

BUILDING BLOCKS OF POLARIZED ENDOMORPHISMS OF NORMAL PROJECTIVE VARIETIES

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ABSTRACT. An endomorphism f of a projective variety X is polarized (resp. quasi-polarized) if $f^*H \sim qH$ (linear equivalence) for some ample (resp. nef and big) Cartier divisor H and integer $q > 1$. First, we use cone analysis to show that a quasi-polarized endomorphism is always polarized, and the polarized property descends via any equivariant dominant rational map. Next, we show that a suitable maximal rationally connected fibration (MRC) can be made f -equivariant using a construction of N. Nakayama, that f descends to a polarized endomorphism of the base Y of this MRC and that this Y is a Q -abelian variety (quasi-étale quotient of an abelian variety). Finally, we show that we can run the minimal model program (MMP) f -equivariantly for mildly singular X and reach either a Q -abelian variety or a Fano variety of Picard number one.

As a consequence, the building blocks of polarized endomorphisms are those of Q -abelian varieties and those of Fano varieties of Picard number one.

Along the way, we show that f always descends to a polarized endomorphism of the Albanese variety $\text{Alb}(X)$ of X , and that the pullback of a power of f acts as a scalar multiplication on the Neron-Severi group of X (modulo torsion) when X is smooth and rationally connected.

Partial answers about X being of Calabi-Yau type, or Fano type are also given with an extra primitivity assumption on f which seems necessary by an example.

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1. INTRODUCTION

We work over an algebraically closed field k which has characteristic zero, and is uncountable (only used to guarantee the birational invariance of the rational connectedness property). Let f be a surjective endomorphism of a projective variety X . We say that f is *polarized* (resp. *quasi-polarized*), if there is an ample (resp. nef and big) Cartier divisor H such that $f^*H \sim qH$ (linear equivalence) for some integer $q > 1$. If X is a point, then the only trivial endomorphism is polarized by convention.

Let X be a projective variety of dimension n . We refer to Definition 2.1 for the numerical equivalence (\equiv) of \mathbb{R} -Cartier divisors and Definition 2.2 for the weak numerical equivalence (\equiv_w) of r -cycles with real coefficients. Denote by $N^1(X) := \text{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ for the Néron-Severi group $\text{NS}(X)$. One can also regard $N^1(X)$ as the quotient vector space of \mathbb{R} -Cartier divisors modulo the numerical equivalence; see Definition 2.1. Denote by $N_r(X)$ the quotient vector space of r -cycles modulo the weak numerical equivalence.

Suppose further X is normal. Then the numerical equivalence and the weak numerical equivalence are the same for \mathbb{R} -Cartier divisors; in particular, the natural map $N^1(X) \rightarrow N_{n-1}(X)$ is well defined and an injection (cf. Definition 2.2 and Lemma 2.3). A Weil \mathbb{R} -divisor F is said to be *big* if $F = A + E$ for some ample \mathbb{Q} -Cartier divisor $A \in N^1(X)$ and pseudo-effective Weil \mathbb{R} -divisor E ; see Definition 2.4.

A surjective endomorphism $f : X \rightarrow X$ of a projective variety X is a finite morphism. In fact, f induces an automorphism $f^* : N^1(X) \rightarrow N^1(X)$. So an ample divisor is the pull back of some divisor, which, together with the projection formula, imply the finiteness of f . Suppose further $f^*H \sim qH$ for some nef and big Cartier divisor H and $q > 0$, then, by taking top self-intersection, the projection formula implies the relation between $\deg f$ and q : $\deg f = q^{\dim(X)}$.

Now we state our main results.

Proposition 1.1. (cf. Proposition 3.6) *Let $f : X \rightarrow X$ be a surjective endomorphism of a projective variety X and $q > 0$ a rational number. Assume one of the following two conditions.*

- (1) $f^*H \equiv qH$ for some big \mathbb{R} -Cartier divisor H .
- (2) X is normal and $f^*H \equiv_w qH$ for some big Weil \mathbb{R} -divisor H .

*Then q is an integer and $f^*A \equiv qA$ for some ample Cartier divisor A . Further, if $q > 1$, then f is polarized. In particular, quasi-polarized endomorphisms are polarized.*

Given a normal projective variety X , denote by $\text{Aut}(X)$ the full automorphism group of X and $\text{Aut}_0(X)$ its neutral connected component. Let B be a Cartier divisor. Denote by $\text{Aut}_{[B]}(X) := \{g \in \text{Aut}(X) \mid g^*B \equiv B\}$. If X is smooth and B is ample, then $[\text{Aut}_{[B]}(X) : \text{Aut}_0(X)] < \infty$ by [27, Proposition 2.2]. Generally, let G be a subgroup of $\text{Aut}(X)$, such that for any $g \in G$, $g^*B_g \equiv B_g$ for some big Cartier divisor B_g . Then $[G : G \cap \text{Aut}_0(X)] < \infty$ by [9, Theorem 2.1]. Now applying Proposition 1.1, we can further generalize this result to the following.

Theorem 1.2. *(cf. Theorem 3.7) Let X be a normal projective variety. Let G be a subgroup of $\text{Aut}(X)$, such that for any $g \in G$, $g^*B_g \equiv_w B_g$ for some big Weil \mathbb{R} -divisor B_g . Then $[G : G \cap \text{Aut}_0(X)] < \infty$.*

The polarized property descends via any equivariant dominant rational map. Indeed, we prove:

Theorem 1.3. *(cf. Theorem 3.11) Let $\pi : X \dashrightarrow Y$ be a dominant rational map between two projective varieties and let $f : X \rightarrow X$ and $g : Y \rightarrow Y$ be two surjective endomorphisms such that $g \circ \pi = \pi \circ f$. Suppose f is polarized. Then g is polarized; and $(\deg f)^{\dim(Y)} = (\deg g)^{\dim(X)}$.*

Given a projective variety X , pick any smooth model $p : X' \rightarrow X$, we define the Albanese map alb_X of X as $\text{alb}_{X'} \circ p^{-1}$:

$$X \xrightarrow{p^{-1}} X' \xrightarrow{\text{alb}_{X'}} \text{Alb}(X') =: \text{Alb}(X).$$

Clearly, alb_X and $\text{Alb}(X)$ are independent of the choice of X' . By the universal property of the Albanese map, any surjective endomorphism (or even dominant rational self-map) f of X descends to a surjective endomorphism $f|_{\text{Alb}(X)}$ of $\text{Alb}(X)$. The following Corollary 1.4 is an application of Theorem 1.3.

Corollary 1.4. *Let X be a projective variety with a polarized endomorphism $f : X \rightarrow X$ and let $\text{alb}_X : X \dashrightarrow \text{Alb}(X)$ be the Albanese map of X . Then the following are true.*

- (1) alb_X is a dominant rational map.
- (2) The endomorphism $f|_{\text{Alb}(X)}$ of $\text{Alb}(X)$ induced from f is polarized.

Remark 1.5. Corollary 1.4 affirmatively answers Krieger - Reschke [26, Question 1.10].

We refer to [24, Chapters 2 and 5] for the definitions and the properties of log canonical (lc), Kawamata log terminal (klt), canonical and terminal singularities. A normal projective variety Y is Q -abelian if there exists a finite surjective morphism $A \rightarrow Y$ étale in codimension one (or *quasi-étale* in short) with A an abelian variety. By the ramification divisor formula, $K_Y \sim_{\mathbb{Q}} 0$.

Let V be a projective variety of dimension n . V is said to be *uniruled* if there is a dominant rational map $\mathbb{P}^1 \times U \dashrightarrow X$ with $\dim(U) = n - 1$. V is said to be *rationally connected*, in the sense of Campana and Kollar-Miyaoka-Mori, ([6], [23]), if any two points of V are connected by a rational curve, which is equivalent to saying that two general points of V are connected by a rational curve since our ground field is uncountable (see [21]). Given a uniruled projective variety X , there is a fibration: $\pi : X \dashrightarrow Y$, such that Y is a non-uniruled normal projective variety (cf. [16]) and the graph of π over Y has the general fibre rationally connected. We call it an *MRC* (maximal rationally connected) fibration in the sense of Campana and Kollar-Miyaoka-Mori and this fibration is unique up to birational equivalence (cf. [21]). The Albanese map of X always factors through the MRC fibration; see Lemma 4.2.

However, in general, fixing one MRC fibration, a surjective endomorphism of X descends only to a dominant rational self-map of Y . Nevertheless, we have the next result.

Proposition 1.6. *Let X be a normal projective variety with a polarized endomorphism $f : X \rightarrow X$. Then there is a special MRC fibration $\pi : X \dashrightarrow Y$ in the sense of Nakayama [31] (which is the identity map when X is non-uniruled) together with a (well-defined) surjective endomorphism g of Y , such that the following are true.*

- (1) $g \circ \pi = \pi \circ f$; g is polarized.
- (2) Y is Q -abelian (with only canonical singularities). Hence there is a finite surjective morphism $T \rightarrow Y$ étale in codimension one with T an abelian variety and g lifts to a polarized endomorphism g_T of T .
- (3) Let $\bar{\Gamma}_{X/Y}$ be the normalization of the graph of π . Then the induced morphism $\bar{\Gamma}_{X/Y} \rightarrow Y$ is equi-dimensional with each fibre (irreducible) rationally connected.
- (4) If X has only klt singularities, then π is a morphism.

Remark 1.7. (1) By N. Fakhruddin (cf. [10]), the set of g -periodic points $\text{Per}(Y, g) := \{y \in Y \mid g^s(y) = y \text{ for some } s > 0\}$ is Zariski dense in Y . Thus the fibre $\bar{\Gamma}_y$ of the normalization of the graph of π over each $y \in \text{Per}(Y, g)$ is rationally connected and admits a polarized endomorphism.

(2) By virtue of Proposition 1.6, the building blocks of polarized endomorphisms are those on Q -abelian varieties and those on rationally connected varieties.

In view of Remark 1.7, we still need to consider an arbitrary polarized endomorphism f of a rationally connected variety X . If X is mildly singular, we can run the minimal model program (MMP) and reach a Fano fibration provided that K_X is not pseudo-effective, and continue the MMP for the base of the Fano fibration and so on. Now the problem is that we need to guarantee the equivariance of the MMP with respect to f (or its positive

power), i.e., to make sure that the extremal rays contracted during the MMP are fixed by f or its positive power. This is not easy, because there might be infinitely many extremal rays, being divisorial type, or flip type, or Fano type.

In Theorem 1.8, we manage to show the equivariance of the MMP with respect to a positive power of f , generalizing results in [35] for lower dimensions to all dimensions. This is done in Section 6.

For a surjective endomorphism $f : X \rightarrow X$ of a projective variety X , we say that $f^*|_{N^1(X)}$ is a *scalar multiplication* if there is some $q \in \mathbb{R}$ such that $f^*x = qx$ for all $x \in N^1(X)$. When alb_X is a surjective morphism, if $f^*|_{N^1(X)}$ is a scalar multiplication, then so is $(f|_{\text{Alb}(X)})^*|_{N^1(\text{Alb}(X))}$. The converse may not hold even if we assume f to be polarized and replace f by any positive power; see Example 7.2. Nevertheless, we have Theorem 1.8 (4) below.

Theorem 1.8. *Let $f : X \rightarrow X$ be a polarized endomorphism of a \mathbb{Q} -factorial klt projective variety X . Then, replacing f by a positive power, there exist a \mathbb{Q} -abelian variety Y , a morphism $X \rightarrow Y$, and an f -equivariant relative MMP over Y*

$$X = X_1 \dashrightarrow \cdots \dashrightarrow X_i \dashrightarrow \cdots \dashrightarrow X_r = Y$$

(i.e. $f = f_1$ descends to f_i on each X_i), with every $X_i \dashrightarrow X_{i+1}$ a divisorial contraction, a flip or a Fano contraction, of a K_{X_i} -negative extremal ray, such that we have:

- (1) If K_X is pseudo-effective, then $X = Y$ and it is \mathbb{Q} -abelian (see Proposition 1.6 or Lemma 4.6 for the lifting of f).
- (2) If K_X is not pseudo-effective, then for each i , $X_i \rightarrow Y$ is equi-dimensional holomorphic with every fibre (irreducible) rationally connected and f_i is polarized by some ample Cartier divisor H_i . The $X_{r-1} \rightarrow X_r = Y$ is a Fano contraction.
- (3) $N^1(X)$ is spanned by the pullbacks of $N^1(Y)$ and those $\{H_i\}_{i < r}$ which are f^* -eigenvectors corresponding to the same eigenvalue $q = (\deg f_i)^{1/\dim(X_i)}$ (independent of i).
- (4) $f^*|_{N^1(X)}$ is a scalar multiplication: $f^*|_{N^1(X)} = q \text{ id}$, if and only if so is f_r^* .

Remark 1.9. In Theorem 1.8, if we weaken the klt singularities assumption on X to lc singularities, then our lemmas assert that we still can run f -equivariant MMP. We refer to [14] for the LMMP of \mathbb{Q} -factorial log canonical pair. However, we could not show that this MMP terminates and could not claim assertions (1) - (2) in Theorem 1.8.

A normal projective variety X is of *Calabi-Yau type* (resp. *Fano type*), if there is a boundary \mathbb{Q} -divisor $\Delta \geq 0$, such that the pair (X, Δ) has at worst lc (resp. klt) singularities and $K_X + \Delta \sim_{\mathbb{Q}} 0$ (resp. $-(K_X + \Delta)$ is ample). Applying Theorem 1.8 and working a bit more, we have the following result.

