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Fuzzy logic: An analysis of logical connectives and their characterizations

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FUZZY LOGIC: AN ANALYSIS OF LOGICAL CONNECTIVES AND THEIR CHARACTERIZATIONS

An Abstract of a Thesis

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In Partial Fulfillment

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ABSTRACT

The focus of this thesis is to determine exactly which functions serve as appropriate fuzzy negation, conjunction and disjunction functions. To this end, the first chapter serves as motivation for why fuzzy logic is needed, and includes an original demonstration of the inadequacy of many valued logics to resolve the sorites paradox. Chapter 2 serves as an introduction to fuzzy sets and logic. The canonical fuzzy set of tall men is examined as a motivating example, and the chapter concludes with a discussion of membership functions.

Four desirable conditions of the negation function are given in Chapter 3, but it is shown that they are not independent. It suffices to take two of these conditions, monotonicity and involutiveness, as negation axioms. Two characterization proofs are given, one with an increasing generator and the other with a decreasing generator. An example of a general class of negation functions is studied, along with their corresponding increasing and decreasing generators.

Chapters 4 and 5 provide an analysis of fuzzy conjunction and disjunction functions, respectively. Five axioms for each are given: boundary conditions, commutativity, associativity, monotone non-decreasing, and continuity. Yager's class of conjunction and disjunction functions are each shown to satisfy all five of these axioms. The additional assumption of strict monotonicity is added to obtain pseudo-characterizations analogous to the characterizations of the negation function. Finally, it is shown that although the min function is a conjunction function, it does not have a decreasing or an increasing generator. Similar results are obtained in Chapter 5 for disjunction functions, with a concluding theorem that the max function has no generators.

The interactions of these three connectives is the content of Chapter 6. In this chapter, negation, conjunction, and disjunction triples are considered that satisfy both of DeMorgan's laws. Distributivity of conjunction and disjunction over each other is examined. It is then shown that the only conjunction and disjunction pair that satisfies the distributivity axiom is the min, max pair.

In conclusion, Chapter 7 discusses why having unique functions serve as conjunction and disjunction is desirable. It also contains a brief discussion of the implication connective and some areas for further investigation.

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CHAPTER 1

INTRODUCTION

Although logic is one of the basic underlying areas of mathematics, it has undergone relatively few changes throughout the years. Consequently, there are some aspects of mathematics which have surpassed the logic available to provide the corresponding solid foundations. Therefore, it is important to understand how far traditional logic (classical logic) has taken us, and to see where and why fuzzy logic began as well as why it is needed.

Classical logic had its roots in antiquity with Aristotle. In the fourth century B.C. Aristotle laid down some of the framework of logic that has remained unchanged for centuries. Most of Aristotle's logic was in the form of syllogisms. The most famous of these is the following: Socrates is a man; all men are mortal; therefore, Socrates is mortal. In the late seventeenth century Gottfried Wilhelm Leibniz made one of the first significant contributions to logic since the time of Aristotle. He wished to develop a universal system in which all mathematical and scientific knowledge could be encoded into symbolic language. Leibniz was unsuccessful in his attempt to do so, but the early nineteenth century brought a new form of logic which took on the symbolic and mathematical structure that was the beginning of Leibniz's dream.

As logicians were trying to realize a part of Leibniz's dream by codifying all of mathematics into symbolic logic, some of them began to see that their task was in vain. One of the more notable personalities to realize some of the intricate problems involved was Bertrand Russell, originally a strong proponent of classical logic. He had been working on an exhaustive work, entitled Principia Mathematica, with Alfred North Whitehead [9]. While the goal of the book was to reduce all of mathematics to symbolic forms, Russell had found a set that could not be encoded into the first order language of the time. Russell first wrote of this set to another leading logician, Gottlob Frege, in 1901 **[14, p.125]**. The statement, now referred to as Russell's paradox, is framed by the following question: Is the set of all sets which are not members of themselves a member of itself **[4, p.663]**? Clearly, there are logical problems with this question, but mathematically it made sense to ask. Here, therefore, was a clear example of a place where classical logic could not support the mathematics. In Principia Mathematica, Russell introduced a theory of types which would not allow any statement (or set) to refer to itself. In other words, any statement that was self-referential was no longer considered to be in the realm of mathematics. However, rather than improving the logic, this was restricting the mathematics. If the codification of all mathematics was still the desired goal, this was a poor solution.

Another troublesome paradox which predated Russell's paradox is the sorites paradox, attributed to the ancient Greek Eubulides **[5, p.328]**. In this paradox, a single grain of sand is removed from a heap of sand. At some point, the number of grains of sand remaining is so small that it could not be considered a heap anymore. At what point, then, does this heap cease to be a heap? Formally, this paradox can be phrased as follows: If the heap contains x grains of sand, clearly x-1 grains of sand constitute a heap as well. By the principle of mathematical induction, it is possible to conclude that any number of sand grains constitutes a heap. Since even a single grain of sand would be a heap, this is clearly contrary to what the term "heap" would mean. Several other paradoxes and equivalent ones were soon found to further support the idea that there was something missing from the supporting logic.

Russell experimented with what he referred to as "vagueness," in which certain statements might not be either true or false. However, in 1920 Jan Łukasiewicz pioneered the solution of these paradoxes by allowing truth values other than just "true" and "false." He developed what is descriptively called *many-valued logic*. With this type of logic the user is allowed to choose the number of truth values. The number of truth values can range from two to an infinite number. Łukasiewicz's dislike of the law of the excluded middle was one of the contributing factors that led to the discovery of many-valued logics. The law of the excluded middle dates back to Aristotle and the origins of logic. It simply states that for any statement P, "P or not P" must be true. Aristotle and many logicians after him believed that this was a fundamental law of logic that was necessarily true. Łukasiewicz disagreed and therefore, in his many-valued logic, this law did not always hold. For example, if a statement A was neither true nor false, then the statement "A or not A" was not necessarily true.

Many-valued logics (MVLs) gave a solution to Russell's paradox by simply giving the statement "the set of all sets which are not members of themselves is a member of itself" a truth value of 0.5. Since this statement is as true as it is false, its negation also receives a truth value of one half. Once this statement was not forced to be either true or false, but could be thought of as both partially true *and* partially false, the contradiction is resolved. However, MVLs did not completely resolve all known paradoxes, as can be seen by the following analysis of the sorites paradox.

To fully understand why MVLs do not resolve the sorites paradox, it is necessary to understand the basic operations and structure of a MVL. As noted above, it is possible to create a MVL with any desired number of truth values. Consider a MVL that has n truth values, remembering that, if desired, n can be infinite. Typically, if n is finite, the truth values are assigned the values $\frac{i}{n-1}$ where i ranges from zero to n-1. Traditionally, 'absolute false' is taken as 0 and 'absolute true' as 1. Historically, if n is chosen to be countably infinite, then rational numbers are used as truth values since the set of rational numbers is countably infinite. Occasionally, some authors will use the set of reals instead of rationals for an uncountably infinite set. In either case, each statement is assigned a unique truth value. Rules are then defined for negation, conjunction, disjunction, and implication. From these rules the success of a MVL at resolving the paradox can be determined. The rules used here are from Łukasiewicz as found in **[3]**. The negation of a statement P, denoted ~P, has the truth value of one minus the truth value of P. If we let [P] denote the truth value of P, then the symbolic notation for the truth value of negation is $[\sim P] = 1 - [P]$. The truth value for the conjunction of P and Q is the minimum of the two truth values. Symbolically, $[P \wedge Q] = \min([P], [Q])$. The truth value for the disjunction of two statements P and Q, on the other hand, is the maximum of the two truth values. Thus, $[P \vee Q] = \max([P], [Q])$. Finally, the truth value of "P implies Q" is given as one minus the truth value of P plus the truth value of Q. If this sum is greater than one, "P implies Q" is simply given the value of one. Thus, $[P \rightarrow Q] = \min(1, 1-[P]+[Q])$.

It is easy to verify that any MVL is closed under these operations. That is, the truth value of any statement or any combination of statements is of the form $\frac{i}{n-1}$ where i ranges from zero to n-1, or is a rational number if n is countably infinite.

Logically, the sorites paradox is expressible as a long string of implications. Each component of the implication has the following form: if x is a heap then x-1 is a heap. Letting P(x) represent the phrase "x grains of sand is a heap" gives the following form to each component of the paradox: $P(x) \to P(x\text{-}1)$. The point of the paradox is that it is impossible to distinguish the truth of P(x) from the truth of P(x-1). In other words, in a MVL, $[P(x)]=[P(x\text{-}1)]$. The variable can therefore be disregarded as it does not effect the truth value of the statement. Thus, the whole string of implications expressing the paradox can be symbolized as follows: $(\ldots(((P \to P) \to P) \to P) \to \ldots \to P)$. Now, what is the truth value of $P \to P$? It is simply min(1, 1-[P]+[P]), which clearly equals one. On the second iteration, the truth value of the paradox would be the truth value of $(P \to P) \to P$. As before, the truth value of $(P \to P) \to P$ is calculated by min(1, 1-$[P \to P]$+[P]). However, $[P \to P] = 1$. Hence, the truth value of $(P \to P) \to P$ is equal to min(1, 1-1+[P]) which is [P]. The third iteration, $((P \to P) \to P) \to P$, would yield the number one again. Thus, the truth values would form an alternating sequence of ones and [P]'s. However, this is clearly an extremely unsatisfactory answer. The truth value of the statement "x grains of sand is a heap" should depend not solely on whether x is even or odd but rather on the magnitude of x itself.

Thus, MVLs could not resolve the sorites paradox, and hence the problem of finding a complete solution to the known paradoxes of mathematics was left unsolved. To address this problem, Max Black wrote an article in 1937 entitled "Vagueness: An Exercise in Logical Analysis," in which he hinted at an idea which was later to become what is now called a "fuzzy set" [8, p.33]. The true birth of fuzzy logic came almost thirty years later in a paper simply called "Fuzzy Sets." In this paper, Lotfi Zadeh spells out the basis for this new type of set [18]. With this new type of set and its corresponding logic, these paradoxes were resolved as will be discussed in Chapters 2 and 7.

The intention of this paper is to provide an analysis of the logical connectives in fuzzy logic. In order to follow the arguments given, it is assumed that the reader has an understanding of how the logical connectives operate in first order predicate logic, as well as an understanding of increasing, decreasing, continuous and monotone functions. The initial step for understanding fuzzy logical connectives is to understand where the problems of classical logic and MVLs lie, which was discussed above. Chapter 2 introduces fuzzy sets and fuzzy logic with some examples and explanations. Chapter 3 starts with the simplest logical operator of negation. Negation is the only unary logical operator. The negation functions have an important characterization theorem in which each negation function has a certain kind of generator and every generator of this kind has a corresponding negation function. The focus of Chapters 4 and 5 is to develop similar results for the conjunction and disjunction functions, respectively. It is necessary, however, to add the additional restriction of strict monotonicity to guarantee a generator for both of these functions. In Chapter 6, the interactions of the connectives, such as distributivity of conjunction over disjunction and disjunction over conjunction, are considered. Finally, the conclusion contains a brief discussion of implication and a summary of the work presented in this paper.

The general organization and several proofs contained in this paper follow the style given by Klir and Yuan [6]. A new set of axioms for the negation, conjunction and disjunction functions is given, as well as a proof that the max and min functions are the unique functions which satisfy all these axioms and the additional axiom of distributivity.

CHAPTER 2

WHAT ARE FUZZY SETS AND FUZZY LOGIC?

Like their classical analog, fuzzy sets are simply collections of objects. Unlike elements of classical sets, however, some objects belong partly to a fuzzy set and partly to its complement. The canonical example of a fuzzy set is the set of tall men. In classical set theory, the set of tall men would have a strict restriction on height. For example, any man over 6 feet tall would qualify for membership. Hence, if some man were 6 feet and 0.5 inches tall he would be in the set of tall men, and any man who is 5 feet and 11.5 inches tall would not be. Furthermore, a man who is 5.99999 feet tall would not be a tall man, but a man who is 6.00001 feet tall would be a tall man. Thus, classical sets have a rigid boundary that is often much more rigid than the language which the sets are made to represent.

Most often when a phrase such as "tall men" is used, its intention is much more lenient than classical sets allow. In fuzzy sets each element has a *degree* of membership. The degree is the extent to which an element belongs to a set. There is a corresponding degree function whose domain is a set of possible elements, and whose range is any interval of the real number line (more generally, any partially ordered set is sufficient). Commonly this interval is taken to be [0,1], where 1 represents full membership and 0 represents no membership. If only the two endpoints 0 and 1 are considered, then fuzzy sets become two-valued and are identical to classical sets where everything is either completely in the set (a membership value of 1) or completely not in the set (a membership value of 0). Using the unit interval aids in interpreting the membership function as an extension of classical set theory. Unfortunately, however, it often leads to confusion between fuzzy logic and probability. In this example, every man has a degree of membership to the set of tall men. Some man who is 8 feet tall would have a degree of membership of 1, while someone who is 6 feet tall would have a degree of membership of perhaps .8, and someone who is under 4 feet tall would have a degree of membership of .1 or lower. To represent this idea of an element x having fuzzy membership of .8 to a fuzzy set A, the following notation is used: $A(x) = 0.8$. These numbers are arbitrarily chosen as an example, and depend upon what is chosen as the membership function for the set. The important concept is that an element need not fully belong to a set or to its complement but rather an element can belong, with a certain degree of membership, to both a set *and* its complement. Thus, the law of the excluded middle does not hold for fuzzy sets.

The concept of subsethood can be handled in a way analogous to the way elementhood is handled. Given two fuzzy sets A and B, A is a subset of B if $A(x) \leq B(x)$ for all x in the common domain of A and B. Thus, the fuzzy set of men who are tall and bald would be a subset of the fuzzy set of tall men. The most common type of subset of a fuzzy set is created by simply adding a modifier to the description of a fuzzy set which is called a *hedge*. In the set of tall men, adding the hedge "very" creates the set of very tall men which is obviously a subset of the set of tall men.

Fuzzy logic follows as an immediate consequence of fuzzy sets. For example, in classical logic the statement "A man 6 feet tall belongs to the set of tall men" receives a truth value of either true or false. This would depend upon the arbitrary cutoff point determined by the classical set of tall men. In fuzzy logic, though, the statement "A man 6 feet tall belongs to the set of tall men" has a truth value that belongs to the unit interval. In the example discussed above, this truth value would be .8. This number does not depend upon a rigid cutoff point but instead upon a membership function. Hence, a statement's truth value is dependent solely upon the associated fuzzy set and its membership function.

To continue this idea of a membership function for the set of tall men, suppose that the following graph is the graph of a membership function (Figure 1).

