Proceedings of the Iowa Academy of Science

Volume 47 | Annual Issue Article 67

1940

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At a meeting of the Bohemian philosophical society in December 1921 Dr. Jasek reported the discovery among the papers of Bernhard Bolzano of a manuscript in which is described the construction of a continuous non-differentiable function. This function was devised more than thirty years before Weierstrass constructed his now well-known function. But while Weierstrass proved that his function possessed a derivative at no point, Bolzano showed the non-differentiability of his function only at a countable set of points everywhere dense. It is possible however to prove that the Bolzano function possesses a derivative at no point of the interval.

Bolzano's method of constructing his function consists essentially in breaking up a line into a zig-zag line of four segments. He bisected a line PQ at M and then divided PM into four equal parts PP_1, P_1P_2, P_2P_3, P_3M and MQ into four equal parts MQ_1, Q_1Q_2, Q_2Q_3, Q_3Q. (Figure 1) Then he determined Q_3' by reflecting Q_3 in the horizontal line drawn through Q and P_3' by reflecting P_3 in the horizontal line drawn through M. Thus he obtained the zig-zag line made up of 4 segments PP_3', P_3' M, MQ_3' and $Q_3'Q$ (figure 1). He then applied the same process to each

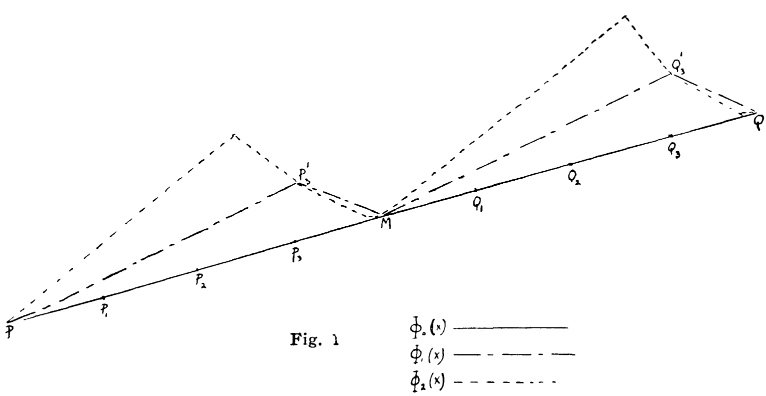

Fig. 1

of these four segments and obtained a zig-zag line composed of 4^2 segments, and so forth. The continuation of this process gives a function which converges to a continuous function which is nowhere differentiable.

If the coordinates of P and Q referred to a rectangular set of axes are (a, A) and (b, B) then the equation of PQ is

$$y = \frac{(b - x)A + (x - a)B}{b - a} = \Phi_0(x)$$

For the line with 4 segments $y = \Phi_1(x)$
" " " " 4^2 " $y = \Phi_2(x)$
" " " " 4^n " $y = \Phi_n(x)$

In order to simplify our work we will bisect the line P_nQ_n then break each half into *two* segments (instead of *four*) having a breadth equal to $\frac{3}{4}$ the breadth of P_nQ_n and a height equal to 3/2 the breadth of P_nQ_n as shown in figure 2.

If h, k and s are the width, breadth and slope of P_nQ_n
then $\frac{3h}{4}$, $\frac{3k}{2}$ and 2s are the width, breadth and slope of P_nR_n
and $\frac{k}{4}$, $\frac{k}{2}$ and $-2s$ are the width, breath and slope of R_nP_{n+1}
Then each rising chord has a slope equal to 2s and each descending chord has a slope equal to $-2s$.

Thus we have an infinite series of continuous functions

$$\Phi_0(x) + [\Phi_1(x) - \Phi_0(x)] + [\Phi_2(x) - \Phi_1(x)] \text{———} + [\Phi_n(x) - \Phi_{n-1}(x)]$$

which is continuous in the interval (a, b) including the boundary. It is evident from the construction that

$$\begin{aligned} |\Phi_n(x) - \Phi_{n-1}(x)| &\leqq |(\tfrac{3}{4})^n k - (\tfrac{3}{4})^{n-1} k| \\ &< (\tfrac{3}{4})^n k \\ &< (\tfrac{3}{4})^n (B - A) \end{aligned}$$

Hence the series converges uniformly for values of x in the given interval by the Weierstrass M test and the Bolzano function $\Phi(x)$ is defined to be

$$\Phi(x) = \Phi_0(x) + \sum_{n=1}^{\infty} [\Phi_n(x) - \Phi_{n-1}(x)].$$

Since the number of terms in each bracket is fixed and the general term approaches zero, the brackets may be removed and the new series will converge to the value of the old one.

$$\therefore \qquad \Phi(x) = \lim_{n \to \infty} \Phi_n(x).$$

In order to study the differentiability of $\Phi(x)$ it is necessary to investigate the existence of a derivative at

(1) the end points P and Q
(2) the "angle-points" such as P_3' and Q_3'
(3) points other than angle points.

1. We have found that the last segment to the right has a

slope $-2^n s$. Hence it is evident that at Q there exists a series of left-hand derivatives having a limiting value $-\infty$. Hence there is no derivative at Q.

