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Exploration of counter examples of balanced sets

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EXPLORATION OF COUNTER EXAMPLES OF BALANCED SETS

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Exploration of Counter Examples of Balanced Sets

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Abstract

Mathematicians are often intrigued with patterns, many times finding themselves looking for pieces of structure within a data set. This research project is no different in that we have explored our vast data set for substructure.

Our goal is to identify the following: how many data points are necessary to guarantee our set has a balanced set/substructure? Naturally rephrasing the previous question, we also ask ourselves what is the largest set that does not have a balanced subset/substructure? This alternate phrasing set us down our current path. We focused on the largest sets with no balanced substructure and what they look like.

After brute force checking all $\mathbb{Z}_5 \times \mathbb{Z}_5$ maximal sets, we found 7 non-isomorphic graphs that did not have balanced substructure. Using those examples as starting points, we then extended into $\mathbb{Z}_7 \times \mathbb{Z}_7$. When successful, our goal was to classify examples in $\mathbb{Z}_p \times \mathbb{Z}_p$ which have no balanced substructure.

Currently, we believe there are four classifications of maximal sets with no balance substructure for any $\mathbb{Z}_p \times \mathbb{Z}_p$. The main proof to follow focuses on one of these classifications called *Kick It*.

Contents

Chapter 1

Reflection

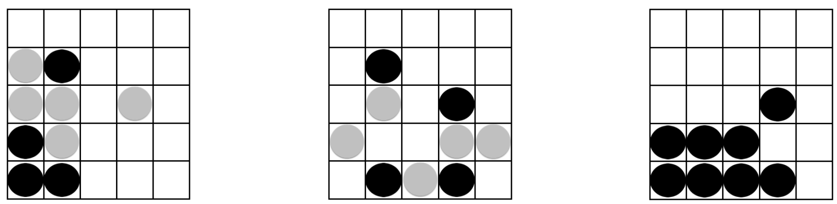
The project I decided to take on for my Honor's Thesis is built upon research I started in Summer 2017 with Dr. Adrienne Stanley. In the summer, we explored our particular data set for substructure. Substructure in this case is called balance. The initial launch question that set us down this path was: how many data points are necessary to guarantee our set has a balanced subset/substructure? Naturally rephrasing this question, we also asked ourselves what is the largest set that does not have a balanced subset/substructure? This alternative phrasing is what set us down our current path of research. We focused on the largest sets with no balanced substructure and what they look like. From there, we identified four different classifications (named Superman, Taxi, Kick It, and Score) of sets that do not have substructure. My thesis picks up our research at this point. We currently believe there are only four classifications of sets of this maximal size that do not have substructure. This thesis begins to confirm the previous statement.

Already having researched this topic for the past half a year and thoroughly enjoying it, I thought this would be an excellent topic for me to continue to explore. This topic lives somewhere in the realm of combinatorics, a field I believe I have interest in after taking the undergraduate course in combinatorics offered at UNI. The problem stood out to me because it seemed like a fun game or a mad puzzle; I could visualize the sets and fiddle with its components in order to obtain a desired outcome. Not only did the problem look intriguing, but Dr. Stanley is easily one of the best math professors on this campus. I could not pass up the opportunity to work with and be advised by such a talented mind. She cares about student learning and student success, and she emphasizes communication skills, clean proof technique, and creative thinking. By working with Dr. Stanley, I have had a holistic educational experience; not only have I gained knowledge in the subject area of balanced sets, but I believe I have improved my creative problem solving ability, grown in my ability to effectively communicate (in writing and orally), as well as learned how to persevere through periods of struggle.

In order to complete this thesis, Dr. Stanley and I have been meeting twice a week for as long as two hours each meeting. In these meetings, we work to prove there are exactly four classifications of sets of this maximal size that do not have substructure. To begin, however, we made a large initial assumption that will be tackled at a later time; we assumed our maximal set without balance has a specific form. This set will be later defined as S. In our efforts to prove there are only four classifications of sets, we started with the easiest classification (Superman) and worked our way to the most difficult (Score). Finding ourselves short on time, we have not yet proven our desired result for Score, though significant progress has been made. However, in tackling the most difficult classification, we were able to identify alternative perspectives on how to prove the desired result for Superman, Taxi, and Kick It. Thus, we used the useful techniques from Score in order to more concisely prove the desired result of Kick It. As a reminder, the result is that maximal size sets with a specific form, defined later, are isomorphic to Kick It.

The final work found in this paper is a proof. There are multiple chapters that lead up to the proof. Chapter two consists of a list of definitions that clearly define some of our new notation as well as motivation for why new notation is introduced. This chapter is important because it tells the reader what sets/objects are being worked with. Chapter three consists of supporting lemmas and proofs. These results lay a ground work from which the proof can be built upon. This chapter is important because it tells the reader what can be done with the sets/objects and how they relate to one another. The transformation lemmas allow us to make our main two suppositions, and the configuration lemmas help draw attention to the sets of points we consider in our main proof.

The origins of this problem date all the way back to 1972 when Heiko Harborth initially questioned what the minimum number of points required in $C_p \oplus C_p$ to guarantee a zero-sum subset of size p for any given prime p [7]. Since 1972, there are have been several talented minds to attempt this question. In 1983, Kemntiz [8] made some progress and determined a range for the minimum number of points for any prime p. This range was $[2p-1, 4p-3]$. Then, in 2004, Gao and Thangadurai [6] confirmed for primes larger than 67 the minimum number of points was $2p-1$. In 2007, Gao, Geroldinger, and Schmid [5] improved the previous claim by confirming it for all primes larger than 47. The idea that $2p-1$ points is the minimum number of points to guarantee a zero-sum is called the *Other Kemnitz Conjecture*. In 2013, Dr. Stanley [9] proved the *Other Kemnitz Conjecture* for all primes, $p > 2$.



