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D-spaces in infinite products of ordinals

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D-SPACES IN INFINITE PRODUCTS OF ORDINALS

An Abstract of a Thesis

Submitted

in Partial Fulfillment

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ABSTRACT

William Fleissner and Adrienne Stanley showed that, in finite products of ordinals, the following are equivalent:

1. X is a D-space.
2. X is metacompact.
3. X is metalindelöf.
4. X does not contain a closed subset which is homeomorphic to a stationary subset of a regular, uncountable cardinal.

In this paper we construct a counterexample that shows that this equivalence does not extend to infinite products of ordinals. We also introduce a new property, club-separable, which we show implies D for subsets of ω_1^ω. We hope that club-separable will be able to replace property (4) above in order to generalize the equivalence to infinite products of ordinals.

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Date	Dr. Adrienne Stanley, Chair, Thesis Committee
Date	Dr. Douglas Mupasiri, Thesis Committee Member
Date	Dr. Theron Hitchman, Thesis Committee Member
Date	Dr. Michael J. Licari, Dean, Graduate College

I would like to dedicate my thesis to my parents.

Without you, I would never have made it this far.

I would like to thank my advisor, Dr. Stanley, for guiding me these last few years. You have always been there to help put me back on track whenever I have lost the way.

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CHAPTER 1

INTRODUCTION

The notion of a D-space was first introduced in a paper by E.K. van Douwen and W. Pfeffer in 1979. The D property is a covering property; it is easy to show that any compact space is necessarily a D-space. However; aside from a couple easy facts, little else is known about the relationship between the D property and most other well-known covering properties. For example, it is not known whether or not covering properties as strong as hereditarily Lindelöf imply D and yet it may be that only very weak properties such as submetacompact or submetalindelöf imply D.

Although many questions about covering properties remain open, there has still been quite a bit of interesting work on D-spaces. In particular, William Fleissner and Adrienne Stanley [2] proved an interesting result on finite products of ordinals which provided the basis for many of the results in this paper.

THEOREM 1.1. *If α is an ordinal and $X \subset \alpha^n$ for some $n \in \omega$, then the following are equivalent:*

1. *X is a D-space.*
2. *X is metacompact.*
3. *X is metalindelöf.*
4. *X does not contain a closed subset which is homeomorphic to a stationary subset of a regular, uncountable cardinal.*

The main goal of the research for this paper was to extend these results to infinite products of ordinals. We created a counterexample which shows that the equivalence no longer holds even in countably infinite products of ordinals. We created a subset of ω_1^ω which does not contain a closed subset which is homeomorphic to a stationary subset of a regular, uncountable cardinal, and is also not D, not metacompact, and not metalindelöf.

THEOREM 1.2. *There exists a space $X \subset \omega_1^\omega$ such that X does not contain a closed subset homeomorphic to a stationary subset of a regular, uncountable cardinal and X is not D, not metacompact, and not metalindelöf.*

You can see that this counterexample only shows that one of the four equivalent statements in the finite case no longer holds. So the question as to whether or not being D, metacompact, and metalindelöf are equivalent in the infinite case is still open. With this goal in mind we have defined a new property, club-separable, which is stronger than property (4) from Theorem 1.1 and we will show that if $X \subset \omega_1^\omega$ is club-separable, then is must be D.

THEOREM 1.3. *If $X \subset \omega_1^\omega$ is club-separable, then X is a D-space.*

In addition, we will show that for any $\alpha < \omega_1$, α^ω is a hereditarily D-space. This is a nice result because, in general, only closed subsets of a D-space must be D.

THEOREM 1.4. *If $\alpha < \omega_1$ and $X \subset \alpha^\omega$, then X is a D-space.*

The question as to whether or not the equivalence between metacompact and D-spaces will hold for arbitrary products of ordinals is still unanswered. An interesting fact is that finite products of ordinals are scattered spaces and it is known that metacompact, scattered spaces are necessarily D-spaces. Countable products of ordinals; however, are not necessarily scattered, although our counterexample is. A natural question from this is whether or not metacompact and D are equivalent for finite and/or arbitrary products of scattered spaces.

