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The Stroboscopic Effect

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MATHEMATICS OF STROBOSCOPY.

I. The Characteristic Stroboscopic Equation.

Consider a straight row of equally spaced white dots on a black belt running at a constant velocity v over two pulleys. Let the dot spacing be D_0. Suppose that the belt is in a darkened room so that even when the belt is at rest the dots are invisible. Let the moving belt be periodically illuminated

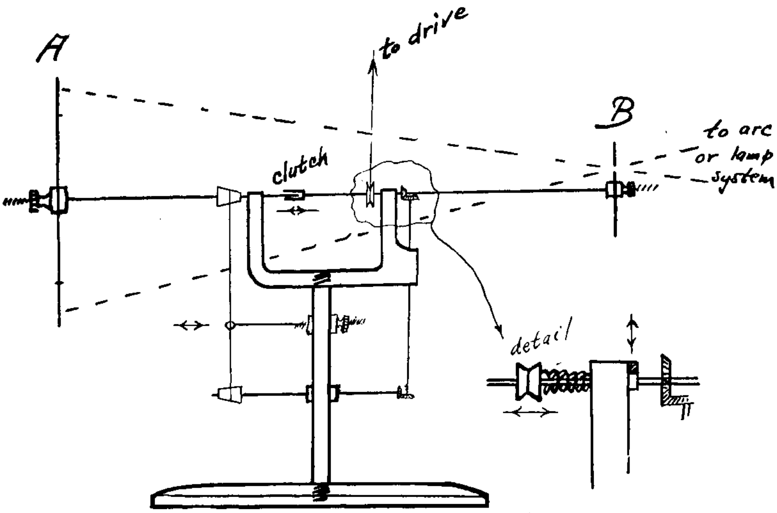

FIG. 37. The strobodeik, an instrument to illustrate the equation
$$V_S = (A - n/m \cdot B)D_0$$

By using two sets of removable discs, A and B, independent values of n and m can be used independent of any angular velocities. With the gear system not in action the angular velocities of discs A and B are equal, and the stationary stroboscopic condition results. With gear system in action the clutch is released, and by means of the adjustable belt system Vs may be given positive or negative values. Each disc of a set is provided with a circular row of equally spaced apertures concentric with the disc axis, but the angular spacing is different for different discs. To avoid phase adjustments of discs A and B the present arrangement for periodic illumination was chosen in preference to a method where discs A and B are of the same size and the illuminations are by coincidences of similar circular rows of apertures with parallel light as illuminant. The instrument may be found useful to illustrate the building up of the various types of stroboscopic images, since lowering the angular velocity of the system does not affect the essential stroboscopic relations. Also, the part played in stroboscopy by retinal lag may be made clear. The stroboscopic effect given by the instrument may be viewed by observation of the front face of disc A (looking from the left in the figure), or the effect may be projected to a screen.

by some source of intermittent and approximately instantaneous illumination of constant frequency B. The dots, or stroboscopic figures, will have a frequency past an imaginary fixed line crossing the moving row, of $A=v/D_0$. At any given single illumination a dot is seen stationary, as a camera snapshot would show it, and as it is impressed on the retina, and its location is say the point P_1. If at the next illumination, or some succeeding illumination not too far removed, there is some other dot at P_2, the two dots will be interpreted by the eye as the same if this condition holds,

$$\text{displacement } P_1 \text{ to } P_2 = \triangle s, \qquad (1)$$

where $\triangle s$ is sufficiently small with the given experimental conditions. The apparently identical dot revealed to the eye at the two separate illuminations, and at succeeding illuminations in a similar way, may be called a simple stroboscopic image. If the illumination has a high enough frequency so that retinal lag bridges over the time interval $1/B$, the image will persist in the field of vision. The single apparent displacement $\triangle s$ is not restricted to the time interval $1/B$; if m is a whole number having a maximum value suitable to the experimental conditions $\triangle s$ may be accomplished in each m illuminations. Thus may be written,

$$\triangle S = m \cdot \triangle s, \qquad (2)$$

where $\triangle S$ is the apparent displacement of a simple image in time m/B, that is the time between two successive illuminations for that image alone. More generally,

$$\triangle s = v/B - n/m \cdot D_0, \qquad (3)$$

where n/m is a fraction at lowest terms. Since v, D_0, B, are all taken constant we have for the stroboscopic velocity,

$$v_s = \triangle s \cdot B = v - n/m \cdot D_0 B. \qquad (4)$$

The number per second of actual appearances of the component dot figures making up a single image is B/m, and this may be called the frequency of the simple image. The period of the image is then m/B. The distance between simple images is $D=D_0/m$. Substituting in (4) the value of v in terms of A,

$$v_s = (A - n/m \cdot B)\, D_0. \qquad (5)$$

which may be called the characteristic equation of stroboscopic velocity.

