 + r_3}, b_2 = \sqrt{r_1 + r_3}, b_3 = \sqrt{r_1 + r_2}$$

in such a way that $r_1^{(1)} = b_2b_3, r_2^{(1)} = b_3b_1$. Since

$$r_1^{(1)}r_2^{(1)}r_3^{(1)} = b_1^2b_2^2b_3^2 = (b_1b_2b_3)^2,$$

we conclude that

$$r_3^{(1)} = \frac{r_1^{(1)}r_2^{(1)}r_3^{(1)}}{r_2^{(1)}r_3^{(1)}} = \frac{(b_1b_2b_3)^2}{(b_2b_3)(b_3b_1)} = b_1b_2.$$

We obtain that

$$(20) \quad r_1^{(1)} = b_2b_3, r_2^{(1)} = b_3b_1, r_3^{(1)} = b_1b_2.$$

Unfortunately, b_i do not have to lie in K . However, all the ratios b_i/b_j lie in K^* . We have

$$r_2 + r_3 = b_1^2, r_1 + r_3 = b_2^2, r_1 + r_2 = b_3^2$$

and therefore

$$(21) \quad \begin{aligned} r_1 &= \frac{-b_1^2 + b_2^2 + b_3^2}{2}, \quad r_2 = \frac{b_1^2 - b_2^2 + b_3^2}{2}, \quad r_3 = \frac{b_1^2 + b_2^2 - b_3^2}{2}, \\ \alpha_1 &= -r_1^2 = \frac{(-b_1^2 + b_2^2 + b_3^2)^2}{4}, \quad \alpha_2 = -r_2^2 = -\frac{(b_1^2 - b_2^2 + b_3^2)^2}{4}, \\ \alpha_3 &= -r_3^2 = -\frac{(b_1^2 + b_2^2 - b_3^2)^2}{4}, \\ P &= (0, -(r_1 + r_2)(r_2 + r_3)(r_3 + r_1)) = (0, -b_1^2b_2^2b_3^2) \in E(K). \end{aligned}$$

Since none of r_i vanishes, we get

$$-b_1^2 + b_2^2 + b_3^2 \neq 0, \quad b_1^2 - b_2^2 + b_3^2 \neq 0, \quad b_1^2 + b_2^2 - b_3^2 \neq 0.$$

Let us put

$$\gamma_1 = -b_1^2 + b_2^2 + b_3^2, \gamma_2 = b_1^2 - b_2^2 + b_3^2, \gamma_3 = b_1^2 + b_2^2 - b_3^2.$$

It follows from Theorem 8.3(i) that all β_i are *distinct* nonzero elements of K . The inequality (19) combined with first formula of (21) gives us

$$\begin{aligned} (-b_1^2 + b_2^2 + b_3^2)(b_1^2 - b_2^2 + b_3^2) + (b_1^2 - b_2^2 + b_3^2)(b_1^2 + b_2^2 - b_3^2) \\ + (b_1^2 + b_2^2 - b_3^2)(-b_1^2 + b_2^2 + b_3^2) \neq 0, \end{aligned}$$

which is equivalent (thanks to (13)) to

$$(b_1 + b_2 + b_3)(b_1 - b_2 - b_3)(b_1 + b_2 - b_3)(b_1 - b_2 + b_3) \neq 0.$$

In particular,

$$b_1 + b_2 + b_3 \neq 0.$$

The equality (18) gives us (thanks to (14))

$$(b_1 + b_2 + b_3)(b_1^3 + b_2^3 + b_3^3 - b_1^2b_2 - b_1b_2^2 - a_2^2b_3 - b_2b_3^2 - b_1^2b_3 - b_1b_3^2 - 2b_1b_2b_3) = 0,$$

i.e.,

$$(b_1^3 + b_2^3 + b_3^3 - b_1^2b_2 - b_1b_2^2 - a_2^2b_3 - b_2b_3^2 - b_1^2b_3 - b_1b_3^2 - 2b_1b_2b_3) = 0.$$

Let us put

$$a_1 = \frac{b_1}{b_3}, \quad a_2 = \frac{b_2}{b_3}, \quad a_3 = \frac{b_3}{b_3} = 1.$$