CHAPTER 2

BASIC DEFINITION AND RESULTS

We will start by making a few definitions. The first concept is that of a neighborhood assignment. The idea for this is simply that for each $x \in X$ we choose some open set $O(x)$ which contains x. Then the collection $\mathcal{O} = \{O(x) : x \in X\}$ is a *neighborhood assignment*. Formally we get;

DEFINITION 2.1. *A neighborhood assignment for a space (X, τ) is a function $O : X \to \tau$ with $x \in O(x)$ for every $x \in X$.*

It is easy to see that any neighborhood assignment determines an open cover. It is also possible to construct a neighborhood assignment from any open cover. If $\mathcal{O}$ is an open cover on X then for every $x \in X$ there exists some $O \in \mathcal{O}$ such that $x \in O$. Let this O be the neighborhood containing x.

There is one more concept that must be covered before we can get into definitions of stationary sets and club sets. That is the concept of refining an open cover.

DEFINITION 2.2. *A refinement for an open cover, $\mathcal{O}$, on a space X is a collection of open sets $\mathcal{U}$ such that for each $U \in \mathcal{U}$ there exists $O \in \mathcal{O}$ such that $U \subset O$ and $\bigcup \mathcal{U}$ covers X.*

Notice that for each U in a refinement of the cover $\mathcal{O}$ we need only find some $O \in \mathcal{O}$ that contains U. This allows there to be an arbitrarily large number of sets from the refinement contained in each $O \in \mathcal{O}$. This is where the main difficulty lies in dealing with refinements: there is always the possibility that there are uncountably many sets from the refinement contained in any $O \in \mathcal{O}$.

This also turns out to be the key difficulty when trying to prove equivalence between metacompactness and the D property. This is the reason why Theorem 1.1 includes (4) as part of the equivalence for finite products of ordinals. There is no known way to directly prove the equivalence of metacompactness and the D property; therefore, we show equiva-

lence by showing each of the preceding two properties is equivalent to (4) of Theroem 1.1 independently.

We are now ready to introduce what it means for a space to be metacompact and metalindelöf. Both are covering properties which are significant weakenings of their counterparts compact and Lindelöf.

DEFINITION 2.3. *A collection of sets, $\mathcal{U}$, is point-finite (point-countable) if for each $x \in \bigcup \mathcal{U}$, $\mathcal{U}(x) = \{U \in \mathcal{U} : x \in U\}$ is finite (countable).*

DEFINITION 2.4. *X is said to be metacompact (metalindelöf) if for any open cover $\mathcal{O}$ of X there exists an open refinement, $\mathcal{U}$, of $\mathcal{O}$ which is point-finite (point-countable).*

It is also worth noting that both metacompact and metalindelöf are properties which are closed hereditarily, i.e. all closed subspaces of a metacompact space are metacompact. This means that any closed subset of a metacompact (metalindelöf) space is also metacompact (metalindelöf).

We are now ready to introduce the more set-theoretic notions. From this point forward we will let $\omega = \{0, 1, \ldots\}$ and we will let ω_1 be an uncountable well-ordered set such that for every $\alpha < \omega_1$, $[0, \alpha]$ is countable.

DEFINITION 2.5. *Let α be an ordinal. Then $C \subset \alpha$ is a club in α if it is both closed and unbounded in α.*

DEFINITION 2.6. *Let α be an ordinal. Then $S \subset \alpha$ is stationary in α if $S \cap C \neq \emptyset$ for every club C in α.*

It is worth noting that any club set is necessarily a stationary set. It is also true that, for a regular, uncountable cardinal κ, the intersection of less than κ many club sets of κ is still club. However; this is not the case for stationary sets. In fact, it is known that any regular, uncountable cardinal κ can be partitioned into κ many pairwise disjoint stationary sets. This illustrates that although stationary sets are big, club sets are much, much bigger. Another interesting property of stationary sets is that for a regular, uncountable cardinal

κ, if $S \subset \kappa$ is stationary and $S = \dot{\bigcup}_{\alpha<\lambda} S_\alpha$ for some $\lambda < \kappa$, then S_α must be stationary in κ for some $\alpha < \lambda$. These are all properties of stationary and club sets which we will use later in this paper. Next we will introduce a very common tool known as the Pressing Down, or Fodor's, Lemma.

