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Stone's Representation Theorem

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

by

Ioan Radu

March 2007

STONE'S REPRESENTATION THEOREM

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Ioan Radu

March 2007

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Abstract

In mathematics, a representation theorem is a theorem that states that every abstract structure with certain properties is isomorphic to a concrete structure. My purpose in this thesis is to analyze some aspects of the theory of distributive lattices - in particular the Representation Theorems:

- Birkhoff's representation theorem for finite distributive lattices
- Stone's representation theorem for infinite distributive lattices

The representation theorem of Garrett Birkhoff establishes a bijection between finite posets and finite distributive lattices. Stone's representation theorem for lattices states that every distributive lattice is isomorphic to a sublattice of the power set lattice of some set. As can be seen, each of these results gives a concrete realization for (abstract) distributive lattices.

Acknowledgements

I would like to express my appreciation to my committee chair, Dr. Gary Griffing, for this guidance, time, energy, and support in producing this thesis. I would also like to thank the professors on my committee, Dr. Zahid Hasan and Dr. Belisario Ventura, for their help.

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Chapter 1

Introduction

In Mathematics, representation theorems help us to investigate unknown abstract structures, by allowing us to consider more well known concrete structure. At the core of every representation theorem is a stucture preserving map from the abstract structure to the concrete one. The original structure is studied via its image under this map.

According to Johnstone [3] the birth of abstract algebra can be traced to a paper of Cayley on group theory (1854). Universal algebra established itself in the 1930's as a unifying tool for the theory of groups, modules, rings and lattices.

The topologist M. Stone introduced in 1934 the notion of topology to algebra and proved his famous representation theorem for finite Boolean algebras [6]. In his work, Stone proved that every finite Boolean algebra can be realized as the full power set of the set of atoms of the Boolean algebra; each element of the Boolean algebra bijectively corresponds to the set of atoms (minimal elements) below it (the join of which is the element). This power set representation can be constructed more generally for any complete atomic Boolean algebra; however in these cases, the image of the representation is not the full power set.

Independently at around the same time (1934), Garrett Birkhoff proved a representation theorem for finite distributive lattices [7] (in this thesis see Theorem 7.4). It states that every finite distributive lattice is isomorphic to a lattice of down-sets of the poset of join-irreducible elements. This establishes a bijection between the class of all finite posets and the class of all finite distributive lattices.

In the same period of time (1936) Stone discovered a method to extend his representation result for finite Boolean algebras to arbitrary Boolean algebras [4], [6] and Birkhoff adapted it to arbitrary distributive lattices [7] (see Theorem 8.6 in this thesis).

In this work, we focused on distributive lattices and give a representation result for both the finite and infinite cases. In each case, the result is due to both, Birkhoff and Stone, and is purely algebraic in that no topology is used.

Throughout this thesis, unless otherwise stated, the references for definitions, lemmas, corrollary, etc, can be found in Davey and Priestley [1] and Grätzer [2]. A nice survey article for the general reader will be G. C. Rota [5].

Chapter 2

Ordered Sets

We begin by introducing the notion of order. When we think about order we refer to more than one objects. An ordering is a binary relation on a set of objects that compares them. Greater than, taller, less or equal, are examples of ordering.

2.1 Ordered Sets

Definition 2.1. *Let P be a set. An* **order** *(or* **partial order***) on P is a binary relation $\leq$ on P such that, for all $x, y, z \in P$,*

(i) $x \leq x$ Reflexivity,

(ii) $x \leq y$ *and* $y \leq x$ *imply* $x = y$ Antisymetry,

(iii) $x \leq y$ *and* $y \leq z$ *imply* $x \leq z$ Transitivity.

A set P equipped with an order relation $\leq$ is said to be an **ordered set**(or **partially ordered set**). We'll refer to the ordered set using the shorthand **poset** and we write $\langle P; \leq \rangle$ when it is necessary to specify the order relation. On any set P, '$=$', the **discret order**, and '$<$' meaning **strict inequality** are order relations. We use $x \leq y$ and $y \geq x$ interchangeably, and write $x \not\leq y$ to mean '$x \leq y$ is false'. Also, we use $x \parallel y$ when x is not comparable with y. We can construct new ordered sets from existing ones. If P is an ordered set and Q a subset of P, then Q inherits an order relation from P; given $x, y \in Q$, $x \leq y$ in Q if and only if $x \leq y$ in P. We say that Q has the **induced order** from P.

2.2 Chains and Antichains

Definition 2.2. *Let P be an ordered set. Then P is a* ***chain*** *if, for all $x, y \in P$, either $x \leq y$ or $y \leq x$. At the opposite extreme from a chain is an antichain. The ordered set P is an* ***antichain*** *if $x \leq y$ in P only if $x = y$.*

The set $\mathbb{R}$ of all real numbers, with its usual order, forms a chain. The natural numbers $\mathbb{N}$, the integers $\mathbb{Z}$, and the rational numbers $\mathbb{Q}$, also have a natural order making them chains.