(a) (b) (c)

Figure 1.1: Notice in (a) and (b) that there are $2p-1$ points in the grid. (a) and (b) have enough points to guarantee a balanced subset. The grey points in (a) and (b) are the balanced subsets. Notice in (c) that there are $2p-2$ points in the grid. (c), based on its number of points, could have a balanced subset but is not guaranteed to have one. In this example, (c) does not have a balanced subset.

This work has been instrumental to starting our discussion because all of the sets S we consider are of size $2p-2$, just small enough to not guarantee a zero-sum set. From there, our research has built on to this by classifying what the sets of size $2p-2$ look like and how they behave. The classifications we found have not been published before, and for that reason are considered to be unknown. The classifications make a good pair with the *Other Kemnitz Conjecture* because it shows that $2p-1$ and $2p-2$ is the true boundary where balance is guaranteed/potentially lost. Figure 1.1 shows two sets of size $2p-1$ and one set of size $2p-2$.

To recapitulate, $2p-1$ points guarantee substructure, but $2p-2$ points do not. We are building upon a conjecture from 1983, providing further information about sets of size $2p-2$ that do not have substructure.

This overall experience has certainly made me a better mathematician. Through this process I have learned to persevere; not all solutions come easily or are simple. It takes time to think and be inspired to try a new technique. This means I have had to think creatively. This could be coming up with a new idea, looking at a problem through a different lens (ex. graphic, algebraic, geometric, analytic), or using an old idea in a new environment. Using old techniques in new situations, is a great example of using what one knows to find something one does not know. Ultimately, that is how new information is discovered, and that is how I have been taught to approach every math problem I encounter.

I would be understating the fact if I said that I have grown in my proof writing and communicating abilities. Before this experience, I had only written a proof of at most three pages; this document is in excess of twenty pages. Never have I ever made such large sections, chapter two and three supporting the work done in chapter four. Before, I did not consider all of the extra text needed to help convince the reader of what was being proven. When it comes to the syntax of the paper, I was shown and recognized a natural flow in which to state assumptions and conclusions. This experience allowed me to get feedback from both an expert and non-expert, regarding the subject of the thesis. Getting this feedback has shown me not only what it takes to write a quality paper but how to make this material both accurate and accessible.

Collaborating with Dr. Stanley, has helped show me that mathematics is a very team oriented endeavor; yes, someone can accomplish/prove something on their own, but it is often more effective, satisfying, and exciting to work as a team. This type of cohesive teamwork is something I hope to continue in my professional career.

Chapter 2

Definitions

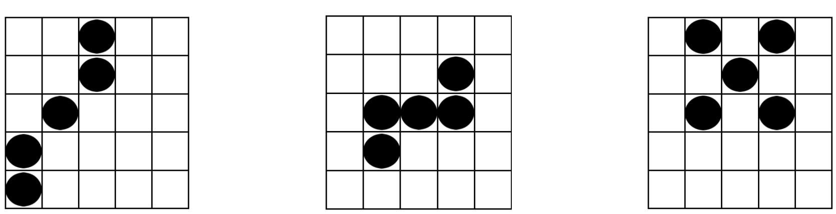
This chapter is filled with definitions useful in the lemmas and proofs to follow.

Definition 1. *Let* $B = \{(x_0, y_0), (x_1, y_1), (x_2, y_2), ..., (x_{p-2}, y_{p-2}), (x_{p-1}, y_{p-1})\}$ *be a set in* $\mathbb{Z}_p \times \mathbb{Z}_p$*. Notice,* $|B| = p$*. We call* B *a* ***Balanced Set*** *if* $\sum_{n=0}^{p-1} x_n = 0$ *and* $\sum_{n=0}^{p-1} y_n = 0$*, all arithmetic in* $\mathbb{Z}_p$*.*

For the duration of the paper, let S satisfy the following:

1. $S \subset \mathbb{Z}_p \times \mathbb{Z}_p$
2. $|S| = 2p - 2$
3. S does not contain a balanced subset
4. S has two points in $p - 1$ columns and no points in 1 column.

The structure of S is significant in this paper; if S takes on a certain form, which will be described later, then S is isomorphic to *Kick It*.



(a) (b) (c)

Figure 2.1: These are examples of possible sets B that are balanced.

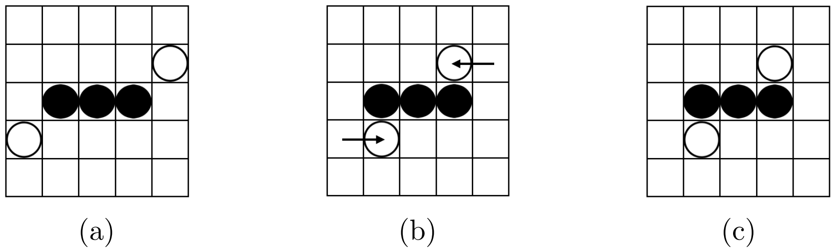


Figure 2.2: Notice how (a) is balanced. In (b), we shift one point left and one point right the same distance. (c) is the new set, and it is also balanced.