PRESSING DOWN LEMMA. *Let κ be a regular, uncountable cardinal and S be a stationary set in κ. If $f : S \to S$ is a function such that $f(\alpha) < \alpha$ for each $\alpha \in S$, then there exists $T \subset S$ which is stationary in κ and $\beta < \kappa$ such that $f(\alpha) = \beta$ for each $\alpha \in T$.*

Often when we want to use the Pressing Down Lemma in this paper we will let $S \subset \omega_1$ be stationary. For each $\alpha \in S$ we will consider a basic open set $(\beta_\alpha, \alpha]$ for some $\beta_\alpha < \alpha$. The Pressing Down Lemma will then give us $T \subset S$ which is stationary in ω_1 and $\beta < \omega_1$ such that for each $\alpha \in T$, $\beta_\alpha = \beta$. In this way, $\gamma = min\{T\}$ will be contained in $(\beta, \alpha]$ for each $\alpha \in T$.

We will now show that any stationary subset of a regular, uncountable cardinal is not metalindelöf and therefore not metacompact. This is a fairly simple result that will come in handy once we begin the discussion about sets which are homeomorphic to stationary subsets of regular, uncountable cardinals as mentioned in Theorem 1.1.

LEMMA 2.1. *If κ is a regular, uncountable cardinal and $S \subset \kappa$ is stationary, then S is not metalindelöf, and therefore not metacompact.*

Proof. Let κ be a regular, uncountable cardinal and $S \subset \kappa$ be stationary. In order to show that S is not metalindelöf, we need to find some open cover of S which has no point-countable refinement. Let $\mathcal{O} = \{[0, \alpha] : \alpha \in S\}$. Then let $\mathcal{U}$ be any refinement of $\mathcal{O}$.

Since $\mathcal{U}$ covers S, for every $\alpha \in S$ such that α is a limit ordinal, there exists some $U \in \mathcal{U}$ and $\beta_\alpha < \alpha$ such that $S \cap (\beta_\alpha, \alpha] \subset U$. By the Pressing Down Lemma there exists some $T \subset S$ which is stationary in κ and $\beta < \kappa$ such that for every $\alpha \in T$, $\beta_\alpha = \beta$. Let γ be the least element in T. Then $|\{U \in \mathcal{U} : \gamma \in U\}| = \kappa$.

Therefore S is not metalindelöf and therefore not metacompact. □

Next we will define what it means to be a D-space. Then we will prove a similar statement as that above, showing that a stationary subset of a regular, uncountable cardinal cannot be D.

DEFINITION 2.7. *X is said to be a D-space if for every neighborhood assignment O, on X, there exists a closed, discrete $D \subset X$ such that $O(D) = \{O(x) : x \in D\}$ covers X.*

Again, it is worth noting that being a D-space is closed hereditarily. This can easily be seen since any open neighborhood assignment on a closed subset of a D-space can be extended to an open neighborhood assignment on the space itself where the neighborhood on every point outside the closed set is just the complement of the closed set. Then by taking our closed, discrete D witnessing that the larger space is a D-space and intersecting it with the closed set, we get a closed, discrete subset of our smaller space which satisfies the requirements for a D-space.

LEMMA 2.2. *If κ is a regular, uncountable cardinal and $S \subset \kappa$ is stationary, then S is not a D-space.*

Proof. Let κ be a regular, uncountable cardinal and $S \subset \kappa$ be stationary. We will use the same open cover as the previous lemma and notice that it is defined in such a way that it is also a neighborhood assignment. Let $O : S \to \mathcal{P}(S)$ be defined such that $O(\alpha) = [0, \alpha] \cap S$ for each $\alpha \in S$. Then any set $E \subset S$ such that $O(E)$ covers S must be unbounded in S. So any closed set $D \subset S$ such that $O(D)$ covers S must be unbounded and closed in S; in other words, D must be a club in S. In which case, D cannot be discrete. So S is not a D-space. □

It is easy to see now why property (4) from Theorem 1.1 may be a good way to ensure equivalence of property D and metacompactness in some topological spaces. We will now show that any space which is metacompact, metalindelöf or a D-space must also have property (4) from Theorem 1.1. The result, along with the counterexample to be presented in the next section, shows us that in order to obtain an equivalence for infinite products of ordinals we will need to strengthen property (4) because it is too weak.