Theorem 1.10. *Let $f : X \rightarrow X$ be a polarized endomorphism of a smooth rationally connected projective variety X . Then, replacing f by a positive power, we have:*

- (1) $f^*|_{\mathbb{N}^1(X)}$ is a scalar multiplication. Namely, $f^*|_{\mathbb{N}^1(X)} = q \text{ id}$ for some $q > 1$.
- (2) The number s of all prime divisors D_i with either $f^{-1}(D_i) = D_i$ or $\kappa(X, D_i) = 0$, satisfies $s \leq \dim(X) + \rho(X)$, where $\rho(X)$ is the Picard number of X .
- (3) The Iitaka D -dimensions satisfy $\kappa(X, -K_X) \geq \kappa(X, -(K_X + \sum_{i=1}^s D_i)) \geq 0$.
- (4) If (i) $s = \dim(X) + \rho(X)$, or (ii) $\kappa(X, -(K_X + \sum_{i=1}^s D_i)) = 0$ or (iii) D_1 is non-uniruled, then $K_X + \sum_{i=1}^s D_i \sim_{\mathbb{Q}} 0$ and X is of Calabi-Yau type (with $s = 1$ in Case (iii)).

Our Theorem 1.10 (3) is slightly stronger than that in [3, Theorem C], where they proved that $-K_X$ is pseudo-effective, but they did not assume X is rationally connected.

Let X be a normal projective variety. A polarized endomorphism $f : X \rightarrow X$ is *imprimitive* if there is a dominant rational map $\pi : X \dashrightarrow Y$ to a normal projective variety Y , with $0 < \dim(Y) < \dim(X)$, such that f descends to a polarized endomorphism f_Y of Y (i.e. $f_Y \circ \pi = \pi \circ f$). We say that f is *primitive* if it is not imprimitive.

Corollary 1.11. *Let $f : X \rightarrow X$ be a polarized endomorphism of a \mathbb{Q} -factorial klt projective variety X such that:*

- (i) X is not a \mathbb{Q} -abelian variety, and
- (ii) f^s is primitive for all $s > 0$.

Then, replacing f by a positive power, we have the following assertions.

- (1) (X, f) is equivariantly birational to a Fano variety X_{r-1} of Picard number one, and $f^*|_{\mathbb{N}^1(X)}$ is a scalar multiplication. Precisely, running the MMP on X we get an f -equivariant sequence $X = X_1 \dashrightarrow \cdots \dashrightarrow X_{r-1}$ of divisorial contractions and flips, such that X_{r-1} is a Fano variety of Picard number one with a polarized endomorphism f_{r-1} of X_{r-1} (induced from f).
- (2) $-K_X$ is big.

Remark 1.12.

- (1) Corollary 1.11 partially answers [39, Question 1.5] about X being of Calabi-Yau type. It also answers [39, Question 1.6 (2)] about X being of Fano type (but only up to f -equivariant birational map) with an extra primitivity assumption on the pair (X, f) which or something similar is needed, otherwise, Example 7.3 gives a negative answer in the general case.

- (2) By Remark 1.7 and Corollary 1.11, we may say the building blocks of polarized endomorphisms are those on $(Q-)$ abelian varieties and Fano varieties of Picard number one. This belief was stated and confirmed in [35] in dimension ≤ 4 .

The following question is natural from Theorem 1.10 without assuming X to be smooth.

Question 1.13. *Let $f : X \rightarrow X$ be a polarized endomorphism of a rationally connected normal projective variety X . Assume that X has at worst \mathbb{Q} -factorial terminal singularities. Is $(f^s)^*|_{\mathbb{N}^1(X)}$ a scalar multiplication for some $s > 0$?*

The above question has a positive answer when $\dim(X) \leq 3$; see [35, Theorem 1.2]. Without the rational connectedness condition, Question 1.13 has a negative answer (see Example 7.1). In view of Example 7.2, though it is a $K3$ surface with canonical singularities, the terminality condition might be needed too.

2. PRELIMINARY RESULTS

Let X be a projective variety. We use Cartier divisor H (a Cartier divisor is integral, unless otherwise indicated) and its corresponding invertible sheaf $\mathcal{O}(H)$ interchangeably. Denote by $\text{Pic}(X)$ the group of Cartier divisors modulo linear equivalence and $\text{Pic}^0(X)$ the subgroup of the classes in $\text{Pic}(X)$ which are algebraically equivalent to 0. Denote by $\mathbb{N}^1(X) := \text{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ for the Néron-Severi group $\text{NS}(X)$, where $\text{NS}(X) = \text{Pic}(X)/\text{Pic}^0(X)$.

Definition 2.1 (equivalences of Cartier divisors). Let X be a projective variety of dimension n and D a Cartier divisor. D is said to be τ -equivalent to 0 if mD is algebraically equivalent to 0 for some integer $m \neq 0$. D is said to be *numerically equivalent* to 0 if $D \cdot C = 0$ for any curve C of X . We extend the above two equivalences to the case of \mathbb{R} -Cartier divisors. Let E be an \mathbb{R} -Cartier divisor. E is said to be τ -equivalent to 0 (denoted by $E \sim_{\tau} 0$) if $E = \sum_i a_i E_i$ where $a_i \in \mathbb{R}$ and E_i is a Cartier divisor τ -equivalent to 0. Similarly, E is said to be *numerically equivalent* to 0 (denoted by $E \equiv 0$) if $E = \sum_i a_i E_i$ where $a_i \in \mathbb{R}$ and E_i is a Cartier divisor numerically equivalent to 0. It is known that $E \sim_{\tau} 0$ if and only if $E \equiv 0$ (cf. [11, Theorem 9.6.3]). Therefore, one can also regard $\mathbb{N}^1(X)$ as the quotient vector space of \mathbb{R} -Cartier divisors modulo the numerical equivalence.

Definition 2.2 (weak numerical equivalence of cycles). Let X be a projective variety of dimension n . Let Z and Z' be two r -cycles with real coefficients. Z is said to be *weakly numerically equivalent* to Z' (denoted by $Z \equiv_w Z'$) if $(Z - Z') \cdot H_1 \cdots H_r = 0$ for any Cartier divisors H_1, \dots, H_r . Denote by $\mathbb{N}_r(X)$ the quotient vector space of r -cycles with real coefficients modulo the weak numerical equivalence.

Suppose further X is normal. $N_{n-1}(X)$ is then the quotient vector space of Weil \mathbb{R} -divisors modulo the weak numerical equivalence. It is known that Cartier divisors of X are Weil divisors. Let D be an \mathbb{R} -Cartier divisor. If $D \equiv 0$, then $D \equiv_w 0$. The converse is true by the lemma below. Therefore, one can regard $N^1(X)$ as a subspace of $N_{n-1}(X)$ and they are the same if X is also \mathbb{Q} -factorial.

Lemma 2.3. (cf. [37, Lemma 3.2]) *Let X be a projective variety of dimension n . Let H_1, \dots, H_{n-1} be ample \mathbb{R} -Cartier divisors and M an \mathbb{R} -Cartier divisor. Suppose that*

$$H_1 \cdots H_{n-1} \cdot M = 0 = H_1 \cdots H_{n-2} \cdot M^2.$$

Then $M \equiv 0$ (i.e., $M \cdot C = 0$ for any curve C of X). In particular, if X is normal, then $N^1(X)$ is a subspace of $N_{n-1}(X)$.

Definition 2.4. Let X be a projective variety of dimension n . We define:

- $\text{Amp}(X)$, the set of classes of ample \mathbb{R} -Cartier divisors in $N^1(X)$.
- $\text{Nef}(X)$, the set of classes of nef \mathbb{R} -Cartier divisors in $N^1(X)$.
- $\text{PEC}(X)$, the closure of the set of classes of effective \mathbb{R} -Cartier divisors in $N^1(X)$.
- $\text{PE}(X)$, the closure of the set of classes of effective $(n-1)$ -cycles with \mathbb{R} -coefficients in $N_{n-1}(X)$.

Clearly, the above sets are all convex cones and contain no lines. Note that $\text{PEC}(X)$ can also be regarded as the closure of the set of classes of big \mathbb{R} -Cartier divisors in $N^1(X)$. Let $f : X \rightarrow X$ be a finite surjective endomorphism. We may define pullback of cycles for f , such that f^* induces an automorphism of $N_r(X)$ and $f_* f^* = (\deg f) \text{id}$; see [35, Section 2]. Note that the above cones are $(f^*)^{\pm 1}$ -invariant.

Suppose further X is normal. Let D be a Weil \mathbb{R} -divisor. Then D is said to be *pseudo-effective* if its class $[D] \in \text{PE}(X)$. In addition, D is said to be *big* if $D = A + E$ for some ample \mathbb{Q} -Cartier divisor and pseudo-effective Weil \mathbb{R} -divisor E , see [15, Theorem 3.5] for equivalent definitions. Note that $\text{PE}(X)$ can also be regarded as the closure of the set of classes of big Weil \mathbb{R} -divisors in $N_{n-1}(X)$.

Definition 2.5. Let X be a normal projective variety. Define:

- (1) $q(X) = h^1(X, \mathcal{O}_X) = \dim H^1(X, \mathcal{O}_X)$ (the irregularity);
- (2) $\tilde{q}(X) = q(\tilde{X})$ with \tilde{X} a smooth projective model of X ; and
- (3) $q^{\natural}(X) = \sup\{\tilde{q}(X') \mid X' \rightarrow X \text{ is finite surjective and étale in codimension one}\}.$

Definition 2.6. Let V be a finite dimensional real normed vector space and $S \subseteq V$ a subset.

The convex hull generated by S is defined as

$$\left\{ \sum_{i \in I} a_i x_i \mid a_i \geq 0, x_i \in S, \sum_{i \in I} a_i = 1, |I| < \infty \right\}.$$

Suppose S is bounded, i.e., there exists some $N > 0$, such that $|s| < N$ for any $s \in S$. Then $|\sum_{i \in I} a_i x_i| \leq \sum_{i \in I} a_i \max_{i \in I} \{|x_i|\} = \max_{i \in I} \{|x_i|\} < N$. So the closure of the convex hull generated by S is bounded. Let m be the dimension of the vector space spanned by S . If S is a finite set, then the convex hull generated by S is a polytope which is covered by finitely many m -simplexes. So for arbitrary S (possibly an infinite set) and the convex hull D generated by S , we may write the element $d \in D$ as $d = \sum_{i=1}^{m+1} a_i x_i$ with $a_i \geq 0$, $x_i \in S$ and $\sum_{i=1}^{m+1} a_i = 1$.

The convex cone generated by S is defined as

$$\left\{ \sum_{i \in I} a_i x_i \mid a_i \geq 0, x_i \in S, |I| < \infty \right\}.$$

Let A and B be two subsets of V . We define $d(A, B) = \inf\{|a - b| : a \in A, b \in B\}$. Denote by $B(x, r) := \{x' \in V : |x - x'| < r\}$.

Let C be a convex cone in V . Let W be the subspace of V spanned by C . Denote by \overline{C} the usual closure of C in V . Note that $\overline{C} \subseteq W$. Denote by C° the interior part of C as the biggest open subset of W contained in C . Denote by $\partial C := \overline{C} - C^\circ$ the boundary of C . Note that $C^\circ = \overline{C}^\circ$ and $\partial C = \partial \overline{C}$.

Let $C_1 \subseteq C_2$ be two convex cones of V . We say C_1 is an *extremal face* of C_2 if $u + v \in C_1$ implies that $u, v \in C_1$ for any $u, v \in C_2$. An extremal face of a closed convex cone is always closed.

Lemma 2.7. *Let V be a positive dimensional real normed vector space. Let $C \subseteq V$ be a closed convex cone which spans V and contains no line. Let $C' \subseteq \partial C$ be a convex subcone. Then C' is contained in a closed extremal face $F \subseteq \partial C$. In particular, C' is contained in a unique minimal closed extremal face in ∂C .*

Proof. Let $F := \{x \in C \mid x + y \in C' \text{ for some } y \in C\}$.

If $x_1, x_2 \in F$, then $x_1 + y_1 \in C'$ and $x_2 + y_2 \in C'$ for some $y_1, y_2 \in C$. For any $a \geq 0$, $ax_1 + ay_1 \in C'$ and $x_1 + x_2 + y_1 + y_2 \in C'$. So F is a convex cone.

If $x \in F \cap C^\circ$, then $x + y \in C'$ for some $y \in C$ and $B(x, r) \subseteq C^\circ$ for some $r > 0$. So $x + y \in B(x + y, r) = y + B(x, r) \subseteq C^\circ$ and hence $C' \cap C^\circ \neq \emptyset$, a contradiction. So $F \subseteq \partial C$.

If $x \in C'$, then $x + 0 = x \in C'$. So $C' \subseteq F$.

If $p, q \in C$ and $p + q \in F$, then $p + q + s \in C'$ for some $s \in C$. By the construction of F , we have both $p, q \in F$. So F is an extremal face.

Since C is closed, F is closed too.