Definition 2. *Let $B \subset S$. We define the x-sum of B as*

$$\sum_x B = \sum \{x \mid \exists y \in \mathbb{Z}_p((x,y) \in B)\}.$$

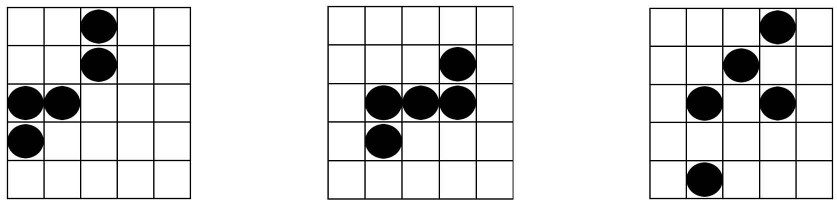
We define the y-sum, $\sum_y B$, similarly.

We now define subsets of $\mathbb{Z}_p \times \mathbb{Z}_p$ that have significant and convenient forms. Observe that a set of p points in $\mathbb{Z}_p \times \mathbb{Z}_p$ where each point is in a different column will balance in the x-direction. However, there is more than one way for a set to balance in the x-direction. Notice, from the p points in different columns, if one point is shifted to the left one column, the x-sum becomes $p-1$, and if one point is shifted to the right one column, the x-sum becomes $p+1$. Thus, if a point is shifted to the left into a different column, another point must be shifted the same amount to the right in order to maintain balance. Notice after the shifts, two columns will now have two points, and two columns will not have any points. The graphs in Figure 2.2 illustrate this fact.

Definition 3. *Let $i, j \in \mathbb{Z}_p$. We say that i, j are* ***cooperative*** *if $i \neq j$ and $i, j, (i+j+1) \neq p-1$.*

Definition 4. *Let i, j be cooperative. Let* ***$\mathcal{B}$(i,j)*** *be the family of sets $B \subset S$ where:*

1. *$|B| = p$*
2. *B contains 2 points in the i and j columns*
3. *B contains no points in the $(i+j+1)$ and $(p-1)$ columns*



(a) $B \in \mathcal{B}(0,2)$ (b) $B \in \mathcal{B}(1,3)$ (c) $B \in \mathcal{B}(1,3)$

Figure 2.3: Notice (a) and (c) do not balance in the y-direction and (b) does.

4. *B contains exactly* 1 *point in every other column.*

Lemma 2.1. For each i and $j \in \mathbb{Z}_p$ that are cooperative and each $B \in \mathcal{B}(i,j)$ we have

$$\sum_x B = 0.$$

That is B balances in the x-direction.

Proof. Note, that selecting one point from every column would create balance in the x-direction:

$$\sum_{i=0}^{p-1} i = 0.$$

Let $B \in \mathcal{B}(i,j)$ be as stated in the lemma. We do not have a point from every column; in two columns we have two points. Let us look at adding the extra points' x values and subtracting the x values that do not add to the sum (the empty columns). Thus, the x-sum is

$$\begin{aligned}\left(\sum_{i=0}^{p-1} i\right) + (i+j) - ((p-1) + (i+j+1)) &= 0 + (i+j) - ((p-1) + (i+j+1)) \\ &= i + j - p + 1 - i - j - 1 \\ &= -p \\ &= 0.\end{aligned}$$

Thus B balances in the x-direction. $\square$

By maintaining balance in the x-direction (using subsets of the form $B \in \mathcal{B}(i,j)$), we are left with discerning if the sum in the y-direction balances (sums to 0).

We use $\mathcal{Y}$ to simplify notation in the main proof. As well, this is where we specify the special form of S that will be shown to be isomorphic to *Kick It*.

Definition 5. *For each $i \in \mathbb{Z}_p$, we define the following.*

1. *Let Y_i be the y-values of the i-th column of S, that is,*

$$Y_i = \{y \in \mathbb{Z}_p : (i, y) \in S\}.$$

2. *For all $i \in \mathbb{Z}_p$ let $\{y_i, y_i + k_i\} = Y_i$ so that k_i is minimal if $Y_i \neq \emptyset$, else let $y_i = 0$.*

3. *Let $\mathcal{Y} = \sum_{i \in \mathbb{Z}_p} y_i$.*

Let us also define some notation that will simplify our main proof. The definition to follow helps us look at the y-sums of our $B \in \mathcal{B}(i,j)$'s.

Definition 6. *Let $i, j \in \mathbb{Z}_p$ be cooperative. We call*

$$y_i + y_j - y_{i+j+1}$$

a ***configuration****.*

A configuration is an algebraic representation of which columns of $B \in \mathcal{B}(i,j)$ contain 2 points (i and j), which column (other than $p-1$) contains 0 points ($i+j+1$), and all remaining columns containing 1 point where $\sum_x B = 0$. When we talk about configurations, we focus on the y-sum because configurations are based off of $B \in \mathcal{B}(i,j)$'s which are already balanced in the x-direction. Recall, we focus on the y-sums because S does not have a balanced subset. So we will consider all possible y-sums for $B \in \mathcal{B}(i,j)$ where i and j are cooperative. As each B is balanced in the x-direction, its y-sum cannot be zero.

This can be visualized as a balance with a fulcrum, where the columns of empty points and 2 points must maintain the balance about the fulcrum.

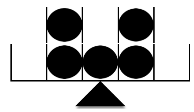


Figure 2.4: $B \in \mathcal{B}(1,3)$ where the y-sum is not balanced.

Definition 7. *For each cooperative $i, j \in \mathbb{Z}_p$ the **minimum y-sum** is*

$$\begin{aligned} \min\left\{\sum_y B \mid B \in \mathcal{B}(i,j)\right\} &= y_i + (y_i + k_i) + y_j + (y_j + k_j) + \sum_{\substack{m \neq i,j,\\ p-1, i+j+1}} y_m \\ &= y_i + y_j - y_{i+j+1} + \mathcal{Y}. \end{aligned}$$

Let us note that the minimum y-sum of configurations will be considered heavily in the main proof. Not only is the minimum y-sum important, but in the next chapter it will be shown that possible y-sums of a given configuration are intervals.