COROLLARY 2.1. *Any topological space X which is metacompact, metalindelöf or a D-space does not contain a closed subset homeomorphic to a stationary subset of a regular, uncountable cardinal.*

This follows directly from the previous lemma and the fact that metacompact, metalindelöf and the D property are all closed hereditarily. If there is a closed subset homeomorphic to a stationary subset of a regular, uncountable cardinal then that closed subset is not metacompact, metalindelöf or D and therefore the entire space cannot be metacompact, metalindelöf or D.

CHAPTER 3

COUNTEREXAMPLE

In this section we will construct a counterexample that shows that it is not possible to extend Theorem 1.1 to infinite products of ordinals. We will begin by proving a result about the structure of stationary sets in ω_1.

LEMMA 3.1. *No stationary set $S \subset \omega_1$ can be partitioned into two uncountable, disjoint, closed (in S) subsets.*

Proof. Let $S \subset \omega_1$ be stationary. Let $S = S_1 \dot{\cup} S_2$ with S_1, S_2 closed in S. Since S is stationary, without loss of generality we can assume that S_1 is stationary. Now since S, S_1 are unbounded, then $\bar{S}, \bar{S_1}$ are both club in ω_1.

By way of contradiction, suppose S_2 is also unbounded in ω_1. Then $\bar{S_2}$ is club in ω_1 and therefore $\bar{S_1} \cap \bar{S_2}$ is club in ω_1. Therefore $(\bar{S_1} \cap \bar{S_2}) \cap S \neq \emptyset$. Let $\alpha \in S \cap (\bar{S_1} \cap \bar{S_2})$. Then since S_1, S_2 are closed in S, we know $\bar{S_1} \cap S = S_1$ and $\bar{S_2} \cap S = S_2$. Therefore $\alpha \in S_1 \cap S_2$, but $S_1 \cap S_2 = \emptyset$. This is a contradiction, so S_2 cannot be unbounded in ω_1. □

Lemma 3.1 above can be extended to the case when S is partitioned into countably many disjoint, closed subset.

COROLLARY 3.1. *If $S \subset \omega_1$ is stationary and is partitioned into countably many disjoint, closed sets $\{S_n : n \in \omega\}$, then there exists $n \in \omega$ such that $|S_n| = \omega_1$ and $|S_i| \leq \omega$ for each $i \neq n$.*

Next is a corollary to the above results which is nice for characterizing subsets of ω_1^ω which are closed and homeomorphic to a stationary subset of ω_1 which will come in handy later.

COROLLARY 3.2. *Let $X \subset \omega_1^\omega$ and let $S \subset X$ be closed and homeomorphic to a stationary subset of ω_1 with $\pi_n(S)$ stationary in ω_1 for some $n \in \omega$. If there exists $m \in \omega$ with $\pi_m(S) \subset \omega$ then there exists $k \in \omega$ such that $\{\vec{s} \in S : \pi_m(\vec{s}) = k\}$ is co-countable, i.e. $|\{\vec{s} \in S : \pi_m(\vec{s}) \neq k\}| \leq \omega$.*

Proof. Let $X \subset \omega_1^\omega$ and let $S \subset X$ be closed and homeomorphic to a stationary subset of ω_1 with $\pi_n(S)$ stationary in ω_1 for some $n \in \omega$. Without loss of generality suppose $n \neq 0$ and $\pi_0(S) \subset \omega$. Then

$$S = \dot{\bigcup}_{k\in\omega}(\pi_0^{-1}(\{k\}) \cap S)$$

where $\pi_0^{-1}(\{k\}) \cap S$ is clopen in S for each $k \in \omega$. Since S is homeomorphic to a stationary subset of ω_1, then by Corollary 3.1 there is a unique $k \in \omega$ such that $|\{\vec{s} \in S : \pi_0(\vec{s}) = k\}| = \omega_1$. Therefore $\{\vec{s} \in S : \pi_0(\vec{s}) = k\}$ is co-countable. □

From this result it is easy to see the idea behind the space we are trying to construct. We want to build a subset of ω_1^ω whose projection onto a single coordinate is stationary in ω_1, with each other projection being countable. We will just let the 0^{th} coordinate be ω_1.

The idea behind this construction is to first divide ω_1 into countably many disjoint stationary sets $S(\{(0, m)\})$. Each of these will then be divided into countably many disjoint stationary sets. We will repeat this process, but at the end we want the intersection of any descending sequence of stationary sets to be at most a singleton, this is the purpose of the last property stated after the construction of our stationary sets.