By taking the intersection of all extremal faces containing C' , we get the minimal one. In fact, F is already the minimal one. Suppose F' is an extremal face containing C' . Then for any $x \in F$, $x + y \in C' \subseteq F'$ for some $y \in C$. So $x \in F'$. \square

Lemma 2.8. *Let V be a positive dimensional real normed vector space. Let $C \subseteq V$ be a closed convex cone containing no lines and $F \subsetneq C$ a positive dimensional extremal face. Fix $d > 0$ and $k > 0$. Let S be the set of all $x \in C$ with $d(x, F) \geq d$ and $|x| \leq k$. Let B be the closure of the convex cone generated by S . Then $d(F, B) > 0$.*

Proof. Let B' be the convex cone generated by S . Let n be the dimension of the vector space spanned by S . Define

$$\eta(p, x_1, \dots, x_{n+1}) := d(p, D_{x_1, \dots, x_{n+1}}),$$

where $p \in F$, $x_i \in S$ and $D_{x_1, \dots, x_{n+1}}$ is the convex polytope generated by x_1, \dots, x_{n+1} . Clearly, η is a continuous function from $F \times S^{\times(n+1)}$ to $\mathbb{R}_{\geq 0}$. If $\eta(p, x_1, \dots, x_{n+1}) = 0$, then $p \in D_{x_1, \dots, x_{n+1}}$. Since F is an extremal face, $x_i \in F$, a contradiction. So $\eta > 0$. Let $F_{>2k} := \{p \in F : |p| > 2k\}$ and $F_{\leq 2k} = F - F_{>2k}$. Then $A := F_{\leq 2k} \times S^{\times(n+1)}$ is compact and hence $\eta|_A \geq d_1$ for some $d_1 > 0$. Note that $D(x_1, \dots, x_{n+1})$ is bounded by k . So for $A' := F_{>2k} \times S^{\times(n+1)}$, $\eta|_{A'} > k$. Since $B' = \cup_{x_1, \dots, x_{n+1} \in S} D_{x_1, \dots, x_{n+1}}$, $d(F, B') = \inf_{p \in F; x_1, \dots, x_{n+1} \in S} \{d(p, D_{x_1, \dots, x_{n+1}})\} \geq \min\{d_1, k\} > 0$ and hence $d(F, B) > 0$. \square

For a linear map $f : V \rightarrow V$ of a finite dimensional real normed vector space V , denote by $\|f\|$ the norm of f .

Proposition 2.9. *Let $f : V \rightarrow V$ be an invertible linear map of a positive dimensional real normed vector space V such that $f^{\pm 1}(C) = C$ for a closed convex cone $C \subseteq V$ which spans V and contains no line. Let q be a positive number. Then (1) and (2) below are equivalent.*

- (1) $f(x) = qx$ for some $x \in C^\circ$ (the interior part of C).
- (2) There exists a constant $N > 0$, such that $\frac{\|f^i\|}{q^i} < N$ for any $i \in \mathbb{Z}$.

Let $W \subseteq V$ be the eigenspace of f corresponding to the eigenvalue q . If (1) or (2) above is true, then f is a diagonalizable linear map with all eigenvalues of modulus q . So $F_\infty := \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \frac{f^i}{q^i}$ is a well-defined linear map onto W and $F_\infty|_W = \text{id}_W$.

Proof. We may assume $q = 1$ after replacing f by f/q .

(1) \Rightarrow (2) We may assume $|x| = 1$. Since $x \in C^\circ$ and C spans V , there exists some $t > 0$, such that $x + tv \in C$ for any $v \in V$ with $|v| \leq 1$. Since C contains no line, there exists some $s > 0$, such that $x - x' \notin C$ for any $x' \in C$ with $|x'| \geq s$. Suppose that $\|f^i\|$ is not bounded. Since C spans V , there exist some $i_0 \in \mathbb{Z}$ and $y \in C$ with $|y| = 1$, such that $|tf^{i_0}(y)| > s$. Since $x - ty \in C$, $f^{i_0}(x - ty) \in C$. However, $f^{i_0}(x - ty) = x - tf^{i_0}(y) \notin C$, a contradiction.

Suppose (2) is true. If either the spectral radius of f is greater than 1 or f has a nontrivial Jordan block whose eigenvalue is of modulus 1. Then $\lim_{n \rightarrow +\infty} \|f^n\| = +\infty$, a contradiction. Similarly, the spectral radius of f^{-1} is 1. Therefore, the last assertion of the proposition follows.

(2) \Rightarrow (1) Set $n = \dim(V)$ and $m = \dim(W)$. If $m = n$, it is trivial. Suppose $m < n$. If $W \cap C^\circ \neq \emptyset$, then we are done. Suppose $W \cap C^\circ = \emptyset$. Since $f^{\pm 1}(C) = C$, $F_\infty(C) \subseteq C$ and hence $F_\infty(C) \subseteq W \cap C$. Note that W is spanned by $F_\infty(C)$. Then $W \cap C$ is an m -dimensional closed convex subcone of C in ∂C . By Lemma 2.7, $W \cap C \subseteq F$ for some minimal closed extremal face F of C in ∂C . Note that $f^{\pm 1}(F)$ are still minimal closed extremal faces in ∂C containing $W \cap C$. By the uniqueness, $f^{\pm 1}(F) = F$. Fix any $y \in C - F$ and $d := d(y, F) > 0$. Then by (2), for any $z \in F$, $d \leq |y - f^{-i}(z)| = |f^{-i}(f^i(y) - z)| < N|f^i(y) - z|$. So $d(f^i(y), F) \geq \frac{d}{N}$ for any $i \in \mathbb{Z}$. Let B be the closure of the convex hull generated by all $f^i(y)$ ($i \in \mathbb{Z}$). By (2), B is bounded, $f^{\pm 1}(B) = B$ and $d(B, F) > 0$ by Lemma 2.8. Since $C \cap W \subseteq F$ and $B \subseteq C$, $B \cap W = B \cap C \cap W \subseteq B \cap F = \emptyset$. However, by Brouwer-fixed point theorem, $f(b) = b$ for some $b \in B$. So we get a contradiction. \square

Lemma 2.10. (cf. [24, Lemma 2.62]) *Let $f : X \rightarrow Y$ be a birational morphism of normal projective varieties. Assume that Y is \mathbb{Q} -factorial. Then there is an effective f -exceptional divisor F such that $-F$ is f -ample.*

Proposition 2.11. (cf. [24, Proposition 1.45]) *Let $f : X \rightarrow Y$ be a morphism of projective varieties with M an ample divisor on Y . If L is an f -ample Cartier divisor on Y , then $L + \nu f^*M$ is ample for $\nu \gg 1$.*

The following lemma slightly extends [32, Lemma 2.2].

Lemma 2.12. *Let X be a projective variety of dimension n and D an \mathbb{R} -Cartier divisor. Assume the following two conditions:*

- (1) $D \cdot G \cdot L_1 \cdots L_{n-2} \geq 0$ for any effective Cartier divisor G and any $L_i \in \text{Nef}(X)$.
- (2) $D \cdot H_1 \cdots H_{n-1} = 0$ for some nef and big \mathbb{R} -Cartier divisors H_1, \dots, H_{n-1} .

Then $D \equiv 0$.

Proof. By the proof of [32, Lemma 2.2], $D \cdot A^{n-1} = 0$ and $D^2 \cdot A^{n-2} = 0$ for some ample divisor A . So $D \equiv 0$ by Lemma 2.3. \square

Lemma 2.13. (cf. [35, Lemma 2.11]) *Let $f : X \rightarrow X$ be a surjective endomorphism of a normal projective variety X with K_X being \mathbb{Q} -Cartier. Let $\overline{\text{NE}}(X)$ be the closure of the convex cone generated by classes of effective 1-cycles in $N_1(X)$. Let $R_C := \mathbb{R}_{\geq 0}[C] \subseteq \overline{\text{NE}}(X)$ be an extremal ray generated by a curve C (not necessarily K_X -negative). Then we have:*

- (1) $R_{f(C)}$ is an extremal ray.
- (2) If C_1 is another curve such that $f(C_1) = C$, then R_{C_1} is an extremal ray.
- (3) Denote by Σ_C the set of curves whose classes are in R_C . Then $f(\Sigma_C) = \Sigma_{f(C)}$.
- (4) If R_{C_1} is extremal, then $\Sigma_{C_1} = f^{-1}(\Sigma_{f(C_1)}) := \{D \mid f(D) \in \Sigma_{f(C_1)}\}$.

Let $f : X \rightarrow Y$ be a surjective morphism between normal projective varieties. Then f has connected fibres if and only if $f_*\mathcal{O}_X = \mathcal{O}_Y$ (cf. [18, Chapter III, Corollary 11.3 and Corollary 11.5]) and hence the composition of two such morphisms still has connected fibres. In particular, the general fibre of f is connected if and only if all the fibres are connected and hence a birational morphism between normal projective varieties always has connected fibres (cf. [18, Chapter III, Corollary 11.4]). Suppose that f has connected fibres. Let X' be a resolution of X with $f' : X' \rightarrow Y$ the induced morphism. Then f' has connected fibres and the general fibre of f' is smooth by generic smoothness (cf. [18, Chapter III, Corollary 10.7]). In particular, the general fibre of f' is irreducible. Note that each fibre of f is an image of f' . So the general fibre of f is irreducible.

Definition 2.14. Let $f : X \dashrightarrow Y$ be a dominant map between two normal projective varieties and $f_{\bar{\Gamma}} : \bar{\Gamma} \rightarrow Y$ the induced morphism with $\bar{\Gamma}$ the normalization of the graph of f . We say that f has *the general fibre rationally connected* if the general fibre of $f_{\bar{\Gamma}}$ is rationally connected.

Let $f_1 : X_1 \rightarrow Y_1$ be a surjective morphism birationally equivalent to f with X_1 and Y_1 being normal projective. Then f has the general fibre rationally connected if and only if so does f_1 . This is because rational connectedness is a birational property (cf. [21, Chapter IV, Proposition 3.6]).

Lemma 2.15. *Let $f : X \dashrightarrow Y$ and $g : Y \dashrightarrow Z$ be two dominant maps between normal projective varieties. Suppose that f and g have the general fibre rationally connected. Then $g \circ f$ has the general fibre rationally connected.*

Proof. We may assume f and g are surjective morphisms between smooth projective varieties. Let U be an open dense subset of Y such that each fibre of f over U is

rationaly connected. Let V be an open dense subset of Z such that $g \circ f$ is smooth over V and each fibre of g over V is rationaly connected and has a nonempty intersection with U . Then for any $z \in V$, we have a natural surjective morphism $f_z : X_z \rightarrow Y_z$, where $X_z = (g \circ f)^{-1}(z)$ while $Y_z = g^{-1}(z)$ is rationaly connected. Since $Y_z \cap U \neq \emptyset$, f_z has the general fibre rationaly connected. By [16, Corollary 1.3], X_z is rationaly connected. So $g \circ f$ has the general fibre rationaly connected. \square

Lemma 2.16. *Let X be a \mathbb{Q} -factorial klt normal projective variety. Suppose that $X = X_1 \dashrightarrow \cdots \dashrightarrow X_r = Y$ is a sequence of divisorial contractions, flips or Fano contractions, of K_{X_i} -negative extremal rays. Let $f : X \dashrightarrow Y$ be the composition. Then f has the general fibre rationaly connected.*

Proof. By Lemma 2.15, it suffices to consider the case when f is a Fano contraction. Then the general fibre of f is a klt Fano variety and hence rationaly connected by [38]. \square

3. PROPERTIES OF (QUASI-) POLARIZED ENDOMORPHISMS

Lemma 3.1. *Let $f : X \rightarrow X$ be a surjective endomorphism of a projective variety X , such that $f^*H \equiv qH$ for some nef and big \mathbb{R} -Cartier divisor H and $q > 0$. Then $\deg f = q^{\dim(X)}$.*

Proof. We may assume $n := \dim(X) > 0$. By the projection formula, $(f^*H)^n = (\deg f)H^n = q^n H^n$. Since $H^n > 0$, $\deg f = q^n$. \square

Lemma 3.2. *Let $f : A \rightarrow A$ be a surjective endomorphism of an abelian variety A . Let Z be a subvariety of A such that $f^{-1}(Z) = Z$. Then $\deg f|_Z = \deg f$.*

Proof. f is étale by the ramification divisor formula and the purity of branch loci. Then $\deg f|_Z = |f^{-1}(z)| = \deg f$ for any $z \in Z$. \square

Lemma 3.3. *Let $f : X \rightarrow X$ be a polarized endomorphism of a projective variety X with $\deg f = q^{\dim(X)}$ and let Z be a closed subvariety of X with $f(Z) = Z$. Then $\deg f|_Z = q^{\dim(Z)}$.*

Proof. We may assume $f^*H \sim qH$ for some ample divisor H of X . Then $H|_Z$ is also an ample divisor of Z and $(f|_Z)^*(H|_Z) \sim q(H|_Z)$. So $\deg f|_Z = q^{\dim(Z)}$ by Lemma 3.1. \square

The following lemma is true for not-necessarily normal projective varieties by using the same proof of [32, Lemma 2.3].

Lemma 3.4. *(cf. [32, Lemma 2.3]) Let $f : X \rightarrow X$ be a surjective endomorphism of a projective variety X such that $f^*H \equiv qH$ for some integer $q > 1$ and ample Cartier divisor H . Then there is an ample Cartier divisor $H' \equiv H$, such that $f^*H' \sim qH'$. In particular, f is polarized.*

Lemma 3.5. *Let $f : X \rightarrow X$ be a surjective endomorphism of a projective variety X . Suppose that there is an ample \mathbb{R} -Cartier divisor H such that $f^*H \equiv qH$ for some rational number $q > 0$. Then q is an integer and $f^*H' \equiv qH'$ for some ample Cartier (integral) divisor H' .*

Proof. By Lemma 3.1, $q^{\dim(X)} = \deg f$. So q is an algebraic integer and also rational by assumption. Hence, q is an integer.

