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Abstract. By considering a boundary value problem of Laplace's differential equation, we construct a gravitational potential function for an oblate spheroid using Newton's law of universal gravitation. We construct this function by retaining only four terms of an absolutely convergent series. The first of these four terms is the contribution due to a sphere while the other three terms contain coefficients which are functions of the oblateness of the spheroid.

INTRODUCTION

Newton's law of universal gravitation for "point" masses states that every two particles of matter attract each other with a force which acts along the line joining the particles and whose intensity varies as the product of their masses and inversely as the square of their mutual distance apart.

An oblate spheroid is the solid formed by revolving an ellipse about its minor axis. We assume that matter is continuously distributed throughout the spheroid.

BOUNDARY VALUE PROBLEM

In spherical coordinates (R,β,α), Laplace's differential equation for a function V, of (R,β,α), is

$$\sin\beta\, D_R(R^2 D_R V) + D_\beta(\sin\beta\, D_\beta V) + \csc\beta\, D_\alpha(D_\alpha V) = 0 \tag{1}$$

where $D_u = \frac{\partial}{\partial u}$, for $u = R,\beta,\alpha$. The oblate spheroid is found by revolving the ellipse

$$\left(\frac{x}{a}\right)^2 + \left(\frac{z}{b}\right)^2 = 1, \qquad a > b > 0 \tag{2}$$

about the z-axis. The spherical coordinates corresponding to the rectangular coordinates (x,y,z) are

$$\begin{aligned} x &= R\sin\beta\cos\alpha \\ y &= R\sin\beta\sin\alpha \\ z &= R\cos\beta \end{aligned} \tag{3}$$

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We require the potential function to be symmetric with respect to the z-axis by setting

$$D_\alpha V = 0.$$

Equation (1) reduces to

$$D_R(R^2 D_R V) + \csc \beta \, D_\beta(\sin \beta \, D_\beta V) = 0. \tag{4}$$

It has been shown in Churchill (1941) that

$$R^{-(m+1)} P_m(\cos \beta) \tag{5}$$

is a solution of (4) for each non-negative integer m where P_m is the Legendre polynomial of order m. Further, a linear combination of terms of type (5) is also a solution to (4). The gravitational potential function that we seek is a linear combination of terms of type (5) which satisfies a boundary condition that we are going to construct next.

The potential function due to an oblate spheroid, given by (2), at a point R ($R \geqslant a$) on the axis of rotation is given by

$$I(R) = 2\pi pG \int_{-b}^{b} \int_{0}^{\bar{x}} \frac{x\,dx\,dz}{\sqrt{(R-z)^2 + x^2}} \tag{6}$$

where G is the Gaussian constant, p is the homogeneous density constant and

$$\bar{x}^2 = \left(\frac{a}{b}\right)^2 (b^2 - z^2), \quad a > b > 0. \tag{7}$$

If we integrate (6) with respect to x, we get

$$I(R) = 2\pi pG \left[\int_{-b}^{b} \sqrt{(R-z)^2 + \left(\frac{a}{b}\right)^2 (b^2 - z^2)}\, dz - \int_{-b}^{b} (R-z)\, dz \right]. \tag{8}$$

If S is a positive real number such that

$$S^2 = \frac{a^2 - b^2}{b^2}, \quad a > b > 0, \tag{9}$$

then the first integral can be written as

$$S^{-1} \int_{-b}^{b} \sqrt{\left(\frac{a}{b}\right)^2 (R^2+b^2S^2) - (S^2z+R)^2}\, dz. \tag{10}$$

For R $\geqslant$ a > b > 0, we make the following change of variable on (10):

$$z = S^{-2}\left[-R + \left(\frac{a}{b}\right)\sqrt{R^2 + b^2S^2}\, \sin E \right] \tag{11}$$