With this in mind we will begin the construction of our counterexample. Let $\{T_f\}_{f\in{}^{<\omega}\omega}$ be a pairwise disjoint collection of stationary sets in ω_1 such that $\omega_1 = \dot{\bigcup}_{f\in{}^{<\omega}\omega} T_f$. Then for each $\alpha \in \omega_1$ we will choose $f_\alpha \in {}^\omega\omega$ such that if $\alpha \in T_f$ then $f \subset f_\alpha$ and for $\alpha \neq \beta$, $f_\alpha \neq f_\beta$. This can clearly be done since $\{g \in {}^\omega\omega : f \subset g\}$ is uncountable for each $f \in {}^{<\omega}\omega$.

For each $f \in {}^{<\omega}\omega$, set $S(f) = \{\alpha \in \omega_1 : f \subset f_\alpha\}$. Then the collection $\{S(f) : f \in {}^{<\omega}\omega\}$ has the following properties:

1. $S(f) \cap S(g) = \emptyset$ whenever $f \neq g$, $f, g \in {}^n\omega$.
2. For each $n \in \omega$, $\omega_1 = \dot{\bigcup}_{f\in{}^n\omega} S(f)$.
3. $S(f)$ is stationary in ω_1.
4. For each $\alpha, \beta \in S(f)$ there is $g \supset f$ such that $\alpha \in S(g)$ and $\beta \notin S(g)$.

The first property comes from the fact that for $f, g \in {}^{n}\omega$ with $f \neq g$, the extensions of f are disjoint from the extensions of g. The second property is clear to see since each $\alpha \in \omega_1$ is assigned a unique f_α. Property 3 is trivial since $T_f \subset S(f)$ for each $f \in {}^{<\omega}\omega$.

Lastly, to see that property 4 holds, let $\alpha, \beta \in S(f)$ for some $f \in {}^{<\omega}\omega$. Then since $f_\alpha \neq f_\beta$, there exists $m \in \omega$ such that $f_\alpha \restriction_m \neq f_\beta \restriction_m$. Clearly since $\alpha, \beta \in S(f)$ we know $m > |f|$. Let $g = f_\alpha \restriction_m$. Then by definition $\alpha \in S(g)$ and $\beta \notin S(g)$.

Now from these stationary sets we will constuct our space $X \subset \omega_1^\omega$. For each $\alpha \in \omega_1$ we have already defined $f_\alpha \in {}^{\omega}\omega$ and so we will define $\vec{x}_\alpha \in \omega_1^\omega$ by

$$\vec{x}_\alpha(n) = \begin{cases} \alpha & \text{if } n = 0 \\ f_\alpha(n-1) & \text{if } n \neq 0 \end{cases}$$

Let $X = \{\vec{x}_\alpha : \alpha \in \omega_1\}$. Next we will use Corollary 3.2 to show that X does not contain a closed subset homeomorphic to a stationary subset of ω_1.

LEMMA 3.2. *X does not contain a closed subset homeomorphic to a stationary subset of ω_1.*

Proof. By way of contradiction, suppose there exists a closed $T \subset X$ which is homeomorphic to a stationary subset of ω_1. Since $\pi_m(X)$ is countable for each $m \geq 1$ and $\pi_0(X)$ is stationary, Corollary 3.2 tells us that for each $m \geq 1$ there exists $n_m \in \omega$ such that $\pi_m^{-1}(\{n_m\}) \cap T$ is co-countable. Let $\hat{T} = \bigcap_{1 \leq m < \omega} \pi_m^{-1}(\{n_m\}) \cap T$. Then $\hat{T}$ is uncountable. So there exists $\alpha, \beta \in \omega_1$ such that $\vec{x_\alpha}, \vec{x_\beta} \in \hat{T}$. But then $f_\alpha = f_\beta$, a contradiction.

□

Now that we have shown that there is no closed subset of X which is homeomorphic to a stationary subset of ω_1, it is clear that there can be no closed subset of X which is homeomorphic to any regular, uncountable cardinal, since $|X| = \omega_1$. So now we need only show that X is not metacompact, metalindelöf or D. We will begin by showing that X is not metalindelöf which implies that X is not metacompact. You will notice that we use a similar open cover to the open cover used in Lemma 2.2.