Let $W \subseteq N^1(X)$ be the eigenspace of $f^*|_{N^1(X)}$ with eigenvalue q and $W_{\mathbb{Q}}$ the set of all \mathbb{Q} -Cartier divisor classes in W . Note that $f^*|_{N^1(X)}$ is determined by $f^*|_{NS_{\mathbb{Q}}(X)}$. So $W_{\mathbb{Q}}$ is dense in W . Set $F_{\infty} := \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \frac{(f^i)^*|_{N^1(X)}}{q^i}$. By Proposition 2.9, F_{∞} is a well-defined projection from $N^1(X)$ onto W . Note that $N^1(X)$ is spanned by $\text{Amp}(X)$, $F_{\infty}(W) = W$, and F_{∞} is an open map from $N^1(X)$ onto W by Proposition 2.9. Then $F_{\infty}(H + \text{Amp}(X)) \cap W_{\mathbb{Q}} \neq \emptyset$. In particular, $H' := F_{\infty}(H + D)$ is an ample \mathbb{Q} -Cartier divisor for some $D \in \text{Amp}(X)$ and $f^*[H'] = q[H']$ in $N^1(X)$. Replacing H' by a multiple, we are done. \square

Proposition 3.6. *Let $f : X \rightarrow X$ be a surjective endomorphism of a projective variety X and $q > 0$ a rational number. Suppose one of the following is true.*

- (1) $f^*H \equiv qH$ for some big \mathbb{R} -Cartier divisor H .
- (2) X is normal and $f^*H \equiv_w qH$ for some big Weil \mathbb{R} -divisor H .

*Then q is an integer and $f^*A \equiv qA$ for some ample Cartier divisor A . Further, if $q > 1$, then f is polarized. In particular, quasi-polarized endomorphisms are polarized.*

Proof. Set $n := \dim(X)$. Clearly, f^* induces automorphisms on $N_{n-1}(X)$ and $N^1(X)$.

For (1), applying Proposition 2.9 to $f^*|_{N^1(X)}$, $\text{PEC}(X)$ and its interior point H , Proposition 2.9(2) holds for $f^*|_{N^1(X)}$.

For (2), applying Proposition 2.9 to $f^*|_{N_{n-1}(X)}$, $\text{PE}(X)$ and its interior point H , Proposition 2.9(2) holds for $f^*|_{N_{n-1}(X)}$ and hence for $f^*|_{N^1(X)}$, since $N^1(X)$ is a subspace of $N_{n-1}(X)$ by Lemma 2.3.

Now for both (1) and (2), applying Proposition 2.9 to $f^*|_{N^1(X)}$ and $\text{Nef}(X)$, $f^*|_{N^1(X)}$ has an eigenvector in $\text{Nef}(X)^{\circ}$. So $f^*A \equiv qA$ for some ample \mathbb{R} -Cartier divisor A . By Lemma 3.5, q is an integer and we may assume A is Cartier.

If $q > 1$, then there exists an ample Cartier divisor $A' \equiv A$, such that $f^*A' \sim qA'$ by Lemma 3.4. \square

Let X be a projective variety. Denote by $\text{Aut}(X)$ the full automorphism group of X and $\text{Aut}_0(X)$ the neutral connected component of $\text{Aut}(X)$.

Theorem 3.7. *Let X be a normal projective variety. Let G be a subgroup of $\text{Aut}(X)$, such that for any $g \in G$, $g^*B_g \equiv_w B_g$ for some big Weil \mathbb{R} -divisor B_g . Then $[G : G \cap \text{Aut}_0(X)] < \infty$.*

Proof. Take an $\text{Aut}(X)$ -equivariant projective resolution $\pi : X' \rightarrow X$. We may regard G and $\text{Aut}_0(X)$ as subgroups of $\text{Aut}(X')$. By [4, Proposition 2.1], we can identify $\text{Aut}_0(X')$ with $\text{Aut}_0(X)$. For any $g \in G$, g fixes some ample class A_X in $\text{Amp}(X)$ by Proposition 3.6. Since π is birational, $A_{X'} = \pi^*A_X$ is big and g fixes $A_{X'}$. By [9, Theorem 2.1], $[G : G \cap \text{Aut}_0(X)] = [G : G \cap \text{Aut}_0(X')] < \infty$. \square

Lemma 3.8. *Let $\pi : X \dashrightarrow Y$ be a dominant rational map between two projective varieties and let $f : X \rightarrow X$ and $g : Y \rightarrow Y$ be two surjective endomorphisms such that $g \circ \pi = \pi \circ f$. Let W be the graph of π and denote by $p_1 : W \rightarrow X$ and $p_2 : W \rightarrow Y$ the two natural projections. Then there is a surjective endomorphism $h : W \rightarrow W$ such that $p_1 \circ h = f \circ p_1$ and $p_2 \circ h = g \circ p_2$. Furthermore, if f is polarized, then h is polarized.*

Proof. Note that f and g induce a natural surjective endomorphism $f \times g : X \times Y \rightarrow X \times Y$ via $(x, y) \mapsto (f(x), g(y))$. Set $h = (f \times g)|_W$. We check that $h(W) = W$. Let U be an open dense subset of X such that $\pi|_U$ and $\pi|_{f(U)}$ are both well-defined morphisms. Then p_1 is isomorphic over U and $f(U)$. For any $(x, y) \in p_1^{-1}(U)$, $y = \pi(x)$ and $h(x, y) = (f(x), g(y)) = (f(x), g(\pi(x))) = (f(x), \pi(f(x))) \in p_1^{-1}(f(U)) \subseteq W$. Then $h(p_1^{-1}(U)) \subseteq W$. Note that $p_1^{-1}(U)$ is open dense in W , W is a projective subvariety of $X \times Y$, and $f \times g$ is finite surjective. So $h(W) = W$. Since $p_1 \circ h(x, y) = f(x) = f \circ p_1(x, y)$ for any $(x, y) \in p_1^{-1}(U)$, we have $p_1 \circ h = f \circ p_1$. Similarly, $p_2 \circ h = g \circ p_2$.

Suppose $f^*H \sim qH$ for some ample divisor H of X and integer $q > 1$. Let $H' = p_1^*H$. Then $h^*H' \sim qH'$. Since p_1 is birational, H' is nef and big. So h is quasi-polarized and hence polarized by Proposition 3.6. \square

Lemma 3.9. *Let $f : X \rightarrow X$ be a surjective endomorphism of a projective variety X . Let $\pi : X_1 \rightarrow X$ be the normalization of X . Then there is a surjective endomorphism $f_1 : X_1 \rightarrow X_1$ such that $\pi \circ f_1 = f \circ \pi$. Furthermore, if f is polarized, then f_1 is polarized.*

Proof. By the universal property of normalization, such f_1 exists. Suppose $f^*H \sim qH$ for some ample divisor H of X and integer $q > 1$. Let $H_1 = \pi^*H$. Then H_1 is ample and $f_1^*H_1 \sim qH_1$. \square

Lemma 3.10. *Let $\pi : X \dashrightarrow Y$ be a dominant rational map between two projective varieties and let $f : X \rightarrow X$ and $g : Y \rightarrow Y$ be two polarized endomorphisms such that $g \circ \pi = \pi \circ f$. Then the eigenvalues of $f^*|_{\mathbb{N}^1(X)}$ are of modulus q if and only if so are the eigenvalues of $g^*|_{\mathbb{N}^1(Y)}$ (if Y is a point, we assume this is always true). In particular, $(\deg f)^{\dim(Y)} = (\deg g)^{\dim(X)}$.*

Proof. Replacing X by the graph of π and by Lemma 3.8, we may assume π is a surjective morphism. Set $m = \dim(X)$ and $n = \dim(Y)$. Suppose that $f^*H_X = pH_X$ and $g^*H_Y \sim qH_Y$ for some ample divisors $H_X \in N^1(X)$, $H_Y \in N^1(Y)$ and $p, q > 1$. Since π is surjective, $\pi^* : N^1(Y) \rightarrow N^1(X)$ is an injection. If Y is not a point, then $N^1(Y)$ is of positive dimension. Applying Proposition 2.9 to the cones $\text{Nef}(X)$ and $\text{Nef}(Y)$, we have $p = q$. The last assertion then follows from Lemma 3.1. \square

Theorem 3.11. *Let $\pi : X \dashrightarrow Y$ be a dominant rational map between two projective varieties and let $f : X \rightarrow X$ and $g : Y \rightarrow Y$ be two surjective endomorphisms such that $g \circ \pi = \pi \circ f$. Suppose f is polarized. Then g is polarized; and $(\deg f)^{\dim(Y)} = (\deg g)^{\dim(X)}$.*

Proof. Replacing X by the graph of π and by Lemma 3.8, we may assume π is a surjective morphism. We may also assume $\dim(Y) > 0$. Since f is polarized, $f^*|_{N^1(X)}$ satisfies Proposition 2.9(2). Since π is surjective, $N^1(Y)$ can be viewed as a subspace of $N^1(X)$ and hence $g^*|_{N^1(Y)}$ also satisfies Proposition 2.9(2); so the eigenvalues of $g^*|_{N^1(Y)}$ are of modulus greater than 1. Therefore, Proposition 2.9(1) and Lemma 3.4 imply that g is polarized. The last formula follows from Lemma 3.10. \square

Corollary 3.12. *Let $\pi : X \dashrightarrow Y$ be a generically finite dominant rational map between two projective varieties and let $f : X \rightarrow X$ and $g : Y \rightarrow Y$ be two surjective endomorphisms such that $g \circ \pi = \pi \circ f$. Then f is polarized if and only if so is g .*

Proof. Replacing X by the graph of π and by Lemma 3.8, we may assume π is a generically finite surjective morphism. If f is polarized, then g is polarized by Theorem 3.11. Suppose $g^*H_Y \sim qH_Y$ for some ample divisor H_Y and $q > 1$. Let $H_X = \pi^*H_Y$. Since π is generically finite surjective, H_X is nef and big. Note that $f^*H_X \sim qH_X$. So f is quasi-polarized and hence polarized by Proposition 3.6. \square

4. SPECIAL MRC FIBRATION AND THE NON-UNIRULED CASE

We refer to [31, Section 4] for the following result.

Lemma 4.1. *(cf. [31, Theorem 4.19]) Let $f : X \rightarrow X$ be a surjective endomorphism of a normal projective variety. Let $\pi : X \dashrightarrow Y$ be the special MRC fibration in the sense of [31, before Theorem 4.18] with Y non-uniruled (cf. [16]). Then there is a surjective endomorphism $h : Y \rightarrow Y$ such that $\pi \circ f = h \circ \pi$.*

Lemma 4.2. *Let $\pi : X \dashrightarrow Y$ be a dominant rational map between two normal projective varieties. Then we have the following commutative diagram:*

$$\begin{array}{ccc} X & \overset{\pi}{\dashrightarrow} & Y \\ \text{alb}_X \downarrow & & \downarrow \text{alb}_Y \\ \text{Alb}(X) & \xrightarrow{p} & \text{Alb}(Y) \end{array}$$

where alb_X and alb_Y are the Albanese maps of X and Y respectively. Suppose π has the general fibre rationally connected (cf. Definition 2.14). Then p is an isomorphism.

Proof. The existence of p follows from the universal property of the Albanese map. For convenience, we may replace X and Y by suitable smooth models such that π , alb_X and alb_Y are well-defined morphisms. Note that π has the general fibre rationally connected and every map from a rationally connected variety to an abelian variety is trivial. So there is a Zariski open dense subset U of Y such that $\text{alb}_X(\pi^{-1}(y))$ is a point for any $y \in U$. By [19, Lemma 14], there is a rational map $s : Y \dashrightarrow \text{Alb}(X)$, such that $s \circ \pi = \text{alb}_X$. Since π is surjective, $p \circ s = \text{alb}_Y$. On the other hand, by the universal property of the Albanese map, $s = t \circ \text{alb}_Y$ for some morphism $t : \text{Alb}(Y) \rightarrow \text{Alb}(X)$. Note that $(p \circ t) \circ \text{alb}_Y = \text{alb}_Y$ and $(t \circ p) \circ \text{alb}_X = \text{alb}_X$. Then $p \circ t$ and $t \circ p$ are both the identity maps, by the universal property of Albanese maps. Hence, p is an isomorphism. \square

Lemma 4.3. *Let $f : X \rightarrow X$ be a polarized endomorphism of a non-uniruled normal projective variety X . Then X is \mathbb{Q} -abelian with canonical singularities and $K_X \sim_{\mathbb{Q}} 0$. Further, there is an abelian variety A , such that the following diagram is commutative:*

$$\begin{array}{ccc} A & \xrightarrow{\tau} & X \\ f_A \downarrow & & \downarrow f \\ A & \xrightarrow{\tau} & X \end{array}$$

where $f_A : A \rightarrow A$ is a polarized endomorphism and τ is a finite surjective morphism which is étale in codimension one.

Proof. By [17, Theorem 1.21] and [32, Theorem 3.2], X is \mathbb{Q} -abelian with only canonical singularities.

By [32, Proposition 3.5], there exist an abelian variety A and a weak Calabi-Yau variety S , such that the following diagram is commutative:

$$\begin{array}{ccc} A \times S & \xrightarrow{\tau} & X \\ f_A \times f_S \downarrow & & \downarrow f \\ A \times S & \xrightarrow{\tau} & X \end{array}$$

where $f_A : A \rightarrow A$, $f_S : S \rightarrow S$ are polarized endomorphisms, and τ is a finite surjective morphism which is étale in codimension one. Since S is non-uniruled, S is \mathbb{Q} -abelian by [17, Theorem 1.21] and hence $\dim(S) = q^\circ(S) = 0$ in the notation of [32] as in their definition of weak Calabi-Yau. \square

Remark 4.4. Assume X is a \mathbb{Q} -abelian variety or just assume there is a finite surjective morphism $A \rightarrow X$ with A an abelian variety. Since A is a homogeneous variety, any effective divisor on A is nef. The same holds on X by the projection formula. Hence if $f : X \rightarrow X$ is quasi-polarized, it is also polarized without using Proposition 3.6.

