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Abstract. This paper considers the problem of a thin cylindrical shell sector clamped on two curved edges and one straight edge, free on the remaining straight edge and loaded by a load similar to a hydrostatic load. The method of solution is that of superimposing the solutions of three separate problems with appropriate boundary conditions.

The type of thin cylindrical shell discussed in this paper is similar in some aspects to that used on certain earth moving equipment; for example, a rigid blade on a cleat tractor.

The method of solution of this problem is that of resolving it into three separate problems, and then superimposing the solution of these under appropriate boundary conditions to obtain the desired solution.

The theory used in solving this problem is based on the following assumptions:

1) The shell material is elastic, homogeneous, and isotropic.
2) The thickness of the shell is small compared to its axial length and radius of curvature; and the displacements are small compared to the dimensions of the shell including the thickness; i.e.,

$$\frac{h}{R} << 1 \quad \text{and} \quad \frac{h}{L} << 1 \ .$$

3) The deflections of the middle surface serve as a good qualitative description of the deflections of the entire shell.
4) All stresses normal to the middle surface are neglected.
5) A rectilinear element normal to the middle surface of the shell remains rectilinear and normal to this surface during deformation. (Hypothesis of Navier).

NOTATION

R	radius of shell measured to middle surface
h	thickness of shell
L	length of shell
a	arclength of shell
x	coordinates measured along a generator
z	coordinate measured normal to the shell surface from middle surface
θ	coordinate measured along an arc in the middle surface of the shell

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N_X, N_θ, $N_{\theta X}$ stress resultants in the middle surface (force/unit length)

M_X, M_θ, $M_{\theta X}$ moment resultants (moment/unit length)

Q_X, $Q\theta$ stress resultants normal to middle surface

P_Z normal load

P_θ tangential load

P_X load along generator

u   deflection in the x direction

v   deflection in the θ direction

w   deflection in the z direction

E   Young's Modulus

$\sqrt{}$ Poisson's Ratio
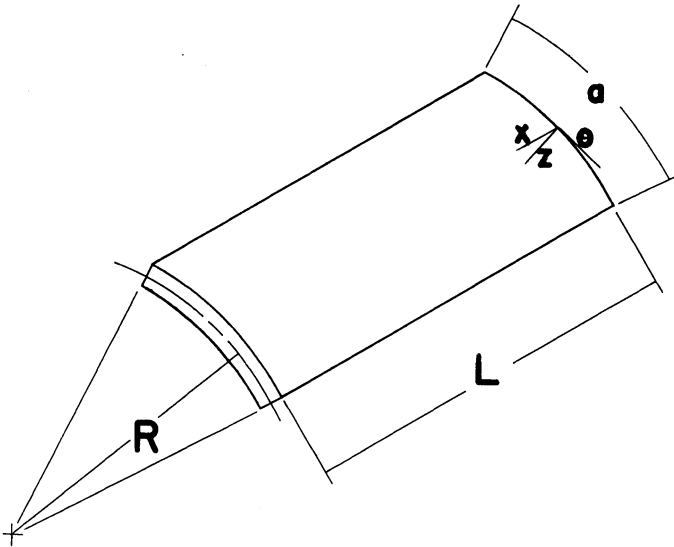
D   Flexural Rigidity

Figure 1.   Shell sector.

BOUNDARY CONDITIONS AND LOADS

The original problem is that of a cylindrical shell sector (Figure 1) clamped on two curved edges and on one straight edge, free on the remaining straight edge, and supporting a load similar to a hydrostatic load.

Boundary Conditions

At the curved edges the shell is clamped but free to move in the x-direction. Then at x=0, L, the boundary conditions are:

$$v = w = 0, \qquad N_X = 0, \qquad \frac{\partial w}{\partial x} = 0.$$

The straight edge $\theta = \eta$ is assumed to be clamped, or built in, so there is no motion in the θ-direction. Then at $\theta = \eta$:

$$u = v = w = 0, \qquad \frac{\partial w}{\partial \theta} = 0.$$

The free edge is at $\theta = -\eta$. The boundary conditions at $\theta = -\eta$ are:

$$N_\theta = 0, \quad M_\theta = 0 \quad N_{\theta x} = \frac{-M_{\theta x}}{R}, \quad Q_\theta = -\frac{\partial M_{\theta x}}{\partial x} \; .$$