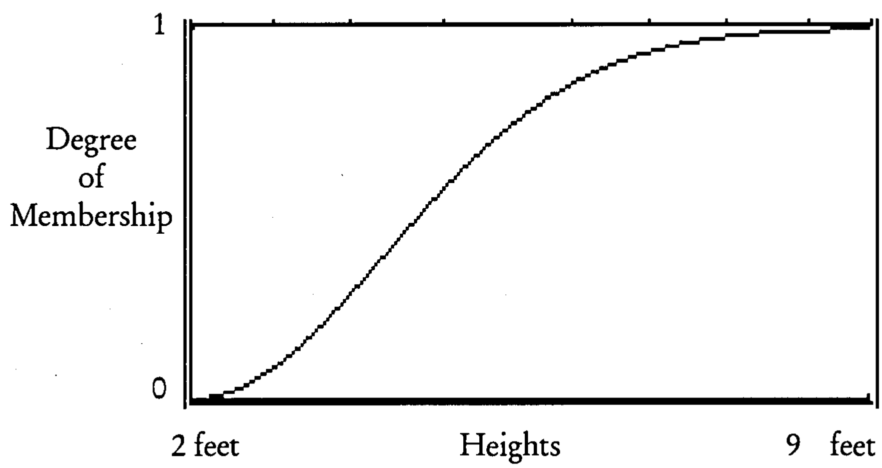


Figure 1: Membership Function for the Set of Tall Men

This graph represents the degree to which each man (based solely upon his height) is a member of the set of tall men. Clearly these same men would also have a degree of membership in the set of not-tall men, which is the focus of the following chapter.

An interesting question arises when considering the domain of a membership function. In the example above our membership function acted on the interval of real numbers from 2 to 9 which represented reasonable adult male heights given in feet. Since fuzzy sets make no distinction between two people with exactly the same height, the domain can be thought of as either an interval of numbers or as a set of people. But what is the universe for this domain? A woman who is seven feet tall would also seem to have a degree of membership in the set of tall men based on the fact that she is definitely tall, but not a man. Hence, just as a short man would have a non-zero degree of membership, she would have a membership less than a man of the same height but a greater membership than a woman who is five feet tall. This gives rise to the question of what the universe is for a fuzzy set. If the universe of the set of tall men is taken to be all living creatures, then it is likely that not only would a woman have a non-zero degree of membership but so would very tall animals such as giraffes. If the universe was taken to be all Homo sapiens, then a woman would have a non-zero degree of membership whereas a giraffe would not. Finally, if the universe was taken to be the set of male Homo sapiens, then a woman would not have a non-zero degree of membership to the set. Thus, the set of objects that constitute the domain of the membership function is the set's universe. If two elements belong to the domain of the same membership function, they are called *comparable elements under the given function* . If one or both do not belong to the domain they are called *incomparable elements under the given function*.

The concept and use of fuzzy sets and fuzzy logic go far beyond this sort of linguistic example, however. Many things in life require a certain amount of vagueness, because they cannot be neatly fit into a bivalent world in which the law of the excluded middle must hold. An excellent example where fuzzy logic helps interpretation of scientific data is in the realm of physics. Physicists are often plagued by interpretations of seemingly contradicting results. For example, when light passes through a double slit it has the properties of a wave. Light also has momentum and properties of a particle. Thus, the question of whether light is a wave or a particle is impossible to answer with classical logic [12]. Fuzzy logic, however, can easily resolve the dilemma of whether light is a particle or a wave by allowing light to belong to both the set of waves and to the set of particles with certain degrees of membership.

Introducing fuzzy sets and fuzzy logic to resolve both practical and theoretical paradoxes leads to interesting questions about how they effect set theory and logic. To begin answering these questions, it is necessary to investigate the logical connectives, namely negation, conjunction, and disjunction. Negation, because it is a unary operator and the easiest to characterize, will be the focus of the following chapter.

CHAPTER 3

NEGATION FUNCTIONS

If the truth value of a statement is known, it seems reasonable to assume that it should be possible to determine a truth value corresponding to the negation of the statement. The truth value of the original statement is somewhat subjective, as it depends on both the context in which the statement is being used and the chosen membership function. Moreover, the truth value of the negation of a statement should be the same for all statements with equal truth values. It is important to note at this time a distinction between two things that at first glance might seem quite similar. The truth value of the negation of a statement of the form "x is an element of A" is actually the degree of membership of the element x to the fuzzy complement of the set A and <u>not</u> the degree of membership of the negation of an element to a set. As a concrete example, take the canonical fuzzy set of tall men. The truth value of the negation of the statement "John is tall" is the degree to which John belongs to the set of not tall men (the fuzzy complement, within a specified universe, of the set of tall men). Its truth value is not the degree to which "not John" belongs to the set of tall men, as "not John" has no specific meaning. Once the membership function of a set has been chosen, however, the related negation membership function of the set should not be subjective as well. Rather, the negation function should be determined by a definite rule based solely upon the membership function of the original set. To determine exactly what this negation rule should be, it is necessary to look at a few conditions that it must satisfy and then consider the functions which satisfy those conditions.

Consider a function N which takes elements from [0,1] to [0,1] and acts as a negation operator. The negation operator should be a function that is independent of any membership function. That is, it should operate on all possible truth values and assign to each possible truth value a real number in the unit interval, which will be a truth value as well. Let "N(x)" be an abbreviation for "N(A(x))." That is, in the earlier example, N(John) is the degree to which John belongs to the fuzzy complement of tall men. N is the negation membership function, and for each value of x its value, N(x), can also be thought of as the truth value of the statement "x is not in A" with respect to a given universe. Since fuzzy logic is to be an extension of classical logic, the boundary conditions of classical logic must be kept. Hence, as a first condition, $N(x) = 1$ if $A(x) = 0$ and $N(x) = 0$ if $A(x) = 1$ where, as is the case in classical logic, 'true' is assigned one and 'false' is assigned zero.

Second, it is reasonable that a change in the truth value of a statement should have an opposite effect on the truth value of the negation of the statement. That is, if the truth value of a statement increases, the truth value of its negation should decrease. To be as general as possible, the demand placed on the negation function is that it will suffice to be non-increasing. To formalize this: if $A(y) \leq A(x)$, then $N(x) \leq N(y)$. Thus, N should be a monotone decreasing function. Note that for it to be meaningful to compare A(x) and A(y), x and y themselves must be comparable, that is, they must belong to the domain of the same membership function.

Third, small changes in the truth value of a statement should correspond to small changes in the truth value of its negation. That is, there should be no jumps or holes in the graph of the negation membership function. Formally, N should be a continuous function.

Fourth, the negation membership function should be involutory. That is, the negation of the negation of a statement should have the same truth value as that of the original statement. That is, $N(N(x)) = A(x)$. Notice that N is being used in two different senses. When the argument is an element of the domain of a fuzzy set, $N(x)$ is taken to mean $N(A(x))$. If the argument is a real number in the unit interval, like $N(A(x))$, then $N(N(x))$ is taken to mean $N(N(A(x))$. This usage will be clear from the context.

In the literature on fuzzy logic it is quite common to simply take $N(x) = 1- A(x)$. While this may be one of the easiest to compute, it is evident that this is not the only function which satisfies the above requirements. In order to determine all the functions which may be candidates for the negation function, it is necessary to convert the above ideas into a set of axioms.

Simply taking each of the above conditions (boundary conditions, monotonicity, continuity, and involutiveness) as axioms may seem appealing. Including all of them, however, is unnecessary as the above conditions are not independent of one another. This is shown in the following proof, adapted from [6, p.52] with notational changes and further explanations.

Theorem 1: If N is an involutory function from [0,1] to [0,1] such that $N(x) \leq N(y)$ whenever $A(y) \leq A(x)$, then N is a continuous function which satisfies the classical boundary conditions. Moreover, N is a bijection.

Proof:

1. N satisfies the boundary conditions.

Since the co-domain of N is $[0,1]$, $N(0) \leq 1$. Furthermore, $N(N(0)) \geq N(1)$ by hypothesis. But, as N is involutory, $N(N(0)) = 0$. Hence, $0 \geq N(1)$. As $N(1) \geq 0$, it follows that $N(1) = 0$. Next, since $N(1) = 0$, applying N to both sides reveals that $N(N(1)) = N(0)$. As $N(N(1)) = 1$, it follows that $N(0) = 1$. Therefore, both boundary conditions are satisfied.

2. N is continuous.

First, observe that N is a bijection. To see surjectiveness, take $a \in [0,1]$. Then, as N is involutory, $N(N(a)) = a$. Hence a has a pre-image, namely $N(a)$, under N. Since a was chosen arbitrarily, N is surjective. For injectiveness, take, $N(a) = N(b)$. Then $N(N(a)) = N(N(b))$ or $a = b$. Since a and b were chosen arbitrarily, N is also injective. To show continuity, suppose that there is a point z at which N is discontinuous. Since N is a monotone function, and using the idea of a greatest lower bound, it can be shown that both the left and right hand limits exist at z (for a full proof of this consult **[15, p.94]**). Since N is discontinuous at z, it must be that either the left or the right hand limit is not equal to $N(z)$. Without loss of generality, assume that the left hand limit does not equal $N(z)$. Then, since N is monotone decreasing,

$y = \lim_{x \to z-} N(x) > N(z)$. Hence, there exists a q (for example, $q = \frac{y + N(z)}{2}$) such that

$y > q > N(z)$. Since N is surjective, there must exist a $p \in [0,1]$ such that $N(p) = q$. As $N(p) > N(z)$, $p < z$. But if $p < z$, then as N is monotone decreasing, $N(p) \geq \lim_{x \to z-} N(x)$.

Hence $q \geq y$, contradicting that $y > q$. Therefore, N cannot have any points of discontinuity, and thus it is a continuous function. ■

As Theorem 1 proves, the four imposed conditions are not independent. It is therefore sufficient to take just the conditions of monotonicity and involutiveness as axioms.

Axioms of Negation

1. N is a monotone decreasing function.
2. N is an involutory function.

Clearly, from Theorem 1, all functions that satisfy Axioms 1 and 2 will necessarily satisfy the boundary conditions and be continuous. There are obviously many functions from [0,1] to [0,1] which satisfy these two axioms. In fact, there are two characterization theorems describing these functions. To help prove the first of these the following lemma will be used. The lemma was also taken from [6, p.57] with notational changes.

Lemma 2: Every negation function N has exactly one fixed point.

Proof:

For α to be a fixed point of N it must satisfy the equation $N(x)-x = 0$. Because of the boundary conditions on N, it is clear that $N(0)-0 = 1$ and $N(1)-1 = -1$. Since N is a continuous function and $-x$ is a continuous function, $N(x)-x$ is a continuous function. Therefore by the intermediate value theorem, it follows that there is a root of $N(x)-x$; call this root α. To see that α is unique, assume that there is another root β. Without loss of generality, assume that that $\alpha < \beta$. Then as N is monotone decreasing, $N(\beta) \leq N(\alpha)$. Combining these results yields that $N(\beta)-\beta < N(\alpha)-\alpha$. This contradicts that $N(\beta)-\beta = 0 = N(\alpha)-\alpha$. Hence α is unique. ■

In the following characterization theorem of fuzzy negation functions, a function g from [0,1] to $\Re$ is needed which satisfies the following criteria: g(0) = 0, g is strictly increasing, and g is continuous. A function which satisfies these criteria is called an *increasing generator*, and will be used in characterizing both conjunction and disjunction functions in subsequent chapters. The following theorem, modified from **[6, p.484]**, has a shorter proof of the direct implication as well as some notational changes.

Theorem 3: N is a fuzzy negation function if and only if there exists a strictly increasing, continuous function g from [0,1] to $\Re$ such that g(0) = 0 and $N(a) = g^{-1}(g(1)-g(a))$ for all $a \in [0,1]$.

Proof:

1. Direct implication

Let N be a fuzzy negation function. It then follows from Lemma 2 that N has a unique fixed point p, where $p < 1$. Let h be any continuous strictly increasing surjective function from [0,p] to [0,c] where c is any fixed positive real number. The function $g: [0,1] \rightarrow \Re$ can then be defined in terms of h as follows: g(a) = h(a) for $a \in [0,p)$; g(a) = 2c-h(N(a)) for $a \in [p,1]$.

The function g is clearly continuous at all points other than p as the functions h and N are both continuous. To see that g is continuous at p, note that in the first interval, $\lim_{x \to p^-} g(x) = \lim_{x \to p^-} h(x) = h(p)$ as h is continuous. Hence, $\lim_{x \to p^-} g(x) = h(p) = c$. In the second interval, g(p) = 2c-h(N(p)) = 2c-h(p) = 2c-c = c. Since h and N are themselves continuous, $\lim_{x \to p^+} g(x) = c$ as well. Hence, g is continuous at p and so it is continuous at all points of [0,1]. Next, g(0) = h(0) = 0. Third, g inherits the property of being strictly increasing from the fact that h is strictly increasing, N is injective and N is therefore strictly decreasing.

Finally, it remains to be shown that $N(a) = g^{-1}(g(1)-g(a))$ for all $a \in [0,1]$. First, from the definition of g it follows that g maps elements from [0,1] to [0,2c]. Since g is strictly increasing, g^{-1} exists and $g^{-1}: [0,2c] \rightarrow [0,1]$. Note that on [0,c), g^{-1} is defined by $g^{-1}(x) = h^{-1}(x)$. Further, since N is involutory, g^{-1} on [c,2c] is defined by $g^{-1}(x) = N(h^{-1}(2c-x))$. Take $a \in [0,1]$. If $a \in [0,p)$, then $2c-h(a) \in (c, 2c]$, as $h(a) < h(p) = c$. Then, by definition of g, $g^{-1}(g(1)-g(a)) = g^{-1}(2c-h(N(1))-h(a)) = g^{-1}(2c-h(a)) = N(h^{-1}[2c-(2c-h(a))]) = N(h^{-1}(h(a))) = N(a)$. Hence, $N(a) = g^{-1}(g(1)-g(a))$ for all $a \in [0,p)$.

If $a \in [p, 1]$, then $g^{-1}(g(1)-g(a)) = g^{-1}[2c-(2c-h(N(a)))] = g^{-1}[h(N(a))]$. Since $N(p) = p$ and $N(1) = 0$, $h(N(a)) \in [0,c]$. Hence, by the definition of g^{-1}, $g^{-1}[h(N(a))] = h^{-1}[h(N(a))] = N(a)$.

Thus, $N(a) = g^{-1}(g(1)-g(a))$ for all $a \in [p,1]$. Hence, for every fuzzy negation function N there exists an increasing generator g such that $N(a) = g^{-1}(g(1)-g(a))$ for all $a \in [0,1]$.