We denote the set $\mathbb{N} \cup \{0\} = \{0, 1, 2, 3, ...\}$ by $\mathbb{N}_0$. This set with the order in which $0 < 1 < 2 < 3 < \ldots$ becomes a chain. In particular, in the finite case we will use the symbol $\underline{\mathbf{n}}$ to mean the chain $0 < 1 < 2 < ... < n - 1$.

The length of a finite chain C is $|C| - 1$. A poset P is said to be of length n, where n is a natural number, iff there is a chain in P of length n and all chains in P are of length $\leq n$. A poset P is of finite length if and only if it is of length n, for some natural number n.

2.3 Order Isomorphism

We say that P and Q are (**order-**) **isomorphic** and write $P \cong Q$, if there exists a map φ from P onto Q such that $x \leq y$ in P if and only if $\varphi(x) \leq \varphi(y)$ in Q. Then φ is called an **order-isomorphism**. This map is neccesarily one-to-one and onto.

2.4 Powersets

Let X be any set. The powerset $\mathcal{P}(X)$, consisting of all subsets of X, is ordered by set inclusion: for $A, B \in P(X)$, we define $A \leq B$ if and only if $A \subseteq B$. Any subset of $\mathcal{P}(X)$ inherits the inclusion order.

For example if we have set X consisting of 3 elements: $X = \{1,2,3\}$ the powerset $\mathcal{P}(X)$ has the following elements shown in Fig. 2.2 (ii):

$$\mathcal{P}(X) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

The power set $\mathcal{P}(X)$, where $X = \{a,b,c,d\}$, has as elements the vertex labels in Fig. 2.2 (iii).

2.5 The Covering Relation

Let P be an ordered set and let $x, y \in P$. We say x is **covered by** y (or y **covers** x), and write $x \prec y$ or $y \succ x$, if $x < y$ and $x \leq z < y$ implies $z = x$. The latter condition is demanding that there be no element z of P with $x < z < y$. If P is finite, $x < y$ if and only if there exists a finite sequence of covering relations $x = x_0 \prec x_1 \prec \ldots \prec x_n = y$. For example:

- In the chain $\mathbb{N}$, we have $m \prec n$ if and only if $n = m + 1$.
- In $\mathbb{R}$, there is no covering relation since there are no pairs x, y such that $x \prec y$.
- In $\mathcal{P}(X)$, we have $A \prec B$ if and only if $B = A \cup \{b\}$ for some $b \in X \backslash A$.

2.6 Diagrams

Let P be a finite ordered set. We can represent P by a configuration of circles (representing the elements of P) and interconnecting lines (indicating the covering relation).

To construct a diagram for a power set, we need first to associate to each point $x \in P$, a point $p(x)$ of the Euclidian plane $\mathbb{R}^2$, depicted by a small circle with center at $p(x)$. Then for each covering pair $x \prec y$ in P, take a line segment $l(x, y)$ joining the circle at $p(x)$ to the circle at $p(y)$. We need to do this in such a way that:

(a) if $x \prec y$, then $p(x)$ is 'lower' than $p(y)$ (that is, in standard cartesian coordinates, has a strictly smaller second coordinate).

(b) the circle at $p(z)$ does not intersect the line segment $l(x, y)$ if $z \neq x$ and $z \neq y$.

Figure 2.1(i) shows two alternative diagrams for the ordered set $P = \{a, b, c, d\}$ in which $a < c$, $b < c$, $a < d$ and $b < d$. We can see also that $a \parallel b$ and $c \parallel d$. In Figure 2.1(ii) we have drawings which are not legitimate diagrams for P. In the first c is 'lower' than b, even in our set $b < c$, so the 3(a) rule in 2.6 is violated. In the second the line ad intersects the circle c but $c \neq a$ and $c \neq d$, so the 3(b) rule is violated. In Figure 2.1(iii) we have the following relations:

- $a < b < d < f$

- $c < e < g$
- $a \| c$, $b \| c$, $d \| e$, $f \| g$, and so on.

