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L. P. Sieg
State University of Iowa

A NOTE ON KATER'S REVERSIBLE PENDULUM

L. P. SIEG

In one of our laboratory classes recently, in connection with a routine experiment with Kater's pendulum, certain of the students were confronted with the situation in which, although the periods of vibration from each of the two knife edges were practically identical, the distance between the knife edges was by no means equal to the length of the equivalent simple pendulum. None of the treatises on dynamics available offered any help in their difficulty. In all the discussions it was virtually stated that when the periods from the two knife edges were equal the distance between knife edges was equal to the length of the simple pendulum of equal period.

When their difficulty was presented to the present writer he was at once reminded of a study he had made some twenty years ago, but which at that time he had deemed unimportant enough for publication. In this study, a paper read before a Sigma Xi meeting, the writer pointed out that the shifting of a knife edge on any compound pendulum causes a variation in the period in which in certain cases it can pass through a minimum. In other words there are two positions of a knife edge with respect to the center of gravity of the system in which the periods of vibration will be the same. Some time later an article by Tatnall[1] appeared and covered almost identically the same ground. As neither of these treatments specifically deals with the present case it is thought worth while to publish a note on the question.

In Fig. 20-A, let C denote the center of gravity (in future abbreviated to c.g.) of the system, and O_1 and O_2 the two points of suspension. Further let h_1 and h_2 be respectively the distances of the c.g. of the whole system from O_1 and O_2, and let T_1 and T_2 be respectively the corresponding periods of vibration. Then we have the well-known expressions,

$$T_1 = 2\pi\sqrt{\frac{I}{M g h_1}} = 2\pi\sqrt{\frac{K^2 + h_1^2}{g h_1}} \qquad (1)$$

$$T_2 = 2\pi\sqrt{\frac{K^2 + h_2^2}{g h_2}} \qquad (2)$$

1 R. R. Tatnall, Phys. Rev., 17, p. 460, 1903.

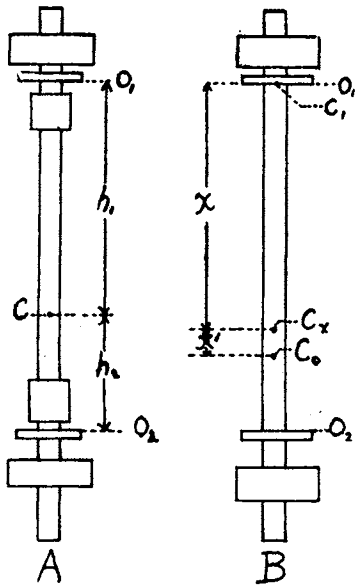

Fig. 20. Kater's pendulum. A, with weightless knife edges; B, with actual knife edges.

where I denotes the moment of inertia of the system about C, M is the total mass, g the acceleration of gravity, and K the radius of gyration of the system about C. Denoting by L_1 and L_2, respectively the lengths of the equivalent simple pendulums in the two cases, we have

$$L_1 = \frac{K^2 + h_1^2}{h_1} \quad \ldots\ldots\ldots\ldots (3)$$

$$L_2 = \frac{K^2 + h_2^2}{h_2} \quad \ldots\ldots\ldots\ldots (4)$$

Let us first, to avoid confusion, make the assumption that the knife edges have no mass, so that their movement will not affect the location of the center of gravity of the system. Later this assumption will be avoided. Let O_1 be at such a distance, x, from C that we have again the same length of the equivalent simple pendulum. Then from (3),

$$\frac{K^2 + h_1^2}{h_1} = \frac{K^2 + x^2}{x}$$

Solving for x, we have the two values

$$x = h_1 \qquad x = \frac{K^2}{h_1} \quad \ldots\ldots\ldots\ldots (5)$$

Thus the knife edge O_1 can be either at a distance h_1 from C, as in the figure, or $\frac{K^2}{h_1}$ from C, and we shall have in both cases the same period of vibration. Similarly, from the other point of suspension, O_2, if y is the distance of O_2 from C we have two other equal periods of vibration if $y = h_2$, or $\frac{K^2}{h_2}$.

We can imagine then that our pendulum (Fig. 20-A) has four weightless knife edges, two on each side of the center, so adjusted with respect to the masses on the pendulum rod that when it is hung in turn from the four, the periods will be the same. The possible distances between knife edges, $x + y$, will then be seen to be,

$$h_1 + h_2,\ h_1 + \frac{K^2}{h_2},\ \frac{K^2}{h_1} + h_2, \text{ and } K^2 \left[\frac{1}{h_1} + \frac{1}{h_2}\right]$$

In general these four distances will be different, and only one of them corresponds to the length of the equivalent simple pendulum. In practice there is no difficulty, for one can by rough calculation determine if the knife edges are in the proper positions. Again, the laboratory forms of Kater's pendulum are usually so constructed that one can not easily attain on the actual pendulum the wrong positions of the knife edges, but in the case referred to in this article, the students had succeeded in doing just this thing.