We refer to the proof of [32, Proposition 3.5] for the following lemma.

Lemma 4.5. *Let X be a normal projective variety with klt singularities and $K_X \sim_{\mathbb{Q}} 0$ and let $\sigma : \hat{X} \rightarrow X$ be the global index-one cover. Then for any surjective endomorphism $f : X \rightarrow X$, there is a surjective endomorphism $\hat{f} : \hat{X} \rightarrow \hat{X}$ such that $\sigma \circ \hat{f} = f \circ \sigma$.*

Lemma 4.6. *Let X be a normal projective variety with klt singularities and $K_X \sim_{\mathbb{Q}} 0$ (this is the case when X is \mathbb{Q} -abelian) and let $f : X \rightarrow X$ be a polarized endomorphism. Then there exist a finite surjective morphism $\tau : A \rightarrow X$ étale in codimension one with A an abelian variety and a polarized endomorphism $f_A : A \rightarrow A$, such that $\tau \circ f_A = f \circ \tau$. In particular, X is \mathbb{Q} -abelian.*

Proof. Let $\sigma : \hat{X} \rightarrow X$ be the global index-one cover, i.e. the minimal quasi-étale cyclic covering satisfying $K_{\hat{X}} \sim 0$. Then there is a polarized endomorphism $\hat{f} : \hat{X} \rightarrow \hat{X}$ satisfying $\sigma \circ \hat{f} = f \circ \sigma$ by Lemmas 4.5 and Corollary 3.12. Since $K_{\hat{X}}$ is Cartier and \hat{X} is klt, \hat{X} has canonical singularities. In particular, $\kappa(\hat{X}) = 0$ and hence \hat{X} is non-uniruled. Now by Lemma 4.3, there exist a finite surjective morphism $\tau' : A \rightarrow \hat{X}$ which is étale in codimension one with A an abelian variety and a polarized endomorphism $f_A : A \rightarrow A$, such that $\tau' \circ f_A = \hat{f} \circ \tau'$. Thus the lemma is proved since $\tau = \sigma \circ \tau'$ is still étale in codimension one by the ramification divisor formula. \square

Lemma 4.7. *Let X be a \mathbb{Q} -abelian variety and $f : X \rightarrow X$ a surjective endomorphism. Assume the existence of a non-empty closed subset $Z \subsetneq X$ and $s > 0$, such that $f^{-s}(Z) = Z$. Then f is not polarized.*

Proof. Replacing f by f^s , we may assume $f^{-1}(Z) = Z$. Suppose that f is polarized. By Lemma 4.6, there exist a finite surjective morphism $\tau : A \rightarrow X$ with A an abelian variety and a polarized endomorphism $f_A : A \rightarrow A$, such that $\tau \circ f_A = f \circ \tau$. Clearly, $f_A^{-1}(\tau^{-1}(Z)) = \tau^{-1}(Z)$. So we may assume that X is an abelian variety; and replacing f by a positive power, we may also assume that Z is irreducible. By Lemma 3.2, $\deg f|_Z =$

$\deg f$; and by Lemma 3.3, $\deg f|_Z = (\deg f)^{\dim(Z)/\dim(X)}$. Since $\dim(Z) < \dim(X)$ and $\deg f > 1$ by Lemma 3.1, we get a contradiction. \square

5. PROOF OF COROLLARY 1.4 AND PROPOSITION 1.6

We begin with the following lemmas.

Lemma 5.1. (cf. [33, Proposition 2.3] or [20, Lemma 8.1]) *Let X be a normal projective variety having only rational singularities (i.e. there exists a resolution $f : Y \rightarrow X$ such that $R^i f_* \mathcal{O}_Y = 0$ for $i > 0$). Then $f^* : \text{Pic}^0(X) \rightarrow \text{Pic}^0(Y)$ is an isomorphism, and alb_X is a morphism. In particular, if $h^1(X, \mathcal{O}_X) \neq 0$, then alb_X is nontrivial.*

Lemma 5.2. *Let $\pi : X \rightarrow Y$ be a surjective morphism between normal projective varieties with connected fibres. Let $f : X \rightarrow X$ and $g : Y \rightarrow Y$ be two polarized endomorphisms such that $g \circ \pi = \pi \circ f$. Suppose that Y is Q -abelian. Then the following are true.*

- (1) *All the fibres of π are irreducible.*
- (2) *π is equi-dimensional.*
- (3) *If the general fibre of π is rationally connected, then all the fibres of π are rationally connected.*

Proof. First we claim that $f(\pi^{-1}(y)) = \pi^{-1}(g(y))$ for any $y \in Y$. Suppose there is a closed point y of Y such that $f|_{\pi^{-1}(y)} : \pi^{-1}(y) \rightarrow \pi^{-1}(g(y))$ is not surjective. Let $S = g^{-1}(g(y)) - \{y\}$. Then $S \neq \emptyset$ and $U := X - \pi^{-1}(S)$ is an open dense subset of X . Since f is an open map, $f(U)$ is an open dense subset of X . In particular, $f(U) \cap \pi^{-1}(g(y))$ is open in $\pi^{-1}(g(y))$. Note that $f(U) = (X - \pi^{-1}(g(y))) \cup f(\pi^{-1}(y))$. So $f(U) \cap \pi^{-1}(g(y)) = f(\pi^{-1}(y))$ is open in $\pi^{-1}(g(y))$. Since f is also a closed map, the set $f(\pi^{-1}(y))$ is both open and closed in the connected fibre $\pi^{-1}(g(y))$ and hence $f(\pi^{-1}(y)) = \pi^{-1}(g(y))$. So the claim is proved.

Let

$$\Sigma_1 := \{y \in Y \mid \pi^{-1}(y) \text{ is not irreducible}\}.$$

Note that $f(\pi^{-1}(y)) = \pi^{-1}(g(y))$. Then $g^{-1}(\Sigma_1) \subseteq \Sigma_1$ and hence $g^{-1}(\overline{\Sigma_1}) \subseteq \overline{\Sigma_1}$. Since $\overline{\Sigma_1}$ is closed and has finitely many irreducible components, $g^{-1}(\overline{\Sigma_1}) = \overline{\Sigma_1}$. By Lemma 4.7, $\Sigma_1 = \emptyset$. So (1) is proved.

Let

$$\Sigma_2 := \{y \in Y \mid \dim(\pi^{-1}(y)) > \dim(X) - \dim(Y)\},$$

and

$$\Sigma_3 := \{y \in Y \mid \pi^{-1}(y) \text{ is not rationally connected}\}.$$

By (1), π is equi-dimensional outside Σ_2 . Since f is finite surjective, $g^{-1}(\Sigma_2) \subseteq \Sigma_2$. By (1), all the fibres of π outside Σ_3 are rationally connected. Note that the image of

a rationally connected variety is rationally connected. So $g^{-1}(\Sigma_3) \subseteq \Sigma_3$. Now the same reason above implies that $\Sigma_2 = \emptyset$. Similarly, $\Sigma_3 = \emptyset$ if the general fibre of π is rationally connected. \square

Lemma 5.3. *Let $\pi : X \dashrightarrow Y$ be a dominant rational map between normal projective varieties. Suppose that (X, Δ) is a klt pair for some effective \mathbb{Q} -divisor Δ and Y is \mathbb{Q} -abelian. Suppose further that the normalization of the graph $\Gamma_{X/Y}$ is equi-dimensional over Y (this holds when $k(Y)$ is algebraically closed in $k(X)$, $f : X \rightarrow X$ is polarized and f descends to some polarized $f_Y : Y \rightarrow Y$; see Lemma 5.2). Then π is a morphism.*

Proof. Let W be the normalization of the graph $\Gamma_{X/Y}$ and $p_1 : W \rightarrow X$ and $p_2 : W \rightarrow Y$ the two projections. Let $\tau_1 : A \rightarrow Y$ be a finite surjective morphism étale in codimension one with A an abelian variety. Let W' be an irreducible component of the normalization of $W \times_Y A$ which dominates W and $\tau_2 : W' \rightarrow W$ and $p'_2 : W' \rightarrow A$ the two projections. Taking the Stein factorization of the composition $W' \rightarrow W \rightarrow X$, we get a birational morphism $p'_1 : W' \rightarrow X'$ and a finite morphism $\tau_3 : X' \rightarrow X$.

$$\begin{array}{ccccc} X' & \xleftarrow{p'_1} & W' & \xrightarrow{p'_2} & A \\ \downarrow \tau_3 & & \downarrow \tau_2 & & \downarrow \tau_1 \\ X & \xleftarrow{p_1} & W & \xrightarrow{p_2} & Y \end{array}$$

Since p_2 is equi-dimensional, by the base change, τ_2 is étale in codimension one. Let $U \subseteq X$ be the domain of $p_1^{-1} : X \dashrightarrow W$. Then, $\text{codim}(X - U) \geq 2$, and the restriction $\tau_3^{-1}(U) \rightarrow U$ of τ_3 is étale in codimension one, since so is τ_2 . Therefore, τ_3 is étale in codimension one. In particular, by the ramification divisor formula, $K_{X'} + \Delta' = \tau_3^*(K_X + \Delta)$ with $\Delta' = \tau_3^*\Delta$ an effective \mathbb{Q} -divisor. Since (X, Δ) is klt, (X', Δ') is klt by [24, Proposition 5.20] and hence X' has rational singularities by [24, Theorem 5.22]. Clearly, $\pi' := p'_2 \circ p'_1^{-1} : X' \dashrightarrow A$ is a dominant rational map, since p'_1 is birational and p'_2 is surjective. Then π' is a surjective morphism (with $p'_2 = \pi' \circ p'_1$) by Lemma 5.1 and the universal property of the Albanese map. Suppose π is not defined over some closed point $x \in X$. Then $\dim(W_x) > 0$ with $W_x = p_1^{-1}(x)$ and $\dim(p_2(W_x)) > 0$ by [8, Lemma 1.15]. Hence, $\dim(p'_2(\tau_2^{-1}(W_x))) > 0$ and then $\dim(p'_1(\tau_2^{-1}(W_x))) > 0$. However, $p'_1(\tau_2^{-1}(W_x)) = \tau_3^{-1}(x)$ has only finitely many points. This is a contradiction. \square

Lemma 5.4. *Let X be a projective variety with a polarized endomorphism $f : X \rightarrow X$. Then the Albanese map $\text{alb}_X : X \dashrightarrow \text{Alb}(X)$ is a dominant rational map.*

Proof. Replacing X by the normalization of the graph of alb_X and by Lemmas 3.8 and 3.9, we may assume X is normal and alb_X is a well-defined morphism. Replacing X by the base of the special MRC fibration of X and by Lemmas 4.1, 4.2 and Theorem 3.11,

it suffices to consider the case when X is a non-uniruled normal projective variety. By Lemma 4.3, X has only canonical singularities with $K_X \sim_{\mathbb{Q}} 0$. In particular, $\kappa(X) = 0$. By [19, Theorem 1] and Lemma 5.1, $\text{alb}_X : X \dashrightarrow \text{Alb}(X)$ is a surjective morphism. \square

Proof of Corollary 1.4. (1) follows from Lemma 5.4. Then (2) follows from (1) and Theorem 3.11. \square

Proof of Proposition 1.6. (1) follows from Lemmas 4.1 and Theorem 3.11; see [31, Corollary 4.20] for a different proof of g being polarized. (2) follows from Lemma 4.3. (3) follows from Lemma 5.2. (4) follows from Lemma 5.3. \square

6. MINIMAL MODEL PROGRAM FOR POLARIZED ENDOMORPHISMS

We follow the approach in [35, Lemma 2.10] and get the following general result:

Lemma 6.1. *Let $f : X \rightarrow X$ be a polarized endomorphism of a projective variety. Suppose $A \subseteq X$ is a closed subvariety with $f^{-i}f^i(A) = A$ for all $i \geq 0$. Then $M(A) := \{f^i(A) \mid i \in \mathbb{Z}\}$ is a finite set.*

Proof. We may assume $n := \dim(X) \geq 1$. By the assumption, $f^*H \sim qH$ for some ample Cartier divisor H and integer $q > 1$. Set $M_{\geq 0}(A) := \{f^i(A) \mid i \geq 0\}$.

We first assert that if $M_{\geq 0}(A)$ is a finite set, then so is $M(A)$. Indeed, suppose $f^{r_1}(A) = f^{r_2}(A)$ for some $0 < r_1 < r_2$. Then for any $i > 0$, $f^{-i}(A) = f^{-i}f^{-sr_1}f^{sr_1}(A) = f^{-i}f^{-sr_1}f^{sr_2}(A) = f^{sr_2-sr_1-i}(A) \in M_{\geq 0}(A)$ if $s \gg 1$. So the assertion is proved.

Next we show that $M_{\geq 0}(A)$ is a finite set by induction on the codimension of A in X . We may assume $k := \dim(A) < \dim(X)$. Let Σ be the union of $\text{Sing}(X)$, $f^{-1}(\text{Sing}(X))$ and the irreducible components in the ramification divisor R_f of f . Set $A_i := f^i(A)$ ($i \geq 0$).