Loading Conditions

The shell under consideration is loaded in the region $0 \leqq \theta \leqq \eta$, $0 \leqq x \leqq L$ by a load P which acts perpendicularly to the chord of the shell sector.
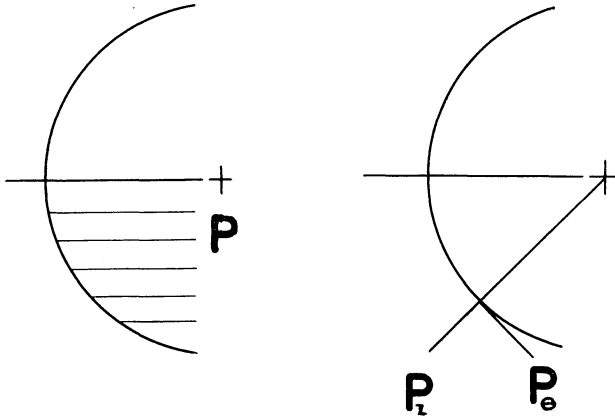

Figure 2.   *Left.* Cross section of shell sector under the applied load. *Right.* Normal and tangential components of the load.

The normal and tangential components of the load are, (Figure 2),

$$P_Z = P \cos \theta, \quad P_\theta = P \sin \theta, P_X = 0.$$

I. The first auxiliary problem is that of a shell sector simply-supported on two curved edges and one straight edge, free on the remaining straight edge, and loaded by the prescribed load of the original problem.

Boundary Conditions I

$$0 \leqq \theta \leqq \eta \qquad 0 \leqq x \leqq L$$
$$w = v = 0 \qquad M_X = 0 \qquad N_X = 0 \quad \text{at } x = 0, L.$$
$$u = w = 0 \qquad M_\theta = 0 \qquad N_\theta = 0 \quad \text{at } \theta = \eta .$$
$$-\eta \leqq \theta \leqq 0 \qquad 0 \leqq x \leqq L$$
$$w_1 = v_1 = 0, \qquad M_{X1} = 0, \qquad N_{X1} = 0, \quad \text{at } x = 0, L.$$

$$N_{\theta 1} = 0, \; M_{\theta 1} = 0, \; Q_{\theta 1} = -\frac{1}{R} \frac{\partial M_{\theta X1}}{\partial \xi}, \quad N_{\theta X1} = -\frac{M_{\theta X1}}{R},$$

at $\theta = -\eta$.

Loading Conditions I

$$0 \leqq \theta \leqq \eta \qquad\qquad 0 \leqq x \leqq L$$
$$P_Z = P \cos, \qquad P_\theta = P \sin \theta, \qquad P_X = 0.$$
$$-\eta \leqq \theta \leqq 0 \qquad\qquad 0 \leqq x \leqq L$$
$$P_Z = P_\theta = P_X = 0.$$

The following conditions must hold at $\theta = 0$ in order to assume the continuity of the deflections and the stresses:

$$w_1 = w \qquad v_1 = v \qquad u_1 = u \qquad M_{\theta 1} = M_\theta$$
$$Q_{\theta 1} = Q_\theta \qquad N_{X\theta 1} = N_{X\theta} \qquad N_{\theta 1} = N_\theta$$

for $0 \leqq x \leqq L$. $\qquad \dfrac{\partial w_1}{\partial \theta} = \dfrac{\partial w}{\partial \theta}$

II. The second auxiliary problem is that of a shell simply-supported on two straight edges and loaded by uniformly distributed moments on the two curved edges. The moment distribution on one edge is equal in magnitude, but opposite in sign to that on the other edge. The solution obtained for this type of shell problem is also a good approximation for a shell sector with one free straight edge.

