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A SIMPLIFIED DEDUCTION OF MAXWELL'S DISTRIBUTION LAW

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It is the object of this paper, not to contribute to the volume of evidence for the validity of Maxwell's molecular velocity law, or to improve upon the cogency of the various lines of argument which have previously led to it; but rather to bring the deduction within easy reach of any student of integral calculus who has knowledge of the merest rudiments of the theory of errors.

The mechanical postulates here used are few and simple. They are:

(1) That we are dealing with a homogeneous body of gas; that is, in any two equal portions of the gas having dimensions large in comparison with the mean free path, there are substantially equal numbers of molecules having equal average speeds.

(2) That in the portion of gas selected, the number of molecules is so enormous as to render the rigorous application of statistical reasoning entirely admissible.

(3) That in any such portion of gas the components of molecular speed in any direction have the same average value and the same sum as in any other direction; that is, there are no currents or eddies, and no external force such as gravity is acting upon the molecules.

(4) That any one molecule chosen at random exerts at any particular instant no influence whatever upon any other molecule likewise chosen at random, except in the remotely possible circumstance that the two may be at that instant in the act of *encounter* — a term chosen as somewhat broader than *collision*. (Of course, if we were to strike one molecule and thus endow it with a large increase in speed, the added energy would *ultimately* distribute itself among the other molecules; but not at the instant in question.)

Now let us assign to each molecule a vector representing its instantaneous velocity, and transfer all these vectors to a common origin O, which is at the same time the origin of a system of rectangular coördinates. (The space defined by these coördinates has nothing to do with the space occupied by the gas; it is a mere

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ideal, a vector space.) The vectors have termini which are points (x, y, z) in this space. We shall study the distribution of these terminal points about O.

Let us first consider the components of all the vectors taken parallel to any line, say the X-axis. They are represented by the perpendicular distances of the vector termini from the YZ plane. If we slice the coördinate space up into laminæ of thickness dx, parallel to the YZ plane, the vector termini will be distributed among these laminæ, and the distribution will be subject to the following conditions:

(1) The number of termini on one side of the YZ plane will be equal to that on the other side, and the algebraic sum of their distances from the YZ plane (Σx) will be zero (postulate 3).

(2) The position of any terminus, i.e., its value of x, will be independent of that of any other terminus; hence, the probability of its x having a given value is the same, whatever the values of x for the other termini (postulate 4).

(3) Small values of x will be more numerous than large values; so that the probability of a terminus lying within a given lamina of the space *decreases* with the distance of the lamina from the YZ plane. This statement is likely to be a bone of contention; but there is one argument for it, at least: large values of the x-component can result only from large actual speeds, while small values may result from either small speeds or from large speeds (in directions making large angles with X). The postulate is, furthermore, supported by the experimental facts.[1]

Now upon reviewing the foregoing conditions we see that *they are precisely those upon which is based Gauss's well known deduction of the law of distribution of accidental errors.* The mathematical expression of this law, as applied to the present case, takes the form

$$\frac{dn}{N} = \frac{k dx}{\sqrt{\pi}} e^{-k^2x^2},$$

or

$$dn = N\, dx \frac{k}{\sqrt{\pi}} e^{-k^2x^2}. \qquad (1)$$

N is the total number of vector termini (and of molecules), dn is the number within the lamina of thickness dx at distance x from the YZ plane. k is the familiar constant *index of precision*, of which more later.

Again let us analyze the lamina at x into columnar elements or strips parallel to Z, having thickness dx and width dy. By precisely

1 Eldridge, Phys. Rev., Vol. 30 (II), p. 931.

the same reasoning as before, dn being the total number of termini within the lamina, the number within the strip at distance y from the XZ plane is

$$d^2n = dn\, dy \frac{k}{\sqrt{\pi}} e^{-k^2y^2}. \qquad (2)$$

And lastly, analyzing the strip into rectangular elements dx dy dz, the number within one element at distance z from the XY plane is

$$d^3n = d^2n\, dz \frac{k}{\sqrt{\pi}} e^{-k^2z^2}. \qquad (3)$$

The value of k in (2) and (3) is the same as in (1), because the distribution of any number of Y or Z vector components is, by postulate 3, the same as that of the X components. Gathering together (1), (2) and (3) by the necessary substitutions, there results

$$d^3n = N \frac{k^3}{\pi^{\frac{3}{2}}} e^{-k^2(x^2+y^2+z^2)}\ dx\ dy\ dz. \qquad (4)$$

It will be recognized at once that $x^2+y^2+z^2$ equals U^2, the square of the actual speed represented by the vector whose terminus is (x, y, z).

The quantity

$$\delta = \frac{d^3n}{dx\ dy\ dz} = \frac{Nk^3}{\pi^{\frac{3}{2}}} e^{-k^2U^2} \qquad (5)$$

is the "numerical density" of the vector termini in the vector space at a distance from the vector origin corresponding to the speed U. Therefore the number Δn of vector termini in a spherical shell of radius U and small thickness ΔU (representing a certain speed range ΔU), is $4\pi U^2 \Delta U . \delta$, or, using (5),

$$\Delta n = \frac{4Nk^3\Delta U}{\sqrt{\pi}} U^2 e^{-k^2U^2}. \qquad (6)$$

This represents the number of molecules having speeds ranging from $U - \frac{1}{2}\Delta U$ to $U + \frac{1}{2}\Delta U$. Finally, the proportion P of the total number N, which have speeds within this range, or the probability that a given molecule selected from the totality N will have a speed within this range, is $\Delta n/N$, or

$$P = \frac{4k^3\Delta U}{\sqrt{\pi}} U^2 e^{-k^2U^2}; \qquad (7)$$

which is Maxwell's law.

The sum of the squares of the speeds of all the molecules is, using dn for Δn and dU for ΔU in (6),

$$\Sigma U^2 = \int U^2\, dn = \frac{4\, N\, k^3}{\sqrt{\pi}} \int_0^\infty U^4 e^{-k^2U^2} dU = \frac{3\, N}{\sqrt{\pi}\, k} \int_0^\infty e^{-k^2U^2} dU.$$

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It is well known that the value of the integral in this last expression is $\frac{\sqrt{\pi}}{2k}$. Therefore

$$\Sigma U^2 = \frac{3N}{2k^2},$$

whence

$$k = \sqrt{\frac{3}{2}}\,\frac{1}{U_e}; \tag{8}$$

in which U_e represents $\sqrt{\frac{\Sigma U^2}{N}}$, the radical mean square speed, the speed corresponding to mean kinetic energy, or what may be appropriately called the *effective* speed. It is this speed which is directly calculable from the density and the pressure of the gas, being equal to $\sqrt{3p/\varrho}$. This gives, from (8),

$$k = \sqrt{\frac{\varrho}{2p}}. \tag{9}$$

Either (8) or (9) may be substituted in (7) to give Maxwell's law in practical working form.

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