Equation (5) shows that v_s may have plus or minus values, or may vanish, at which time the stationary stroboscopic condition is present. V_s is positive in the direction of v positive. When $v_s = O$ we have,

$$A_o / B_o \equiv n/m \tag{6}$$

The fraction n/m in (5) is to a limited degree independent of A/B, although equal to A_0/B_0, for the value A/B may be increased or decreased within certain limits without changing the value of n/m. These limits are the values at which v_s has become so great that the moving row of simple images becomes a blur. Continued change will introduce a new value of n/m. In (5), as seen by (6), n is independent of m just as A is independent of B.

If in (3) v, B, and D_0 are variable, then

$$\triangle s = \int_{t_1}^{t_2} v\,dt - n/m \cdot \int_{t_1}^{t_2} D_o\,dt, \tag{7}$$

where D_0 is a function of the time consistent with the observed $\triangle s$, and the time between two successive illuminations is $(t_2 - t_1)$. The average stroboscopic velocity over this time interval may be taken as the "instantaneous stroboscopic velocity," and its value is

$$v_s = \triangle s/(t_2 - t_1), \tag{8}$$

where the value of $\triangle s$ is given by (7). In practical cases v, though not exactly constant, may have a fairly constant mean value, and equation (5) may be used to find the "mean" stroboscopic velocity, provided B and D_0 are constant.

If the factors v, D_0, B, in (5) are variable and known functions of the time, then by special mathematical treatment of general application (see IV below) instantaneous values may be assigned to v_s.

II. Mathematics of the Tonoscope and the Tonodeik.

The belt illustration in I holds for the stroboscopic screen of the tonoscope and the tonodeik where there are a large number of parallel rows of figures on the same stroboscopic screen, each with its own value of D_0. The value of v for each row is of course the same, likewise B. The stroboscopic velocity given by (5) is the tangential value, as is the value of v substituted in (5), and D_0 is a length of arc. The difference in the value of

A from row to row is unity, or may be two in the tonodeik. The rows are arranged as in Scripture's disc in ascending values of A; the order of ascent in these two instruments is from left to right, corresponding to pitches on the piano keyboard. Because a considerable number of the rows have frequencies near that of the row with zero stroboscopic velocity (for a given value of B), the stroboscopic response is not confined to the one row, but includes others on either side of it. The stationary row is located in the central part of this region of response. The responding rows to the right of the stationary row have stroboscopic velocities in one direction that increase from row to row. The same is true for responding rows at its left except that the stroboscopic velocity is in the opposite direction. This type of symmetrical response facilitates the location of the stationary row, or the one that is nearest stationary. Equation (5) is to be applied to each row of stroboscopic figures. The illumination frequency B is by suitable devices equal to the frequency of a sounded tone of definite pitch. With the values of A for the different rows as nearly constant as possible, depending on the constancy of drive of the stroboscopic screen in the form of drum or disc, the pitch of a tone can be found from the known frequencies A, or vice versa, if the frequency B is known, as it is with a calibrated tuning fork, the velocity of the screen can be found. If the lowest frequency row has an A value equal to the frequency of say C below middle C, then, except in the case of a bass voice at a pitch below that, the value of n is to be taken as unity, and the values of m as 1, 2, 3, depending on the value of m in the expression for the distance between simple images, $D=D_0/m$. The total number of rows on the screen includes but an octave of musical scale, obviously all that is necessary.

To find the range of response we note that $D_0=2\pi r/N$, where r is the radius of the reentrant circle of dots, and N the number of dots in the circle. If the screen rotates once per second then N=A. by (4),

$$v_s = v - n/m \cdot 2\pi\, r/N \cdot B \qquad (9)$$

For the stationary row,

$$N = N_o = n/m \cdot 2\pi\, r\, B/v, \qquad (10)$$

whence, for a row of stroboscopic velocity v_s,

$$N = v/(v-v_s) \cdot N_o \qquad (11)$$

Let V be the maximum value of v_s at which the response vanishes. Then for the lower limiting row,

$$N_1 = v/(v+V) \cdot N_o, \tag{12}$$

and for the upper limiting row,

$$N_2 = v/(v-V) \cdot N_o \tag{13}$$

It follows that the range of N values for the region of response is $2vV/(v^2-V^2) \cdot N_0$, and the range of N values above the stationary row is greater than the range below that row by the amount $2V^2/(v^2-V^2) \cdot N_0$. The taking of V as constant in equations (12) and (13) neglects any variation in that quantity with D_0 and supposes the dots all of equal size. In these equations v is the linear velocity of the drum or disc used as stroboscopic screen. The distribution of stroboscopic velocities in the case of a disc will be of a character modified from that of the drum because of the varying radii of the reentrant circles of figures.

