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Proton Orbits in a Small Cyclotron

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Proton Orbits in a Small Cyclotron[1]

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Abstract. In the theoretical study of a cyclotron, it is possible to make many simplifying approximations and obtain satisfactory results. For example, off-center orbits may be considered as circular in some cases. A method was developed to simulate cyclotron operation by calculating the trajectories of proton motion in cylindrical coordinates. Results of the calculations were then used to determine where simpler calculations may or may not be used. The conclusion is that resonant couplings between axial and radial oscillations should be studied by the calculation of proton trajectories. It is unnecessary to study electric decelerations with this method since circular orbit approximations appear to be suffi:cient.

To study the operation of a cyclotron it is often unnecessary to calculate the position of a proton as a function of time. Instead one may approximate orbits as a series of discontinuously connected semicircles with increasing radii. Studies of this type were carried out by Rose[1]. Mueller[2] performed a theoretical study of the ISU Cyclotron using these methods.

In studying the ISU Undergraduate Physics Cyclotron it became apparent that it would be useful to calculate the coordinates of a proton as functions of time. The magnetic field in this cyclotron is assumed to be azimuthally symmetric with its center at that of the dee box. Protons may begin orbits over a centimeter from the center of the field. Since the field decreases rapidly near the maximum radius of 11.25cm, circular approximations of these orbits may introduce significant errors that would not appear for protons starting closer to the center or moving in a larger machine with a more uniform field. Also a resonant coupling between radial and axial motions in the region of 10.1cm radius causes many protons to lose the positive effects of magnetic focussing and strike the dees. The purpose of this report is to briefly discuss and analyze a method developed to solve the equations of motion of a proton in the cyclotron by using numerical integration.

THE METHOD OF ORBIT CALCULATIONS

The azimuthally symmetric magnetic field is defined as $\mathbf{B} = -B\hat{z} + B_r\ \hat{r}$, where B and B_r are functions of r and z. If a proton with speed v is to travel in a circular orbit about the center of the field in the median plane, the orbital

[1] The project was assisted by a National Science Foundation Undergraduate Research Participation grant.

[2] Physics senior.

radius will be $r_0 = \frac{vm}{eB(r_0)}$. This is called the equilibrium orbit. The proton encounters no radial field and the same axial field at all times; therefore, it must have an angular velocity $\dot{\phi} = \frac{eB(r_0)}{m}$. All other protons with speed v will have their coordinates oscillate about $r = r_0$ and $z = 0$. Here the effects of the electric accelerating field are not considered.

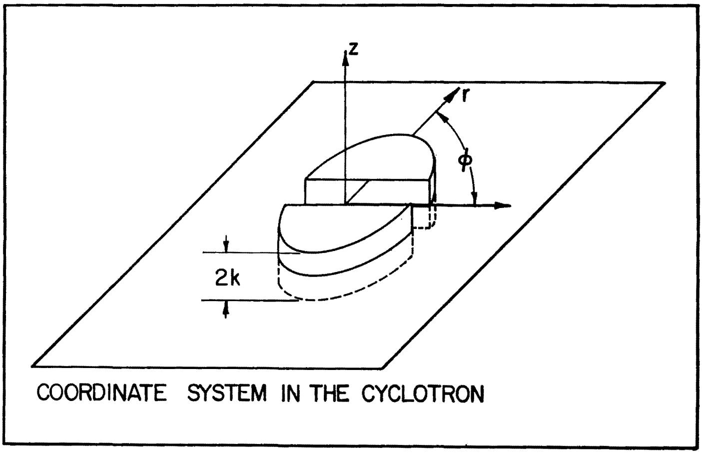

Figure 1. Coordinate system in the cyclotron.