LEMMA 3.3. *X is not metalindelöf.*

Proof. In order to show that X is not metalindelöf we need to construct an open cover which has no point-countable refinement. Let $\mathcal{O} = \{[0, \alpha] \times \omega_1^\omega : \alpha \in \omega_1\}$. Let $\mathcal{U}$ be any refinement of $\mathcal{O}$. For each $\vec{x}_\alpha \in X$, let W_α be some basic open neigborhood of $\vec{x}_\alpha$ such that:

1. $W_\alpha \subset U$ for some $U \in \mathcal{U}$.
2. $\pi_0(W_\alpha) = (\beta_\alpha, \alpha]$ for some $\beta_\alpha < \alpha$.
3. $supp(W_\alpha) = n$ for some $n \in \omega$
   (n here is being considered as a set).

Then for each $n \in \omega$, let $S_n = \{\alpha \in \omega_1 : supp(W_\alpha) = n\}$. Since $\omega_1 = \dot{\bigcup}_{n\in\omega} S_n$, there is some $n \in \omega$ such that S_n is stationary in ω_1.

Fix $n \in \omega$ such that S_n is stationary in ω_1. Then since $\omega_1 = \dot{\bigcup}_{f\in {}^n\omega} S(f)$, we have $S_n = \dot{\bigcup}_{f\in {}^n\omega}(S(f) \cap S_n)$. Since ${}^n\omega$ is countable, there exists some $f \in {}^n\omega$ such that $S(f) \cap S_n$ is stationary in ω_1. By the Pressing Down Lemma on the first coordinate, there exists $\beta \in \omega_1$ and a stationary set $T \subset S(f) \cap S_n$ such that $\beta = \beta_\alpha$ for each $\alpha \in T$.

Let γ be least in T. We will show $\{U \in \mathcal{U} : \vec{x_\gamma} \in U\}$ is uncountable. First note that for each $U \in \mathcal{U}$, $\{\alpha < \omega_1 : W_\alpha \subset U\}$ is countable since $\pi_0(U)$ is bounded above in ω_1. So if we can show that $\{\alpha < \omega_1 : \vec{x_\gamma} \in W_\alpha\}$ is uncountable, then $\{U \in \mathcal{U} : \vec{x_\gamma} \in U\}$ must also be uncountable. Notice now that T is uncountable and $\gamma \in \pi_0(W_\alpha)$ for each $\alpha \in T$. Further, $\alpha \in T \subset S(f) \cap S_n$ implies $\pi_i(\vec{x_\alpha}) = \pi_i(\vec{x_\gamma})$ for each $1 \leq i \leq n-1$. Thus, $\vec{x_\gamma} \in W_\alpha$ for each $\alpha \in T$. Thus, $\{\alpha < \omega_1 : \vec{x_\gamma} \in W_\alpha\}$ is uncountable and hence $\{U \in \mathcal{U} : \vec{x_\gamma} \in U\}$ is uncountable. Therefore X is not metalindelöf and thus not metacompact.

□

Now the only thing we have left to show is that X is not a D-space. We will first show that no discrete subset of X can be stationary on the 0^{th} coordinate and then we will use this to show that X cannot be a D-space.

CLAIM 3.1. *If $D \subset X$ is discrete, then $\pi_0(D)$ is not stationary in ω_1.*

Proof. Let $D \subset X$ be discrete. Let $S = \{\alpha \in \omega_1 : \vec{x}_\alpha \in D\} = \pi_0(D)$. By way of contradiction, suppose S is stationary in ω_1. For each $\alpha \in S$, let W_α be as in the previous lemma with the additional condition that $W_\alpha \cap D = \{\vec{x}_\alpha\}$. Then for each $n \in \omega$, let $D_n = \{\vec{x_\alpha} \in D : supp(W_\alpha) = n\}$. So there exists some $n \in \omega$ such that $\pi_0(D_n)$ is stationary in ω_1.