Again, to motivate the next chapter and use our stated definitions, let it be known that the main strategy in our main proof hinges on supposing balance in the x-direction by using $B \in \mathcal{B}(i,j)$'s and using the fact that the y-sum for any $B \in \mathcal{B}(i,j)$ does not equal zero. From there, we show the values y_i must take on.

Chapter 3

Technical Proofs

3.1 Transformation Lemmas

These proofs are instrumental. In the first section, we show a few functions are bijective. These transformations allow us to make some assumptions. Namely, in the main proof, they allow us to assume $y_0 = 0$ and $\mathcal{Y} = 0$. As well, these help us show that there are many balance sets that are isomorphic to others.

In the second section, we reveal properties that give us more information on the y-sums of our set S. They are quoted and used in the main theorem.

Lemma 3.1 (**Bijective Shift**)**.** Let $a, b \in \mathbb{Z}_p$. Let $f : \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{Z}_p \times \mathbb{Z}_p$ where $f(i, j) = (i + a, j + b)$. f is bijective.

Proof. Let a, b, and f be as stated in the lemma. We will show f is injective. Let $(i_0, j_0), (i_1, j_1) \in \mathbb{Z}_p \times \mathbb{Z}_p$ (domain) such that $f(i_0, j_0) = f(i_1, j_1)$. We will show that $(i_0, j_0) = (i_1, j_1)$. We have,

$$f(i_0, j_0) = (i_0 + a, j_0 + b)$$

and

$$f(i_1, j_1) = (i_1 + a, j_1 + b).$$

Since $f(i_0, j_0) = f(i_1, j_1)$, $(i_0 + a, j_0 + b) = (i_1 + a, j_1 + b)$. So, $i_0 + a = i_1 + a$ and $j_0 + b = j_1 + b$. Thus, $i_0 = i_1$ and $j_0 = j_1$. Thus, $(i_0, j_0) = (i_1, j_1)$. Hence, f is injective.

We will show f is surjective. Let $(x, y) \in \mathbb{Z}_p \times \mathbb{Z}_p$ (co-domain). We will find (i, j) such that $f(i, j) = (x, y)$. Let $i = x - a$ and $j = y - b$. Thus, $f(i, j) = f(x - a, y - b) = (x - a + a, y - b + b) = (x, y)$. Hence, f is surjective. $\square$

Corollary 3.1 (**Preservation of Balance**). Let $a, b \in \mathbb{Z}_p$. Let B be a balanced set in $\mathbb{Z}_p \times \mathbb{Z}_p$ and $f : \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{Z}_p \times \mathbb{Z}_p$ where $f(i, j) = (i+a, j+b)$. f preserves balance; that is, $f(B)$ is a balanced set.

Proof. Let a, b and f be as stated in the corollary. Let

$$B = \{(i_0, j_0), (i_1, j_1), (i_2, j_2), ..., (i_{p-2}, j_{p-2}), (i_{p-1}, j_{p-1})\}$$

be a balanced set in $\mathbb{Z}_p \times \mathbb{Z}_p$, meaning $\sum_{n=0}^{p-1} i_n = 0$ and $\sum_{n=0}^{p-1} j_n = 0$. We will show that $f(B)$ is also a balanced set (B after a shift) where $a, b \in \mathbb{Z}_p$. Notice we have,

$$\begin{aligned}\sum_{n=0}^{p-1}(i_n + a) &= \sum_{n=0}^{p-1} i_n + \sum_{n=0}^{p-1} a \\ &= 0 + p(a) \\ &= 0 + 0 = 0.\end{aligned}$$

Similarly,

$$\begin{aligned}\sum_{n=0}^{p-1}(j_n + b) &= \sum_{n=0}^{p-1} j_n + \sum_{n=0}^{p-1} b \\ &= 0 + p(b) \\ &= 0 + 0 = 0.\end{aligned}$$

Therefore, balance is preserved as the sum of the coordinates is 0. $\square$

Lemma 3.2 (**Bijective Stretch**). Let $a, b \in \mathbb{Z}_p \setminus \{0\}$. Let $f : \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{Z}_p \times \mathbb{Z}_p$ where $f(i, j) = (ai, bj)$. f is bijective.

Proof. Let a, b and f be as stated in the lemma. We will show f is injective. Let $(i_0, j_0), (i_1, j_1) \in \mathbb{Z}_p \times \mathbb{Z}_p$ (domain) such that $f(i_0, j_0) = f(i_1, j_1)$. We will show that $(i_0, j_0) = (i_1, j_1)$. Notice we have,

$$f(i_0, j_0) = (ai_0, bj_0)$$

and

$$f(i_1, j_1) = (ai_1, bj_1).$$

Since $f(i_0, j_0) = f(i_1, j_1)$, $(ai_0, bj_0) = (ai_1, bj_1)$. So, $ai_0 = ai_1$ and $bj_0 = bj_1$. Since $a, b \neq 0$, $i_0 = i_1$ and $j_0 = j_1$. Thus, $(i_0, j_0) = (i_1, j_1)$. Hence, f is injective.