A first order approximation gives the angular frequency of radial oscillations as $\omega_r = \dot{\phi}\sqrt{1-n}$ and that of axial oscillations as $\omega_z = \dot{\phi}\sqrt{n}$ where $n = -\frac{r}{B}\frac{\partial B}{\partial r}$. The degree to which an orbit is off-center is defined as the displacement, $\delta r = r_A - r_0$, where r_A is the maximum orbital radius. Note that throughout this report the radius of the equilibrium orbit (r_0) is related to proton speed by the equation

$$(1) \qquad v = r_0 \frac{eB(r_0)}{m} = r_0 \dot{\phi}(r_0).$$

The calculation of a proton's position as a function of time is the most complete method of anaylzing proton motion, but this requires the largest amount of calculation. This study is designed to use said calculations to determine where they are necessary and where simpler methods may be used.

Calculations in this paper deal mainly with the actions of off-center orbits with $\delta r \leqslant 1$cm. The study should indicate whether or not δr changes significantly as proton energy increases. Studies of magnetic focussing by Rose[1] and Mueller[2] predict that the amplitudes of axial oscillations will decrease as orbital radii increase. A resonant coupling between the radial and axial oscillations occurs where $\omega_r \approx 2\,\omega_Z$, at $r = 10.1$cm with $n = 0.2$. This study should indicate the effects of resonance on the axial motions. Also considered in this paper is the functional relationship between δr and the maximum energy a proton may obtain before being electrically decelerated.

Since the magnetic field is approximately uniform up to 5cm radius, orbits in this region will be considered as circular and as having constant displacement (δr). These orbits will not be discussed here since problems concerning them are basically related to the electric field. Many of the problems involving effects of the electric field on the small orbits have been studied by Cohen[3].

Magnetic Field Approximations

It has been assumed that the field is azimuthally symmetric with respect to the center of the dee box. That is, the field has no angular component and is independent of angular position. Since the pole faces are symmetric with respect to one another, there is no radial field component ($B_r = 0$) on the median plane ($z = 0$). Neglecting the electric field of the dees, Maxwell's equations in the dee box are $\nabla \times \mathbf{B} = \mathbf{0}$ and $\nabla \cdot \mathbf{B} = 0$.

From these one may derive $\frac{\partial B}{\partial r} = -\frac{\partial B_r}{\partial z}$ and $\frac{B_r}{r} + \frac{\partial B_r}{\partial r} = \frac{\partial B}{\partial z}$. Since B_r is equal to zero on the median plane, the following approximation is valid for small axial displacements: $B_r = -z\frac{\partial B}{\partial r}$. In the present study, the axial displacements are considered to be small, and $\frac{\partial B}{\partial z}$ is equal to zero on the median plane. Hence, it was approximated that B is independent of z. Then the magnetic field approximation is $\mathbf{B} = -B(r)\hat{z} - z\frac{dB}{dr}\hat{r}$.

A magnetic field approximation developed by Dr. D. E. Hudson was used in this study. This relationship is shown graphically in Figure 2; the mathematical expression is:

$$B(r) = [17{,}000 + 0.25\text{cm}^{-2}r^2 + 0.232\text{cm}^{-3}r^3 - 0.0118\text{cm}^4 r^4 + 6.21\text{cm}^{-11.6}10^{-10}\,r^{11.6}]\text{ gauss}. \quad (2)$$

Equations of Angular and Radial Motion

Since the axial displacements are considered small and have oscillation frequencies lower than the revolution frequency, the axial component of v is neglected in solving for r(t) and ϕ(t). This is equivalent to solving for the angular and radial components only, on the median plane.