All a_i lie in K . Clearly, the triple $\{a_1, a_2, a_3\}$ satisfies all the conditions of Theorem 8.3 including (15). Let us put

$$\begin{aligned}\beta_1 &= -a_1^2 + a_2^2 + a_3^2 = \frac{\gamma_1}{b_3^2} = \frac{\gamma_1}{r_1 + r_2}, \\ \beta_2 &= a_1^2 - a_2^2 + a_3^2 = \frac{\gamma_2}{b_3^2} = \frac{\gamma_2}{r_1 + r_2}, \\ \beta_3 &= a_1^2 + a_2^2 - a_3^2 = \frac{\gamma_3}{b_3^2} = \frac{\gamma_3}{r_1 + r_2}.\end{aligned}$$

The equation of E is

$$y^2 = \left(x + \frac{\gamma_1^2}{4}\right) \left(x + \frac{\gamma_2^2}{4}\right) \left(x + \frac{\gamma_3^2}{4}\right).$$

Then E is isomorphic to

$$\begin{aligned}E(r_1 + r_2) : y'^2 &= \left(x' + \frac{\gamma_1^2}{4(r_1 + r_2)^2}\right) \left(x' + \frac{\gamma_2^2}{4(r_1 + r_2)^2}\right) \left(x' + \frac{\gamma_3^2}{4(r_1 + r_2)^2}\right) = \\ &= \left(x' + \frac{\beta_1^2}{4}\right) \left(x' + \frac{\gamma_2^2}{4}\right) \left(x' + \frac{\gamma_3^2}{4}\right).\end{aligned}$$

Clearly, $E(r_1 + r_2)$ coincides with $E_{5;a_1,a_2,a_3}$. \square

Remark 8.5. Let $E_{5;a_1,a_2,a_3}$ be as in Theorem 8.3. Clearly, $E_{5;a_1,a_2,a_3}(a_3) = E_{5;a_1/a_3,a_2/a_3,1}$. Let us put $\lambda = a_1/a_3, \mu = a_2/a_3$. Then

$$(22) \quad E_{5;a_1/a_3,a_2/a_3,1} = E_{5;\lambda,\mu,1} : y^2 = \left[x + \left(\frac{-\lambda^2 + \mu^2 + 1}{2} \right)^2 \right] \left[x + \left(\frac{\lambda^2 - \mu^2 + 1}{2} \right)^2 \right] \left[x + \left(\frac{\lambda^2 + \mu^2 - 1}{2} \right)^2 \right].$$

The equation of (isomorphic) $E_{5;\lambda,\mu,1} \left(\frac{\lambda^2 + \mu^2 - 1}{2} \right)$ is as follows.

$$(23) \quad E_{5;\lambda,\mu,1} \left(\frac{\lambda^2 + \mu^2 - 1}{2} \right) : y^2 = \left[x + \left(\frac{1 - \lambda^2 + \mu^2}{\lambda^2 + \mu^2 - 1} \right)^2 \right] \left[x + \left(\frac{\lambda^2 - \mu^2 + 1}{\lambda^2 + \mu^2 - 1} \right)^2 \right] (x+1).$$

The conditions on a_1, a_2, a_3 may be rewritten in terms of λ, μ as follows.

$$(24) \quad \begin{aligned}\lambda^3 + \mu^3 - \lambda^2\mu - \lambda\mu^2 - \lambda^2 - 2\lambda\mu - \mu^2 - \lambda - \mu + 1 &= 0, \\ \lambda \pm \mu &\neq \pm 1, \quad \lambda \neq 0, \quad \mu \neq 0, \quad \lambda \neq \pm\mu, \\ \lambda^2 + \mu^2 &\neq 1, \quad \lambda^2 - \mu^2 \neq \pm 1.\end{aligned}$$

The equality (24) is equivalent to

$$(25) \quad (\lambda + \mu)(\lambda - \mu)^2 - (\lambda + \mu)^2 - (\lambda + \mu) + 1 = 0.$$

Multiplying (25) by (non-vanishing) $(\lambda + \mu)$, we get the equivalent equation

$$(26) \quad (\lambda^2 - \mu^2)^2 - (\lambda + \mu)^3 - (\lambda + \mu)^2 + (\lambda + \mu) = 0.$$