Boundary Conditions II

Since both curved edges are loaded with a bending moment of like magnitude it seems reasonable to assume the following three conditions.

$$w \Big|_{x=0} = w \Big|_{x=L} \qquad ,$$

$$u \Big|_{x=0} = u \Big|_{x=L} \qquad ,$$

$$-\frac{\partial w}{\partial x}\Big|_{x=0} = \frac{\partial w}{\partial x}\Big|_{x=L} \qquad .$$

Also,

$$u = w = 0, \qquad N_\theta = 0, \qquad M_\theta = 0 \qquad \text{at } \theta = \pm \eta$$

Loading Conditions II

$$M_X = -M_1 \qquad \text{at} \quad x = 0.$$
$$M_X = M_1 \qquad \text{at} \quad x = L.$$

III. The third auxiliary problem is that of a shell simply-supported on two curved edges, free on one straight edge and loaded by a uniformly distributed bending moment on the remaining straight edge.

Boundary Conditions III

$$v = w = 0, \qquad M_x = 0, \qquad N_x = 0 \qquad at \quad x = 0, L.$$

$$N_\theta = 0, \quad M_\theta = 0, \quad N_{\theta x} = -\frac{M_{\theta x}}{R}, \quad Q_\theta = -\frac{\partial M_{\theta x}}{\partial x}$$

$$at \quad \theta = 2\eta;$$

where the substitution $\phi = \eta - \theta$ has been made in order to simplify the resulting expressions.

Loading Conditions III

$$M_\theta = M \qquad\qquad at \; \phi = 0.$$

SOLUTIONS OF THE AUXILIARY PROBLEMS

Auxiliary Problem I

Loaded region. The equations used in the analysis of this shell problem are the equilibrium equations written in terms of the displacements. The development of these equations may be found in Timoshenko (1).

Substituting

$$\xi = \frac{x}{R} \qquad \nabla^2 = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \theta^2}$$

$$K = \frac{(1 - \vee^2)}{Eh} R^2, \quad c^2 = \frac{h^2}{12R^2}$$

these equations become

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{(1 - \vee)}{2} \frac{\partial^2 u}{\partial \theta^2} + \frac{(1 + \vee)}{2} \frac{\partial^2 v}{\partial \xi \partial \theta} - \vee \frac{\partial w}{\partial \theta} = 0,$$

(I) $\quad \dfrac{(1 + \vee)}{2} \dfrac{\partial^2 u}{\partial \xi \partial \theta} + \dfrac{\partial^2 v}{\partial \theta^2} + \dfrac{(1 - \vee)}{2} \dfrac{\partial^2 v}{\partial \xi^2} - \dfrac{\partial w}{\partial \theta} = - K P_\theta,$

$$\vee \frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \theta} - w - c^2 \nabla^4 w = - K P_z.$$

Let

$$u = \frac{\partial^3 F}{\partial \xi \partial \theta^3} - \vee \frac{\partial^3 F}{\partial \xi^3} + u_0,$$

(II) $\qquad v = -\frac{\partial^3 F}{\partial \theta^3} - (2 + \vee)\frac{\partial^3 F}{\partial \xi^2 \partial \theta} + v_0,$

$$w = - \nabla^4 F + w_0,$$

where $F = F(\xi,\theta)$ is a stress-strain function. Then F satisfies the following differential equation if $P_X = P_\theta = P_Z = 0$:

$$\nabla^8 F + \frac{1 - V^2}{c^2} \frac{\partial^4 F}{\partial \xi^4} = 0 \; ;$$

and u_o, v_o, and w_o are a system of particular solutions of the non-homogeneous equations.

Choosing

$$F(\xi,\theta) = e^{\alpha\theta} \sin \lambda_m \xi \; ;$$

the most general expression for F may be written in the following form:

$$F(\xi,\theta) = \sum_{m=1}^{\infty} f_m(\theta) \sin \lambda_m \xi \; ,$$

where

$$f_m(\theta) = \sum_{i=1}^{8} c_{im} e^{\alpha i\theta}$$

and c_{1m}, c_{2m}, . . . c_{8m} are arbitrary sets of constants.