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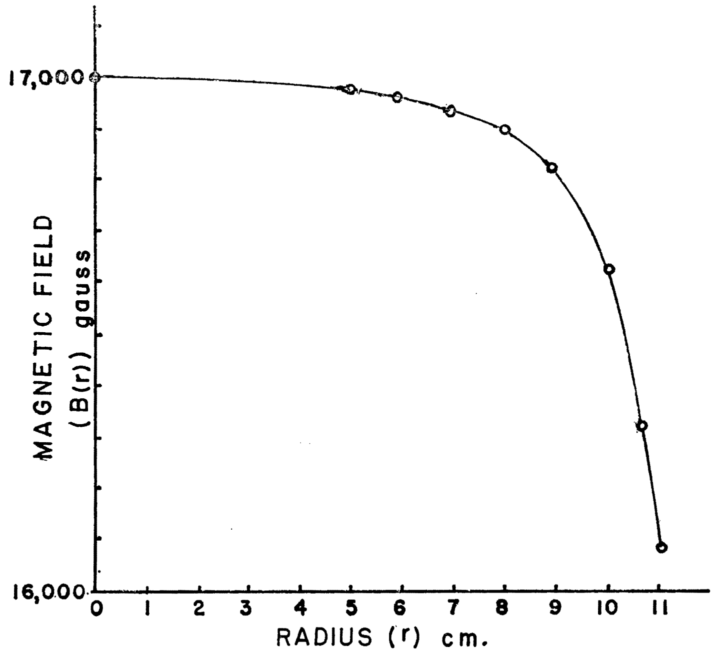

Figure 2. Approximation of the azimuthally symmetric magnetic field.

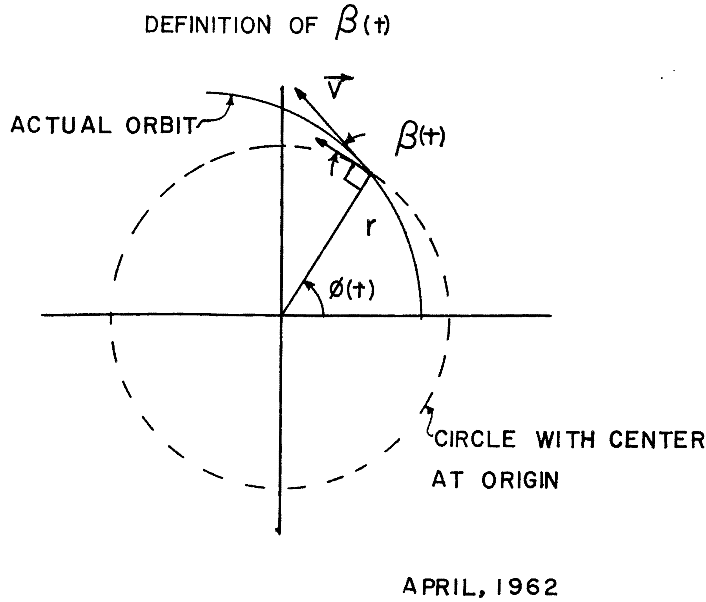

Figure 3. Definition of β (t).

To solve the equations of motion on the median plane by numerical integration, the calculations are simplified by transforming r and ϕ into β, the angle between the velocity vector (v) and the tangential vector ($\hat{\phi}$).

By projecting v on a line parallel to $\hat{\phi}$, one may see that the tangential velocity of the proton is $r\dot{\phi} = v \cos \beta$. The angular acceleration is $\ddot{\phi} = -v\left(\frac{\dot{\beta} \sin\beta}{r} + \frac{\dot{r} \cos\beta}{r^2}\right)$. Likewise, the radial velocity and acceleration are $\dot{r} = v \sin\beta$ and $\ddot{r} = v\dot{\beta} \cos\beta$ respectively. The force on a proton is $\mathbf{F} = \hat{r}$ v e B cos $\beta + \hat{\phi}$ v e B $\sin\beta$. After transforming the acceleration terms to cylindrical coordinates, the equations of motion may be written as follows:

$$-v\,e\,B(r)\cos\beta = m\,(\ddot{r} - r\,\dot{\phi}^2), \text{ and}$$
$$v\,eB(r)\sin\beta = m\,(r\,\ddot{\phi} + 2\,\dot{r}\,\dot{\phi}).$$

The above equations yield the following differential equation:

$$(3) \qquad \dot{\beta} = \frac{v \cos\beta}{r} - \frac{eB(r)}{m}$$

By integration we have $r = \int_0^t v \sin\beta \, dt' + r_1$ where r_1 is the radius at $t = 0$. Note that $\beta = 0$ when r_1 is either a maximum or minimum, then consider r_1 to be a maximum. Then we have

$$(4) \qquad \beta(t) = \int_0^t \frac{v \cos\beta}{\int_0^{t'} v \sin\beta \, dt'' + r_1} - \frac{eB(\int_0^{t'} v \sin\beta dt'' + r_1)}{m} \, dt'.$$

This equation gives β as a function of time. A simplified Runge-Kutta method was used to solve for β by numerical integration.