We will show f is surjective. Let $(x, y) \in \mathbb{Z}_p \times \mathbb{Z}_p$ (co-domain). We will find (i, j) such that $f(i, j) = (x, y)$. Since $a, b \neq 0$, a^{-1} and b^{-1} exist in $\mathbb{Z}_p$. Let $i = a^{-1}x$ and $j = b^{-1}y$. Thus, $f(i, j) = f(a^{-1}x, b^{-1}y) = (a^{-1}x * a, b^{-1}y * b) = (x, y)$. Hence, f is surjective. $\square$

Corollary 3.2 (**Preservation of Balance**). Let $a, b \in \mathbb{Z}_p \setminus \{0\}$. Let B be a balanced set in $\mathbb{Z}_p \times \mathbb{Z}_p$ and $f : \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{Z}_p \times \mathbb{Z}_p$ where $f(i, j) = (ai, bj)$. f preserves balance such that $f(B)$ is a balanced set.

Proof. Let a, b and f be as stated in the corollary. Let

$$B = \{(i_0, j_0), (i_1, j_1), (i_2, j_2), ..., (i_{p-2}, j_{p-2}), (i_{p-1}, j_{p-1})\}$$

be a balanced set in $\mathbb{Z}_p \times \mathbb{Z}_p$, meaning $\sum_{n=0}^{p-1} i_n = 0$ and $\sum_{n=0}^{p-1} j_n = 0$. We will show that $f(B)$ is also a balanced set (B after a stretch). Notice, if $a$or b were 0, the function would no longer be bijective.) Notice we have,

$$\begin{aligned} \sum_{n=0}^{p-1} ai_n &= a \sum_{n=0}^{p-1} i_n \\ &= (a)0 = 0. \end{aligned}$$

Similarly,

$$\begin{aligned} \sum_{n=0}^{p-1} bj_n &= b \sum_{n=0}^{p-1} j_n \\ &= (b)0 = 0. \end{aligned}$$

Therefore, balance is preserved as the sum of the coordinates is 0. $\square$

Lemma 3.3 (**Bijective Slant**). Let $a, b, c, d \in \mathbb{Z}_p$ where $bc \neq ad$. Let $f : \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{Z}_p \times \mathbb{Z}_p$ where $f(i, j) = (ai + bj, ci + dj)$. f is bijective.

Proof. Let a, b, c, d, and f be as stated in the lemma. We will show f is injective. Let $(i_0, j_0), (i_1, j_1) \in \mathbb{Z}_p \times \mathbb{Z}_p$ (domain) such that $f(i_0, j_0) = f(i_1, j_1)$. We will show that $(i_0, j_0) = (i_1, j_1)$. We have

$$f(i_0, j_0) = (ai_0 + bj_0, ci_0 + dj_0)$$

and

$$f(i_1, j_1) = (ai_1 + bj_1, ci_1 + dj_1).$$

Since $f(i_0, j_0) = f(i_1, j_1)$,

$$ai_0 + bj_0 = ai_1 + bj_1$$

and

$$ci_0 + dj_0 = ci_1 + dj_1.$$

This implies

$$aci_0 + bcj_0 = aci_1 + bcj_1$$

and

$$aci_0 + adj_0 = aci_1 + adj_1.$$

Thus,

$$(bc - ad)j_0 = (bc - ad)j_1.$$

As $bc \neq ad$, $(bc - ad)^{-1} \in \mathbb{Z}_p$. Thus, $j_0 = j_1$. Similarly, $i_0 = i_1$, and so f is injective.

Now we will show that f is surjective. Let $(k, m) \in \mathbb{Z}_p \times \mathbb{Z}_p$ (co-domain). We will find (i, j) such that $f(i, j) = (k, m)$. Since $bc \neq ad$, $(bc - ad)^{-1} \in \mathbb{Z}_p$. Let

$$i = (bm - dk)(bc - ad)^{-1}$$

and

$$j = (ck - am)(bc - ad)^{-1}.$$

From here, it can be shown (proof left to the reader) that f is surjective. $\square$

Corollary 3.3 (**Preservation of Balance**). Let $a, b, c, d \in \mathbb{Z}_p$ where $bc \neq ad$. Let B be a balanced set in $\mathbb{Z}_p \times \mathbb{Z}_p$ and $f : \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{Z}_p \times \mathbb{Z}_p$ where $f(i, j) = (ai + bj, ci + dj)$. f preserves balance such that $f(B)$ is a balanced set.

Proof. Let a, b, c, d and f be as stated in the corollary. Let

$$B = \{(i_0, j_0), (i_1, j_1), (i_2, j_2), ..., (i_{p-2}, j_{p-2}), (i_{p-1}, j_{p-1})\}$$

be a balanced set in $\mathbb{Z}_p \times \mathbb{Z}_p$, meaning $\sum_{n=0}^{p-1} i_n = 0$ and $\sum_{n=0}^{p-1} j_n = 0$. We will show that $f(B)$ is also a balanced set (B after a slant). Notice we have,

$$\begin{aligned} \sum_{n=0}^{p-1} ai_n + bj_n &= \sum_{n=0}^{p-1} ai_n + \sum_{n=0}^{p-1} bj_n \\ &= a\sum_{n=0}^{p-1} i_n + b\sum_{n=0}^{p-1} j_n \\ &= a(0) + b(0) = 0 + 0 = 0. \end{aligned}$$

Similarly,

$$\begin{aligned} \sum_{n=0}^{p-1} ci_n + dj_n &= \sum_{n=0}^{p-1} ci_n + \sum_{n=0}^{p-1} dj_n \\ &= c\sum_{n=0}^{p-1} i_n + d\sum_{n=0}^{p-1} j_n \\ &= c(0) + d(0) = 0 + 0 = 0. \end{aligned}$$

Therefore, balance is preserved as the sum of the coordinates is 0.