2. Inverse implication

Assume that N is described as in the hypothesis. Then, since g is strictly increasing, it is easy to see that $N: [0,1] \rightarrow [0,1]$. Next, for any $a, b \in [0,1]$, if $a < b$ then $g(a) < g(b)$ since g is assumed to be strictly increasing. Therefore, $g(1) - g(a) > g(1)-g(b)$. Since g is a strictly increasing function on [0,1], g^{-1} exists on $[0, g(1)]$ and is also strictly increasing.

Hence, $g^{-1}(g(1)-g(a)) > g^{-1}(g(1)-g(b))$. But $g^{-1}(g(1)-g(a)) = N(a)$ and $g^{-1}(g(1)-g(b)) = N(b)$, so that $N(a) > N(b)$. Hence, N satisfies one of the two conditions necessary to be a fuzzy negation function.

Finally, to see that N is involutory, take $a \in [0,1]$. Then, $N(N(a)) = g^{-1}[g(1)-g(N(a))] = g^{-1}\{g(1)-g[g^{-1}(g(1)-g(a))]\} = g^{-1}(g(a)) = a$. Hence, N is involutory. Therefore, N is a fuzzy negation function. ■

In an analogous fashion, as Theorem 4 shows, it is possible to determine a *decreasing generator* for a fuzzy negation membership function. A decreasing generator is a function f from [0,1] to $\Re$ such that $f(1)=0$, f is strictly decreasing, and f is continuous. Decreasing generators will also be used in subsequent chapters. The proof of the following theorem is also taken from **[6, p.61]** with notational changes.

Theorem 4: N is a fuzzy negation function if and only if there exists a strictly decreasing, continuous function f from [0,1] to $\Re$ such that $f(1)=0$ and $N(a) = f^{-1}(f(0)-f(a))$ for all $a \in [0,1]$.

Proof:

1. Direct implication

According to Theorem 3 a function N is a fuzzy negation function if and only if there exists an increasing generator g such that $N(a) = g^{-1}(g(1)-g(a))$ for all $a \in [0,1]$. Let $f: [0,1] \rightarrow \Re$ be given by: $f(x) = g(1)-g(x)$. Then clearly $f(1) = 0$ and, since g is strictly increasing, f must be strictly decreasing. Further, f is also clearly continuous, making f a decreasing generator. Since g(1) is a constant, it is easy to see that $f^{-1}(a) =$

$g^{-1}(g(1)\text{-}a) = g^{-1}(f(0)\text{-}a)$. Therefore, $N(a) = g^{-1}(g(1)\text{-}g(a)) = f^{-1}(g(a)) =$ $f^{-1}(g(1)\text{-}(g(1)\text{-}g(a))) = f^{-1}(f(0)\text{-}f(a))$, as $f(0) = g(1)$.

2. Inverse implication

Given a decreasing generator f described as above, define a function g: $[0,1] \rightarrow \Re$ as follows: $g(x) = f(0)\text{-}f(x)$. Since f is strictly decreasing and continuous, it is easily seen from the definition of g that $g(0) = 0$, g is strictly increasing, and g is continuous. Hence, g is an increasing generator. Now, $g(1) = f(0)$ and $g^{-1}(a) = f^{-1}((f(0)\text{-}a)$. Hence, $N(a) = f^{-1}(f(0)\text{-}f(a)) = g^{-1}(f(a)) = g^{-1}(f(0)\text{-}g(a)) = g^{-1}(g(1)\text{-}g(a))$. By Theorem 3, $g^{-1}(g(1)\text{-}g(a))$ is a fuzzy negation function, and therefore $N(a) = f^{-1}(f(0)\text{-}f(a))$ is also a fuzzy negation function. ■

The question of what are some examples of fuzzy negation functions now arises. The standard negation function originally used by Zadeh is $N(a) = 1\text{-}a$ [18]. Several authors, such as Yager, Sugeno, Schweizer, and Sklar, have found more general classes of which the standard fuzzy negation is a member. A common example of a general class of fuzzy negation functions (that is, a class of functions satisfying the two negation axioms) is the class of functions $N(a) = \left(\frac{1-a^{\omega}}{1+\lambda a^{\omega}}\right)^{1/\omega}$ where ω and λ are real numbers greater than 0 and -1 respectively. Note that if $\omega = 1$ and $\lambda = 0$, then $\left(\frac{1-a^{\omega}}{1+\lambda a^{\omega}}\right)^{1/\omega}$ reduces to the standard negation function 1-a.

These functions can be shown by routine but careful computation to be both involutory and monotone decreasing, and hence are in fact fuzzy negation functions. Like all negation functions, they will also have both an increasing and a decreasing generator. An increasing generator g(a) $= \frac{1}{\lambda}\ln(1+\lambda a^{\omega})$ was given in [6, p.60], but the demonstration was not shown. From the definition of g it is clear that it is an increasing generator. To see that g also in fact does generate the above fuzzy negation function, consider g^{-1}(g(1)-g(a)), as suggested by Theorem 3. To find g^{-1}(a), let

$y = \frac{1}{\lambda}\ln(1+\lambda a^{\omega})$. Then, $\lambda y = \ln(1+\lambda a^{\omega})$, so that $e^{\lambda y} = 1+\lambda a^{\omega}$. Hence,

$a = \left(\frac{e^{\lambda y}-1}{\lambda}\right)^{1/\omega}$. After interchanging the variables y and a, it follows that

$g^{-1}(a) = [\frac{1}{\lambda}(e^{\lambda a}-1)]^{1/\omega}$. Thus, $g^{-1}(g(1)\text{-}g(a)) = g^{-1}\ (\frac{1}{\lambda}\ln(1+\lambda) - \frac{1}{\lambda}\ln(1+\lambda a^{\omega}))$

$$= \left[\frac{1}{\lambda}(e^{\ln(1+\lambda) - \ln(1+\lambda a^{\omega})} - 1)\right]^{1/\omega} = \left(\frac{1-a^{\omega}}{1+\lambda a^{\omega}}\right)^{1/\omega} = N(a).$$

The decreasing generator for this negation function can be found, as the proof of Theorem 4 suggests, by letting f(a) = g(1)-g(a). This yields

$f(a) = \frac{1}{\lambda}\ln(1+\lambda) - \frac{1}{\lambda}\ln(1+\lambda a^{\omega}) = \frac{1}{\lambda}\ln\left(\frac{1+\lambda}{1+\lambda a^{\omega}}\right)$. With a little computation, it can be seen that f^{-1} is given by: $f^{-1}(a) = \left[\frac{1}{\lambda}\left(\frac{1+\lambda}{e^{\lambda a}} - 1\right)\right]^{1/\omega}$. To see that f also generates $\left(\frac{1-a^{\omega}}{1+\lambda a^{\omega}}\right)^{1/\omega}$, consider $f^{-1}(f(0)\text{-}f(a))$:

$$f^{-1}(f(0)\text{-}f(a)) = f^{-1}\left(\frac{1}{\lambda}\ln(1+\lambda) - \frac{1}{\lambda}\ln\left(\frac{1+\lambda}{1+\lambda a^{\omega}}\right)\right)$$

$$= f^{-1}\left(\frac{1}{\lambda}\ln\left(\frac{1+\lambda}{\left(\frac{1+\lambda}{1+\lambda a^{\omega}}\right)}\right)\right)$$

$$= f^{-1}\left(\frac{1}{\lambda}\ln(1+\lambda a^{\omega})\right)$$

$$= \left[\frac{1}{\lambda}\left(\frac{1+\lambda}{e^{\lambda(\frac{1}{\lambda}\ln(1+\lambda a^{\omega}))}} - 1\right)\right]^{1/\omega}$$

$$= \left[\frac{1}{\lambda}\left(\frac{1+\lambda}{1+\lambda a^{\omega}} - 1\right)\right]^{1/\omega}$$

$$= \left[\frac{1}{\lambda}\left(\frac{1+\lambda}{1+\lambda a^{\omega}} - \frac{1+\lambda a^{\omega}}{1+\lambda a^{\omega}}\right)\right]^{1/\omega}$$

$$= \left[\frac{1}{\lambda}\left(\frac{\lambda - \lambda a^{\omega}}{1+\lambda a^{\omega}}\right)\right]^{1/\omega}$$

$$= \left(\frac{1-a^{\omega}}{1+\lambda a^{\omega}}\right)^{1/\omega} = N(a).$$

Hence it has been shown that the function $N(a) = \left(\frac{1-a^{\omega}}{1+\lambda a^{\omega}}\right)^{1/\omega}$ has both an increasing and a decreasing generator.

The graph below (Figure 2) shows (1) this fuzzy negation function when $\lambda = \omega = 1$, (2) the traditional negation function $N(a) = 1\text{-}a$ (which is the general negation function when $\lambda = 0$ and $\omega = 1$) and (3) this fuzzy negation function when $\lambda = 10$ and $\omega = 4$.

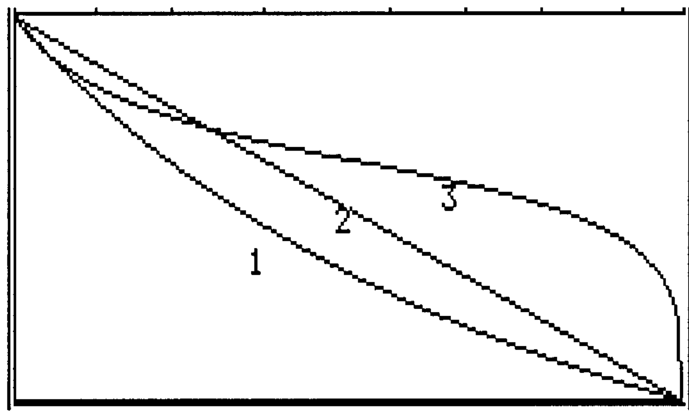


Figure 2: Negation Functions

A list of examples of negation functions is given in Appendix A. This large number of negation functions gives rise to the question of which function may best serve as the negation membership function. The answer depends not only upon the context of the question but also upon what the other fuzzy operators are.

After finding generators for the negation function, it is natural to ask whether similar results can be obtained for the other logical operators. The focus of the following chapter is to determine whether fuzzy conjunction functions can be characterized in terms of increasing and decreasing generators as well.

CHAPTER 4

CONJUNCTION FUNCTIONS

In the last chapter a class of negation functions was defined from the co-domain of a membership function to an interval of the real number line. That is, they were maps from an interval onto an interval. The focus of this chapter is to construct a function from a pair of intervals to a single interval in a way that can be interpreted as fuzzy conjunction functions. Again, for simplicity of interpretation, the intervals will always be taken to be the unit interval [0,1]. However, any interval on the real number line would work with the appropriate boundary adjustments.

The first consideration that the fuzzy conjunction function, C, must satisfy is a specification of the type of elements in the domain of C. The function C is to be a means of measuring how much a particular element belongs to a combination of two fuzzy sets. For example, if the statement "John is tall" has a truth value of .8 and the statement "John is thin" has a truth value of .7, then C measures the extent to which John is tall *and* thin. Thus, C is a function from the Cartesian product of two membership functions' co-domains to the unit interval, that is from $[0,1] \times [0,1]$ to $[0,1]$. The notation "$C(a, b)$" is taken to mean "$C(A(x), B(x))$" where $A(x) = a$ and $B(x) = b$ are the degrees to which x belongs to the fuzzy sets A and B, respectively. Thus, x must be an element of both A and B for it to be in the domain of C.

Since C is to be a generalization of the classical conjunction, it must also satisfy the classical boundary conditions. It must satisfy the minimal conditions of $C(0, 1) = 0$,

$C(1, 0) = 0$, $C(0, 0) = 0$ and $C(1, 1) = 1$. However, the classical conditions imply more than just these four relations. Classically, conjoining a statement with a true statement has no effect on the truth value of the statement. Formally, $C(a, 1) = C(1, a) = a$ for all $a \in [0,1]$. Similarly, conjoining any statement with a false statement is false. That is, for all $a \in [0,1]$, $C(a, 0) = C(0, a) = 0$.

Third, to comply with the classical conjunction, C must be symmetric (or commutative). That is, $C(a, b) = C(b, a)$ for all $a, b \in [0,1]$. The final property inherited from the classical conjunction is associativity. That is, $C(a, C(b, c)) = C(C(a, b), c)$ for all $a, b, c \in [0,1]$.

Additionally, if the truth value of either one of the arguments is increased, the truth value of the conjunction should not decrease. That is, the function C should be monotone non-decreasing in both arguments. Formally, $a_1 \leq a_2$ and $b_1 \leq b_2$ imply $C(a_1, b_1) \leq C(a_2, b_2)$.

Finally, small changes in the arguments of the conjunction function should result in small changes in the functional values. Thus, the function C should be continuous. Each of these conditions will serve as an axiom of conjunction, which are summarized below.

Axioms of Conjunction:

1. Boundary Conditions: $C(a, 1) = a = C(1, a)$ and $C(0, a) = 0 = C(a, 0)$.
2. Commutativity: $C(a, b) = C(b, a)$.
3. Associativity: $C(a, C(b, c)) = C(C(a, b), c)$.
4. Monotone Non-Decreasing: $a_1 \leq a_2$ and $b_1 \leq b_2$ imply $C(a_1, b_1) \leq C(a_2, b_2)$.
5. Continuity.

Functions satisfying conditions 1 through 4 are called *triangular norm functions* or simply *t-norms*. These functions were originally studied independently of fuzzy sets. Schweizer and Sklar [10] and [11], as well as Ling [7], for example, worked with characterizing t-norms in various ways. A *strict t-norm* is a t-norm that is continuous and satisfies strict monotonicity. Strict monotonicity is defined as follows: $a_1 < a_2$ and $b_1 < b_2$ imply $C(a_1, b_1) < C(a_2, b_2)$. Thus, every fuzzy conjunction is a t-norm and every strict t-norm is a fuzzy conjunction function.

Several authors, including Yager, Ling, Schweizer, and Sklar, have been trying to create general classes of functions which satisfy the conjunction axioms. The idea is that in various applications one should have a variety of conjunction functions to choose from and then use the one that best fits the data. A general class of continuous t-norms is Yager's class defined as follows: $C_p(a, b) = 1\text{-min}(1, [(1\text{-}a)^p + (1\text{-}b)^p]^{1/p})$ for $p \geq 1$ [16] and [17]. To be a conjunction function, $C_p(a, b)$ must satisfy the five axioms of conjunction. This is shown in the following theorem.

Theorem 5: Any function in the Yager class, defined by
$C_p(a, b) = 1\text{-min}(1, [(1\text{-}a)^p + (1\text{-}b)^p]^{1/p})$ for $p \geq 1$, satisfies the axioms for fuzzy conjunction functions.

Proof:

1. Boundary conditions

If $b = 1$, then $C_p(a, 1) = 1\text{-min}(1, [(1\text{-}a)^p + (1\text{-}1)^p]^{1/p}) = 1\text{-min}(1, (1\text{-}a)) = 1\text{-}(1\text{-}a) = a$.