We claim that A_i is contained in Σ for infinitely many i . Otherwise, replacing A by some A_{i_0} , we may assume that A_i is not contained in Σ for all $i \geq 0$. So we have $f^*A_{i+1} = a_i A_i$ with $a_i \in \mathbb{Z}_{>0}$ and

$$\begin{aligned} q^n H^k \cdot A_{i+1} &= (f^*H)^k \cdot f^*A_{i+1} = a_i q^k H^k \cdot A_i, \\ 1 \leq H^k \cdot A_{i+1} &= \frac{a_i}{q^{n-k}} \cdots \frac{a_1}{q^{n-k}} H^k \cdot A_1. \end{aligned}$$

Thus for infinitely many i , $a_i \geq q^{n-k} > 1$. Hence $A_i \subseteq \Sigma$. This proves the claim.

If $k = n - 1$, by the claim, $f^{r_1}(A) = f^{r_2}(A)$ for some $0 < r_1 < r_2$. Then $|M_{\geq 0}(A)| < r_2$.

If $k \leq n - 2$, assume that $|M_{\geq 0}(A)| = \infty$. Let B be the Zariski-closure of the union of those A_{i_1} contained in Σ . Then $k + 1 \leq \dim(B) \leq n - 1$, and $f^{-i}f^i(B) = B$ for all $i \geq 0$. Choose $r \geq 1$ such that $B' := f^r(B), f(B'), f^2(B'), \dots$ all have the same number of irreducible components. Let X_1 be an irreducible component of B' of maximal dimension. Then $k + 1 \leq \dim(X_1) \leq n - 1$ and $f^{-i}f^i(X_1) = X_1$ for all $i \geq 0$. By induction, $M_{\geq 0}(X_1)$

is a finite set. So we may assume that $f^{-1}(X_1) = X_1$, after replacing f by a positive power and X_1 by its image. Note that $f|_{X_1}$ is polarized. Now the codimension of A_{i_1} in X_1 is smaller than that of A in X . By induction, $M_{\geq 0}(A_{i_1})$ and hence $M_{\geq 0}(A)$ are finite. \square

Let X be a log canonical (lc) normal projective variety. We refer to [13, Theorem 1.1] for the cone theorem and [1, Corollary 1.2] for the existence of log canonical flips.

Lemma 6.2. *Let X be a lc normal projective variety and $f : X \rightarrow X$ a surjective endomorphism. Let $\pi : X \rightarrow Y$ be a contraction of a K_X -negative extremal ray $R_C := \mathbb{R}_{\geq 0}[C]$ generated by some curve C . Suppose that $E \subseteq X$ is a subvariety such that $\dim(\pi(E)) < \dim(E)$ and $f^{-1}(E) = E$. Then replacing f by a positive power, $f(R_C) = R_C$; hence, π is f -equivariant.*

Proof. Since $\dim(\pi(E)) < \dim(E)$, we may assume $C \subseteq E$. Denote by $N_{\mathbb{C}}^1(X) := N^1(X) \otimes_{\mathbb{R}} \mathbb{C}$. By the cone theorem (cf. [13, Theorem 1.1(4)iii], or [24, Corollary 3.17]), we have the linear exact sequence

$$0 \rightarrow N_{\mathbb{C}}^1(Y) \xrightarrow{\pi^*} N_{\mathbb{C}}^1(X) \xrightarrow{\cdot C} \mathbb{C} \rightarrow 0.$$

So $\pi^* N_{\mathbb{C}}^1(Y)$ is a subspace in $N_{\mathbb{C}}^1(X)$ of codimension 1. Let $i : E \hookrightarrow X$ be the inclusion map. For any \mathbb{R} -Cartier divisor D of X , denote by $D|_E := i^*D \in N_{\mathbb{C}}^1(E)$ the pullback. Let $N_{\mathbb{C}}^1(X)|_E := i^*(N_{\mathbb{C}}^1(X))$ which is a subspace of $N_{\mathbb{C}}^1(E)$. Let $L := \pi^* N_{\mathbb{C}}^1(Y)|_E := i^* \pi^*(N_{\mathbb{C}}^1(Y))$. Then L is a subspace in $N_{\mathbb{C}}^1(X)|_E$ of codimension at most 1. Note that for an ample divisor H in X , $H|_E \cdot C = H \cdot C \neq 0$. Therefore, $H|_E \notin L$ and hence L has codimension 1 in $N_{\mathbb{C}}^1(X)|_E$. Let $S := \{D|_E \in N_{\mathbb{C}}^1(X)|_E : (D|_E)^{\dim(E)} = 0\}$.

We claim that S is a hypersurface (an algebraic set defined by a non-zero polynomial) in the complex affine space $N_{\mathbb{C}}^1(X)|_E$ and L is an irreducible component of S in the sense of Zariski topology. Indeed, let $\{e_1, \dots, e_k\}$ be a fixed basis of $N_{\mathbb{C}}^1(X)|_E$. Then $S = \{(x_1, \dots, x_k) \mid (\sum_{i=1}^k x_i e_i)^{\dim(E)} = 0\}$ is determined by a homogeneous polynomial of degree $\dim(E)$ and the coefficient of the term $\prod_i x_i^{\ell_i}$ is the intersection number $e_1^{\ell_1} \cdots e_k^{\ell_k}$. Note that for an ample divisor H in X , $H|_E \in N_{\mathbb{C}}^1(X)|_E$ and $(H|_E)^{\dim(E)} = H^{\dim(E)} \cdot E > 0$. So $e_1^{\ell_1} \cdots e_k^{\ell_k} \neq 0$ for some ℓ_i . In particular, S is determined by a non-zero polynomial. Since $\dim(\pi(E)) < \dim(E)$, $\pi_* E = 0$. For any $P \in N_{\mathbb{C}}^1(Y)$, $(\pi^*(P)|_E)^{\dim(E)} = \pi^*(P)^{\dim(E)} \cdot E = P^{\dim(E)} \cdot (\pi_* E) = 0$ by the projection formula. So $\pi^*(P)|_E \in S$. Hence $L \subseteq S$. Since L and S have the same dimension, L is an irreducible component of S . The claim is proved.

The pullback f^* induces an automorphism of $N_{\mathbb{C}}^1(X)|_E$. Note that $f^*E = aE$ (as cycles) for some $a > 0$, and $(f^*D)^{\dim(E)} \cdot E = \frac{\deg f}{a} D^{\dim(E)} \cdot E$. Hence, $D \in S$ if and only if $f^*D \in S$. This implies that S is f^* -invariant. Replacing f by a positive power, L is

f^* -invariant. In particular, for any $P \in N_{\mathbb{C}}^1(Y)$, $\pi^*P|_E = (f|_E)^*(\pi^*P|_E) = (f^*\pi^*P)|_E$ for some $P' \in N_{\mathbb{C}}^1(Y)$. By Lemma 2.13 and since $f^{-1}(E) = E$, we have $f^{-1}(R_C) = R_{C'}$ for some curve $C' \subseteq E$ with $f(C') = C$. Write $f_*(C') = eC$ where $e > 0$. Now we have

$$\pi^*P \cdot C' = f^*\pi^*P' \cdot C' = f_*(f^*\pi^*P' \cdot C') = \pi^*P' \cdot eC = 0.$$

Thus, $R_{C'} = R_C$ and hence $f(R_C) = R_C$. The last assertion is true since the contraction π is uniquely determined by the ray R_C . \square

Remark 6.3. In Lemma 6.2, if $E = X$ i.e., if π is a Fano contraction, then π is f^s -equivariant for some $s > 0$. This is also a corollary of [34, Theorem 2.2] by showing that X has only finitely many Fano contractions.

Lemma 6.4. *Let $f : X \rightarrow X$ be a polarized endomorphism of a \mathbb{Q} -factorial lc projective variety X . Suppose that $\pi : X \rightarrow X_1$ is a divisorial contraction of a K_X -negative extremal ray $R_C := \mathbb{R}_{\geq 0}[C]$ generated by some curve C . Then f^s descends to a surjective endomorphism of X_1 for some $s > 0$.*

Proof. Let E be the exceptional divisor. Then E is irreducible (cf. [24, Proposition 2.5]). By Lemma 2.13, $f^{-i}f^i(E) = E$ for all $i \geq 0$. By Lemma 6.1, $M(E)$ is a finite set. So we may assume $f^{-1}(E) = E$ after replacing f by its positive power. Then π is f^s -equivariant for some $s > 0$ by Lemma 6.2. \square

Lemma 6.5. *Let $f : X \rightarrow X$ be a polarized endomorphism of a lc projective variety X . Let $\sigma : X \dashrightarrow X^+$ be a flip with $\pi : X \rightarrow Y$ the corresponding flipping contraction of a K_X -negative extremal ray $R_C := \mathbb{R}_{\geq 0}[C]$ generated by some curve C . Then the commutative diagram*

$$\begin{array}{ccc} X & \dashrightarrow & X^+ \\ & \searrow \pi & \swarrow \pi^+ \\ & Y & \end{array}$$

is f^s -equivariant for some $s > 0$.

Proof. Let E be the exceptional locus of π . By Lemma 2.13, $f^{-i}f^i(E) = E$ for all $i \geq 0$. Choose $i_0 \geq 0$ such that $E' := f^{i_0}(E), f(E'), f^2(E'), \dots$ all have the same number of irreducible components. Then $f^{-i}f^i(E'(k)) = E'(k)$ for every irreducible component $E'(k)$ of E' . By Lemma 6.1, $M(E'(k))$ is a finite set. Then $f^r(E'(k)) = f^s(E'(k))$ for some $r > s \geq 0$ and hence $f^{-i_0}(E'(k)) = f^{-i_0}(f^{-r}f^r(E'(k))) = f^{-i_0}(f^{-r}f^s(E'(k))) = f^{-(r-s)}(f^{-i_0}(E'(k)))$. So $f^{-(r-s)}$ permutes the irreducible components of $f^{-i_0}(E'(k))$. Let $E(k)$ be an irreducible component of E such that $f^{i_0}(E(k)) = E'(k)$. Then $f^{-t}(E(k)) = E(k)$ for some $t > 0$. Since $\dim(\pi(E(k))) < \dim(E(k))$, we have $f^s(R_C) = R_C$ for some

$s > 0$ by applying Lemma 6.2 to f^t and $E(k)$. Hence, the rational maps on Y and X^+ induced from f^s are well-defined morphisms by the following Lemma 6.6. \square

The following lemma is true by using the same proof of [35, Lemma 3.6] since log canonical flips are now known to exist (cf. [1, Corollary 1.2]).

Lemma 6.6. (cf. [35, Lemma 3.6]) *Let X be a normal projective variety with at worst lc singularities, $f : X \rightarrow X$ a surjective endomorphism, and $X \dashrightarrow X^+$ a flip with $\pi : X \rightarrow Y$ the corresponding flipping contraction of a K_X -negative extremal ray $R_C := \mathbb{R}_{\geq 0}[C]$ generated by some curve C . Suppose that $R_{f(C)} = R_C$. Then the dominant rational map $f^+ : X^+ \dashrightarrow X^+$ induced from f , is holomorphic. Both f and f^+ descend to one and the same endomorphism of Y .*

Definition 6.7. (cf. [29]) Let X be a normal projective variety and D an \mathbb{R} -Cartier divisor. We say D is *movable* if: for any $\epsilon > 0$, any ample divisor H and any prime divisor Γ , there is an effective \mathbb{R} -Cartier divisor Δ such that $\Delta \equiv D + \epsilon H$ and $\Gamma \not\subseteq \text{Supp } \Delta$.

Lemma 6.8. *Let $f : X \rightarrow X$ be a polarized endomorphism of a \mathbb{Q} -factorial klt projective variety X . Suppose that K_X is movable. Then X is \mathbb{Q} -abelian.*

Proof. First we claim that $K_X \sim_{\mathbb{Q}} 0$. Since K_X is movable, K_X is pseudo-effective and satisfies the first condition of Lemma 2.12. Suppose that $f^*H \sim_{\mathbb{Q}} qH$ for some ample divisor H of X and $q > 1$. Taking intersection numbers with $(f^*H)^{n-1} = f^*H \cdots f^*H$ of the both sides of the ramification divisor formula $K_X = f^*K_X + R_f$, we obtain

$$(q-1)K_X \cdot H^{n-1} + R_f \cdot H^{n-1} = 0.$$

Since K_X and R_f are pseudo-effective, $K_X \cdot H^{n-1} = 0$. So by Lemma 2.12, $K_X \equiv 0$ and hence $K_X \sim_{\mathbb{Q}} 0$ by [29, Chapter V, Corollary 4.9].

Now the lemma follows from Lemma 4.6. \square

Lemma 6.9. *Let $f : X \rightarrow X$ be a polarized endomorphism of a \mathbb{Q} -factorial klt projective variety X . Assume that K_X is pseudo-effective. Then X is \mathbb{Q} -abelian.*

Proof. We run MMP on X . Since K_X is pseudo-effective, we will never arrive at a non-birational contraction. By running finitely many steps, we get a birational map $\pi : X \dashrightarrow Y$ such that any MMP starting from Y will always be a sequence of flips. Replacing f by a positive power, f descends step by step by Lemmas 6.4, 6.5, and Theorem 3.11. Let $g = f|_Y$ and $g^*H \sim_{\mathbb{Q}} qH$ for some ample Cartier divisor H of Y and $q > 1$.