□

3.2 Configurations

Below is the Cauchy-Davenport Inequality [3] [4]; it gives us the range of possible y-sums for each $B \in \mathcal{B}(i, j)$. Later, this is coupled with the Interval Lemma in order to state that the y-sums that give balance are in an interval. Recall, the main strategy in our main proof hinges on supposing balance in the x-direction by using $B \in \mathcal{B}(i, j)$'s and using the fact that the y-sum for any $B \in \mathcal{B}(i, j)$ does not equal zero. From there, we show the values y_i must take on.

Lemma 3.4 (Cauchy-Davenport Inequality)**.** If $A_0, A_1, ..., A_n$ are non-empty subsets of $\mathbb{Z}_p$, then

$$\left|\sum_{i=0}^{n} A_i\right| \geq \min\{p, \sum_{i=0}^{n} |A_i| - n\}.$$

The following is a useful lemma from Dr. Stanley [9]. It shows that the possible y-sums of the counter example sets (namely *Kick It*) are in an interval.

Lemma 3.5 (Interval Lemma)**.** Suppose $A_1, ..., A_i \subset \mathbb{Z}_p$ such that each A_i has the form $\{y_i, y_i + 1\}$. Then,

$$\sum_{i=1}^{n} A_i = \left[\sum_{i=1}^{n} y_i, \left(\sum_{i=1}^{n} y_i\right) + n\right].$$

In particular, $\sum_{i=1}^{n} A_i$ is an interval in $\mathbb{Z}_p$ of length $n + 1$.

Proof. Let $A_1, ..., A_n$ be as stated in the lemma. Let $k \in \mathbb{Z}$ so that $0 \leq k \leq n$. Then,

$$\left(\sum_{i=1}^{n} y_i\right) + k = \sum_{i=1}^{k}(y_i + 1) + \sum_{i=k+1}^{n} y_i \in \sum_{i=1}^{n} A_i.$$

Thus,

$$\sum_{i=1}^{n} A_i \subset \left[\sum_{i=1}^{n} y_i, \left(\sum_{i=1}^{n} y_i\right) + n\right].$$

This proves the lemma. □

The following corollary is significant because it helps determine the y-sum interval for most of the configurations used in the main proof. However, there is a time in the main proof when the space bonus is removed, and the y-sum interval for that configuration is smaller.

Corollary 3.4 (Space Bonus)**.** Suppose $A_1, ..., A_n$ be as stated in Lemma 3.5. Let A_n have the form $\{y_n, y_n + 2\}$. Then,

$$\sum_{i=1}^{n} A_i = \left[\sum_{i=1}^{n} y_i, \left(\sum_{i=1}^{n} y_i\right) + n + 2\right].$$

In particular, $\sum_{i=1}^{n} A_i$ is an interval in $\mathbb{Z}_p$ of length $n + 2$.

Proof. Let $A_1, ..., A_n$ be as stated in the corollary. By Lemma 3.5, let

$$\sum_{i=1}^{n-1} A_i = \left[\sum_{i=1}^{n-1} y_i, \left(\sum_{i=1}^{n-1} y_i\right) + (n-1)\right].$$

Then,

$$\begin{aligned}\sum_{i=1}^{n} A_i &= \left[\sum_{i=1}^{n-1} y_i + y_n, \left(\sum_{i=1}^{n-1} y_i\right) + (n-1) + (y_n + 2)\right] \\ &= \left[\sum_{i=1}^{n} y_i, \left(\sum_{i=1}^{n} y_i\right) + n + 1\right]\end{aligned}$$

□

The following is a direct consequence of the Interval Lemma from Stanley [9].

Lemma 3.6. Let $p > 7$ be prime. Let $|Y_k| = 2$ for $k \in \mathbb{Z}_p$ with $k \neq p-1$. Let $i, j \in \mathbb{Z}_p$ be cooperative. Then, the set of all possible y-sums below,

$$\left\{\sum_{y} B \mid B \in \mathcal{B}(i,j)\right\}$$

is an interval.

Chapter 4

KICK IT

We now prove our main theorem. For this counter example, we will show that S can take on two forms, both of which are isomorphic to *Kick It*. In the proof, a common thread of techniques is based on algebraic manipulations. In the statement of the theorem, columns correspond to a particular first coordinate of a point in S. When intervals of points in a column are mentioned, the points' second coordinates are being discussed.

Theorem 4.1. Let $S \subset \mathbb{Z}_p \times \mathbb{Z}_p$ satisfying the following:

1. $|S| = 2p - 2$
2. S does not have a balanced subset
3. S has one empty column, the remaining columns have 2 points each
4. S has $p - 2$ columns whose points are non-trivial intervals
5. S has one column of the form $\{a, a + 2\}$.

Then, S is isomorphic to *Kick It*.

Proof. By the transformation lemmas, without loss of generality, we can let $y_0 = 0$ and $\mathcal{Y} = 0$. By the Cauchy-Davenport Inequality [3] [4], the length of the interval of y-sums will be

$$\begin{aligned} 2(p-4) - (p-5) + 1 &= 2p - 8 - p + 5 + 1 \\ &= p - 2. \end{aligned}$$

Thus, in order to avoid balance, the y-sums must be in the set $\{1, 2\}$. If the y-sum was not in $\{1, 2\}$, then the interval of y-sums would include 0, allowing for balance to occur. Using our usual configuration, we have

$$\begin{aligned}(y_i + 1) + (y_j + 1) - y_{i+j+1} + \mathcal{Y} &\in \{1, 2\}\\ y_i + y_j - y_{i+j+1} &\in \{-1, 0\}.\end{aligned}$$

This is useful; notice the configuration $y_0 + y_j - y_{j+1} \in \{-1, 0\}$ simplifies to

$$y_j - y_{j+1} \in \{-1, 0\} \qquad j \neq 0, p-3, p-2.$$

We call this property of consecutive points: *snakiness*. When *snakiness* is used, it is natural to talk about taking a step. Taking one step would be applying *snakiness* to one j value. However, notice that it can be applied across multiple j values, resulting in a "step" for each j value. In order to indicate when *snakiness* is being used across multiple j values, a starting y_j will be stated.