III. Mathematics of the Movies.

In motion pictures the stroboscopic screen is in reality the film that is intermittently illuminated, and the stroboscopic figures are the film images. The stroboscopic images may be regarded as projected to the movie screen where they appear magnified. The resultant moving picture as a whole is considered for purposes of analysis as made up of a very large number of small areas (method of the calculus). Each small area contains its own moving picture, which is treated as a simple stroboscopic image, with its corresponding stroboscopic figures on the film. Unlike the images in the case of the belt in I, the elementary images in the movies have a component of motion at right angles to the velocity of the stroboscopic screen as well as in its own direction. The component of motion at right angles is due to the manner of distribution of the component elementary figures on the film.

Equations (7) and (8) are applicable if n/m is put equal to unity. For the component of motion at right angles to the film velocity the second term in (7) is zero, and the first term is replaced, since there is no velocity of the film in this direction, by $(\triangle s)_x$ the distributional displacement at right angles to the film motion. For the component in the film direction,

$$(\triangle s)_y = \int_{t_1}^{t_2} v\,dt - \int_{t_1}^{t_2} D_o\,dt \tag{14}$$

Thus,

$$\triangle s = \sqrt{(\triangle s)_x{}^2 + \left[\int_{t_1}^{t_2} v\,dt - \int_{t_1}^{t_2} D_o\,dt\right]^2} \tag{15}$$

The average instantaneous velocity over the common periods of the elementary simple images is obtained by dividing Δs in (15) by $(t_2—t_1)$, the interval between two flashes on the movie screen. For simplicity the film velocity v may be taken as constant, and the first term in (14) becomes v/B, where B is the number of projections per second. To get the stroboscopic velocity on the movie screen, v_s, obtained from (15), must be multiplied by the magnification.

It may be convenient to consider the distributional displacement in the y-direction, of the elementary figures. In this event let D_1 be the distance between centers of successive pictures on the film, then $D_0 = D_1 + \Delta y$, which is to be substituted in (15).

For the stationary condition lengthwise of the film we have A=B, and for the rectangular direction, A=O. For motion in the latter direction A may still be regarded as zero and only $(\triangle s)_x$ considered. If v_s is to be expressed in terms of A then when $v_s = O$ the A for either direction may be taken as any value or any function of the time consistent with the observed stroboscopic motion.

Many stroboscopic devices (toys) are similar to motion pictures, particularly in having n/m=1.

IV. A General Mathematical Treatment of the Simple Stroboscopic Image.

Stroboscopic velocity cannot in general be treated as an ordinary velocity for it deals with a moving object which, in its physical phase, is discontinuous. We can divide a definite displacement $\triangle s$ by a definite time 1/B and obtain a quantity that has the nature of an average velocity, but beyond the interval of time 1/B we cannot go to carry the ratio to the limit as in the calculus. We are at liberty, however, to treat the simple stroboscopic image as an object, say a small circular dot (geometrical point), having continuous existence. Further, there can be assigned to it any velocity, as a continuous function of the time, that is consistent with the observed behaviour of the image itself. Call the geometrical point that represents the image the "index point," and write,

$$\triangle s = \int_{t_1}^{t_2} D_t s \, dt \tag{16}$$

where $D_t s$ is the instantaneous velocity of the index point, a vector function, and $(t_2—t_1)$ is the period of the stroboscopic image. Evidently $D_t s$ may be any function of the time whose integral between the limits t_1, t_2, will give $\triangle$'s.

The most general treatment of the stroboscopic velocity is for three dimensions. Let the stroboscopic figures have a known distribution along a space curve, which may be called the curve of location. This curve is attached to the stroboscopic screen and moves with it. Let it be defined in terms of cartesian coördinates fixed on the screen (moving axes), and express the motion of the index point by the time functions,

$$\begin{aligned} x' &= f(t) \\ y' &= g(t) \\ z' &= h(t) \end{aligned} \tag{17}$$

These functions must be such that at any given "participating" illumination ("participating" in any simple image) the index point is at one of the point stroboscopic figures on the curve of location. Elimination of t from equations (17) will leave two independent relations,

$$\begin{aligned} F(x', y', z') &= 0 \\ G(x', y', z') &= 0, \end{aligned} \tag{18}$$

which represent two geometrical surfaces whose line of intersection is the curve of location.