We claim that K_Y is movable. Take an ample divisor A of Y and a sequence of positive numbers t_j approaching 0. By [2], we can run $(K_Y + t_j A)$ -MMP on Y with scaling of A

to obtain a birational map

$$\pi_j : (Y, t_j A) \dashrightarrow (Y_j, t_j A_j),$$

such that $K_{Y_j} + t_j A_j$ is nef and hence movable. Since Y and Y_j are isomorphic in codimension 1 and they are both \mathbb{Q} -factorial, we have a natural isomorphism $\pi_j^* : N^1(Y_j) \rightarrow N^1(Y)$ and $D \in N^1(Y_j)$ is movable if and only if $\pi_j^* D$ is movable. So $K_Y + t_j A = \pi_j^*(K_{Y_j} + t_j A_j)$ is movable for each j . In particular, K_Y is movable.

Note that Y is Q -abelian by Lemma 6.8 and g is polarized. Then $X \rightarrow Y$ is a birational equi-dimensional morphism by Lemma 5.3. This is possible only when $X \rightarrow Y$ is an isomorphism by the Zariski's main theorem (cf. [18, Chapter V, Theorem 5.2]). \square

7. EXAMPLES OF POLARIZED ENDOMORPHISMS

In the example below, $f : Z \rightarrow Z$ is polarized, but, $(f^i)^*|_{N^1(Z)}$ is not a scalar multiplication for any $i > 0$.

Example 7.1. Let E be an elliptic curve with no complex multiplication and $Z = E \times E$ an abelian surface with an endomorphism f corresponding to the matrix $\begin{pmatrix} 1 & -5 \\ 1 & 1 \end{pmatrix}$. Then $\rho(Z) = 3$ and $(f^i)^*|_{N^1(Z)}$ is not a scalar multiplication for any $i > 0$; see [30, Example 4.1.5]. In particular, $f^*|_{N^1(Z)}$ has only one real eigenvalue (counting multiplicities) and the spectral radius of $f^*|_{N^1(Z)}$ is 6; further $f^*H \equiv 6H$ for some nef divisor $H \neq 0$ by applying the Perron-Frobenius theorem to $\text{Nef}(Z)$. We claim that $H^2 > 0$ and hence H is ample; see Remark 4.4. If $H^2 = 0$, then $H^\perp := \{D \in N^1(Z) \mid D \cdot H = 0\}$ is a f^* -invariant 2-dimensional subspace. Note that $H \in H^\perp$. So $f^*|_{N^1(Z)}$ has two real eigenvalues, counting multiplicities. This is a contradiction. Hence f is polarized.

Next we show the existence of a polarized endomorphism $g : S \rightarrow S$, such that $(g^i)^*|_{N^1(S)}$ is not a scalar multiplication for any $i > 0$ while $(g|_{\text{Alb}(S)})^*|_{N^1(\text{Alb}(S))}$ is a scalar multiplication.

Example 7.2. We use the notation in Example 7.1. Let G be a group generated by $\text{diag}[-1, -1]$ and denote by $S = Z/G$ which is a normal K3 surface. Since $q(S) = 0$ and S has rational singularities, the Albanese map is trivial by Lemma 5.1. Note that f is G -equivariant, and $\pi : Z \rightarrow S$ is a finite surjective morphism. So $g = f|_S$ is also polarized by Theorem 3.11. Next we claim that $(g^i)^*|_{N^1(S)}$ is not a scalar multiplication for any $i > 0$. Clearly, it suffices to show that $\rho(S) \geq 2$ (then it follows that $\rho(S) = 3$ since $f^*|_{N^1(Z)}$ has only one real eigenvalue, counting multiplicities). Suppose $\rho(S) = 1$. A fibre E_0 of $Z \rightarrow E$ has $E_0^2 = 0$. Since E_0 is G -invariant, $\pi^*\pi(E_0) \equiv aE_0$ for some $a > 0$. Then $0 = a^2 E_0^2 = (\pi^*\pi(E_0))^2 = 4\pi(E_0)^2$ and hence $\pi(E_0)$ is not ample, a contradiction.

Example 7.3. We construct polarized endomorphisms $f : X \rightarrow X$ such that:

- (1) $\dim(X) = m + n - 1$ with $m \in \{4, 6\}$ and $0 < n < m$,
- (2) X has \mathbb{Q} -factorial terminal (quotient) singularities and is rationally connected,
- (3) the smooth locus X_{reg} of X has infinite algebraic fundamental group $\pi_1^{\text{alg}}(X_{\text{reg}})$,
- (4) $q^{\natural}(X) = n > 0$ (see Definition 2.5),
- (5) the Iitaka D -dimension satisfies $\kappa(X, -K_X) = m - 1$, and
- (6) the ramification divisor $R_f \subseteq X$ of f is non-trivial.

Indeed, let $G \cong \mathbb{Z}/(m)$ act on \mathbb{P}^{m-1} as a (coordinates) permutation subgroup of S_m so that G has no non-trivial pseudo-reflections (i.e. for any non-trivial $g \in G$, g fixes at most a codimension 2 subset), and \mathbb{P}^{m-1}/G has only canonical singularities. This is guaranteed if the age $a(h) \geq 1$ at every point fixed by a non-trivial h in G , e.g. if $m = 4, 6$; see the proof of [25, Lemma 3]. Let $G = \langle g = \exp(\frac{2\pi\sqrt{-1}}{m}) \rangle$ act diagonally on the abelian variety $A = E^n = E \times \cdots \times E$, with E being an elliptic curve such that G has no non-trivial pseudo-reflection (and hence A/G is Q -abelian) and A/G is rationally connected. This is achievable by letting $E = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\zeta_m)$ and choosing suitable $m > n > 0$ (e.g., $m = 4, 6$ and $0 < n < m$); see [7] and [22, Corollary 25]. Let G act diagonally on $W = \mathbb{P}^{m-1} \times A$. Then $X = W/G$ projects to rationally connected A/G with the general fibre \mathbb{P}^{m-1} and hence it is also rationally connected by [16, Corollary 1.3]. For any non-trivial $g \in G$, $g|_A$ contributes a positive value to the age $a(g)$ and hence $a(g) > 1$. So X is \mathbb{Q} -factorial terminal. Now the multiplication map $\mu_r : A \rightarrow A$, $a \mapsto ra$, with $r \geq 2$, is polarized such that $\mu_r^*H = r^2H$ for any symmetric ample divisor H on A . The power map $q_P : \mathbb{P}^{m-1} \rightarrow \mathbb{P}^{m-1}$, $[X_0 : \cdots : X_{m-1}] \mapsto [X_0^q : \cdots : X_{m-1}^q]$ with $q = r^2$, is also polarized. Thus $f_W = (q_P, \mu_r)$ is a polarized endomorphism of W and it descends to a polarized endomorphism f on X (f_W commutes with the G -action). Since G also has no non-trivial pseudo-reflections on W , the quotient map $\gamma : W \rightarrow X$ is quasi-étale, $K_W = \gamma^*K_X$, and $\kappa(X, -K_X) = \kappa(W, -K_W) = m - 1$. Hence the topological fundamental group $\pi_1(X_{\text{reg}})$ of the smooth locus of X is the extension of $\mathbb{Z}/(m)$ by $\mathbb{Z}^{\oplus 2 \dim(A)}$, and $q^{\natural}(X) = \dim(A) > 0$. For (6), we take D'_i as the pullback to W of the coordinate hyperplane $\{X_i = 0\} \subseteq \mathbb{P}^{m-1}$. Then $f_W^*D'_i = qD'_i$. Now $R_f \geq (q - 1)D_i$ with $D_i \subseteq X$ the image of D'_i .

8. PROOF OF THEOREM 1.8

Proof of Theorem 1.8. If K_X is pseudo-effective, then (1) follows from Lemma 6.9 and (3) and (4) are trivial. Next we consider the case where K_X is not pseudo-effective.

By [2, Corollary 1.3.3], since K_X is not pseudo-effective, we may run MMP with scaling for a finitely many steps: $X = X_1 \dashrightarrow \cdots \dashrightarrow X_j$ (divisorial contractions and flips) and end up with a Mori's fibre space $X_j \rightarrow X_{j+1}$. Note that X_{j+1} is again \mathbb{Q} -factorial (cf. [24,

Corollary 3.18] with klt singularities (cf. [12, Corollary 4.5]). So by running the same program several times, we may get the following sequence:

$$(*) \quad X = X_1 \dashrightarrow \cdots \dashrightarrow X_i \dashrightarrow \cdots \dashrightarrow X_r = Y,$$

such that K_{X_r} is pseudo-effective. Replacing f by a positive power, suppose $f = f_1$ descends to a polarized endomorphism $f_{i-1} : X_{i-1} \rightarrow X_{i-1}$ via the above sequence. By Lemmas 6.4, 6.5 and Remark 6.3, one can further descend f_{i-1} to a surjective endomorphism $f_i : X_i \rightarrow X_i$ after replacing f_{i-1} by a positive power (i.e. replacing f by a positive power). By Theorem 3.11, f_i is again polarized by some ample Cartier divisor H_i . So the sequence $(*)$ is f^s -equivariant for some $s > 0$. Since K_{X_r} is pseudo-effective, $Y = X_r$ is \mathbb{Q} -abelian by Lemma 6.9.

By Lemma 5.3, the composition $X_i \dashrightarrow Y$ is a morphism for each i . If $X_i \dashrightarrow X_{i+1}$ is a flip, then for the corresponding flipping contraction $X_i \rightarrow Z_i$, (Z_i, Δ_i) is klt for some effective \mathbb{Q} -divisor Δ_i by [12, Corollary 4.5]. Hence $Z_i \dashrightarrow Y$ is also a morphism by Lemma 5.3 again. Together, the sequence $(*)$ is a relative MMP over Y .

By Lemmas 2.16 and 5.2, $X_i \rightarrow Y$ is equi-dimensional with every fibre being (irreducible) rationally connected. Note that K_{X_i} is not pseudo-effective for any $i < r$ by (1). Then the final map $X_{r-1} \rightarrow X_r$ is a Fano contraction. So (2) is proved.

Via the pullback, $N^1(X_{i+1})$ can be regarded as a subspace of $N^1(X_i)$ and hence a subspace of $N^1(X)$. Then $f_i^*|_{N^1(X_i)} = f^*|_{N^1(X_i)}$. If $X_i \dashrightarrow X_{i+1}$ is a flip, then $N^1(X_i) = N^1(X_{i+1})$. If $X_i \rightarrow X_{i+1}$ is a divisorial contraction or a Fano contraction, then $N^1(X_{i+1})$ is a codimension one subspace of $N^1(X_i)$ by the cone theorem and $H_i \notin N^1(X_{i+1})$. So $N^1(X_i)$ is spanned by $N^1(X_{i+1})$ and H_i . Together, $N^1(X)$ is spanned by $N^1(Y)$ and those $\{H_i\}_{i < r}$. Clearly, if $i < r$, then $\dim(X_i) > 0$. By Proposition 2.9 and Lemmas 3.1 and 3.10, the eigenvalue of H_i is the same $q = (\deg f_i)^{1/\dim(X_i)}$. So (3) is proved.

(4) is straightforward from (3). □

9. PROOF OF THEOREM 1.10 AND COROLLARY 1.11

Lemma 9.1. *Let $f : X \rightarrow X$ be a polarized endomorphism of a \mathbb{Q} -factorial klt projective variety X . Suppose either one of the following conditions (which are not automatic even for rationally connected terminal X ; see Example 7.3).*

(i) $q^h(X) = 0$ (see Definition 2.5).

(ii) *The smooth locus X_{reg} of X has finite algebraic fundamental group $\pi_1^{\text{alg}}(X_{\text{reg}})$.*

Then X is rationally connected and $(f^s)^|_{N^1(X)}$ is a scalar multiplication for some $s > 0$.*

Proof. By Theorem 1.8, for some $s > 0$, there is an f^s -equivariant equi-dimensional fibration $\pi : X \rightarrow Y$ with irreducible fibres and Y being a \mathbb{Q} -abelian variety. Let $A \rightarrow Y$

be the finite cover étale in codimension one with A an abelian variety. Denote by X' the normalization of $X \times_Y A$ which is irreducible since π has irreducible fibres. Then the projection $p_1 : X' \rightarrow X$ is a finite surjective morphism étale in codimension one.

Note that $q(A) \leq q^h(X') = q^h(X)$. So Condition (ii) implies that $\dim(A) = 0$. Assume Condition (i) and $\dim(Y) > 0$. Replacing A by its étale cover, we may assume $\deg p_1 > |\pi_1^{\text{alg}}(X_{\text{reg}})|$. Since p_1 is étale in codimension one, we get a contradiction by the purity of branch loci. In particular, both two conditions imply that $A = Y$ is a point and hence X is rationally connected by Theorem 1.8.

Now $(f^s)^*|_{\mathbb{N}^1(X)}$ is a scalar multiplication by Theorem 1.8. \square

Proposition 9.2. *Let $f : X \rightarrow X$ be a polarized endomorphism of a \mathbb{Q} -factorial klt projective variety X with the irregularity $q(X) = 0$ (this is the case when X is rationally connected). Suppose that $f^*|_{\mathbb{N}^1(X)} = q \text{ id}$ for some $q > 1$. Then the following are true.*

- (1) *If the Itaka D -dimension $\kappa(X, F) = 0$ for a prime divisor F , then $f^{-1}(F) = F$.*
- (2) *Let D_i ($1 \leq i \leq s$) be all the prime divisors with $f^{-1}(D_i) = D_i$ and let $D = \sum_{i=1}^s D_i$. Then $s \leq \dim(X) + \rho(X)$ with $\rho(X)$ the Picard number of X . The equality holds true only when $K_X + D \sim_{\mathbb{Q}} 0$ and hence X is of Calabi-Yau type.*
- (3) *We have the ramification divisor $R_f = (q-1)D + \Delta$ for some effective divisor Δ , such that Δ and D have no common irreducible component and $\kappa(X, \Delta_j) > 0$ for every irreducible component Δ_j of Δ .*
- (4) *We have $-(K_X + D) \sim_{\mathbb{Q}} \frac{1}{q-1}\Delta$. So either $\Delta \neq 0$ and $\kappa(X, -(K_X + D)) > 0$, or $K_X + D \sim_{\mathbb{Q}} 0$ and hence X is of Calabi-Yau type.*

Proof. We follow the approach of [35, Claim 3.15]. Since $q(X) = 0$, numerical equivalence implies \mathbb{Q} -linear equivalence.