Now, in order to prove that *Kick It* must occur, we will proceed to look at how configurations relate, use our assumptions of $y_0 = 0$ and $\mathcal{Y} = 0$, and use *snakiness*. We will now gather information about $y_{\frac{p-3}{2}}$, $y_{\frac{p-1}{2}}$, and y_{p-2}.

Using configuration $y_{\frac{p-3}{2}} + y_{\frac{p+1}{2}} - y_0 \in \{-1, 0\}$, we have

$$y_{\frac{p-3}{2}} + y_{\frac{p+1}{2}} \in \{-1, 0\}.$$

Using *snakiness* across two steps (two different j values), we have the difference

$$y_{\frac{p-3}{2}} - y_{\frac{p+1}{2}} \in \{-2, -1, 0\}.$$

Now, add the two previous equations:

$$\begin{aligned}(y_{\frac{p-3}{2}} + y_{\frac{p+1}{2}}) + (y_{\frac{p-3}{2}} - y_{\frac{p+1}{2}}) &\in \{-1, 0\} + \{-2, -1, 0\}\\ 2(y_{\frac{p-3}{2}}) &\in \{-3, -2, -1, 0\}\\ y_{\frac{p-3}{2}} &\in \left\{\frac{p-3}{2}, -1, \frac{p-1}{2}, 0\right\}\end{aligned}$$

Now let us find $y_{\frac{p-1}{2}}$. By using the general configuration,

$$\begin{aligned}y_i + y_{p-i-1} - y_0 &\in \{-1, 0\}\\ y_i + y_{p-i-1} &\in \{-1, 0\}.\end{aligned}$$

Notice that each y_i has a partner (y_{p-i-1}) and their sum is either -1 or 0 for $1 < i < \frac{p-1}{2}$. Sacrificing the space bonus while applying the Cauchy-Davenport Inequality [3] [4], we have the configuration $y_1 + y_{p-2} - y_0 = y_1 + y_{p-2} \in \{-2, -1, 0\}$. This is another pair but will be considered on its own since its sum is in a different set.

Exactly how many pairs of points are there? Well, y_{p-1} is nonexistent, y_{p-2} and y_1 are in their own unique pair, $y_{\frac{p-1}{2}}$ is the middle/does not have a configuration where y_0 is subtracted, and y_0 is not able to be used in a pair configuration because y_0 cannot be both added and subtracted (although it can be thought of as the pair of y_{p-1}). Since there are five y values that are not in a partnership, there are a total of $(p-5)/2$ pairs. Using this setup, let us find the possible $y_{\frac{p-1}{2}}$ values.

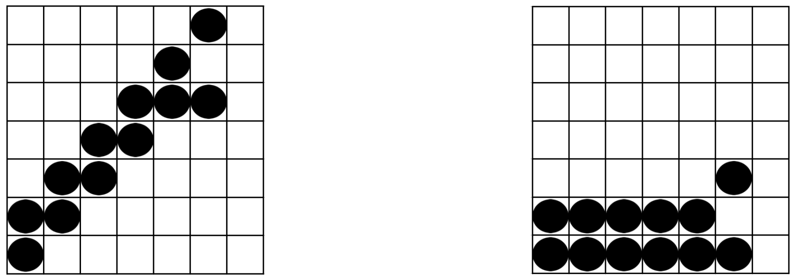
$$\begin{aligned} \mathcal{Y} &= pairs + y_0 + y_{\frac{p-1}{2}} + (y_1 + y_{p-2}) \\ 0 &\in \left[-\frac{p-5}{2}, 0\right] + 0 + y_{\frac{p-1}{2}} + [-2, 0] \\ -y_{\frac{p-1}{2}} &\in \left[\frac{-p+1}{2}, 0\right] \\ y_{\frac{p-1}{2}} &\in \left[0, \frac{p-1}{2}\right] \end{aligned} \tag{4.1}$$

Now let us find y_{p-2}. Let us consider the following configuration:

$$\begin{aligned} y_2 + y_{p-2} - y_1 &\in \{-2, -1, 0\} \\ y_{p-2} - (y_1 - y_2) &\in \{-2, -1, 0\} \\ y_{p-2} - \{-1, 0\} &\in \{-2, -1, 0\} \\ y_{p-2} + \{1, 0\} &\in \{-2, -1, 0\} \\ y_{p-2} &\in \{-3, -2, -1, 0\} \end{aligned}$$

Now that we have some values to work with, let us see what conclusions we can draw.

Case 1. Suppose $y_{\frac{p-1}{2}} = \frac{p-1}{2}$. Since $y_{\frac{p-1}{2}} = \frac{p-1}{2}$ is the extreme negative case (see equation 4.1), the pairs must sum to $-\frac{p-5}{2}$ and $y_1 + y_{p-2} = -2$. Since $y_1 + y_{p-2} = -2$ and $y_{p-2} \in \{-3, -2, -1, 0\}$, then $y_1 \in \{-2, -1, 0, 1\}$. By *snakiness* from $y_{\frac{p-1}{2}}$, $y_1 = 1$; note that *snakiness* forces each $y_i = i$ for



(a) *Kick It* before slant and shift (b) *Kick It* after slant and shift

Figure 4.1: Example shown in $\mathbb{Z}_7 \times \mathbb{Z}_7$

$1 < i < \frac{p-1}{2}$. Since $y_i = i$ and $y_i + y_{p-i-1} = -1$ for $1 < i < \frac{p-1}{2}$, then $y_{p-i-1} = p - i - 1$. Since $y_1 + y_{p-2} = -2$ and $y_1 = 1$, $y_{p-2} = -3$.