At point P, there is a series of right-hand differential quotients of form $2^n s$ which has the limiting value $+\infty$. Therefore there is no derivative at P.

2. At angle points the right-hand derivatives are $-2^n s$ and the corresponding left-hand derivatives are $2^n s$. So there is no derivative at any angle point.

3. Now there remains to consider what happens at any point that is not an angle point.

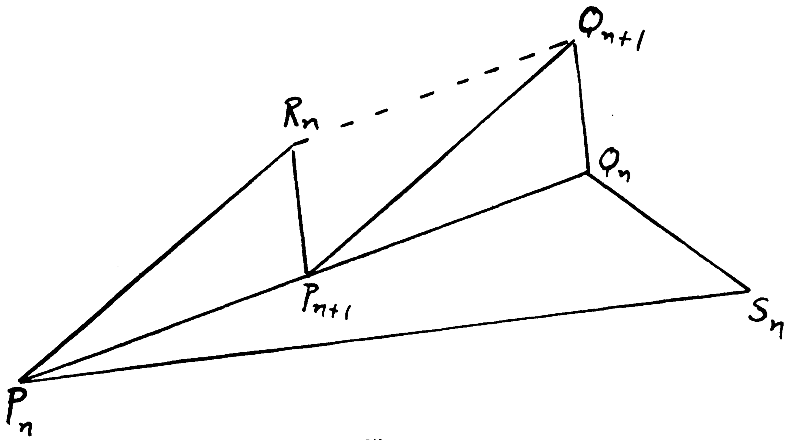

Fig. 2.

Let us take a point A of the curve $y = \Phi(x)$ that lies above $P_{n+1}Q_n$ the upper half of a segment P_nA_n of the curve $\Phi_n(x)$ the length of which approaches zero as n increases (figure 2). If P_nQ_n is rising, then P_nA will have a greater slope than P_nQ_n because A is above P_nQ_n. When these relations are satisfied for infinitely many values of n, then it is said that at the point A there is a series of difference quotients with limiting value $+\infty$. Similarly there is a series of difference quotients, for infinitely many values of n, with limiting value $-\infty$. If a few of the chords P_nQ_n are rising and a few falling then it will happen that almost all of the accompanying chords P_nQ_n are rising or nearly all are falling. By reflection in the y-axis the second case can be brought into the first so it is necessary to investigate only the latter. Then there exists, as before, a series of left-hand difference quotients with limiting value $+\infty$. In order to make sure that there is not perhaps a fixed derivative with value $+\infty$, consideration of figure 2 is sufficient.

It certainly happens for infinitely many values of n that

$P_{n+1}Q_{n+1}$ lies infinitely often above the right half of P_nQ_n. If that were not so, then the chord P_nQ_n, for a certain index, would have an angle point coinciding with point A, which is contrary to our hypothesis that A is not an angle point. Let us think of A as lying above the chord $P_{n+1}Q_{n+1}$. If we designate the coordinates of P_nS_n by h_n and k_n, then the coordinates of P_nQ_n $\frac{3h_n}{4}, \frac{3k_n}{2}$, the coordinates P_nR_n are $\frac{3^2h_n}{2 \cdot 4^2}, \frac{3^2k_n}{2 \cdot 2^2}$ and the coordinates of R_nS_n which equals $P_nS_n - P_nR_h$ are $\frac{23h_n}{32}$ and $-\frac{1k_n}{8}$. The slope of R_nS_n will then be $-\frac{4k_n}{23h_n}$ and this tends to $-\infty$ as n increases. Point A lies infinitely often above R_nS_n so that slope of AS_n is numerically greater than that of R_nS_n, and we have at the point A a well defined series of right-hand difference quotients which converge to $-\infty$. There remains to be considered what happens when Point A, (fig. 2) than corresponding to infinitely many values of n, lies almost always below the chord R_nS_n. The chord AR_n extending from A to the left is steeper than S_nR_n and to the right the chord AQ_{n+1} is steeper than chord R_nQ_{n+1} which is, in turn, steeper than P_nQ_n. Thus we have at the point A a series of left-hand difference quotients with limiting value $-\infty$ and a series of right-hand difference quotients with limiting value $+\infty$.

It has been shown that the Bolzano function $\Phi(x)$, constructed as described above, has a derivative for no value of x on the interior of the interval (a, b) and at the ends of the interval it possesses a one-sided derivative with the limiting value $+\infty$ or $-\infty$. The curve can be reflected in the ordinates drawn at the ends of the interval and thus the function can be extended to a periodic function whose behavior in the interior of all intervals has been established.

REFERENCES

1. KNOPP, K. Stetige nirgends differenzierbare Funktionen Mathematische Zeits. 2 (1), (1918).
2. KOWALEWSKI, G. Bolzanos Verfahren zer Herstellung einer nirgends differenzierbaren stetigen Funktion. Berichte der Wissenschaften zu Leipzig, 1922: 91.
3. PERKINS, F. W. An Elementary Example of a Continuous Non-Differentiable Function. American Mathematical Monthly, 34: 476, (1927).

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