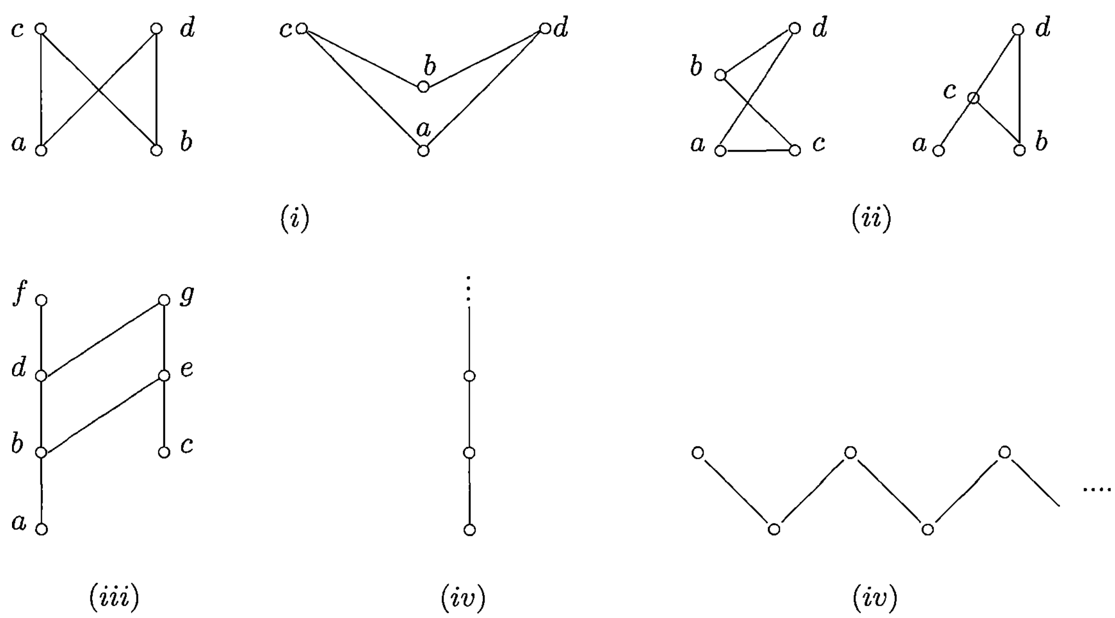

Figure 2.1: The Construction of Diagrams

We have defined diagrams only for finite ordered sets. It is not possible to represent the whole of an infinite ordered set by a diagram, but if its structure is sufficiently regular, like in Figure 2.1(iv), we can sugest how the ordered set looks like. Figure 2.2 contains diagrams for a variety of ordered sets:

(i) all possible ordered sets with three elements.

(ii) the power set $\mathcal{P}\{1,2,3\}$

(iii) the power set $\mathcal{P}\{a,b,c,d\}$

2.7 The Dual of an Ordered Set

For any ordered set P there exists a new ordered set P^δ (the **dual** of P) defined by $x \leq y$ to hold in P^δ if and only if $y \leq x$ holds in P. For finite ordered sets, a diagram