Since it was known that $r = \int_0^t v \sin\beta \, dt' + r_1$ and $\phi = \int_0^t \frac{v \cos \beta}{r} dt'$, and these values are contained in equation 4, it is now possible to compute the radial and angular coordinates of a proton as functions of time.

Equation of Axial Motion

From the relation ma = e v $\times$ **B**, one may derive the equation of motion in the axial direction, $m\,\ddot{z} = -r\,\dot{\phi}\,e\,B_r$. If $r\,\dot{\phi}$ is approximated as v, only second order accuracy is lost since $r\,\dot{\phi} = v\,(\cos\beta)\,(\cos \gamma)$, where γ is the angle between the velocity vector and the median plane. Since β and γ are of the order of 0.1 radian, the error is considered insignificant. When the approximation of the radial magnetic field is used, the equation of axial motion becomes

$$(5) \qquad \ddot{z} - \frac{ev}{m}\frac{dB}{dr} z = 0.$$

To solve this equation of axial motion, it was combined with the solution of the equations of angular and radial motion. Then a numerical integration was performed to obtain z along with r and ϕ.

Electric Accelerations

In order to simulate the operation of the cyclotron, it is necessary to study the effects of the dee-to-dee electric field. For the present study, the electric field is considered as concentrated in a region of zero width at the center of the dee gap. The dee-to-dee voltage is defined as V; therefore, a proton will receive a boost of energy, $\triangle E = eV$, when it crosses the dee gap. It is assumed that the momentum is altered only in the direction perpendicular to the dee gap and parallel to the median plane. From the above, approximations for the changes in v and β were derived using Taylor's series expansions:

(6) $$\triangle v = \frac{\triangle E}{mv} - \frac{(\triangle E)^2}{2m^2v^3} + \frac{(\triangle E)^3}{2m^3v^5}, \text{ and}$$

(7) $$\Delta\beta = \tan\beta \left\{ -\frac{\Delta v}{v} + \left[1 + \frac{\tan^2\beta}{2} \right] \left(\frac{\Delta v}{v} \right)^2 \right\}.$$

During this instantaneous acceleration r, $\dot{z}$ and z are not altered. Hence, if the coordinates and velocities of a proton are known just before acceleration, they may be determined for the instant directly afterwards.

The electric field oscillates at an angular frequency of ω. The angular position of the electric vector is defined as $\omega t + \theta_0$. The voltage drop at the dee gap is defined as $V_0 \cos(\omega t + \theta_0)$ where V_0 is the maximum dee-to-dee voltage. The angle from r to the electric vector is defined as $\theta = \omega t - \phi + \theta_0$. The number of half-revolutions is defined as $\nu = \frac{\phi}{\pi}$; therefore, $V = V_0 \cos(\theta + \pi\nu)$. Since $\pi\nu$ is approximately an integral multiple of π when the proton is at the dee gap, it is possible to state that the potential drop in the direction of proton motion across the dee gap is $V = V_0 \cos\theta$.

If θ becomes greater than $\frac{\pi}{2}$ or less than $-\frac{\pi}{2}$, the proton will be decelerated. If an orbital radius were to exceed that of an initial deceleration, the proton energy would increase since velocity and radius are proportional; however, this contradicts the loss of energy in deceleration. Hence, it is assumed that once a proton suffers an electric deceleration it will never reach a greater energy.