Let us make the change of variables

$$\xi = \lambda + \mu, \eta = \lambda^2 - \mu^2.$$

Then (26) may be rewritten as

$$(27) \quad \eta^2 = \xi(\xi^2 + \xi - 1),$$

which is an (affine model of an) elliptic curve if $\text{char}(K) \neq 5$ and a singular rational plane cubic (Cartesian leaf) if $\text{char}(K) = 5$. Since

$$(28) \quad \lambda^2 + \mu^2 = \frac{(\lambda + \mu)^2 + (\lambda - \mu)^2}{2} = \frac{\xi^2 + \frac{\eta^2}{\xi^2}}{2} = \frac{\xi^2 + \frac{\xi^2 + \xi - 1}{\xi}}{2} = \frac{\xi^3 + \xi^2 + \xi - 1}{2\xi},$$

the only restrictions on (ξ, η) besides the equality (27) are the inequalities

$$\xi(\xi^2 + \xi - 1) \neq 0, \pm 1; \quad \xi^3 + \xi^2 + \xi - 1 \neq 2\xi, \quad \pm 1 \neq \frac{\eta}{\xi} = \sqrt{\frac{\xi(\xi^2 + \xi - 1)}{\xi^2}},$$

i.e.

$$(29) \quad \xi \neq 0, \pm 1, \frac{-1 \pm \sqrt{5}}{2}.$$

This means that

$$(30) \quad (\xi, \eta) \notin \{(0, 0), (\pm 1, \pm 1), \left(\frac{-1 \pm \sqrt{5}}{2}, 0\right)\}.$$

In light of (28), the equation (22) may be rewritten with coefficients being rational functions in ξ, η (rather than (λ, μ)) as follows.

$$\mathcal{E}_{5,\xi,\eta} : y^2 = \left[x + \left(\frac{2(1-\eta)}{\xi^3 + \xi^2 + \xi - 3} \right)^2 \right] \left[x + \left(\frac{2(\eta+1)}{\xi^3 + \xi^2 + \xi - 3} \right)^2 \right] (x+1).$$

Theorem 8.6. *Let E be an elliptic curve over K . Then the following conditions are equivalent.*

- (i) $E(K)$ contains a subgroup isomorphic to $\mathbb{Z}/10\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.
- (ii) There exist $(\xi, \eta) \in K^2$ that satisfy the equation (27) and inequalities (30) and such that E is isomorphic to $\mathcal{E}_{5,\xi,\eta}$.

Proof. The result follows from Theorem 8.4 combined with Remark 8.5. □

Remark 8.7. In Theorem 8.6 we do *not* assume that $\text{char}(K) \neq 5$!

Corollary 8.8. *Let E be an elliptic curve over \mathbb{F}_q with $q = 13, 17, 19, 23, 25, 27$. Then $E(\mathbb{F}_q)$ is isomorphic to $\mathbb{Z}/10\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ if and only if E is isomorphic to one of $\mathcal{E}_{5,\xi,\eta}$.*

Proof. Suppose that $E(\mathbb{F}_q)$ is isomorphic to $\mathbb{Z}/10\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. By Theorem 8.6, E is isomorphic to one of elliptic curves $\mathcal{E}_{5,\xi,\eta}$.

Conversely, suppose that E is isomorphic to one of these curves. We need to prove that $E(\mathbb{F}_q)$ is isomorphic to $\mathbb{Z}/10\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. By Theorem 8.6, $E(\mathbb{F}_q)$ contains a subgroup isomorphic to $\mathbb{Z}/10\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$; in particular, 20 divides $|E(\mathbb{F}_q)|$. In order to finish the proof, it suffices to check that $|E(\mathbb{F}_q)| < 40$, but this inequality follows from the Hasse bound (10)

$$|E(\mathbb{F}_q)| \leq q + 2\sqrt{q} + 1 \leq 27 + 2\sqrt{27} + 1 < 40.$$

□

Corollary 8.9. *Let E be an elliptic curve over \mathbb{F}_q with $q = 31, 37, 41, 43$. Then $E(\mathbb{F}_q)$ is isomorphic to $\mathbb{Z}/20\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ if and only if E is isomorphic to one of $\mathcal{E}_{5,\xi,\eta}$.*

Proof. Suppose that $E(\mathbb{F}_q)$ is isomorphic to $\mathbb{Z}/20\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$; the latter contains a subgroup isomorphic to $\mathbb{Z}/10\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. By Theorem 8.6, E is isomorphic to one of elliptic curves $\mathcal{E}_{5,\xi,\eta}$.