In order to obtain a particular solution, P_θ and P_Z are expanded in a Fourier sine and cosine series respectively, and the u_o, v_o, and w_o are chosen as

$$u_o = \sum_{m=1}^{\infty} A_{om} \cos \theta \cos \lambda_m \xi \; ,$$

$$v_o = \sum_{m=1}^{\infty} B_{om} \sin \theta \sin \lambda_m \xi \; ,$$

$$w_o = \sum_{m=1}^{\infty} C_{om} \cos \theta \sin \lambda_m \xi \; .$$

Substituting the loads and the particular solutions into (I), and equating coefficients the values of A_{om}, B_{om}, C_{om} are obtained.

With the use of equations (II) and the boundary conditions, four of the eight constants of $f_m(\theta)$ may be evaluated.

Unloaded region. Equations (I) are again used, with $P_Z = P_\theta = 0$, and with the following substitutions:

$$u_1 = \frac{\partial^3 G}{\partial \xi \partial \theta^3} - V \frac{\partial^3 G}{\partial \xi^3}$$

(II) $$v_1 = - \frac{\partial^3 G}{\partial \theta^3} - (2 + V) \frac{\partial^3 G}{\partial \xi^2 \partial \theta}$$

$$w_1 = - \nabla^4 G \; .$$

Then (I) reduces to

$$\nabla^8 G + \frac{(1 - V^2)}{c^2} \frac{\partial^4 G}{\partial \xi^4} = 0$$

where $G = G(\xi, \theta)$ is a stress-strain function. In an analogous manner as in the loaded region the most general expression for G is

$$G(\xi, \theta) = \sum_{m=1}^{\infty} g_m(\theta) \sin \lambda_m \xi$$

where

$$g_m(\theta) = \sum_{i=1}^{8} D_{im} e^{\alpha i \theta} .$$

With the use of equations (III) and the boundary conditions, four of the eight constants of $g_m(\theta)$ may be evaluated.

With the use of the eight conditions at $\theta = 0$ the remaining eight constants may be determined.

The deflections are then completely determined for both the loaded and unloaded regions.

Auxiliary Problem II

For this type of loading it seems reasonable to assume that $v = 0$ and that $u = u(x)$, $w = w(x)$; and that all quantities are independent of θ.

With the above assumptions and with $P_X = P_\theta = P_Z = 0$, the displacement equations reduce to the following two equations:

(IV)
$$\frac{d^2 w}{dx^4} + 4\beta^4 w = \frac{VN_x}{DR}$$

(V)
$$\frac{du}{dx} - \frac{Vw}{R} = C$$

where

$$\beta^4 = \frac{3(1 - V^2)}{R^2 h^2} , \quad D = \frac{Eh^2}{12(1 - V^2)}$$

according to Timoshenko (I) the solution to the differential equation (IV) is

$$w = e^{-\beta x}(C_1 \cos \beta x + C_2 \sin \beta x)$$
$$+ e^{\beta x}(C_3 \cos \beta x + C_4 \sin \beta x) + W_0 ,$$

where

$$W_0 = \frac{Eh \sqrt{C}}{4DR\beta^4(1 - V^2)}$$

and C is a constant yet to be determined.

Making use of the boundary conditions for this problem the values of C_1, C_2, C_3, and C_4 are obtained.
From equation (V), u may be written as

$$u = Cx + \frac{\sqrt{}}{2R\beta} \left\{ C_1 \left[e^{-\beta x} (\sin \beta x - \cos \beta x) \right] \right.$$

$$+ C_2 \, e^{-\beta x} (- \cos \beta x - \sin \beta x) + C_3 e^{\beta x} (\cos \beta x + \sin \beta x)$$

$$+ \left. C_4 \, e^{\beta x} (\sin \beta x - \cos Bx) \right\} + K,$$

where K is a constant.

Using the condition

$$u \Big|_{x = 0} = u \Big|_{x = L}$$

the value of C is obtained.

Since K contributes nothing to the stresses, it can only correspond to a rigid body motion in the x direction superimposed on the entire shell; and therefore K may be neglected.

To superimpose solutions of the auxiliary problems, it is convenient to expand u and w in appropriate Fourier series. Then

$$u = \sum_{m=1}^{\infty} A_m \cos \frac{m\pi x}{L},$$

$$w = \sum_{m=1}^{\infty} D_m \sin \frac{m\pi x}{L}$$

where A_m and D_m are the Fourier coefficients.