Also, $C_p(1, a) = a$. Similarly, $C_p(0, a) = 1\text{-min}(1, [(1\text{-}0)^p + (1\text{-}a)^p]^{1/p}) =$

$1\text{-min}(1, [1 + (1\text{-}a)^p]^{1/p}) = 1\text{-}1 = 0$. Finally, $C_p(a, 0) = 0$.

2. Commutativity

The symmetry in the definition of $C_p(a, b)$ clearly shows that C_p is commutative.

3. Associativity.

Assume, for this case, that $[(1-b)^p + (1-c)^p]^{1/p} \leq 1$ and $[(1-a)^p + (1-b)^p]^{1/p} \leq 1$.

$$
\begin{aligned}
\text{Then } C_p(a, C_p(b, c)) &= 1\text{-min}(1, [(1-a)^p + (1-C_p(b, c))^p]^{1/p}) \\
&= 1\text{-min}(1, [(1-a)^p + (1-(1\text{-min}(1, [(1-b)^p + (1-c)^p]^{1/p})))^p]^{1/p}) \\
&= 1\text{-min}(1, [(1-a)^p + [(1-b)^p + (1-c)^p]]^{1/p}) \\
&= 1\text{-min}(1, [(1-a)^p + (1-b)^p + (1-c)^p]^{1/p}) \\
&= 1\text{-min}(1, [([(1-a)^p + (1-b)^p]^{1/p})^p + (1-c)^p]^{1/p}) \\
&= 1\text{-min}(1, [(1-(1-[(1-a)^p + (1-b)^p]^{1/p})^p + (1-c)^p]^{1/p}) \\
&= 1\text{-min}(1, [(1-(1\text{-min}(1,[(1-a)^p + (1-b)^p]^{1/p}))^p + (1-c)^p]^{1/p}) \\
&= 1\text{-min}(1, [(1-C_p(a, b))^p + (1-c)^p]^{1/p}) \\
&= C_p(C_p(a, b), c).
\end{aligned}
$$

The other three cases follow in an analogous manner.

4. Monotone Non-Decreasing

Choose a and b such that $a \leq b$. Then $1-a \geq 1-b$ and $(1-a)^p \geq (1-b)^p$. Thus,

$[(1-a)^p + (1-c)^p]^{1/p} \geq [(1-b)^p + (1-c)^p]^{1/p}$. From this it follows that

$1\text{-min}(1, [(1-a)^p + (1-c)^p]^{1/p}) \leq 1\text{-min}(1, [(1-b)^p + (1-c)^p]^{1/p})$. Hence,

$C_p(a, c) \leq C_p(b, c)$. Since C_p is commutative, it follows that it is monotone non-decreasing in each argument. Finally, if $a_1 \leq a_2$ and $b_1 \leq b_2$, then

$C_p(a_1, b_1) \leq C_p(a_1, b_2) \leq C_p(a_2, b_2)$.

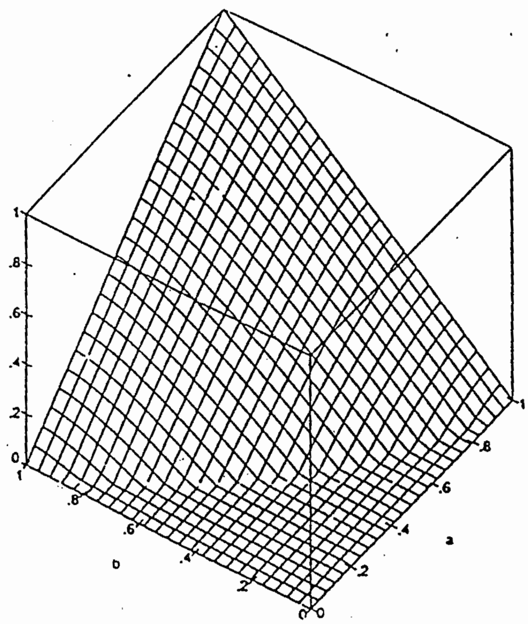
5. Continuity.

Continuity of C_p follows from the fact that each of the functions in the definition of C is itself continuous. ■

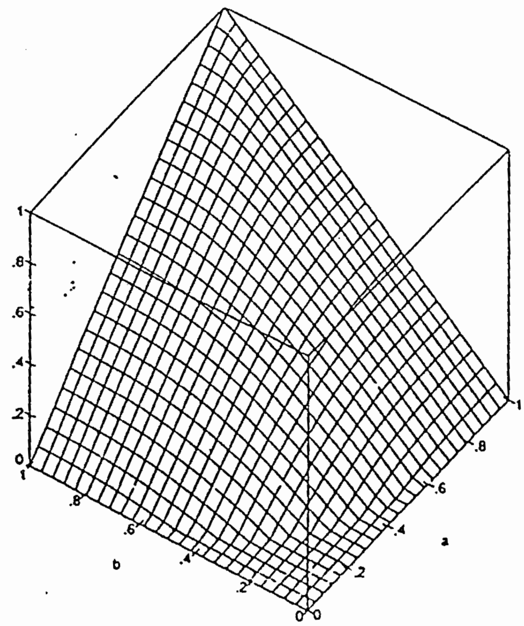
To understand what these functions represent, it is important to view them graphically. Since C_p is a function from [0,1] x [0,1] to [0,1], the graph of C_p is a three dimensional surface. The truth values of the two arguments are given along the x and y axes and the truth value of their conjunction is given along the z axis. Figure 3 shows the graphs of four different Yager conjunction functions. As the value of p increases the membership value of the conjunction increases (given along the z-axis). This can be seen by noting that the first derivative of the functions in the Yager class with respect to p are all positive.

A natural question to ask when considering these graphs is what is the interpretation of the value p. In his article "On a General Class of Fuzzy Connectives," Ronald Yager [16, p.241] suggests an answer to this question as follows:

> Noting first that by the logical statement 'S_1 and S_2' we are requiring or demanding the simultaneous satisfaction of both conditions S_1 and S_2. It is a common phenomenon in spoken language to strongly emphasis the 'and' when we are demanding strong satisfaction to these two conditions. Thus, by the 'strength' of an 'and' we shall mean how strongly we are demanding this simultaneous satisfaction. We suggest that in the 'and' operator we can consider the parameter p as inversely related to the strength of the 'and'.



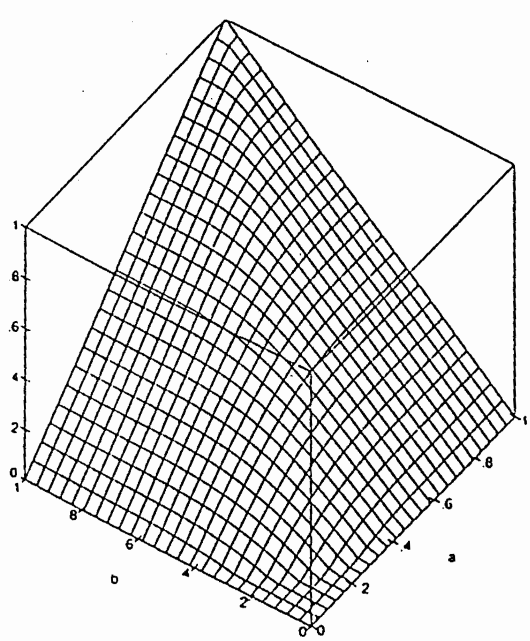
p = 1.5



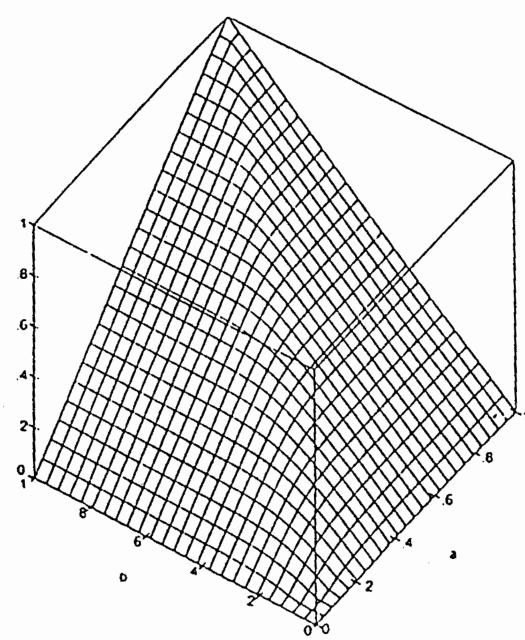
p = 3

Figure 3: Graphs of Functions in Yager's Class of Conjunctions.

Taken from [6].



p = 5



p = 10

If $p = \infty$, then the function C_p reduces simply to the min function, which is the least "demanding" or weakest conjunction function (for a proof of this, see Appendix D). To understand what Yager means by strength, consider how close the truth values of a and b have to be to 1 before the truth value of the conjunction is nearly 1. For example, if $a = b$ and $C_p(a, b) \geq 0.9$, what is the minimum value of a when $p = 1$? A simple calculation reveals that $a \geq .95$. With the same conditions and $p = 10$, $a \geq .9067$, a lower (weaker) condition. Hence, the *strength* of a Yager conjunction function C_p can be defined as follows: $s(C_p) = 1/p$ where $s(C_p) = 0$ when $p = \infty$. Of course, the Yager class is just one example of a class of conjunction functions. Appendix B lists several other examples of conjunction functions.

If an additional axiom is added to the list of conjunction axioms to make them strict t-norms, namely the axiom of strict monotonicity, then it is possible to find generators for conjunction functions in a fashion similar to what was done for fuzzy negation functions. In other words, as with negation functions, there will be two pseudo-characterization theorems, one involving decreasing generators and one involving increasing generators, for a fuzzy conjunction function. The statement and proof of the pseudo-characterization theorem for decreasing generators is easier and is therefore presented first. As noted in the previous chapter, a decreasing generator is a function f from [0,1] to $\Re$ such that $f(1) = 0$, f is strictly decreasing, and f is continuous. In the proof of Theorem 6 and several later proofs the idea of the pseudoinverse, $f^{(-1)}$, of a function f is useful and is defined as follows:

$$f^{(-1)}(a) = \begin{cases} 1 & \text{for } a \in (-\infty, 0) \\ f^{-1}(a) & \text{for } a \in [0, f(0)] \\ 0 & \text{for } a \in (f(0), \infty) \end{cases} \quad \text{where } f^{-1} \text{ is the ordinary inverse of } f.$$

Note that since f is continuous, $f^{(-1)}$ will also be continuous. In all subsequent discussions involving the pseudoinverse $f^{(-1)}$ of a function f, the reader should check that the correct part of the definition of $f^{(-1)}$ is used. Returning now to the issue of generators, the proof of the following pseudo-characterization theorem is unique to this paper although it uses ideas presented in [1], [10], and [11]. While Schweizer and Sklar [10] consider a binary operator on [0,1] x [0,1], their proof assumes the existence of a decreasing generator f with domain [0,1] which may allow $f(0)$ to be infinite. Since generators are to be continuous functions, this is clearly contradictory. Therefore, in Theorem 6, the domain of the fuzzy conjunction function C will be (0,1] x (0,1]. The reader will see the importance of this restriction by noting that the definition of f involves the ln function which is not defined at zero. A function, such as f in the following theorem, that serves as an decreasing generator but does not have the closed unit interval as its domain, is called an *open-ended decreasing generator.*

Theorem 6: 1. If a binary operation C on (0,1] x (0,1] to [0,1] is a conjunction function with strict monotonicity then there exists an open-ended decreasing generator $f:(0,1] \to \Re$ such that $C(a, b) = f^{(-1)}(f(a)+f(b))$. 2. If $f: [0,1] \to \Re$ is a decreasing generator, and if $C: [0,1] \times [0,1] \to [0,1]$ is given by $C(a, b) = f^{(-1)}(f(a)+f(b))$, then C is a fuzzy conjunction function.

Proof:

1. In [10, p.71] the following theorem is proven: Given a strict fuzzy conjunction function C, there exists a continuous, strictly increasing function h from [0,1] onto [0,1]

(so that $h(0) = 0$ and $h(1) = 1$) such that $C(a, b) = h^{(-1)}(h(a) \bullet h(b))$ for all $a, b \in [0,1]$, where $h^{(-1)}$ is, as before, the pseudoinverse of h. Now, remembering that $C: (0,1] \times (0,1] \rightarrow [0,1]$, define a function $f:(0,1] \rightarrow \Re$ by $f(a) = -\ln h(a)$. From this and the definition of pseudoinverse it can easily be shown that $f^{(-1)}(a) = h^{(-1)}(e^{-a})$.

Clearly, $f(1) = 0$. Next, f inherits the property of being continuous from the fact that both h and ln are continuous functions. Finally, f is strictly decreasing as both ln and h are increasing, so that -ln h must be strictly decreasing. Thus, f is an open-ended decreasing generator. Further, it is then easy to verify that $h(a) = e^{-f(a)}$ and $h^{(-1)}(a) = f^{(-1)}(-\ln a)$. Thus, $C(a, b) = h^{(-1)}(h(a) \bullet h(b)) = f^{(-1)}(-\ln(e^{-f(a)} \bullet e^{-f(b)}) = f^{(-1)}(-\ln(e^{-f(a)-f(b)})) = f^{(-1)}(f(a) + f(b))$. Therefore, every strictly increasing fuzzy conjunction function has an open-ended decreasing generator.

2. Suppose f exists as described above. To see that $C(a, b) \in [0,1]$ it is helpful to split the interval $[0, 2f(0)]$ into the two subintervals $[0, f(0)]$ and $(f(0), 2f(0)]$, and then use the second and third parts of the definition of the pseudoinverse. To show that $C(a, b) = f^{(-1)}(f(a) + f(b))$ is now a conjunction function the five conjunction axioms must be shown to be satisfied.

A. Boundary Conditions

$C(a, 1) = f^{(-1)}(f(a) + f(1)) = f^{(-1)}(f(a) + 0) = f^{(-1)}(f(a)) = f^{-1}(f(a)) = a$ as $f(a) \in [0, f(0)]$.

If $a < 1$, then $C(0, a) = f^{(-1)}(f(0) + f(a)) = 0$ as $f(0) + f(a) \in (f(0), \infty)$. If $a = 1$, then $C(0, a) = 0$ from the above boundary condition.

B. Commutativity

$C(a, b) = f^{(-1)}(f(a) + f(b)) = f^{(-1)}(f(b) + f(a)) = C(b, a)$.

C. Associativity

$C(a, C(b, c)) = f^{(-1)}(f(a) + f(C(b, c))) = f^{(-1)}(f(a) + f(f^{(-1)}(f(b) + f(c)))) =$

$f^{(-1)}(f(a) + f(b) + f(c)) = f^{(-1)}(f(f^{(-1)}(f(a) + f(b))) + f(c)) = f^{(-1)}(f(C(a, b)) + f(c)) =$

$C(C(a, b), c)$.