Since $\pi_0(D_n)$ is stationary in ω_1 and $\pi_m(D) \subset \pi_m(X)$ is countable for every $1 \leq m < \omega$, Corollary 3.2 says that for each $1 \leq m \leq n-1$ there exists $n_m \in \omega$ such that $\pi_m^{-1}(\{n_m\}) \cap D_n$ is co-countable in D_n. Let $\hat{T} = \bigcap_{m=1}^{n-1} \pi_m^{-1}(\{n_m\}) \cap D_n$. Then $D_n \backslash \hat{T}$ is countable and therefore $\pi_0(\hat{T})$ is stationary in ω_1. Then the Pressing Down Lemma says there exists a stationary subset $T \subset \pi_0(\hat{T})$ and $\beta < \omega_1$ such that $\beta_\alpha = \beta$ for each $\alpha \in T$. Let γ be least in T. Then for each $\alpha \in T$, we know $supp(W_\alpha) = supp(W_\gamma)$ and $\pi_m(\vec{x_\alpha}) = \pi_m(\vec{x_\gamma})$ for each $1 \leq m \leq n-1$. Thus $\vec{x_\gamma} \in W_\alpha \cap D$ for each $\alpha \in T$, a contradiction. □

LEMMA 3.4. *X is not a D-space.*

Proof. Let O be the open neighborhood assignment defined by $O(\vec{x}_\alpha) = [0, \alpha] \times \omega_1^\omega$ for each $\alpha \in \pi_0(X) = \omega_1$. Let D be a closed subset of X such that $O(D)$ covers X. Then $X \backslash D$ is open. Then $\pi_0(X \backslash D)$ is open since π_0 is an open mapping. Since π_0 is also a one to one mapping in this case, $\pi_0(D) = \pi_0(X) \backslash \pi_0(X \backslash D)$ which is closed. In order for $O(D)$ to cover X, $\pi_0(D)$ must be unbounded in $\pi_0(X)$ and therefore D must be uncountable. But if $\pi_0(D)$ is uncountable and closed, then $\pi_0(D)$ is a club. But no discrete subset of X can be stationary in ω_1. Since every club is stationary, D must be countable and therefore $O(D)$ cannot cover X. Thus X is not a D-space.

□

Now as a culmination of everything from this section, we have the following theorem.

THEOREM 1.2. *There exists a space $X \subset \omega_1^\omega$ such that X does not contain a closed subset homeomorphic to a stationary subset of a regular, uncountable cardinal and X is not D, not metacompact, and not metalindelöf.*

As you can see, a lot of the reason that X fails to be a D-space comes from the fact that π_0 is a one-to-one mapping from X to $\pi_0(X)$. This is a very nice property which you can not hope to have in the general case for infinite product spaces of ordinals.

In the next section we will introduce a new technical property which we hope will be able to replace property (4) from Theorem 1.1 in the general case for infinite products of ordinals. We will also show that for every $\alpha < \omega_1$, α^ω is a D-space.

CHAPTER 4

D-SPACES IN ω_1^ω

We will begin this section by observing that for each $\alpha < \omega_1$, α^ω has a countable base. We will then use this fact to begin the discussion on D-spaces in ω_1^ω.

FACT 4.1. *If $\alpha < \omega_1$, then α^ω has a countable base.*

This can clearly be seen from the fact that basic open sets in α^ω have finite support and the fact that ${}^{<\omega}\omega$ is countable.

Now we will introduce a couple of interesting results that we will use to prove the Theorems mentioned in the Introduction. The first being a result by A.V. Arhangel'skii [1]:

THEOREM 4.1. *If X is a topological space with a point-countable base, then X is a D-space.*

The last important result we will use in this paper was proved by H. Guo and H.J.K. Junilla [3] about the infinite union of closed D-subspaces.

THEOREM 4.2. *Suppose $X = \bigcup_{\alpha<\lambda} X_\alpha$, where each X_α is D, and for each $\beta < \lambda$, $\bigcup_{\alpha<\beta} X_\alpha$ is closed. Then X is a D-space.*

We will use Theorem 4.1 immediately to prove Theorem 1.4.

THEOREM 1.4. *Let $\alpha < \omega_1$ and $X \subset \alpha^\omega$. Then X is a D-space.*

Proof. Let $\alpha < \omega_1$ and $X \subset \alpha^\omega$. From the preceding fact we can say that α^ω has a countable base and therefore X has a countable base. By Theorem 4.1, X must be a D-space. □

Now that we have shown that countable products of countable ordinals are D-spaces, we are ready to introduce our property which we hope will be able to replace property (4) from Theorem 1.1 when extended to the case of infinite products. In order to do this we will need to first introduce some notation.