Arrangement of Calculations For Complete Solution

A series of programs were written to study off-center orbits using the Cyclone Computer at Iowa State University. The initial conditions of position and velocity are specified, and then the computer takes over to simulate cyclotron operation by calculating the coordinates of a proton as functions of time. The equations used during the computations are mainly Equations 4, 5, 6 and 7. The maximum dee-to-dee voltage (V_o) and the magnetic field approximation must be supplied for the program to operate.

Initial Conditions

It is most convenient to start an orbit at an apogee. The initial speed (v_i), phase lag (θ_o), orbital displacement (δr_i) and angular position of the apogee (σ_{ri}) are specified. Also the initial axial position and velocity (z_i and $\dot{z}_i$) are entered.

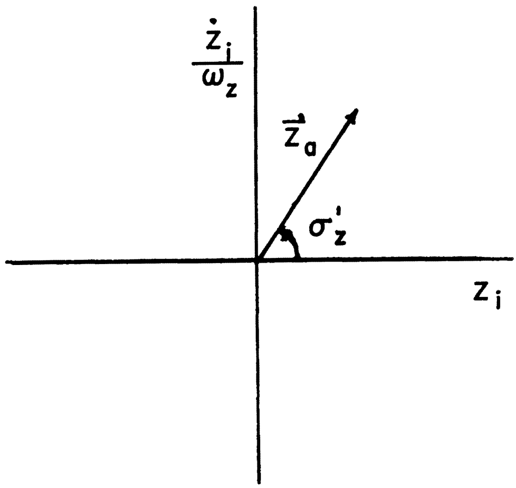

Figure 4. Vector representation of initial axial displacement and velocity.

For a general study of axial motion, a uniform distribution of z_i and $\dot{z}_i$ must be considered. The angular frequency of axial motion is approximated as $\omega_Z = \dot{\phi}\sqrt{n(r_o)}$. Note that z_i and $\dot{z}_i$ may be represented in a two dimensional vector space by z_a as in Figure 4. The magnitude of z_a is the amplitude of the axial oscillations. Note that Equation 5 is linear; therefore, the units of z_a are arbitrary. Hence, only the direction of z_a is significant in forming a uniform distribution. The angular position of z_a is defined as σ_Z'. Then we have $z_i = z_a \cos \sigma_Z'$ and $\dot{z}_i = \omega_Z z_a \sin \sigma_Z'$. To study a uniform distribution of z_i and $\dot{z}_i$, values of σ_Z' were chosen to be $\sigma_Z' =$

$N\frac{\pi}{8}$ where $N = 0, 1, 2, \cdots 7$. Eight calculations were performed while only σ_Z' was varied. Other integral values of N would give the same results with a possible reversal in sign. The true axial motion is not quite simple harmonic, but this gives a reasonable distribution approximation.

Computation

As the calculations are carried out, the computer outputs r and ϕ at apogees and perigees in radial positions. It gives z and ϕ at maxima of $|z|$. It also outputs r, ϕ and θ at selected dee gap crossings. Note that nowhere is t mentioned. Even though r, ϕ and z are functions of t, their relations to one another are more useful to study since these have more experimental significance. The basic results of the calculations may be graphed as in Figure 5. The overlapping plots of z vary only in the initial condition σ_Z'. Since the approximations of the magnetic field and the solutions for radial and angular motion give r and ϕ as independent of z, only one graph of r versus ϕ is needed when variations in σ_Z' are considered. Here $\sigma_Z' = N\frac{\pi}{4}$ where $N = 0, 1, 2,$ and 3.