Conversely, suppose that E is isomorphic to one of these curves. We need to prove that $E(\mathbb{F}_q)$ is isomorphic to $\mathbb{Z}/20\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. By Theorem 8.6, $E(\mathbb{F}_q)$ contains a subgroup isomorphic to $\mathbb{Z}/10\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$; in particular, 20 divides $|E(\mathbb{F}_q)|$. It follows from the Hasse bound (10) that

$$20 < 31 - 2\sqrt{31} + 1 \leq |E(\mathbb{F}_q)| \leq 43 + 2\sqrt{43} + 1 < 60.$$

This implies that $|E(\mathbb{F}_q)| = 40$, and therefore $E(\mathbb{F}_q)$ is isomorphic to a direct sum of $\mathbb{Z}/5\mathbb{Z}$ and the order 8 abelian group $E(\mathbb{F}_q)(2)$; in addition, the latter group is isomorphic to a direct sum of two cyclic groups of even order (because it contains a subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$). This implies that $E(\mathbb{F}_q)(2)$ is isomorphic to $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. It follows that $E(\mathbb{F}_q)$ is isomorphic to a direct sum

$$\mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/20\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

□

Corollary 8.10. *Let E be an elliptic curve over \mathbb{F}_q with $q = 59$ or 61 . Then $E(\mathbb{F}_q)$ is isomorphic to $\mathbb{Z}/30\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ if and only if E is isomorphic to one of $\mathcal{E}_{5,\xi,\eta}$.*

Proof. Suppose that $E(\mathbb{F}_q)$ is isomorphic to $\mathbb{Z}/30\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$; the latter contains a subgroup isomorphic to $\mathbb{Z}/10\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. By Theorem 8.6, E is isomorphic to one of elliptic curves $\mathcal{E}_{5,\xi,\eta}$.

Conversely, suppose that E is isomorphic to one of these curves. We need to prove that $E(\mathbb{F}_q)$ is isomorphic to $\mathbb{Z}/30\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. By Theorem 8.6, $E(\mathbb{F}_q)$ contains a subgroup isomorphic to $\mathbb{Z}/10\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$; in particular, 20 divides $|E(\mathbb{F}_q)|$. It follows from the Hasse bound (10) that

$$40 < 59 - 2\sqrt{59} + 1 \leq |E(\mathbb{F}_q)| < 61 + 2\sqrt{61} + 1 < 80.$$

This implies that $|E(\mathbb{F}_q)| = 60$; in particular, $E(\mathbb{F}_q)$ contains a subgroup isomorphic to $\mathbb{Z}/3\mathbb{Z}$. This implies that $E(\mathbb{F}_q)$ contains a subgroup isomorphic to

$$(\mathbb{Z}/10\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}/3\mathbb{Z} \cong \mathbb{Z}/30\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z};$$

the order of this subgroup is 60, i.e., it coincides with the order of the whole group $E(\mathbb{F}_q)$. □

Theorem 8.11. *Let K be a quadratic field and E be an elliptic curve over K . Then the following conditions are equivalent.*

- (i) *The torsion subgroup $E(K)_t$ of $E(K)$ is isomorphic to $\mathbb{Z}/10\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.*
- (ii) *There exist $(\xi, \eta) \in K^2$ that satisfy the equation (27) and inequalities (30) and such that E is isomorphic to $\mathcal{E}_{5,\xi,\eta}$.*

Proof. By Theorem 4.3, if $E(K)$ contains a subgroup isomorphic to $\mathbb{Z}/10\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ then $E(K)_t$ is isomorphic to $\mathbb{Z}/10\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Now the desired result follows from Theorem 8.6. □

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