(1) If $f^{-1}(F) \neq F$, then $f^*F \sim_{\mathbb{Q}} qF$ but $f^*F \neq qF$. So $\kappa(X, F) > 0$.

(2) follows from [36, Theorem 1.3].

(3) Note that $f^*D_i = qD_i$ for each i . So $R_f = (q-1)D + \Delta$ for some effective divisor Δ . Clearly, D_i is not an irreducible component of Δ for each i . So, for each j , $f^{-1}(\Delta_j) \neq \Delta_j$ and hence $\kappa(X, \Delta_j) > 0$ by (1).

(4) By the ramification divisor formula, $K_X = f^*K_X + R_f \sim_{\mathbb{Q}} qK_X + (q-1)D + \Delta$. Therefore, $-(K_X + D) \sim_{\mathbb{Q}} \frac{1}{q-1}\Delta$. If $\Delta \neq 0$, then $\kappa(X, -(K_X + D)) > 0$ by (3). If $\Delta = 0$, then $K_X + D \sim_{\mathbb{Q}} 0$. Therefore, (X, D) is log canonical and hence X is of Calabi-Yau type; see [5, Theorem 1.4] or [36, Theorem 1.3]. \square

Lemma 9.3. *Let X be a rationally connected normal projective variety and D a non-uniruled prime divisor such that $K_X + D$ is \mathbb{Q} -Cartier. Then $K_X + D$ is pseudo-effective.*

Proof. Replacing X by a log resolution X' of the pair (X, D) and D by its proper transform D' , we may assume that both X and D are smooth, so that the argument in [28, Theorem 3.7] is applicable. Suppose that $K_X + D$ is not pseudo-effective. Then there is a dominant rational map $X \dashrightarrow Y$ such that D birationally dominates Y . In particular, Y is non-uniruled. This contradicts that X is rationally connected. \square

Proof of Theorem 1.10. Since X is smooth, any quasi-étale morphism onto X is étale by purity of branch loci. Since X is smooth and rationally connected, X has no non-trivial étale cover; see [8, Corollary 4.18]. In particular, $q^{\sharp}(X) = q(X) = 0$. Then (1) follows from Lemma 9.1. The assertions (2), (3), and (4) cases (i) and (ii) are straightforward by Proposition 9.2.

For (4) case (iii), by Lemma 9.3, $K_X + D_1$ is pseudo-effective and hence $K_X + D = -\frac{1}{q-1}\Delta$ is pseudo-effective by Proposition 9.2(4). So $\Delta = 0$ and $D = D_1$. Now $K_X + D \sim_{\mathbb{Q}} 0$ and X is of Calabi-Yau type by Proposition 9.2(4). \square

Proof of Corollary 1.11. (1) Since X is not Q -abelian, by Theorem 1.8, K_X is not pseudo-effective. Further, replacing f by a positive power, we may run f -equivariant MMP

$$X = X_1 \dashrightarrow \cdots \dashrightarrow X_i \dashrightarrow \cdots \dashrightarrow X_r = Y,$$

such that $X_{r-1} \rightarrow X_r$ is a Fano contraction, Y is Q -abelian, and $f|_Y$ is polarized. Since (X, f) is primitive, Y is a point and $X_{i-1} \dashrightarrow X_i$ is either a divisorial contraction or a flip for each $i < r$. Clearly, X_{r-1} is then a Fano variety of Picard number one, so X_{r-1} and hence X are rationally connected; see [38]. By Theorem 1.8(4), $f^*|_{\mathbb{N}^1(X)}$ is a scalar multiplication. Thus Proposition 9.2 is applicable.

(2) By taking pullback, we may regard $\mathbb{N}^1(X_i)$ as a subspace of $\mathbb{N}^1(X)$ for each i . Let $0 < i_1 < i_2 < \cdots < i_s < r$ (s can be taken as 0) be all the indexes such that $X_i \rightarrow X_{i+1}$ is a divisorial contraction if $i = i_t$ for some $1 \leq t \leq s$. Denote by E_t the exceptional divisor of $X_{i_t} \rightarrow X_{i_t+1}$. Note that $\mathbb{N}^1(X_{i_t+1})$ is a codimension 1 subspace of $\mathbb{N}^1(X_{i_t})$ by the cone theorem and $-E_t$ is relative ample over X_{i_t+1} by Lemma 2.10. So $\mathbb{N}^1(X_{i_t})$ is spanned by $\mathbb{N}^1(X_{i_t+1})$ and E_t . Together, $\mathbb{N}^1(X)$ is spanned by $\mathbb{N}^1(X_{r-1})$ and those E_t (E_t may not be a prime divisor in $\mathbb{N}^1(X)$).

Note that the MMP is f^s -equivariant for some $s > 0$. So replacing f by a positive power, we may assume $f^{-1}(E_t(k)) = E_t(k)$ for each irreducible component $E_t(k)$ of E_t . Let R_f be the ramification divisor of f , $R_{f_{r-1}}$ the ramification divisor of f_{r-1} , and R' the strict transform of $R_{f_{r-1}}$ on X . Clearly, $R_f \geq R'$. If $R_{f_{r-1}} = 0$, then $K_{X_{r-1}} = f^*K_{X_{r-1}} \equiv qK_{X_{r-1}}$ as we see in (1), and hence $(q-1)K_{X_{r-1}} \equiv 0$. This is absurd because X_{r-1} is a Fano variety. So $R_{f_{r-1}} \neq 0$ and hence $R' \neq 0$. In particular, $\mathbb{N}^1(X)$ is spanned by R' and those E_t , since X_{r-1} is of Picard number 1.

Let A be an ample divisor in $N^1(X)$. We may write $A = \sum_{t=1}^s a_t E_t + bR'$ for some $a_t, b \in \mathbb{R}$. Note that $R_f \geq R'$ and $eR_f \geq \sum_{t=1}^s E_t$ for some $e \gg 1$ since $R_f \geq E_t(k)$ for each t and k . Then $cR_f = A + F$ for some pseudo-effective \mathbb{R} -Cartier divisor F if $c \gg 1$. So R_f is big and hence $-K_X \equiv \frac{1}{q-1}R_f$ is big. \square

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REFERENCES

- [1] C. Birkar, Existence of log canonical flips and a special LMMP, *Publ. Math. Inst. Hautes Études Sci.* **115** (2012), 325-368.
- [2] C. Birkar, P. Cascini, C. D. Hacon and J. McKernan, Existence of minimal models for varieties of log general type. *J. Amer. Math. Soc.*, **23**(2):405-468, 2010.
- [3] S. Boucksom, T. de Fernex and C. Favre, The volume of an isolated singularity, *Duke Math. J.* **161** (2012), no. 8, 1455-1520.
- [4] M. Brion, On automorphism groups of fiber bundles, *Publ. Mat. Urug.* **12** (2011), 39-66.
- [5] A. Broustet and A. Höring, Singularities of varieties admitting an endomorphism, *Math. Ann.* **360** (2014), no. 1-2, 439-456.
- [6] F. Campana, Connexité rationnelle des variétés de Fano, *Ann. Sci. École Norm. Sup. (4)* **25** (1992), no. 5, 539-545.
- [7] F. Catanese, K. Oguiso and A. Verra, On the unirationality of higher dimensional Ueno-type manifolds, *Rev. Roumaine Math. Pures Appl.* **60** (2015), no. 3, 337-353.
- [8] O. Debarre, Higher-dimensional algebraic geometry, Universitext. Springer-Verlag, New York, 2001.
- [9] T.-C. Dinh, F. Hu, and D.-Q. Zhang, Compact Kähler manifolds admitting large solvable groups of automorphisms, *Adv. Math.* **281** (2015), 333-352.
- [10] N. Fakhruddin, Questions on self-maps of algebraic varieties, *J. Ramanujan Math. Soc.*, **18**(2):109-122, 2003.
- [11] B. Fantechi, L. Göttsche, L. Illusie, S. L. Kleiman, N. Nitsure and A. Vistoli, Fundamental algebraic geometry, Grothendieck's FGA explained, *Mathematical Surveys and Monographs*, **123**, American Mathematical Society, Providence, RI, 2005.
- [12] O. Fujino, Applications of Kawamata's positivity theorem, *Proc. Japan Acad. Ser. A Math. Sci.* **75** (1999), no. 6, 75-79.

- [13] O. Fujino, Fundamental theorems for the log minimal model program, *Publ. Res. Inst. Math. Sci.* **47** (2011), no. 3, 727-789.
- [14] O. Fujino, Some remarks on the minimal model program for log canonical pairs, *J. Math. Sci. Univ. Tokyo* **22** (2015), no. 1, 149-192.
- [15] M. Fulger, J. Kollár, B. Lehmann, Volume and Hilbert function of \mathbb{R} -divisors, *Michigan Math. J.* **65** (2016), no. 2, 371-387.
- [16] T. Graber, J. Harris and J. Starr, Families of rationally connected varieties, *J. Amer. Math. Soc.*, **16**(1):57-67 (electronic), 2003.
- [17] D. Greb, S. Kebekus and T. Peternell, Étale fundamental groups of Kawamata log terminal spaces, flat sheaves, and quotients of Abelian varieties, *Duke Math. J.* **165** (2016), no. 10, 1965-2004.
- [18] R. Hartshorne, *Algebraic Geometry*, Grad. Texts in Math. **52**, Springer, 1977.
- [19] Y. Kawamata, Characterization of abelian varieties, *Compos. Math.* **43** (1981), no. 2, 253-276.
- [20] Y. Kawamata, Minimal models and the Kodaira dimension of algebraic fiber spaces, *J. Reine Angew. Math.* **363** (1985), 1-46.
- [21] J. Kollár, Rational curves on algebraic varieties, *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*, **32**. Springer-Verlag, Berlin, 1996.
- [22] J. Kollár and M. Larsen, Quotients of Calabi-Yau varieties, in : *Algebra, arithmetic, and geometry, In honor of Y. I. Manin on the occasion of his 70th birthday*, Birkhäuser Progress in Mathematics **270**, 179-211 (2009).
- [23] J. Kollár, Y. Miyaoka and S. Mori, Rationally connected varieties, *J. Alg. Geom.* **1** (1992), 429-448.
- [24] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Math., **134** Cambridge Univ. Press, 1998.
- [25] J. Kollár and C. Xu, Fano varieties with large degree endomorphisms, [arXiv:0901.1692](https://arxiv.org/abs/0901.1692).
- [26] H. Krieger and P. Reschke, Cohomological conditions on endomorphisms of projective varieties, [arxiv:1505.07088](https://arxiv.org/abs/1505.07088).
- [27] D. I. Lieberman, Compactness of the Chow scheme: applications to automorphisms and deformations of Kähler manifolds, *Fonctions de plusieurs variables complexes, III (Sém. François Norguet, 1975-1977)*, pp. 140-186, *Lecture Notes in Math*, **670**, Springer, Berlin, 1978.
- [28] S. S. Y. Lu and D.-Q. Zhang, Positivity criteria for log canonical divisors and hyperbolicity, *J. Reine Angew. Math.* **726** (2017), 173-186.
- [29] N. Nakayama, Zariski-decomposition and abundance, *MSJ Memoirs Vol.* **14**, Math. Soc. Japan, 2004.
- [30] N. Nakayama, On complex normal projective surfaces admitting non-isomorphic surjective endomorphisms, Preprint 2 September 2008.
- [31] N. Nakayama, Intersection sheaves over normal schemes, *J. Math. Soc. Japan* **62** (2010), no. 2, 487-595.
- [32] N. Nakayama and D.-Q. Zhang, Polarized endomorphisms of complex normal varieties, *Math. Ann.* **346** (2010), no. 4, 991-1018.
- [33] M. Reid, Projective morphisms according to Kawamata, preprint, Univ. of Warwick, 1983.
- [34] J. Wiśniewski, On contractions of extremal rays of Fano manifolds, *J. Reine Angew. Math.* **417** (1991), 141-157.
- [35] D.-Q. Zhang, Polarized endomorphisms of uniruled varieties. *Compos. Math.* **146** (2010), no. 1, 145-168.

- [36] D.-Q. Zhang, Invariant hypersurfaces of endomorphisms of projective varieties, *Adv. Math.*, 2013, **252**(3):185-203.
- [37] D.-Q. Zhang, n -dimensional projective varieties with the action of an abelian group of rank $n - 1$, *Trans. Amer. Math. Soc.* **368** (2016), no. 12, 8849-8872.
- [38] Q. Zhang, Rational connectedness of log Q -Fano varieties, *J. Reine Angew. Math.* **590** (2006), 131-142.
- [39] Y. Zhang, On isolated singularities with noninvertible finite endomorphism, *Adv. Math.* **308** (2017), 859-878.

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