D. Monotone Non-Decreasing

Assume $0 \leq a_1 \leq a_2$ and $0 \leq b_1 \leq b_2$. Then $f(a_1) \geq f(a_2)$ and $f(b_1) \geq f(b_2)$ since f is monotone decreasing. Hence $f(a_1) + f(b_1) \geq f(a_2) + f(b_2)$. However, since f is monotone decreasing, the fact that $f^{(-1)}$ is also decreasing follows from the definition of pseudoinverse. Therefore, $f^{(-1)}(f(a_1) + f(b_1)) \leq f^{(-1)}(f(a_2) + f(b_2))$. Thus, by the definition of C, $C(a_1, b_1) \leq C(a_2, b_2)$.

E. Continuity

The continuity of C is evident from the continuity of both f and $f^{(-1)}$.

Hence, $C(a, b) = f^{(-1)}(f(a) + f(b))$ satisfies the axioms necessary to become a conjunction function. ■

The following pseudo-characterization theorem, whose proof is original to this paper, shows that every fuzzy conjunction function also has an increasing generator. Since Part 1 uses Theorem 6 the domain of C will again be (0,1] x (0,1], and since the increasing function g is defined in terms of the open-ended decreasing function f the domain of g will be (0,1]. An increasing function, such as g in the following theorem, which satisfies the properties of being an increasing generator except that the domain is not the closed unit interval, is defined to be an *open-ended increasing generator.*

Theorem 7: 1. If a binary operation C on $(0,1] \times (0,1]$ to $[0,1]$ is a fuzzy conjunction function with strict monotonicity then there exists an open-ended increasing generator $g: (0,1] \rightarrow \Re$ such that $C(a, b) = g^{(-1)}(g(a) + g(b) - g(1))$ for all $a,b \in (0,1]$. 2. If $g: [0,1] \rightarrow \Re$ is an increasing generator and if $C: [0,1] \times [0,1] \rightarrow [0,1]$ is given by $C(a, b) = g^{(-1)}(g(a) + g(b) - g(1))$ then C is a fuzzy conjunction function.

Proof:

1. Given a strict fuzzy conjunction function C [10] guarantees that there exists a continuous, strictly increasing function h from $[0,1]$ onto $[0,1]$ (so that $h(0) = 0$ and $h(1) = 1$) such that $C(a, b) = h^{(-1)}(h(a) \bullet h(b))$ for all $a, b \in [0,1]$ where $h^{(-1)}$ is, as before, the pseudoinverse of h. Define a function $g:(0,1] \rightarrow \Re$ by $g(a) = \ln(h(a))$. To see that g is in fact an open-ended increasing generator, note that g is strictly increasing as both ln and h are strictly increasing, and g is continuous as ln and h are continuous. The pseudoinverse of g, $g^{(-1)}(a) = h^{(-1)}(e^{a})$ can be found from the definition of g. Also note that $h(a) = e^{g(a)}$ and $h^{(-1)}(a) = g^{(-1)}(\ln a)$. Thus, $C(a, b) = h^{(-1)}(h(a) \bullet h(b)) = h^{(-1)}(e^{g(a)} \bullet e^{g(b)}) = h^{(-1)}(e^{g(a)+g(b)}) = g^{(-1)}(\ln(e^{g(a)+g(b)})) = g^{(-1)}(g(a)+g(b))$. Since $g(1) = 0$, $C(a, b) = g^{(-1)}(g(a)+g(b)) = g^{(-1)}(g(a)+g(b)-g(1))$. Therefore, every strictly increasing fuzzy conjunction function has an open-ended increasing generator.

2. A straightforward check of the five axioms, similar to the proof of Part 2 in Theorem 6, shows that if $C(a, b) = g^{(-1)}(g(a)+g(b)-g(1))$, then $C: [0,1] \times [0,1] \rightarrow [0,1]$ and is a fuzzy conjunction function. ■

There are, however, some conjunction functions which are not strict t-norms. Consequently, it is not possible to find generators for all fuzzy conjunction functions. Consider the min function. It clearly satisfies all the axioms of conjunction, but it is not strictly increasing. Therefore, it is not a strict t-norm. The first part of the following theorem, originally given in [7, p.197] and with notational changes here, shows that the min function has no decreasing generator h where $\min(a, b) = h^{(-1)}(h(a)+h(b))$. The second part, which is original to this paper, shows that the min function cannot have an increasing generator h where $\min(a, b) = h^{(-1)}(h(a)+h(b)-h(1))$.

Theorem 8: The min function has no decreasing or increasing generator. That is, there does not exist any continuous monotone function $h:[0,1] \rightarrow \Re$ such that $\min(a, b) = h^{(-1)}(h(a)+h(b))$ or $\min(a, b) = h^{(-1)}(h(a)+h(b)-h(1))$.

Proof :

Suppose that a function h exists such that $\min(a, b) = h^{(-1)}(h(a)+h(b))$. Since $\min(a, 0) = 0 = h^{(-1)}(h(a) + h(0))$ for all $a \in [0,1]$, $h(0) = h(a)+h(0)$, or $h(a) = 0$ for all $a \in [0,1]$. Thus h must be the constant function 0. However, $\min(1, 1) = 1$, while $h^{(-1)}(h(1) + h(1)) = h^{(-1)}(0+ 0) = 0$. Since $1 \neq 0$,this is a contradiction. Therefore the min function cannot have a generator h where $\min(a, b) = h^{(-1)}(h(a) + h(b))$.

Next, suppose that a function h exists such that $\min(a, b) = h^{(-1)}(h(a)+h(b)-h(1))$. Since $\min(a, 0) = 0 = h^{(-1)}(h(a) + h(0)-h(1))$ for all $a \in [0,1]$, $h(0) = h(a)+h(0)-h(1)$, or $h(a) = h(1)$ for all $a \in [0,1]$. However, if $a<b$, then $\min(a, b) = h^{(-1)}(h(a)+h(b)-h(1)) = h^{(-1)}(h(1)+h(b)-h(1))$ as $h(a) = h(1)$. Thus, $\min(a, b) = h^{(-1)}(h(b)) = b$, a contradiction as $a<b$. Therefore the min function cannot have a generator h where $\min(a, b) = h^{(-1)}(h(a)+h(b)-h(1))$. ■

This theorem shows us that it is impossible to have a general result about all conjunction functions identical to Theorem 3, in which every negation function was found to have both a decreasing and an increasing generator. The other reasonable restriction to place on conjunction functions is in how they operate with other logical connectives. To this end, the next chapter investigates the disjunction function. Chapter 6 will then study the interactions between fuzzy negation, conjunction, and disjunction functions.

CHAPTER 5

DISJUNCTION FUNCTIONS

In Chapter 4 t-norms were studied as a general class of conjunction functions. In this chapter an analog for the disjunction function ("or") will be discussed in a similar manner.

The first condition that the fuzzy disjunction function, D, must satisfy is a specification of the type of elements in the domain of D. The function D is to be a means of measuring how much a particular element belongs to a union of two fuzzy sets. For example, if the statement "John is tall" has a truth value of .8 and the statement "John is thin" has a truth value of .7, then D measures the extent to which John is tall *or* thin. Hence, D is a function from the Cartesian product of the co-domains of two membership functions to the unit interval, that is from $[0,1] \times [0,1]$ to $[0,1]$. "D(a, b)" is taken to mean "D(A(x), B(x))" where $A(x) = a$ and $B(x) = b$ are the degrees to which x belongs to fuzzy sets A and B respectively. Thus, x must be in either the domain of A or in the domain of B for it to be in the domain of D(a, b).

Since D is to be a generalization of the classical conjunction, it must also satisfy the classical boundary conditions. It must satisfy the minimal conditions of $D(0, 1) = D(1, 0) = D(1, 1) = 1$ and $D(0, 0) = 0$. As in Chapter 4, the classical conditions imply more than just these four relations. Classically, combining a statement and a false statement with a disjunction has no effect on the truth value of a statement. Formally, $D(a, 0) = D(0, a) = a$ for all $a \in [0,1]$. Similarly, combining a statement with a true statement is necessarily true. That is, for all $a \in [0,1]$, $D(a, 1) = D(1, a) = 1$.

Further, to comply with the classical disjunction, D must be symmetric (or commutative). That is, D(a, b) = D(b, a) for all $a,b \in [0,1]$. The final property inherited from the classical disjunction is associativity. That is, D(a, D(b, c)) = D(D(a, b), c) for all $a, b, c \in [0,1]$.

Additionally, if the truth value of either one of the arguments is increased, the truth value of the disjunction should also be increased. That is, the function D should be monotone non-decreasing in both arguments. Formally, $a_1 \le a_2$ and $b_1 \le b_2$ imply $D(a_1, b_1) \le D(a_2, b_2)$.

Finally, small changes in the arguments of the disjunction function should result in small changes in the functional values. Thus, the function D should be a continuous function of two variables. Each of these conditions will serve as an axiom of disjunction, which are summarized below.

<u>Axioms of Disjunction</u>

1. Boundary Conditions: D(a, 1) = 1 = D(1, a) and D(0, a) = a = D(a, 0).
2. Commutativity: D(a, b) = D(b, a).
3. Associativity: D(a, D(b, c)) = D(D(a, b), c).
4. Monotone Non-Decreasing: $a_1 \le a_2$ and $b_1 \le b_2$ imply $D(a_1, b_1) \le D(a_2, b_2)$.
5. Continuity.

A function that satisfies conditions 1 through 4 is called a *triangular conorm function* or simply a *t-conorm*. T-conorms, like t-norms, were studied independently of fuzzy sets by Schweizer and Sklar [10] and [11], as well as Ling [7]. If a function is a continuous t-conorm with strict monotonicity it is called a *strict t-conorm*.

Authors have been trying to generalize the functions which satisfy the axioms of disjunction for the same reasons they have been working with conjunction functions. One example of a general class of continuous t-conorms is Yager's class defined as follows: $D_p(a, b) = \min(1, [(a)^p + (b)^p]^{1/p})$ for $p \geq 1$ [16] and [17]. To be a disjunction function, $D_p(a, b)$ must satisfy the five axioms of disjunction. This is shown in the following theorem.

Theorem 9: Any function in the Yager class, defined by $D_p(a, b) = \min(1, [(a)^p + (b)^p]^{1/p})$ for $p \geq 1$, satisfies the axioms for fuzzy disjunction functions.

1. Boundary conditions.

   If $b = 1$, then $D_p(a, 1) = \min(1, [(a)^p + (1)^p]^{1/p}) = \min(1, [(a)^p + 1]^{1/p}) = 1$, as $[(a)^p + 1]^{1/p} \geq 1$. Also, $D_p(1, a) = 1$. Similarly, $D_p(0, a) = \min(1, [(0)^p + (a)^p]^{1/p}) = \min(1, a) = a$. Finally, $D_p(a, 0) = a$.

2. Commutativity.

   The symmetry in the definition of $D_p(a, b)$ clearly shows that D_p is commutative.

3. Associativity.

   Assume, for this case, that $[(b)^p + (c)^p]^{1/p} \leq 1$ and $[(a)^p + (b)^p]^{1/p} \leq 1$. Then

$$
\begin{aligned}
D_p(a, D_p(b, c)) &= \min(1, [(a)^p + (D_p(b, c))^p]^{1/p}) \\
&= \min(1, [(a)^p + (\min(1, [(b)^p + (c)^p]^{1/p}))^p]^{1/p}) \\
&= \min(1, [(a)^p + [(b)^p + (c)^p]]^{1/p}) \\
&= \min(1, [(a)^p + (b)^p + (c)^p]^{1/p}) \\
&= \min(1, [([(a)^p + (b)^p]^{1/p})^p + (c)^p]^{1/p})
\end{aligned}
$$

$$= \min(1, [(\min(1,[(a)^p + (b)^p]^{1/p}))^p + (c)^p]^{1/p})$$

$$= \min(1, [(D_p(a, b))^p + (c)^p]^{1/p}) = D_p(D_p(a, b), c).$$

The other three cases follow in an analogous manner.

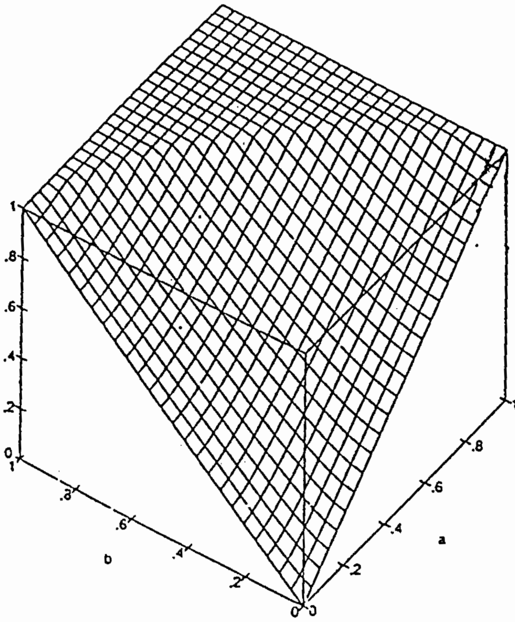
4. Monotone Non-Decreasing.

Choose a and b such that $a \leq b$. Then $(a)^p \leq (b)^p$. Thus, $[(a)^p + (c)^p]^{1/p} \leq [(b)^p + (c)^p]^{1/p}$. From this it follows that $\min(1, [(a)^p + (c)^p]^{1/p}) \leq \min(1, [(b)^p + (c)^p]^{1/p})$. Therefore, $D_p(a, c) \leq D_p(b, c)$. Since $D_p(a, b)$ is commutative, it follows that it is monotone non-decreasing in each argument. Finally, as in the proof of Theorem 5, if $a_1 \leq a_2$ and $b_1 \leq b_2$ then $D_p(a_1, b_1) \leq D_p(a_1, b_2) \leq D_p(a_2, b_2)$.

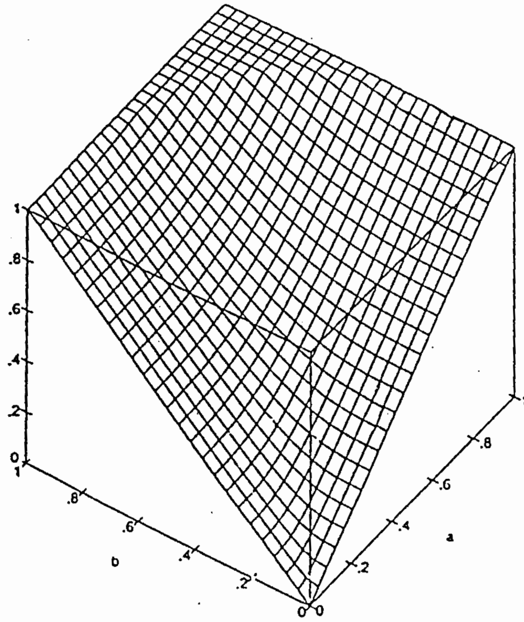
5. Continuity.

