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Abstract. Whenever we have a measure function α defined on some set M of subsets of a set T, we may determine a binary relation Q between the elements of M by defining, for all members A and B of M, AQB if and only if $\alpha(A) \leqslant \alpha(B)$. Using such a binary relation, we may derive certain measure theoretic properties independently of the real number system. In particular, if we use what might be termed a process of completion, we may construct, from a system of Borel sets, not only a system of Lebesgue measurable sets, but, in general, a somewhat larger system.

In the following paper, we shall state certain measure theoretic results, without proof, which can be derived using a binary relation rather than a measure function. For typographical reasons, we shall denote the intersection and union of two sets, A and B, by $A \cdot B$ and $A + B$ respectively. If A is a subset of B, we shall write $A < B$. The empty set will be denoted by θ.

A measure function α on a set T induces a partial order Q on the set of measurable subsets M of T, if we define, for all A and $B \varepsilon M$, AQB if and only if $\alpha(A) \leqslant \alpha(B)$. Such a binary relation can easily be shown to satisfy the following axioms.

Axiom 1. If A and $B \varepsilon M$, then either AQB or BQA or both.
Axiom 2. If A and $B \varepsilon M$ and $A < B$, then AQB.
Axiom 3. If A,B, and $C \varepsilon M$, AQB, and BQC, then AQC.
Axiom 4. If A,B, and $\theta \varepsilon M$ and $AQ\theta$, then $(A+B)QB$.
Axiom 5. If A_i and $\theta \varepsilon M$ and $A_i Q \theta$, for all positive integers i, then $\sum_i A_i \varepsilon M$ and $(\sum_i A_i) Q \theta$.

A binary relation which satisfies the above axioms will be termed a measure relation.

In the remainder of the paper, we shall assume that M is closed under countable unions and set differences, and contains θ. We note this implies that M is closed under countable intersections. We shall also suppose that a measure relation Q is defined on M.

Definition 1. If A and $B \varepsilon M$, then $A(=)B$ if and only if AQB and BQA.

Definition 2. If A and $B < T$, then $A(=)_1 B$ if and only if there is a $D \varepsilon M$ such that $DQ\theta$ and $(A-B)+(B-A) < D$.

$A(=)B$ corresponds to saying that A and B have the same measure. $A(=)_1 B$ is analogous to saying that A and B are equal

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almost everywhere, that is, except on a set of measure zero.

We may now prove the following theorem.

Theorem 1. (i) $(=)$ and $(=)_1$ are equivalence relations on their respective domains.

(ii) If A and $B\varepsilon M$ and $A(=)_1B$, then $A(=)B$.

In general, however, we may have $A(=)_1B$ without having $A(=)B$, even though A is a member of M. To eliminate this possibility, we shall now extend the domain of definition of Q.

Definition 3. (i) $H(M)=\{A \mid \text{There is a } B\varepsilon M \text{ such that } A<B.\}$.

(ii) If $A\varepsilon H(M)$, then $\overline{P}(A)=\{B \mid A<B \text{ and } B\varepsilon M.\}$.

(iii) If $A\varepsilon H(M)$, then $P(A)=\{B \mid B<A \text{ and } B\varepsilon M.\}$.

Definition 4. $A\varepsilon L(M)$ if and only if $\sum_{B\varepsilon P(A)} B(=)_1 \pi_{B\varepsilon\overline{P}(A)} B$.

L(M) will be called the set of Q-measurable sets with respect to M.

Theorem 2. L(M) is closed with respect to countable unions and set differences.

We will now impose a measure relation on L(M), with the aid of the following definition.

Definition 5. If A and $B\varepsilon L(M)$, then $A\overline{Q}B$ if and only if for every $C\varepsilon\overline{P}(B)$ there is a $D\varepsilon\overline{P}(A)$ such that DQC.

Theorem 3. (i) $\overline{Q}$ is a measure relation on L(M).

(ii) $\overline{Q}$ corresponds with Q on M.

The following theorem is now derivable.

Theorem 4. If $A(=)_1B$ and $A\varepsilon L(M)$, then $B\varepsilon L(M)$ and $A(\overline{=})B$. ($(\overline{=})$ is the equivalence relation corresponding to $\overline{Q}$.)

At this point, we shall note a few things. If Q had been induced by a Borel measure α on a set M of Borel sets of a topological space T, then a set $A<T$ would be Lebesgue measurable if and only if there existed a set $B\varepsilon M$ such that $A(=)_1B$. Thus, by the previous theorem, L(M) would contain the Lebesgue measurable sets. However, in general, L(M) will be somewhat larger.

Also, one might ask whether it is possible to extend L(M) by using the above procedure. The answer is contained in the following "closure" theorem.

Theorem 5. $L(M)=L(L(M))$.

Needless to say, the previous paragraphs left many questions unanswered. Some of these questions are answered in (1) and, it is hoped, a subsequent series of papers will answer many more.

Literature Cited

1. Sanderson, Donovan F. 1961. "On the Construction of the Measurable Sets," Masters thesis. Department of Mathematics, Iowa State University, Ames, Iowa.