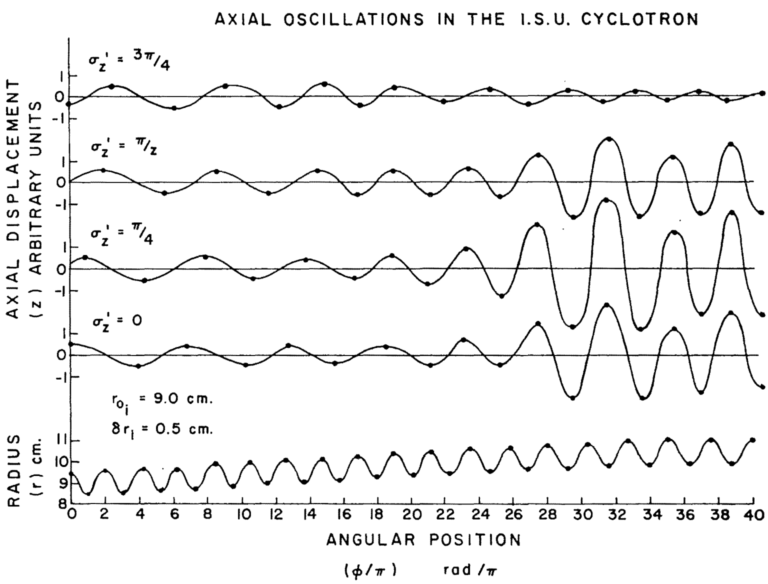

Figure 5. Orbit plot, as calculated on the Cyclone Computer at Iowa State University.

Checks on the Computational Method

Since the process of numerical integration is an approximation itself, it is necessary to be certain that it yields the desired accuracy. Since the solution of the actual cyclotron problem is not known, a problem was devised for which a solution could be found simply. The program calculates the angular and radial coordinates of a proton in a magnetic field unaffected by an electric field. It was checked by entering a uniform magnetic field. The results should be circular motion with $r_o = \frac{vm}{eB}$ There should be no change in the coordinates of the apogees and perigees (no precession), and the average angular velocity should be that predicted by the cyclotron equation ($\bar{\dot{\phi}} = \frac{eB}{m}$). The program for axial motion was checked by entering a constant term for the gradient ($\frac{dB}{dr}$). The maximum of $|z|$ should be constant and the frequency of oscillations should be that predicted ($\omega_Z = \bar{\dot{\phi}}\sqrt{n(r_o)}$). The program that simulates the electric accelerations was checked by entering it many times, as in a lengthy orbit study. The results of each acceleration were used as initial conditions for the next. Then the cyclotron equation and momentum transfer hypotheses were used to predict the final results in one step. The two sets of results were then compared for accuracy. All of the tests made on the program indicated that the errors resulting from computational methods would not be as significant as those due to the experimentally determined quantities such as field values. This does not include the errors resulting from the magnetic field approximation for points away from the median plane of the dee box. These errors can be studied when more is known about the magnetic field shape.

Results

Figure 5 gives a general picture of proton motion. Note the amplitude expansion of axial oscillations in the region of 10.1cm. This is the predicted resonance at $n = 0.2$. Also at $r = 10.3$cm where $n = 0.25$ ($\omega_Z \approx 2\dot{\phi}$) there is a coupling due to the shifting of the orbit as a result of the electric accelerations. Since the two resonances are only about 0.2cm apart, the amplitude expansion cannot be considered as a result of only one of the two factors. In one plot of z the coupling had the opposite effect by reducing the amplitude of axial oscillation.

At the left of Figure 5 each axial oscillation is approximately one-eighth of an oscillation out of phase with respect to the next one, since σ_Z' varies by $\frac{\pi}{4}$. At the region of resonance, some of the oscillations have encountered phase localization; that is, all but the reduced oscillations are very nearly in phase.

In an orbit study which began with r_{oi} = 5cm and δr_i =

0.5cm, the phase distribution remained approximately constant until the radius entered the region of resonance. Then the axial motions obtained were very much like those graphed here. This indicates that phase localization of axial oscillations at radii greater than 5 cm is generally caused by resonances.

The study beginning with $r_{oi} = 5$cm also indicated that δ r increased to approximately 0.7cm at maximum radius; therefore, off-center orbits do not become circular as their radii increase. Improved field approximations entering dependence of r and ϕ on z may alter this statement.