If we use (11) in (8) and integrate the second integral directly then we get

$$I(R) = 2\pi \rho G\left[\frac{a^2(R^2+b^2S^2)}{b^2S^3} \int_{E_1}^{E_2} \cos^2 E\, dE - 2Rb \right] \tag{12}$$

where

$$\sin E_1 = \frac{Rb - b^2S^2}{a\sqrt{R^2 + b^2S^2}} \quad \text{and}$$

$$\sin E_2 = \frac{Rb + b^2S^2}{a\sqrt{R^2 + b^2S^2}}. \tag{13}$$

If we let

$$K = \frac{bS}{R} \tag{14}$$

and integrate (12), we get

$$I(R) = 2\pi \rho G\left[\frac{a^2b(1+K^2)}{2RK^3} (E_2 - E_1 + \sin E_2 \cos E_2 - \sin E_1 \cos E_1) - 2Rb \right], \tag{15}$$

where

$$\sin E_1 = \frac{b - RK^2}{a\sqrt{1 + K^2}} \quad \text{and}$$

$$\sin E_2 = \frac{b + RK^2}{a\sqrt{1 + K^2}}. \tag{16}$$

From (16), we find that

$$\sin E_2 \cos E_2 - \sin E_1 \cos E_1 = \frac{2K(a^2 - 2b^2)}{a^2(1 + K^2)} \tag{17}$$

and

$$E_2 - E_1 = \arcsin \left[\frac{2K}{1 + K^2}\right]. \tag{18}$$

From (15), (17), and (18), we get

$$I(R) = \frac{3MG}{2RK^2}\left\{\left[\frac{1 + K^2}{2K}\right]\arcsin \left[\frac{2K}{1 + K^2}\right] - 1\right\}. \tag{19}$$

We define e, the coefficient of oblateness, such that

$$b^2 = a^2(1 - e^2), \ a > b > 0. \tag{20}$$

From (9), (14), and (20), we see that

$$K = (\frac{ae}{R}), \tag{21}$$

and for $R \geqslant a$ and $e < 1$,

$$0 < K < \frac{2K}{1 + K^2} < 1. \tag{22}$$

If we expand the arcsine in (19) in powers of $\frac{2K}{1 + K^2}$ and retain the first four terms, then

$$I(R) = \frac{3MG}{2RK^2}\left\{\frac{1}{6}\left[\frac{2K}{1+K^2}\right]^2 + \right.$$

$$\frac{3}{40}\left[\frac{2K}{1+K^2}\right]^4 + \frac{15}{336}\left[\frac{2K}{1+K^2}\right]^6 \Bigg\} . \tag{23}$$

If we expand each of the terms containing $[\frac{2K}{1+K^2}]$ according to the appropriate binomial expansion and retain only powers of K less than or equal to 8 then we are left with

$$I(R) = \frac{3MG}{R}\left\{ \frac{1}{3} - \frac{1}{15}\left(\frac{ae}{R}\right)^2 + \frac{1}{35}\left(\frac{ae}{R}\right)^4 - \frac{1}{63}\left(\frac{ae}{R}\right)^6 \right\} . \tag{24}$$

Equation (24) is the boundary condition that the gravitational potential function must satisfy. Along the positive z-axis, we have $\cos \beta = 1$ and

$$P_m (\cos \beta) = 1 \tag{25}$$

for $m = 0, 2, 4, 6$. Therefore if $\underline{R} = (R,\beta,\alpha)$ is any point of "free space" then

(26)

$$I(\underline{R}) = \frac{3MG}{R}\left\{ \frac{1}{3} - \frac{1}{15}\left(\frac{ae}{R}\right)^2 P_2(\cos \beta) + \frac{1}{35}\left(\frac{ae}{R}\right)^4 P_4(\cos \beta) - \frac{1}{63}\left(\frac{ae}{R}\right)^6 P_6(\cos \beta) \right\} , \tag{26}$$

where $P_0 (\cos \beta) = 1$. (26) is the gravitational potential function for the oblate spheroid since it is a solution to (4) and satisfies the boundary condition, (24), along $\beta = 0$.

Literature Cited

Churchill, R. V. (1941) Fourier series and boundary value problems. McGraw-Hill Book Company, Inc., New York, N. Y.