Continuity of $D_p(a, b)$ follows from the fact that each of the functions in the definition of D_p is itself continuous. ■

To understand what these functions represent, it is important to view them graphically. Since D_p is a function from [0,1] x [0,1] to [0,1], the graph of D_p is a three dimensional surface. In Figure 4, as in Figure 3, the truth values of the two arguments are given along the x and y axes and the truth value of their disjunction is given along the z axis. Figure 4 shows the graphs of four different Yager disjunction functions with p values of 1.5, 3, 5, and 10. Note that as the value of p increases the value of the disjunction decreases. This is exactly opposite to what happens with the Yager class of conjunction functions. For a proof of this, note that the first derivative of D_p with respect to p is negative.



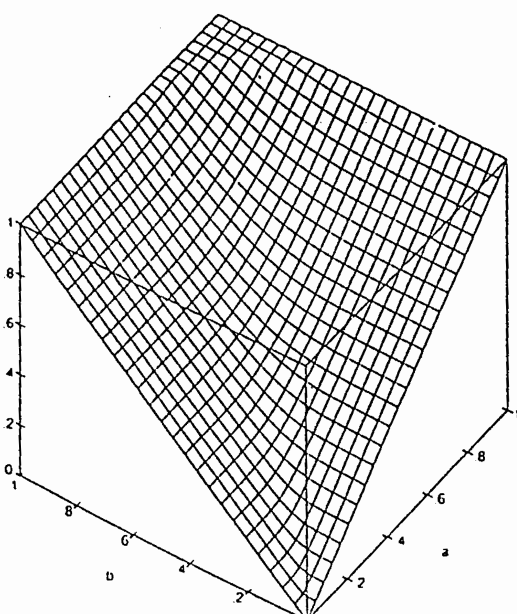
p = 1.5



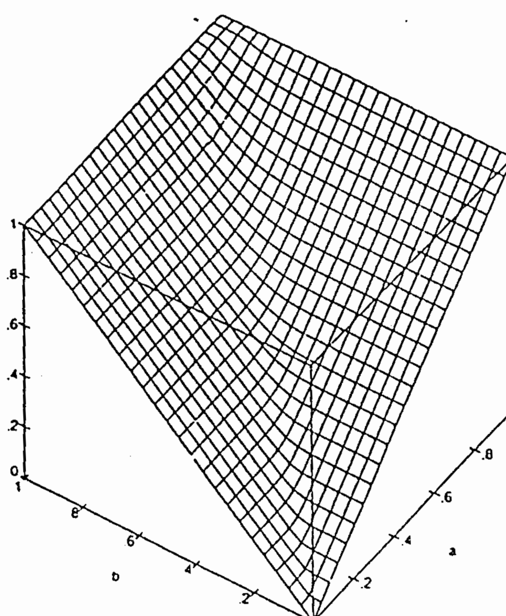
p = 3

Figure 4: Graphs of Functions in Yager's Class of Disjunctions.

Taken from [6].



p = 5



p = 10

If $p=\infty$, the function D_p reduces simply to the max function (for a proof of this see Appendix D), which is the most "demanding" disjunction function. The reason for this is that in the case of disjunction functions, the interpretation of the value p is the opposite of the interpretation for p in the conjunction functions. A strong disjunction function means that the truth values of both a and b must be close to zero for the disjunction truth value to be close to zero. That is, for disjunctions, the strength is directly proportional to the value of p. The strength function s, defined in Chapter 4, can be applied to the Yager disjunction as well. Here, $s(D_p) = p$. Of course, the Yager class is just one class of disjunction functions. Appendix C lists several other examples of classes of disjunction functions.

It is possible to find generators for *some* disjunction functions as was done for the conjunction functions. As noted earlier, an increasing generator is a function g from [0,1] to $\Re$ such that $g(0)=0$, g is strictly increasing, and g is continuous. An open-ended increasing generator is an increasing generator that does not have the unit interval as its domain. In the following proof and in several later proofs the idea of the pseudoinverse, $g^{(-1)}$, of a function g will again be useful. To recall, it is defined as follows:

$$g^{(-1)}(a)=\begin{cases}0 & \text{for } a\in(-\infty,0)\\ g^{-1}(a) & \text{for } a\in[0,g(1)]\\ 1 & \text{for } a\in(g(1),\infty)\end{cases}\quad \text{where } g^{-1} \text{ is the ordinary inverse of g.}$$

Note that since g is continuous, $g^{(-1)}$ will also be continuous. As noted in Chapter 4, in all discussions involving the pseudoinverse, the reader should check that the correct part of the definition is being used. The proof of the following pseudo-characterization theorem is original to this paper but combines ideas found in [10], [11], and [1].

Theorem 10: 1. If a binary operation D on $[0,1) \times [0,1)$ to $[0,1]$ is a strictly increasing fuzzy disjunction function then there exists an open-ended increasing generator $g: [0,1) \to \Re$ such that $D(a, b) = g^{(-1)}(g(a)+g(b))$. 2. If $g: [0,1] \to \Re$ is an increasing generator, and if $D:[0,1] \times [0,1] \to [0,1]$ is given by $D(a, b) = g^{(-1)}(g(a)+g(b))$, then D is a fuzzy disjunction function.

Proof:

1. In **[11, p.179]** Schweizer and Sklar show that every strict t-norm determines a strict t-conorm and every strict t-conorm determines a strict t-norm. This is seen by the following rule: $S(a, b) = 1 - T(1\text{-}a, 1\text{-}b)$, where S is a strict t-conorm and T is a strict t-norm. By use of this theorem, we see that for an arbitrary strict fuzzy disjunction function D(a, b), $D(a, b) = 1 - T(1\text{-}a, 1\text{-}b)$ for some strict t-norm T. Theorem 6, however, established that every strict t-norm on $(0,1]\times(0,1]$ could be written in the form $T(a, b) = f^{(-1)}(f(a)+f(b))$ for some open-ended decreasing generator f. Hence, $D(a, b) = 1 - f^{(-1)}(f(1\text{-}a)+f(1\text{-}b))$. Now define a function $g:[0,1) \to \Re$ by $g(a) = f(1\text{-}a)$. Then it is easy to verify that $f(a) = g(1\text{-}a)$, $g^{(-1)}(a) = 1 - f^{(-1)}(a)$ and $f^{(-1)}(a) = 1\text{-}g^{(-1)}(a)$. Note that g is continuous as f is continuous, and that g is strictly increasing since f is strictly decreasing. Finally, since $g(0) = f(1) = 0$, g is an open-ended increasing generator. To conclude, note that $D(a, b) =$ $1\text{-}f^{(-1)}(f(1\text{-}a)+f(1\text{-}b)) = g^{(-1)}(f(1\text{-}a)+f(1\text{-}b) = g^{(-1)}(g(a)+g(b))$.

2. Suppose g exists as defined above. To see that $D(a, b) \in [0,1]$, it is helpful to split the interval $[0, 2g(1)]$ into the two subintervals $[0, g(1)]$ and $(g(1), 2g(1)]$, and use the second and third parts of the definition of the pseudoinverse. To show that $g^{(-1)}(g(a)+g(b))$ is a fuzzy disjunction function, the five disjunction axioms must be shown to be satisfied.

A. Boundary Conditions

If $a>0$, then $D(a, 1) = g^{(-1)}(g(a)+g(1)) = 1$ as $g(a)>g(0) = 0$ so that $g(a)+g(1) \in (g(1), \infty)$. If $a = 0$, then $D(a, 1) = g^{(-1)}(g(0)+g(1)) = g^{(-1)}(0+g(1)) = g^{(-1)}(g(1)) = g^{-1}(g(1)) = 1$.

$D(0, a) = g^{(-1)}(g(0)+g(a)) = g^{(-1)}(0+g(a)) = g^{(-1)}(g(a)) = g^{-1}(g(a)) = a$.

B. Commutativity

$D(a, b) = g^{(-1)}(g(a)+g(b)) = g^{(-1)}(g(b)+g(a)) = D(b, a)$.

C. Associativity

$D(a, D(b, c)) = g^{(-1)}(g(a) + g(D(b, c))) = g^{(-1)}(g(a) + g(g^{(-1)}(g(b) + g(c)) = g^{(-1)}(g(a) + g(b)+g(c)) = g^{(-1)}(g(g^{(-1)}(g(a) + g(b))) +g(c)) = g^{(-1)}(g(D(a, b)) +g(c)) = D(D(a, b), c)$.

D. Monotone Non-Decreasing

Assume $0\leq a_1 \leq a_2$ and $0\leq b_1 \leq b_2$. Then $g(a_1)\leq g(a_2)$ and $g(b_1)\leq g(b_2)$. Hence $g(a_1) + g(b_1)\leq g(a_2)+g(b_2)$. Since g is monotone increasing, so is $g^{(-1)}$. Thus, $g^{(-1)}(g(a_1) +g(b_1))\leq g^{(-1)}(g(a_2)+g(b_2))$, or $D(a_1,b_1)\leq D(a_2,b_2)$.

E. Continuity

The continuity of D is evident from the continuity of g and hence that of $g^{(-1)}$.

Thus, $D(a, b) = g^{(-1)}(g(a)+g(b))$ satisfies the axioms necessary to become a conjunction function. ■

The following pseudo-characterization theorem shows that a fuzzy disjunction function also has an open-ended decreasing generator.

Theorem 11: 1. If a binary operator D on [0,1) x [0,1) to [0,1] is a strictly increasing fuzzy disjunction function then there exists an open-ended decreasing generator $f: [0,1) \to \Re$ such that $D(a, b) = f^{(-1)}(f(a)+f(b)-f(0))$. 2. If $f: [0,1] \to \Re$ is a decreasing generator, and if $D: [0,1] \times [0,1] \to [0,1]$ is given by $D(a, b) = f^{(-1)}(f(a)+f(b)-f(0))$, then D is a fuzzy disjunction function.

Proof:

1. Given a strict t-norm T, **[10, p.71]** guarantees that $T(a, b) = h^{(-1)}(h(a) \bullet h(b))$ for all $a, b \in [0,1]$, where h is a continuous, strictly increasing function from [0,1] onto [0,1]. Schweizer and Sklar show that every strict t-conorm D is of the following form: $D(a, b) = 1-T(1-a, 1-b)$ for some strict t-norm T. Therefore, given a strictly increasing disjunction function D, $D(a, b) = 1-T(1-a, 1-b) = 1-h^{(-1)}(h(1-a) \bullet h(1-b))$. Define a function $f: [0,1) \to \Re$ by $f(a) = \ln(h(1-a))$. Then $h(a) = e^{f(1-a)}$, $f^{(-1)}(a) = 1-h^{(-1)}(e^{a})$, and $h^{(-1)}(a) = 1-f^{(-1)}(\ln a)$. Note that since 1-a is decreasing f is decreasing, and since both h and ln are continuous f is continuous. Therefore, f is an open-ended decreasing generator. Next, $D(a, b) = 1-h^{(-1)}(h(1-a) \bullet h(1-b)) = 1-h^{(-1)}(e^{f(a)} \bullet e^{f(b)}) = 1-h^{(-1)}(e^{f(a)+f(b)}) = f^{(-1)}(f(a)+f(b))$. Now, since $f(0) = 0$, $f^{(-1)}(f(a)+f(b)) = f^{(-1)}(f(a)+f(b)-f(0))$. Hence, every strictly increasing fuzzy disjunction function D can be written as follows: $D(a, b) = f^{(-1)}(f(a)+f(b)-f(0))$.

2. A straightforward check of the five axioms, similar to the proof of Part 2 in Theorem 10, shows that $D: [0,1] \times [0,1] \to [0,1]$ and that D is a fuzzy disjunction function. ■

As expected, there are some disjunction functions which are not strict t-conorms. Thus, just as in the case of conjunction functions, there exists disjunction functions which do not have either an increasing or decreasing generator. The following theorem, original to this paper, shows that the max function has no generator.

Theorem 12: The max function has no increasing or decreasing generator. More generally, there does not exist any continuous monotone function $h:[0,1] \rightarrow \Re$ such that $\max(a, b) = h^{(-1)}(h(a)+h(b))$ or $\max(a, b) = h^{(-1)}(h(a)+h(b)-h(0))$.

Proof:

Suppose that a continuous monotone function h exists such that $\max(a, b) = h^{(-1)}(h(a)+h(b))$. Since $\max(1, a) = 1 = h^{(-1)}(h(1)+h(a))$ for all $a \in [0,1]$, $h(1)=h(1)+h(a)$. Thus, $h(a) = 0$ for all $a \in [0,1]$. Hence, $h^{(-1)}(h(1)+h(a)) = h^{(-1)}(0+0) = 0$ as $h(0) = 0$. But, $h^{(-1)}(h(1)+h(a)) = h^{(-1)}(h(1)+0) = 1 \neq 0$, providing a contradiction. Thus, the supposition that h exists must be false.

Next, suppose that a function h exists such that $\max(a, b) = h^{(-1)}(h(a)+h(b)-h(0))$. Then $\max(a, 1) = 1 = h^{(-1)}(h(a)+h(1)-h(0))$. Thus, $h(1) = h(a)+h(1)-h(0)$ or $h(a) = h(0)$. Hence, h is a constant function. Now if $b < a$, $a = \max(a, b) = h^{(-1)}(h(a)+h(b)-h(0)) = h^{(-1)}(h(0)+h(b)-h(0)) = h^{(-1)}(h(b)) = b$. Since, $b > a$, this is a contradiction. Therefore, the supposition that such an h exists must be false. ■

As was the case for conjunction functions, it is not possible to characterize all fuzzy disjunction functions by generators. As is shown in Chapter 6, however, it is possible to reduce the number of functions to be considered as disjunction functions. In Chapter 6, additional restrictions will be placed on the conjunction and disjunction functions, namely that of DeMorgan's laws and distributivity.

CHAPTER 6

INTERACTIONS OF THE CONNECTIVES

After studying each of the connectives individually, the only remaining restrictions are in how the fuzzy negation, conjunction, and disjunction functions interact with each other. Because of the desire to make fuzzy logic have some of the same properties as classical logic, satisfying DeMorgan's laws is a useful feature. DeMorgan's laws state that $N(C(a, b)) = D(N(a), N(b))$ and $N(D(a, b)) = C(N(a), N(b))$. Any conjunction function C and disjunction function D that satisfy these two relations is called a *dual triple* with respect to the negation function N. More concisely, $<N, C, D>$ is called a dual triple.

The question arises as to how to create triples of functions that will be dual triples. The process is straightforward as the next two theorems show. Part 1 of the proof of Theorem 13 was taken from **[6, p.84]** with notational changes, while Part 2 is original to this paper.