This is isomorphic to *Kick It* (in the line of slope 1). Thus, in order to get it into its recognizable form, we will perform a slant and shift $f : (i, j) \rightarrow (i, i - j + 1)$. See Figure 4.1 (a).

Case 2. Suppose $y_{\frac{p-1}{2}} = \frac{p-3}{2}$. Since $y_{\frac{p-1}{2}} = \frac{p-3}{2}$ and $y_{p-2} \in \{-3, -2, -1, 0\}$, by *snakiness* from $y_{\frac{p-1}{2}}$, $y_{p-2} = -3$. Since $y_{p-2} = -3$ and $y_1 + y_{p-2} \in \{-2, -1, 0\}$, $y_1 \in \{1, 2, 3\}$. However, let us consider a group of configurations that all involve adding y_{p-2} :

$$\begin{aligned} y_{i+1} + y_{p-2} - y_i &\in \{-2, -1, 0\} \qquad 0 < i < \frac{p-1}{2} \\ y_{i+1} - y_i &\in \{1, 2, 3\} \qquad 0 < i < \frac{p-1}{2} \end{aligned}$$

This means y_i must increase by 1, 2, or 3 . By *snakiness* however, we know that each y_i can be the same or one greater than the previous. Thus, $y_{i+1} - y_i = 1$ for $0 < i < \frac{p-1}{2}$. Since $y_{i+1} - y_i = 1$ for $0 < i < \frac{p-1}{2}$ (using this *snakiness* property from $y_{\frac{p-1}{2}}$), $y_1 = 0$. Contradiction.

Case 3. Suppose $y_{\frac{p-1}{2}} \in [1, \frac{p-5}{2}]$. The distance between $y_{\frac{p-1}{2}}$ and y_{p-2} is

$$(p-2) - (\frac{p-1}{2}) = \frac{p-3}{2}.$$

We can apply *snakiness* to the $\frac{p-3}{2}$ steps from $y_{\frac{p-1}{2}}$ to y_{p-2}. Thus, by *snakiness*, $y_{p-2} \in [1, -4]$. Contradiction.

Case 4. Suppose $y_{\frac{p-1}{2}} = 0$. We will use a *snakiness* argument that will force all y_i to equal 0, thus making S isomorphic to *Kick It*.

Let us recall that $y_{\frac{p-1}{2}} = 0$ is an extreme case (see equation 4.1) in which each set of pairs sum to zero, including the y_1, y_{p-2} pair. Again, let us look at y_{p-2}.

Since $y_{p-2} \in \{-3, -2, -1, 0\}$ and $y_1 + y_{p-2} = 0$, $y_1 \in \{0, 1, 2, 3\}$. Since $y_1 \in \{0, 1, 2, 3\}$, by *snakiness* from y_1, $y_{\frac{p-1}{2}} \in [0, \frac{p+3}{2}]$. Thus, the only way for $y_{\frac{p-1}{2}} = 0$, is for $y_1 = 0$. Since $y_1 = 0$ and $y_1 + y_{p-2} = 0$, $y_{p-2} = 0$. Since $y_1 = 0$ and $y_{\frac{p-1}{2}} = 0$, by *snakiness*, $y_i = 0$ for $1 < i < \frac{p-1}{2}$. Since $y_i = 0$ and $y_i + y_{p-i-1} = 0$ for $1 < i < \frac{p-1}{2}$, $y_{p-i-1} = 0$ for $1 < i < \frac{p-1}{2}$. Thus $y_i = 0$ for $0 \leq i \leq p-2$. Thus *Kick It* occurs.

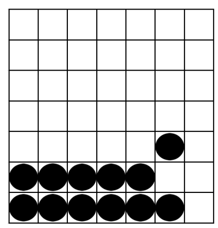


Figure 4.2: *Kick It* example in $\mathbb{Z}_7 \times \mathbb{Z}_7$

$\square$

Chapter 5

Conclusion and Future Work

As mentioned before, Kick It is only one of four classifications of counter examples (Superman, Taxi, Kick It, and Score). Even though their proofs were not included in this thesis, we have proven that Superman and Taxi occur when S is in the proper form. We have yet to prove that Score is obtained when set S has the proper form, but it should be noted that significant progress has been made.

Notably, we must also prove if S does not have a balanced subset, then S is guaranteed to have the desired form, this desired form being isomorphic to our counter examples. In general, S has one empty column, the remaining columns have 2 points each. The number of spaces and which columns have "spaces" are what change amongst the four counter examples. If S is isomorphic to Superman, then S has one column of the form $\{y_i, y_i+3\}$. If S is isomorphic to Taxi, then S has two columns of the form $\{y_i, y_i+2\}$. If S is isomorphic to Kick it, then S has one column of the form $\{y_i, y_i+2\}$. Lastly, if S is isomorphic to Score, then all columns of S are of the form $\{y_i, y_i+1\}$.

It is also possible to expand our research of balanced sets and its counter examples into $(\mathbb{Z}_p)^3$; there has been little to no progress made beyond $(\mathbb{Z}_p)^2$.

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