DEFINITION 4.1. *Let α be an ordinal and C a club in α. We will let*

$$l(C) = \bigcup_{n\in\omega} \bigcup_{\beta\in C} [0,\beta]^n \times \{\beta\} \times [0,\beta]^\omega$$

So let C be a club in α and fix $\beta \in C$ and $n \in \omega$. Then $[0,\beta]^n \times \{\beta\} \times [0,\beta]^\omega$ is a subset of α^ω which is $\{\beta\}$ only on the n^{th} coordinate and $[0,\beta]$ everywhere else. So $l(C)$ is the union of these sets over $\beta \in C$.

DEFINITION 4.2. *Let $X \subset \alpha^\omega$ for some ordinal α. We say that X is club-separable if there exists $C \subset \alpha$ which is club such that $X \cap l(C) = \emptyset$.*

Next we will show that club-separability is enough to imply D for subsets of ω_1^ω. We hope that the converse can also be shown, but we have not been able to show this thus far. It may be that we have strengthened property (4) from Theorem 1.1 too much.

THEOREM 1.3. *If $X \subset \omega_1^\omega$ is club-separable, then X is a D-space.*

Proof. Let $X \subset \omega_1^\omega$ be club-separable and let C be a club in ω_1 such that $X \cap l(C) = \emptyset$. For each $\beta \in C$, define $X_\beta = X \cap [0,\beta]^\omega = X \cap [0,\beta)^\omega$. Then for each $\beta \in C$, X_β is closed in X. By Theorem 1.4, we can say that X_β is a D-space since $\beta < \omega_1$.

We will show $X = \bigcup_{\beta\in C} X_\beta$. To see this suppose $\vec{x} \in X$. Then $\{\pi_m(\vec{x}) : m < \omega\}$ is bounded in ω_1. Therefore there exists $\beta \in C$ such that $\beta > \sup_{m\in\omega}\{\pi_m(\vec{x})\}$. Thus $\vec{x} \in X_\beta$. Therefore $X = \bigcup_{\beta\in C} X_\beta$.

Furthermore, for each $\beta \in C$, we will show $\bigcup_{\gamma<\beta} X_\gamma$ is closed. To see this, let $\beta \in C$. If there exists $\delta \in C$ such that $\delta < \beta$ and $(\delta,\beta) \cap C = \emptyset$, then $\bigcup_{\gamma<\beta} X_\gamma = X_\delta$, which is closed. If no such δ exists, then $\bigcup_{\gamma<\beta} X_\gamma = X_\beta$, which is closed. Therefore $\bigcup_{\gamma<\beta} X_\gamma$ is closed for each $\beta \in C$ and thus by Theorem 4.2, X is a D-space. □

This is a nice result because it tells us something about what D-spaces look like in subsets of ω_1^ω. However, there is trouble in extending this result to ω_2^ω. The natural way of trying to extend this result would be with a proof by induction. During the induction

step, when $\alpha > \omega_1$ is an ordinal with countable cofinality and $X \subset \alpha^\omega$, we can no longer say that if $\vec{x} \in X$, then $\{\pi_m(\vec{x}) : m \in \omega\}$ is bounded in α.

CHAPTER 5

CONCLUSION

There is still plenty of work to be done to characterize D-spaces in ω_1^ω. I will state some of the remaining questions which should be answered to continue this work.

QUESTION 5.1. *If $X \subset \omega_1^\omega$ is a D-space, then is X club-separable?*

QUESTION 5.2. *If $X \subset \omega_1^\omega$ is club-separable, then is X metacompact or metalindelöf?*

QUESTION 5.3. *If $X \subset \omega_1^\omega$ is metacompact or metalindelöf, then is X club-separable?*

Furthermore if we want to show that club-separable is equivalent to D, metacompact and metalindelöf with respect to subspaces of α^ω then we want a positive result to the following question.

QUESTION 5.4. *If $X \subset \alpha^\omega$ for some ordinal α and X contains a closed subset homeomorphic to a stationary subset of a regular, uncountable cardinal, then is X club-separable?*

BIBLIOGRAPHY

[1] Arhangel'skii, A.V. and R. Buzyakova, "Addition theorems and D-spaces." *Comment. Math. Univ. Carolin.* 43(2002), 653-663.

[2] Fleissner, W. and A. Stanley, "D-Spaces." *Topology and its Applications* 114(2001), 261-271.

[3] Guo, H. and H.J.K. Junnila, "On D-spaces and their unions." preprint.