Theorem 13: Given any conjunction function C and any negation function N then, if $D(a, b) = N(C(N(a), N(b)))$, $<N, C, D>$ is a dual triple.

Proof:

1. D is a disjunction function.

   Note first that since $N: [0,1] \rightarrow [0,1]$ and $C:[0,1] \times [0,1] \rightarrow [0,1]$, it follows from the definition of D and the fact that N is involutory that $D: [0,1] \times [0,1] \rightarrow [0,1]$.

   A. Boundary Conditions

      $D(a, 1) = N(C(N(a), 0)) = N(0) = 1$ and $D(0, a) = N(C(1, N(a))) = N(N(a)) = a$.

B. Commutativity

$D(a, b) = N(C(N(a), N(b))) = N(C(N(b), N(a))) = D(b, a)$.

C. Associativity

$D(a, D(b, c)) = D(a, N(C(N(b), N(c)))) = N(C(N(a), N(N(C(N(b), N(c)))))) =$

$N(C(N(a), C(N(b), N(c)))) = N(C(C(N(a), N(b)), N(c))$, as C is a conjunction function and therefore associative. But, $N(C(C(N(a), N(b)), N(c)) =$

$N(C(N(N(C(N(a), N(b)))), N(c))) = N(C(N(D(a, b)), N(c))) = D(D(a, b), c)$.

Therefore, $D(a, D(b, c)) = D(D(a, b), c)$.

D. Monotone Non-Decreasing

Note that $a_1 \leq a_2$ and $b_1 \leq b_2$ imply that $N(a_1) \geq N(a_2)$ and $N(b_1) \geq N(b_2)$.

Thus, $C(N(a_1), N(b_1)) \geq C(N(a_2), N(b_2))$. Hence

$N(C(N(a_1), N(b_1))) \leq N(C(N(a_2), N(b_2)))$, or $D(a_1, b_1) \leq D(a_2, b_2)$.

E. Continuity

Since both N and C are continuous functions, $N(C(N(a), N(b)))$ is continuous and thus D is continuous.

2. $<N, C, D>$ is a dual triple

By definition, $D(a, b) = N(C(N(a), N(b)))$. Letting the negation function operate on each side of the equality yields $N(D(a, b)) = C(N(a), N(b))$ since N is involutory. This is one of DeMorgan's laws. To obtain the other law, simply note that $D(N(a), N(b)) = N(C(N(N(a)), N(N(b)))) = N(C(a, b))$. ■

As expected, an identical result is obtained if disjunction and negation functions are the two initially prescribed functions: a corresponding conjunction function can then be found so that all three functions form a dual triple. The statement of Theorem 14 was given in **[6, p.86]**, but the proof is original to this paper.

Theorem 14: Given any disjunction function D and any negation function N then, if $C(a, b) = N(D(N(a), N(b)))$, $<N, C, D>$ forms a dual triple.
Proof:

1. C is a conjunction function.

   As in the proof of Theorem 12, it can be seen that $C: [0,1] \times [0,1] \to [0,1]$.

   A. Boundary Conditions

   $C(a, 1) = N(D(N(a), 0)) = N(N(a)) = a$ and $C(0, a) = N(D(1, N(a))) = N(1) = 0$.

   B. Commutativity

   $C(a, b) = N(D(N(a), N(b))) = N(D(N(b), N(a))) = C(b, a)$.

   C. Associativity

   $C(a, C(b, c)) = C(a, N(D(N(b), N(c)))) = N(D(N(a), N(N(D(N(b), N(c)))))) =$ $N(D(N(a), D(N(b), N(c)))) = N(D(D(N(a), N(b)), N(c)))$, as D is a disjunction function and therefore associative. However, $N(D(D(N(a), N(b)), N(c))) =$ $N(D(N(N(D(N(a), N(b)))), N(c))) = N(D(N(C(a, b)), N(c))) = C(C(a, b), c)$. Hence, $C(a, C(b, c)) = C(C(a, b), c)$.

   D. Monotone Non-Decreasing

   As before, $a_1 \leq a_2$ and $b_1 \leq b_2$ imply that $N(a_1) \geq N(a_2)$ and $N(b_1) \geq N(b_2)$. Thus, $D(N(a_1), N(b_1)) \geq D(N(a_2), N(b_2))$. Hence,

$N(D(N(a_1), N(b_1))) \leq N(D(N(a_2), N(b_2)))$ or $C(a_1, b_1) \leq C(a_2, b_2)$.

E. Continuity

Since both N and D are continuous functions, N(D(N(a), N(b))) is continuous and thus C is continuous.

2. <N, C, D> is a dual triple

By definition, C(a, b) = N(D(N(a), N(b))). Letting the negation function operate on each side of the equality yields N(C(a, b)) = D(N(a), N(b)) since N is involutory. This is one of DeMorgan's laws. To obtain the other law, simply note that C(N(a), N(b)) = N(D(N(N(a)), N(N(b)))) = N(D(a, b)). ■

In addition to being relatively easy to create, these dual triples have an interesting relation to the generator functions studied in previous chapters. If a single generator is chosen (either increasing or decreasing), the negation, conjunction, and disjunction functions that are created will form a dual triple. This is shown in the following theorem, taken from **[6, p.86]** with modifications.

Theorem 15: If an increasing generator $g: [0,1] \rightarrow \Re$ generates a negation function, a conjunction function and a disjunction function, then they form a dual triple.

Proof:

The negation, conjunction, and disjunction functions generated by the increasing generator g are given respectively as follows:

$N(a) = g^{-1}(g(1)\text{-}g(a))$; $C(a, b) = g^{(-1)}(g(a)+g(b)\text{-}g(1))$; $D(a, b) = g^{(-1)}(g(a)+g(b))$.

To see that DeMorgan's first law holds, first consider N(C(a, b)) where $a>0$.

$N(C(a, b)) = g^{-1}(g(1)\text{-}g(C(a, b))) = g^{-1}(g(1)\text{-}g(g^{(-1)}(g(a)+g(b)\text{-}g(1)))) =$

$g^{-1}(g(1)\text{-}(g(a)+g(b)\text{-}g(1))) = g^{-1}(2g(1)\text{-}g(a)\text{-}g(b)) = g^{(-1)}(2g(1)\text{-}g(a)\text{-}g(b)) =$

$g^{(-1)}(g(1)\text{-}g(a)+ g(1)\text{-}g(b)) = g^{(-1)}(g(g^{-1}(g(1)\text{-}g(a)))+g(g^{-1}(g(1)\text{-}g(b)))) =$

$g^{(-1)}(g(N(a))+g(N(b))) = D(N(a), N(b))$. If $a = 0$, then both $N(C(0, b)) = 1$ and $D(N(0), N(b)) = 1$ as $N(0) = 1$.

To see that DeMorgan's other law holds, consider $N(D(a, b))$.

$N(D(a, b)) = g^{-1}(g(1)\text{-}g(D(a, b))) = g^{-1}(g(1)\text{-}g(g^{(-1)}(g(a)+g(b)))) =$

$g^{-1}(g(1)\text{-} g(a)\text{-}g(b)) = g^{(-1)}(g(1)\text{-}g(a)\text{-}g(b)) = g^{(-1)}((g(1)\text{-}g(a))+ (g(1)\text{-}g(b))\text{-}g(1)) =$

$g^{(-1)}(g(g^{-1}(g(1)\text{-}g(a)))+g(g^{-1}(g(1)\text{-}g(b)))\text{-}g(1)) = g^{(-1)}(g(N(a))+g(N(b))\text{-}g(1)) =$

$C(N(a), N(b))$. Therefore, by definition, $<N,C,D>$ is a dual triple. ■

Since each connective has a decreasing generator as well as an increasing generator, an analogous theorem holds for decreasing generators. The proof is original to this paper.

Theorem 16: If a decreasing generator f generates a negation function, a conjunction function and a disjunction function, then they form a dual triple.

Proof:

The negation, conjunction, and disjunction functions generated by a decreasing generator f are given below.

$N(a) = f^{-1}(f(0)\text{-}f(a))$; $C(a, b) = f^{(-1)}(f(a)+f(b))$; $D(a, b) = f^{(-1)}(f(a)+f(b)\text{-}f(0))$.

To see that DeMorgan's first law holds, consider $N(C(a, b))$.

$N(C(a, b)) = f^{-1}(f(0)\text{-}f(C(a, b))) = f^{-1}(f(0)\text{-}f(f^{-1}(f(a)+ f(b)))) = f^{-1}(f(0)\text{-}f(a)\text{-}f(b)) =$

$f^{(-1)}(f(0)\text{-}f(a)\text{-}f(b)) = f^{(-1)}(f(0) - f(a)+ f(0)\text{-}f(b)\text{-}f(0)) =$

$f^{(-1)}(f(f^{-1}(f(0)-f(a))) + f(f^{-1}(f(0)-f(b)))-f(0)) = f^{(-1)}(f(N(a)) + f(N(b)) - f(0)) =$ D(N(a), N(b)).

For DeMorgan's second law, consider N(D(a, b)).

$N(D(a, b)) = f^{-1}(f(0)-f(D(a, b))) = f^{-1}(f(0)-f(f^{(-1)}(f(a)+f(b)-f(0)))) =$

$f^{-1}(f(0)-(f(a)+f(b)-f(0))) = f^{-1}(2f(0)-f(a)-f(b)) = f^{(-1)}(2f(0)-f(a)-f(b)) =$

$f^{(-1)}(f(0)-f(a)+f(0)-f(b)) = f^{(-1)}(f(f^{-1}(f(0)-f(a)))+f(f^{-1}(f(0)-f(b)))) =$

$f^{-1}(f(N(a)) + f(N(b))) = C(N(a), N(b))$. Therefore, by definition, <N,C, D> is a dual triple. ■

There are other ways in which these fuzzy operations interact with each other. Three of the most common ways are: the law of the excluded middle, the law of contradiction, and the laws of distributivity. Each of these laws hold in classical logic. As discussed in the first chapter, the law of the excluded middle states that D(a, N(a)) = 1. One version of the law of contradiction states that C(a, N(a)) = 0. Third, there are two laws of distributivity: conjunction distributes over disjunction, and disjunction distributes over conjunction, that is, C(a, D(b, c)) = D(C(a, b), C(a, c)) and D(a, C(b, c)) = C(D(a, b), D(a, c)). It is straightforward (again using the definition of the pseudoinverse) to verify that the dual triples constructed in Theorems 15 and 16 satisfy the law of the excluded middle and the law of contradiction. As discussed in Chapter 1, however, it is not reasonable or necessarily desirable to assume that the law of the excluded middle holds. Distributivity, however, is one of the ways in which the conjunction and disjunction interact in a very useful way. The following theorem shows that if we have a dual triple which satisfies the law of the excluded middle and the law of contradiction, then it cannot satisfy the distributive laws. The proof below, taken with modifications from [6, p.87-88], shows that no dual triple can satisfy all three of the above mentioned laws.

Theorem 17: Let $<N, C, D>$ be a dual triple which satisfies the law of the excluded middle and the law of contradiction. Then $<N, C, D>$ does not satisfy the distributive laws.

Proof:

Assume that the distributive law $C(a, D(b, c)) = D(C(a, b), C(a, c))$ holds. Lemma 2 in Chapter 2 guarantees that the negation function N will have a unique fixed point e. Clearly e cannot be 0 or 1, as $N(0) = 1$ and $N(1) = 0$. Thus, $e \in (0,1)$. Now, $D(e, e) = D(e, N(e)) = 1$ by the law of the excluded middle. Similarly, $C(e, e) = C(e, N(e)) = 0$ by the law of contradiction. Hence, $C(e, D(e, e)) = D(C(e, e), C(e, e))$ or $C(e, 1) = D(0, 0)$. Consequently, $e = 0$. This is a contradiction, since $e \in (0,1)$. ■

In fact, there is only *one pair* of functions satisfying the fuzzy conjunction and disjunction axioms where each distributes over the other. The following proof, original to this paper, shows that the min, max pair is the only such pair.

Theorem 18: The min and max functions are the unique functions such that the conjunction function distributes over the disjunction function and the disjunction function distributes over the conjunction function.

Proof:

Let a and b be elements of $[0,1]$, where $a \leq b$. $C(a, b) \geq C(a, a) = C(D(a, 0), D(a, 0)) = D(a, C(0, 0)) = D(a, 0) = a$. Thus, $C(a, b) \geq a$. But $a = C(a, 1) \geq C(a, b)$. Hence $C(a, b) = a$ whenever $a \leq b$. This result, along with the fact that C is commutative, shows that C must be the min function.

Let a and b be elements of $[0,1]$ where $a \leq b$. $D(a, b) \leq D(b, b) = D(C(b, 1), C(b, 1)) = C(b, D(1, 1)) = C(b, 1) = b$. Thus, $D(a, b) \leq b$. But $b = D(0, b) \leq D(a, b)$. Hence $D(a, b) = b$ whenever $a \leq b$. This result, along with the fact that D is commutative, shows that D must be the max function.

Finally, it is necessary to show that the min and max functions do in fact distribute over each other. Assume that $a \leq b \leq c$ and $a,b,c \in [0,1]$. Then $\min(a, \max(b, c)) = \min(a, c) = a = \max(a, a) = \max(\min(a, b), \min(a, c))$. Also, $\max(a, \min(b, c)) = \max(a, b) = b = \min(b, c) = \min(\max(a, b), \max(a, c))$. The other cases follow in an analogous manner. Thus, the min and max functions distribute over each other. ■

If the additional axiom of distributivity is added to the list of axioms for fuzzy conjunction and disjunction functions, Theorem 18 shows that there is a unique function which satisfies each of the conditions. This corresponds exactly with what Zadeh had originally taken to be the conjunction and disjunction functions in his original paper on fuzzy sets [18].

Chapter 7 will reveal why this result is useful, and will discuss some areas that are still open to investigation.

CHAPTER 7

CONCLUSION

In Chapter 1 some motivational ideas for fuzzy logic were given. Chapter 2 then provided an introduction to fuzzy sets and fuzzy logic. Chapters 3, 4, and 5 provided axioms and some pseudo-characterization theorems about fuzzy negation, conjunction, and disjunction functions, respectively. In Chapter 6, the interactions of these three classes of fuzzy functions were investigated. Now it is important to summarize into a single listing the desirable axioms for the basic connectives in fuzzy logic.

Axioms for the Basic Fuzzy Logic Connectives

1. The Negation Function is monotone decreasing.
2. The Negation Function is involutory.
3. The Conjunction Function satisfies the classical boundary conditions.
4. The Conjunction Function is commutative.
5. The Conjunction Function is associative.
6. The Conjunction Function is monotone non-decreasing.
7. The Conjunction Function is continuous.
8. The Disjunction Function satisfies the classical boundary conditions.
9. The Disjunction Function is commutative.
10. The Disjunction Function is associative.
11. The Disjunction Function is monotone non-decreasing.
12. The Disjunction Function is continuous.
13. The Negation, Conjunction, and Disjunction Function form a dual triple.
14. The Conjunction and Disjunction Functions distribute over each other.