Next refer the moving axes to a set of cartesian axes fixed in position. The transformation equations are,

$$\begin{aligned} x &= l_1 x' + l_2 y' + l_3 z' + h \\ y &= m_1 x' + m_2 y' + m_3 z' + j \\ z &= n_1 x' + n_2 y' + n_3 z' + k \end{aligned} \tag{19}$$

where l, m, n, are direction cosines, h, j, k, are coördinates of the origin of moving axes referred to the fixed axes, and x', y', z', are coördinates of any position of the index point referred to the moving axes,—all functions of the time. The direction cosines, which are not all independent, together with h, j, k, are known functions of the time for they describe the motion of the stroboscopic screen. There follows,

$$\triangle s = \sqrt{(\triangle x)^2 + (\triangle y)^2 + (\triangle z)^2}\ , \qquad (20)$$

where

$$\triangle x = \int_{t_1}^{t_2} \partial x / \partial t \, dt$$

and similar expressions for $\triangle y$ and $\triangle z$. The partial derivatives give the three instantaneous components of the velocity of the index point. The displacements $\triangle x, \triangle y, \triangle z$, are those of the index point with reference to the fixed axes, and they have occured by motion of the index point along the curve of location during the time interval $(t_2—t_1)$. The stroboscopic velocity is,

$$v_s = \sqrt{(\partial x/\partial t)^2 + (\partial y/\partial t)^2 + (\partial z/\partial t)^2}\ , \qquad (21)$$

where the values of the partial derivatives may be obtained from (19).

For the stationary stroboscopic condition we have,

$$\begin{aligned} l_1\ x' + l_2\ y' + l_3\ z' + h &= C_1 \\ m_1\ x' + m_2\ y' + m_3\ z' + j &= C_2 \\ n_1\ x' + n_2\ y' + n_3\ z' + k &= C_3, \end{aligned} \qquad (22)$$

where the C's are constants.

For the tonoscope and the tonodeik, if the tangent plane of the drum is taken as the stroboscopic screen, the conditions are,

$$\begin{aligned} & l_1 = 1, \quad l_2 = l_3 = 0 \\ & m_2 = 1, \quad m_3 = m_1 = 0 \\ & n_3 = 1, \quad n_1 = n_2 = 0 \\ & k = 0, \end{aligned} \qquad (23)$$

provided that the plane stroboscopic screen moves in its own plane without rotation and along the y-axis, and that the axis of x' is parallel to the axis of x, the axes of y' and y lie in the same straight line, and the curve of location is a straight line parallel to the y′ axis. Then,

$$
\begin{aligned}
(v_s)_x &= \partial/\partial t\,(x'+h)=0 \\
(v_s)_y &= \partial/\partial t\,(y'+j) \\
(v_s)_z &= \partial/\partial t\,(z'+0)=0
\end{aligned}
\tag{24}
$$

By (21) $v_s = \partial/\partial t\,(y'+j)$. With the stationary condition

$$\partial y'/\partial t = -\partial j/\partial t\,, \tag{25}$$

which shows that for this condition the index point has a velocity with respect to the moving axes equal and opposite to that of the screen with respect to the fixed axes.

Similar treatment holds for the vertical component of stroboscopic velocity for an elementary image in the movies, although the general equation (21) is applicable to them if the curve of location is suitably chosen. This curve may be considered as confined to the surface of the film, thus reducing the problem to one of two dimensions. The movie audience interprets the motions in three dimensions as they actually occurred in nature, and the curves of location in two dimensions on the film are projections of curves of location in three dimensions.

THEORY OF THE STROBOSCOPIC EFFECT BY DIRECT REFLECTION OF LIGHT FROM VIBRATING MIRRORS.

An examination of the values of D in the equation, $D=D_0/m$, for the distance between simple stroboscopic images in the tonodeik when the stroboscopic effect is produced first by manometric flame and then by vibrating mirror, reveals that with a given frequency of vibration the values of D are identical in the two cases. This indicates that the chief determining factor in the effect by vibrating mirror, is the intensity difference on a small area of the screen at the two half-period pauses. Intensity maxima occurring during the half-periods should, on the contrary, have the effect of doubling the frequency, which would make the value of D by vibrating mirror equal to 1/2 D by manometric flame, and this is not in harmony with observed fact.

The importance of the half-period pause of a vibrating light pencil of constant intensity is strikingly seen in photographs of oscillating beams where there is relative motion between the plane of vibration of the beam and the plate during the photographing. Some excellent photographs of this character, taken with the aid of the phonodeik, are given in Professor D. C. Miller's recent book, "Science of Musical Sounds."