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ON THE CHARACTERIZATION OF THE ALGEBRA OF FOUR FUNDAMENTAL BOOLEAN FUNCTIONS

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Let 0 and 1 denote the null and unit elements of a Boolean algebra B. The following four functions with argument in B and values in B are of fundamental importance in Boolean algebra. They are: the null function, defined by $0x = 0$; the unit function, defined by $1x = 1$; the identity function j, defined by $jx = x$, and the complementary function c, defined by the rule, cx is the complement of x. These four functions, 0, 1, j, c, form a tri-operational algebra in the sense of Karl Menger.*

They are easily seen to admit the following tables relative to the combinations; sum represented by $f+g$; product, represented by $f \bullet g$; substitution represented by fg, where f and g are any functions.

I: $f+g$

	0	1	j	c
0	0	1	j	c
1	1	1	1	1
j	j	1	j	1
c	c	1	1	c

II: $f \bullet g$

	0	1	j	c
0	0	0	0	0
1	0	1	j	c
j	0	j	j	0
c	0	c	0	c

III: fg

	0	1	j	c
0	0	0	0	0
1	1	1	1	1
j	0	1	j	c
c	1	0	c	j

These tables suggest the problem: Characterize this algebra of four elements abstractly. Denoting the algebra by B4, we assume (1) that it is tri-operational in the sense of Karl Menger and satisfies his axioms; (2) the algebra is Boolean with respect to the properties of sum and product; (3) there is an element c such that the four elements c, cc, $c + cc$, $c \bullet cc$ are all distinct and are all the elements of B4; (4) setting $j = cc$, and $0 = j \bullet c$, $1 = c + j$, we assume that $cj = c$, $c0 = 1$, $c1 = 0$.

These conditions are obviously necessary. We shall show that they are sufficient. The question of independence, or minimum assumptions is open.

Tables I and II are immediate consequences of (1) and (2). We proceed to verify III. The fourth row of this table is given by (4). Since substitution is an associative operation, we have:

$$\begin{aligned} jc &= (cc)c = c(cc) = cj = c, \\ jj &= (cc)cc = c(ccc) = cc = j, \\ j0 &= c(c0) = c1 = 0, \\ j1 &= c(c1) = c0 = 1. \end{aligned}$$

The first two rows of the third table follow readily from the formulas:

$$0x = (c \bullet j)x = cx \bullet jx, \quad 1x = (c+j)x = cx + jx,$$

*) Algebra of Analysis, Notre Dame Mathematical Lectures, Number 3 (1944).

the third and fourth rows, tables I and II, together with assumption I, which includes the hypothesis: $(f+g)h = fh+gh$, $(f \bullet g)h = fh \bullet gh$.

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