Recall from Theorem 17 that if Axiom 13 is accepted, and if $<N, C, D>$ satisfies both the law of the excluded middle and the law of contradiction, then Axiom 14 cannot be accepted. Further, Theorem 18 shows that the min and max functions are the unique functions which satisfy all fourteen axioms. In fact, Axiom 13 is not needed to demonstrate the uniqueness of these functions, but it does place a restriction on the type of negation functions allowed.

There are two main reasons for selecting the min and max functions as the unique conjunction and disjunction functions. The first is that when fuzzy logic is used to model a problem, it is quite common to try to "best fit" the data by changing both the membership functions involved and the functions used for the connectives. If, however, the only suitable conjunction and disjunction functions are the min and max functions, respectively, then one needs only to adjust the membership function. By doing so, it is much easier to evaluate whether fuzzy logic is indeed the best way to model a situation. Second, the min and max functions have the desirable property of being idempotent. That is, conjoining a statement to itself with either a conjunction or disjunction does not alter the truth value of the statement. Thus, a statement A has the same truth value as "A and A" and as "A or A." The following theorem, taken from **[6, p.63]**, shows that the min function is the only idempotent conjunction function.

Theorem 18: The min function is the only idempotent conjunction function.

Proof:

Clearly, $\min(a, a) = a$ for all $a \in [0,1]$, and is thus idempotent. Assume that there exists another conjunction function C such that C is idempotent. Then, for any $a,b \in [0,1]$, $a \leq b$ implies $a = C(a, a) \leq C(a, b) \leq C(a, 1) = a$ by monotonicity and the boundary condition.

Hence, C(a, b) = a = min (a, b). Similarly, b $\leq$ a implies C(a, b) = b = min(a, b). Therefore, the min function is the unique idempotent conjunction function. ■

In an identical fashion, it is easy to see that the max function is the only idempotent disjunction function. Thus, in regard to reducing the conjunction and disjunction classes to just the min and max functions, taking idempotentcy as an axiom for the conjunction and disjunction functions has the same effect as distributivity. In this sense, idempotency and distributivity, in the presence of the other axioms, each imply the other.

There are other reasons for desiring the min and max functions as the conjunction and disjunction functions. For example, in [13] Dubois and Prade give an interesting discussion of the suitability of the min and max functions as the best conjunction and disjunction functions because the correspondence between these two functions and the way in which various population samples think of "and" and "or."

These are the axioms only for the basic connectives of negation, conjunction, and disjunction. The fuzzy connective implication is more open to interpretation as to which properties are desirable and which are not. J. F. Baldwin and B. W. Pilsworth give an excellent axiomatic approach to implication in [2, p.216], in which they conclude:

> The motivation for this paper has been the recognition that there are two complementary principles in determining what is an appropriate rule for implication when used for approximate reasoning with fuzzy logic. In the first place it is important to consider what are the essential properties of implication that satisfy an intuitive understanding of its meaning so that inappropriate implication rules can be rejected. In the second place, it is not necessary to specify a unique rule for implication. On the contrary, the choice of implication law, from the set of those which satisfy the above-mentioned intuitive properties is a problem of modeling to suit the needs of particular applications.

The implication for MVLs was discussed in the analysis of the sorites paradox in Chapter 1. Although a similar rule for implication can be used in fuzzy logic, the paradox does not arise. The reason is not because of a difference in the implication function, but rather in the way truth values are considered. In a MVL, the initial stage of the paradox assumes that the heap of sand is a heap is totally true and not a heap is totally false. The problem arises when considering at what stage does the heap cease being totally a heap? In fuzzy logic, even at the initial stage there is a certain degree of "non-heapness," and removing a single grain raises this degree of non-heapness. In a MVL, the sum of the degree of heapness and the degree of non-heapness must be one. This is not so in fuzzy logic. It is possible that the degree of non-heapness increases while the degree of heapness stays at one. It is this sort of thinking that avoids this and similar paradoxes.

Thus, fuzzy logic clearly plays a role in expanding the ideas of classical logic to resolve the paradoxes found in classical logic. The first step towards the general acceptance of fuzzy logic is to provide a solid foundation for the connectives of fuzzy logic. In doing so, having found unique functions which can serve as reasonable conjunction and disjunction functions eases the task of modeling real world situations with fuzzy logic. This, of course, is a useful way to show the applicability of fuzzy logic. Later steps in the development and applications of fuzzy logic might involve such matters as determining what might be appropriate fuzzy functions corresponding to a semantics for quantification theory (the predicate calculus) and to both the syntax and semantics of first-order, possibly fuzzy, mathematical theories.

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APPENDIX A

NEGATION FUNCTIONS

Name	N(a) =	Decreasing generator $f(a)$ =	Increasing generator g(a) =
Klir & Yuan [6]	$\frac{\lambda^2(1-a)}{a+\lambda^2(1-a)}$ $(\lambda > 0)$	$1-\frac{a}{\lambda+(1-\lambda)a}$	$\frac{a}{\lambda+(1-\lambda)a}$
Standard Negation [18]	1-a	$-ka+k \quad (k>0)$	ka
Sugeno [6]	$\left(\frac{1-a^{\omega}}{1+\lambda a^{\omega}}\right)^{1/\omega}$	$\frac{1}{\lambda}\ln\left(\frac{1+\lambda}{1+\lambda a^{\omega}}\right)$	$\frac{1}{\lambda}\ln\left(1+\lambda a^{\omega}\right)$
Yager [17]	$(1-a^{\omega})^{1/\omega}$	$1-a^{\omega}$	a^{ω}

APPENDIX B

CONJUNCTION FUNCTIONS

Name	C(a, b) =	Decreasing generator $f(a)$ =	Increasing generator g(a) =
Dombi [6]	$\dfrac{1}{\left[\left(\frac{1}{a}-1\right)^{\lambda}+\left(\frac{1}{b}-1\right)^{\lambda}\right]^{1/\lambda}}$ $(\lambda>0)$	$\left(\frac{1}{a}-1\right)^{\lambda}$	None
Dubois & Prade [6]	$\dfrac{ab}{\max(a,b,\lambda)}$ $(\lambda\in[0,1])$	None	None
Frank [6]	$\log_{\lambda}\left[1+\dfrac{(\lambda^{a}-1)(\lambda^{b}-1)}{\lambda-1}\right]$ $(\lambda>0,\lambda\neq 1)$	$-\ln\left(\dfrac{\lambda^{a}-1}{\lambda-1}\right)$	None
Hamacher [6]	$\dfrac{ab}{\lambda+(1-\lambda)(a+b+-ab)}$ $(\lambda\geq 0)$	$-\ln\left(\dfrac{a}{\lambda+(1-\lambda)a}\right)$	None
Schweizer & Sklar[1] [6]	$\max\left(0,(a^{\lambda}+b^{\lambda}-1)^{1/\lambda}\right)$ $(\lambda\neq 0)$	$1-a^{\lambda}$	a^{λ}
Schweizer & Sklar[2] [6]	$1-\left[(1-a)^{\lambda}+(1-b)^{\lambda}-(1-a)^{\lambda}(1-b)^{\lambda}\right]^{1/\lambda}$ $(\lambda>0)$	$\ln\left[1-(1-a)^{\lambda}\right]^{1/\lambda}$	None
Schweizer & Sklar[3] [6]	$\exp\left(-\left(\lvert\ln a\rvert^{\lambda}+\lvert\ln b\rvert^{\lambda}\right)^{1/\lambda}\right)$ $(\lambda>0)$	$\lvert\ln a\rvert^{\lambda}$	None
Schweizer & Sklar[4] [6]	$\dfrac{ab}{(a^{\lambda}+b^{\lambda}-a^{\lambda}b^{\lambda})^{1/\lambda}}$ $(\lambda>0)$	$a^{-\lambda}-1$	$-a^{-\lambda}$
Weber [6]	$\max\left(0,\dfrac{a+b+\lambda ab-1}{1+\lambda}\right)$ $(\lambda>-1)$	$\dfrac{1}{\lambda}\ln[1+\lambda(1-a)]$	$\dfrac{1}{\lambda}\ln\left[\dfrac{1+\lambda}{1+\lambda(1-a)}\right]$
Yager [17]	$1-\min\left(1,\left[(1-a)^{\lambda}+(1-b)^{\lambda}\right]^{1/\lambda}\right)$ $(\lambda>0)$	$(1-a)^{\lambda}$	$1-(1-a)^{\lambda}$
Yu [6]	$\max[0,(1+\lambda)(a+b-1)-\lambda ab]$ $(\lambda>-1)$	$\dfrac{1}{\lambda}\ln\left(\dfrac{1+\lambda}{1+\lambda a}\right)$	$\dfrac{1}{\lambda}\ln(1+\lambda a)$

APPENDIX C

DISJUNCTION FUNCTIONS

Name	D(x,y) =	Decreasing generator $f(a)$ =	Increasing generator $g(a)$ =
Dombi [6]	$\dfrac{1}{1+\left[\left(\frac{1}{a}-1\right)^{\lambda}+\left(\frac{1}{b}-1\right)^{\lambda}\right]^{-1/\lambda}}$ $(\lambda>0, \lambda\neq 1)$	$-\left(\frac{1}{a}-1\right)^{-\lambda}$	$\left(\frac{1}{a}-1\right)^{-\lambda}$
Dubois & Prade [6]	$1-\dfrac{(1-a)(1-b)}{\max((1-a),(1-b),\lambda)}$ $(\lambda\in[0,1])$	None	None
Frank [6]	$1-\log_{\lambda}\left[1+\dfrac{\left(\lambda^{1-a}-1\right)\left(\lambda^{1-b}-1\right)}{\lambda-1}\right]$ $(\lambda>0)$	None	$-\ln\left(\dfrac{\lambda^{1-a}-1}{\lambda-1}\right)$
Hamacher [6]	$\dfrac{a+b+(\lambda-2)ab}{\lambda+(\lambda-1)ab}$ $(\lambda>0)$	None	$-\ln\left(\dfrac{1-a}{\lambda+(1-\lambda)(1-a)}\right)$
Schweizer & Sklar[1] [6]	$1-\max(0,(1-a)^{\lambda}+(1-b)^{\lambda}-1)^{1/\lambda}$ $(\lambda\neq 0)$	$-(1-a)^{\lambda}$	$1-(1-a)^{\lambda}$
Schweizer & Sklar[2] [6]	$\left[a^{\lambda}+b^{\lambda}+a^{\lambda}b^{\lambda}\right]^{1/\lambda}$ $(\lambda>0)$	None	$\ln\left[1-a^{\lambda}\right]^{1/\lambda}$
Schweizer & Sklar[3] [6]	$1-\exp\left(-\left(\lvert\ln(1-a)\rvert^{\lambda}+\lvert\ln(1-b)\rvert^{\lambda}\right)^{1/\lambda}\right)$ $(\lambda>0)$	None	$\lvert\ln(1-a)\rvert^{\lambda}$
Schweizer & Sklar[4] [6]	$\dfrac{(1-a)(1-b)}{\left((1-a)^{\lambda}+(1-b)^{\lambda}-(1-a)^{\lambda}(1-b)^{\lambda}\right)^{1/\lambda}}$ $(\lambda>0)$	$-(1-a)^{-\lambda}$	$(1-a)^{-\lambda}-1$
Yager [17]	$\min\left(1,\left(a^{\lambda}+b^{\lambda}\right)^{1/\lambda}\right)$ $(\lambda>0)$	$-a^{\lambda}$	a^{λ}
Yu [6]	$\min[1,a+b+\lambda ab]$ $(\lambda>-1)$	$\dfrac{1}{\lambda}\ln\left(\dfrac{1+\lambda}{1+\lambda a}\right)$	$\dfrac{1}{\lambda}\ln(1+\lambda a)$

APPENDIX D

PROOF THAT C_p = MIN AND D_p = MAX WHEN $p=\infty$

Lemma 19: $\lim_{p\to\infty}\min[1, (a^p + b^p)^{1/p}] = \max(a, b)$ where $a, b \in [0,1]$.

Proof (original to this paper):

If a or b is equal to zero, the result clearly holds. If a = b, then, as $2^{1/p} \to 1$ as $p \to \infty$, the result holds. Assume, without loss of generality, that $0 < a < b$Let $(a^p + b^p)^{1/p} = Q$.

$$\text{Then } \lim_{p\to\infty} \ln Q = \lim_{p\to\infty} \frac{\ln(a^p + b^p)}{p} = \lim_{p\to\infty} \frac{a^p \ln a + b^p \ln b}{a^p + b^p} \quad \text{(l'Hospital's rule)}$$

$$= \lim_{p\to\infty} \frac{(a/b)^p \ln a + \ln b}{(a/b)^p + 1}$$

$$= \ln b \quad \text{as } 0 < (a/b) < 1.$$

From this it follows that $\lim_{p\to\infty} Q = \lim_{p\to\infty}(a^p + b^p)^{1/p} = b = \max(a, b)$. Therefore,

$\lim_{p\to\infty}\min[1, (a^p + b^p)^{1/p}] = \min[1, \lim_{p\to\infty}(a^p + b^p)^{1/p}] = \min(1, b) = b = \max(a, b)$ as $0 < a < b$. ■

Theorem 20: The Yager conjunction class $C_p(a, b) = 1\text{-}\min(1, [(1\text{-}a)^p + (1\text{-}b)^p]^{1/p})$ $(p \ge 1)$ reduces to the min function when $p = \infty$.

Proof:

Take $a, b \in [0,1]$. Then $\lim_{p\to\infty} C_p(a, b) = \lim_{p\to\infty} 1\text{-}\min(1, [(1\text{-}a)^p + (1\text{-}b)^p]^{1/p})$

$$= 1\text{-}\min(1, \lim_{p\to\infty}[(1\text{-}a)^p + (1\text{-}b)^p]^{1/p})$$

$$= 1\text{-}\min(1, \max((1\text{-}a), (1\text{-}b))) \quad \text{(Lemma 19)}$$

$$= \min(a, b). \ ■$$

Theorem 21: The Yager disjunction class $D_p(a, b) = \min(1, [a^p + b^p]^{1/p})$ $(p \geq 1)$ reduces to the max function when $p = \infty$.

Proof:

Take $a, b \in [0,1]$. Then, $\lim\limits_{p\to\infty} D_p(a, b) = \lim\limits_{p\to\infty} \min(1, [a^p + b^p]^{1/p})$

$$= \min(1, \lim_{p\to\infty}[a^p + b^p]^{1/p})$$

$$= \min(1, \max(a, b)) \qquad \text{(Lemma 19)}$$

$$= \max(a, b). \quad \blacksquare$$