It is evident that if there are two values, x, of the distances from O_1 to C that yield the same period of vibration, there must be some x beween them which is the only distance corresponding to a single period, and that a maximum or a minimum. Our function of x to be studied is $\frac{K^2 + x^2}{x}$, and a simple examination shows that this becomes a minimum for $x = \pm K$. Only the plus value of K need be considered, since the other value relates to the other support. The relation becomes more understandable on platting a typical curve connecting x and T. We have

$$T = 2\pi\sqrt{\frac{K^2 + x^2}{g\,x}} \quad \text{.....................(6)}$$

or

$$T^2 = c\left[\frac{K^2 + x^2}{x}\right] \quad \text{...................(7)}$$

where C is a constant. In figure 21 we have such a graph for the special case where $C = 1$, and $K = 2$ (not supposed at all to represent the facts, but merely to represent the nature of the function). On the curve the point D represents the minimum. It is the only value of x corresponding to the period represented by the ordinate at that point. From A we draw the line ABC parallel to the X-axis. The two intersections B and C represent the two

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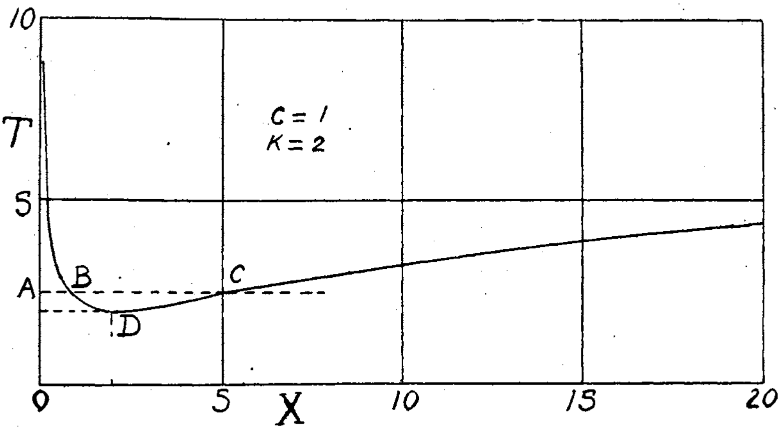

Fig. 21. Variation of the period of vibration, *I*, with the distance, x, from knife edge to center of gravity of the whole pendulum.

values of x corresponding to the period A. To revert for a moment to the first paragraph of this article, my students had chanced in their experiment on the distance AB, when they should have employed the greater distance AC for h_1.

Let us assume now that the knife edges have mass, and let us refer to Fig. 20-B. We can assume, without any loss of generality, that we can attain equality of periods by moving solely O_1. Hence the mass of O_2 can be considered as merged with the remainder of the pendulum mass.

Let $M =$ the mass of all the pendulum excepting O_1.

$m =$ the mass of O_1.

$C_x =$ the center of mass of the whole.

$C_0 =$ the center of mass of M.

$C_1 =$ the center of mass of m_1 (assume C_1 to be coincident with the near edge of O_1. The error will be only a small constant correction).

$x =$ the distance of O_1 from C_x.

Then we have

$$T_1 = 2\pi\sqrt{\frac{I}{(M+m)g\,x}} = 2\pi\sqrt{\frac{(M+m)\,K^2 + (M+m)x^2}{(M+m)g\,x}}$$

$$= 2\pi\sqrt{\frac{K^2+x^2}{g\,x}} \qquad (8)$$

where K is the radius of gyration about C_x, and I is the moment of inertia about O_1.

The only difference between equations (8) and (6) is that in the former K is not a constant, but is a function of x. Now let $C_0C_x = x'$, then

$$mx = Mx' \qquad (9)$$

and

$$x' = \frac{mx}{M} \qquad (10)$$

Let K_0 be the radius of gyration of M about C_0, and k_0 that of m about O_1. Then the moment of inertia about C_x is given by

$$I_0 = (M + m)k^2 = Mk_0^2 + Mx'^2 + mk_0^2 + mx^2 \qquad (11)$$

Substituting for x' from (10) we have

$$K^2 = \frac{MK_0^2 + mk_0^2 + m^2x^2/M + mx^2}{M + m} \qquad (12)$$

Let

$$M = aM \qquad (13)$$

then

$$K^2 = \frac{K_0^2 + ak_0^2}{1 + a} + ax^2 \qquad (14)$$

or

$$K^2 = B + ax^2 \qquad (15)$$

where

$$B = \frac{K_0^2 + ak_0^2}{1 + a} \qquad (16)$$

Substituting (15) in (8) we have

$$T_1 = 2\pi \sqrt{\frac{B + ax^2 + x^2}{gx}} = 2\pi \sqrt{\frac{B + (1 + a)x^2}{gx}} \qquad (17)$$

Again for every value of T_1 there are two values for x, except at the minimum point. Determining from (17) the value of x to make T_1 a minimum, we find

$$x = \sqrt{\frac{B}{1 + a}}$$

This readily reduces to the former value $x = K$, if $m = O$ (and hence $a = O$).

The form of (17) is seen to be the same as that of (1), and so here too we have the possibility in the most general case of four positions of the knife edges which would yield the same period of vibration.

In conclusion then we can state the following: The equality of periods, when a pendulum is suspended in turn from two knife edges, is a necessary condition that the length between the knife edges may be equal to the equivalent simple pendulum, but it can not be said to be a sufficient condition.

State University Of Iowa.