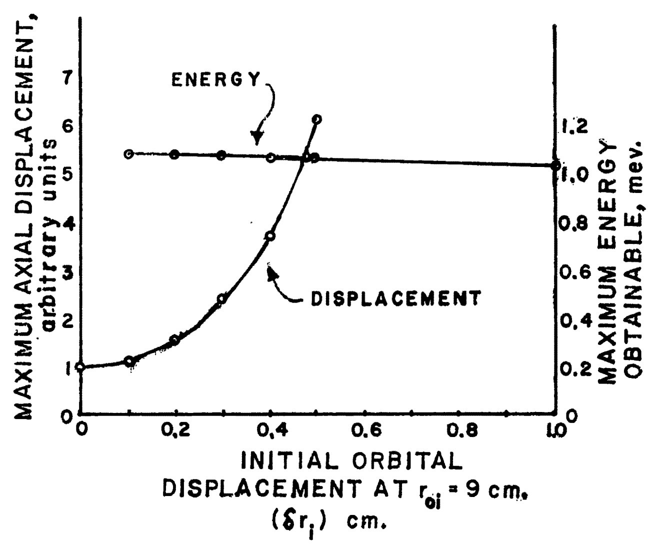

Figure 6. Maximum axial displacement and maximum energy obtainable as functions of initial orbital displacement.

Amplitudes of Axial Oscillations

To obtain Figure 6, a series of calculations was performed with $r_{oi} = 9$cm, $\theta_o = 0$, $\sigma_Z' = N\frac{\pi}{8}$ and δr_i varying from 0.1cm to 1.0cm. For each value of δr_i the maximum value of $|z|$ was obtained. For example, if a proton entered the orbit with $\delta r_i =$ 0.5cm, its axial amplitude should be less than $\frac{1}{6.1}$times the height of the dees if there is to be 100% certainty that the proton will not strike the dees.

Maximum Energies

The graph of maximum energies in Figure 6 was made from the same calculations as the graph of maximum $|z|$ except that the axial motion was neglected in the former case. The orbits were discontinued at maximum energies because θ became greater than $\frac{\pi}{2}$; this caused an electric deceleration in each case. Note that the maximum energy decreased by about 3% as δr_i increased from 0.1cm to 1cm.

CONCLUSION

A study by the author indicated that the average angular velocities of off-center orbits are lower than those of centered orbits of the same energy. Since $\theta = \omega t - \phi + \theta_o$, θ will reach $\frac{\pi}{2}$ sooner for off-center orbits and will terminate them at lower energies. The variations in ϕ due to changes in δr_i are much more pronounced at large radii in the less uniform portion of the field. Therefore, it is expected that orbits with $r_{oi} \leqslant 9$cm would show only slightly greater variations in maximum energy than those of Figure 6.

An orbit is considered prematurely terminated if the proton does not reach the maximum energy obtained by another proton with the same initial conditions except $\delta r_i = 0$. On comparing the two graphs in Figure 6 it was concluded that resonances have far more effect on premature termination of off-center orbits than do electric decelerations.

The results of this study indicate that the resonant conditions must be considered in making studies of proton motions in the region of 10.1cm radius. The effects of off-center orbits on the energies at which electric decelerations occur appear to be negligible. For nonresonant orbits it would be advisable to develop a simpler method for studying off-center orbits.

In the case of a centered orbit, the proton will strike the target when r_o exceeds r_t, the target radius. Target energies would be approximately single-valued for centered orbits for which $E = \frac{mv^2}{2} = \frac{[r_o e B(r_o)]^2}{2m}$. However, off-center protons may strike the target whenever $r_o + \delta r$ exceeds r_t. It is then possible to have heterogeneous target energies since r_o, and consequently E, may vary. Therefore, any simplified method of off-center orbit study must include a means for considering heterogeneous target energies.

The approximations of the magnetic field and the axial motion studies are valid for small axial displacements. It would be wise to determine the maximum amplitude of axial oscillations that this method can handle before significant errors arise. It is necessary to have a more accurate means of approximating the magnetic field before this can be done.

Acknowledgements

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