Auxiliary Problem III
With a shell supported in this manner Timoshenko (1) obtains the following partial differential equation for M_θ:

$$\frac{\partial^8 M_\theta}{\partial \phi^8} + (2 + \sqrt{})R^2 \frac{\partial^8 M_\theta}{\partial x^2 \partial \phi^6} + 2 \frac{\partial^6 M_\theta}{\partial \phi^6}$$

$$+ (1 + 2\sqrt{})R^4 \frac{\partial^8 M_\theta}{\partial x^4 \partial \phi^4} + 2(2 + \sqrt{})R^2 \frac{\partial^6 M_\theta}{\partial x^2 \partial \phi^4}$$

(VI)

$$+ \frac{\partial^4 M_\theta}{\partial \phi^4} + \sqrt{} R^6 \frac{\partial^8 M_\theta}{\partial x^6 \partial \phi^2} + (1 + \sqrt{})^2 R^4 \frac{\partial^6 M\theta}{\partial x^4 \partial \phi^2}$$

$$+ (2 + \nu)R^2 \frac{\partial^4 M_\theta}{\partial x^2 \partial \phi^2} + \frac{12(1 - \nu^2)R^6}{h^2} \frac{\partial^4 M_\theta}{\partial x^4} = 0,$$

where $\phi = \eta - \theta$.

A solution of this equation is given by the expression

$$M_\theta = \sum_{m = 1, 3, 5}^{\infty} A_m\, e^{\epsilon_m \phi}\, \sin\, \frac{m\pi x}{L}.$$

Substituting this expression into (VI) an eighth degree algebraic equation in ε_m is obtained. According to (1) it is not unreasonable to assume that the moment M_θ diminishes rapidly as ϕ increases from 0. Therefore only those four roots which satisfy this assumption are used. Combining the four corresponding solutions, M_θ becomes

$$M_\theta = \sum_{m = 1,3,5}^{\infty} \left\{ e^{-\mu_1 \phi}\, (C'_{1m} \cos \tau_1 \phi + C'_{2m} \sin \tau_1 \phi) \right.$$

$$\left. + e^{-\mu_2 \phi}\, (C'_{3m} \cos \tau_2 \phi + C'_{4m} \sin \tau_2 \phi) \right\}\, \sin\, \frac{m\pi x}{L}$$

The uniformly distributed bending moment is expanded in a Fourier series as follows:

$$M = \sum_{m = 1, 3}^{\infty} B_m \sin \frac{m\pi x}{L},$$

where $B_m = \dfrac{4M}{m\pi}$.

The boundary conditions on the moments are then used to evaluate C'_{im}.

Timoshenko (1) has a set of equations which give the displacements in terms of M_θ. From these equations and the boundary conditions on the deflections the values of u, v, w are obtained.

SOLUTION TO THE ORIGINAL PROBLEM

The solution of this shell problem is obtained by superimposing the solutions of the three auxiliary problems under the appropriate boundary conditions. There are two sets of deflections, one in the loaded region and one in the unloaded region.

The deflections are:
$$U = u_I + u_{II} + u_{III}$$
$$V = v_I + v_{III}$$
$$W = w_I + w_{II} + w_{III}$$

where u_I, v_I, w_I are the deflections in the loaded region for auxiliary problem one, v_I, v_{III} are the deflections for auxiliary problem two and u_{III}, v_{IIII}, w_{III} are the deflections for auxiliary problem three.

Applying the boundary conditions at $\theta = \eta$ the values of M and M_1 are obtained.

Unloaded Region

The deflections are:

$$U = u'_I + u_{II} + u_{III}$$
$$V = v'_I + v_{III}$$
$$W = w'_I + w_{II} + w_{III}$$

where u'_I, u'_{II}, u'_{III} are the deflections in the unloaded region for auxiliary problem one, and the other deflections are the same as in the loaded region above.

Applying the boundary conditions at x = 0, L, the value of M_1 is obtained.

With the values of M and M_1 obtained in the above two sections the displacements for the shell are not known.

Reference

1. Timoshenko, S. and S. Woinowsky-Krieger. 1959. *Theory of Plates and Shells,* 2nd edition. McGraw-Hill Book Company, New York, N. Y.