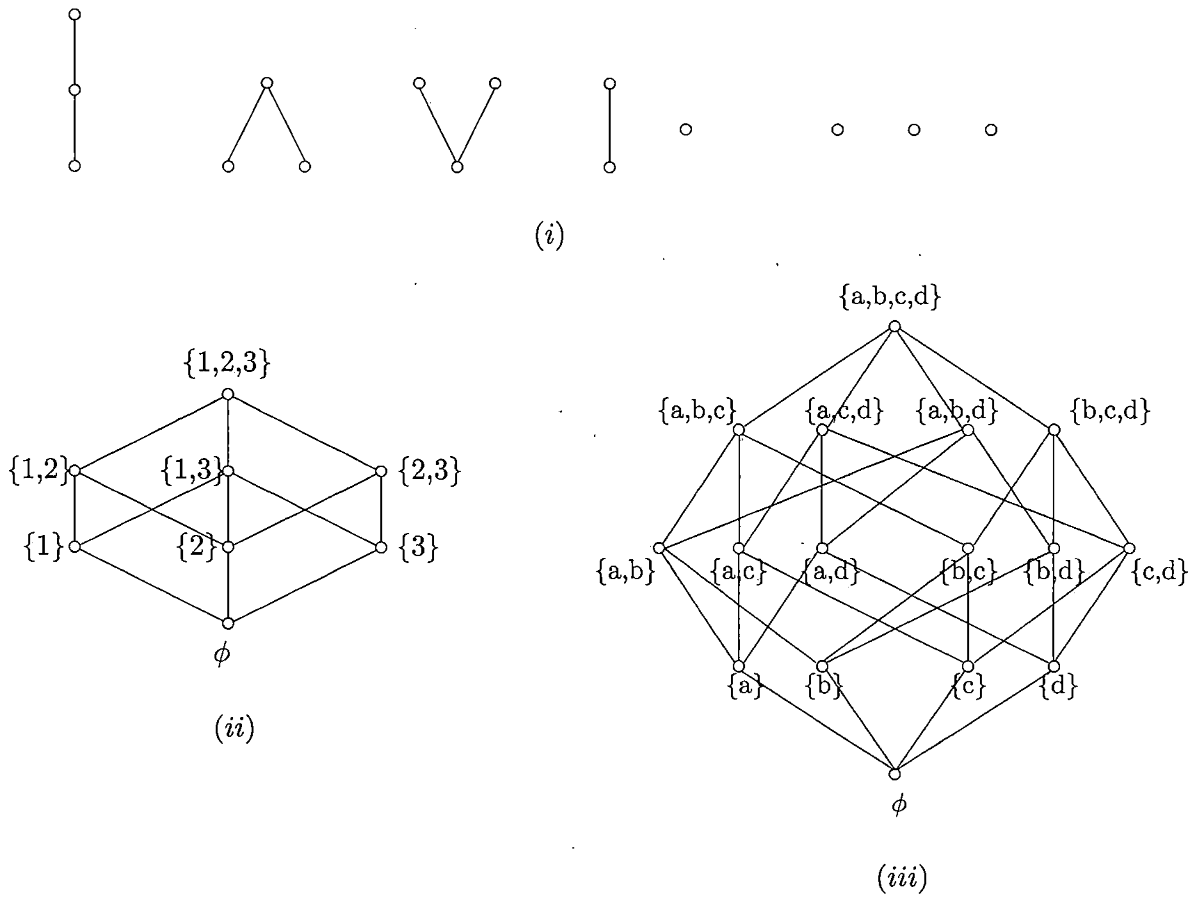

Figure 2.2: Examples of Diagrams

for the dual can be obtained simply by 'turning upside down' the original diagram, as we can see in Figure 2.3.

We can observe that for each statment about the ordered set P there coresponds a statement about P^δ. For example, in Figure 2.3 we can say that in P there exists a unique element e covering exactly three others elements b, c, d, while in P^δ there exists a unique element a covered by exactly three others elements b, c, d. In general, given any statement Φ about ordered sets, we obtain the **dual statement** Φ^δ by replacing each occurrence of $\leq$ by $\geq$ and viceversa.

The duality principle. Given a statement Φ about ordered sets which is true in all ordered sets, the dual statement Φ^δ is also true in all ordered sets.

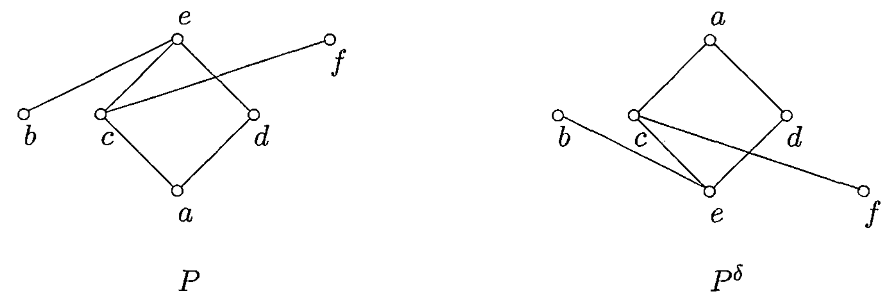

Figure 2.3: The Dual Ordered Set

2.8 Bottom and Top

Definition 2.3. *Given an ordered set P, we say P has a **bottom** element if there exists $\bot \in P$ with the property that $\bot \leq x$ for all $x \in P$.*

Using the duality principle, we get the definition of top element

Definition 2.4. *Given an ordered set P, we say P has a **top** element if there exists $\top \in P$ with the property that $\top \geq x$ for all $x \in P$.*

Lemma 2.5. *If an ordered set P has bottom, then this is unique.*

Proof. Let P be an ordered set, and let $\bot_1$ and $\bot_2$ be two bottoms of the ordered set. If $\bot_1 \in P$ is a bottom of P then $\bot_1 \leq x$ for all $x \in P$. That implies $\bot_1 \leq \bot_2$, since $\bot_2 \in P$. If $\bot_2 \in P$ is a bottom of P then $\bot_2 \leq x$ for all $x \in P$, in particular, $\bot_2 \leq \bot_1$ since $\bot_1$ is an element of P. Therefore $\bot_1 = \bot_2$ by the antisymmetry of $\leq$. □

As a consequence of the duality principle, if an ordered set P has top, then this is unique. In $\langle \mathcal{P}(X); \subseteq \rangle$, we have $\bot = \Phi$ and $\top = X$. A finite chain allways has bottom and top elements, but an infinite chain need not have. For example the chain $\langle \mathbb{N}; \leq \rangle$ has bottom element 1 and no top element, while $\langle \mathbb{Z} \,; \leq \rangle$, the chain of integers has neither top nor bottom.

2.9 Maximal and Minimal Elements

Definition 2.6. *Let P be an ordered set and let $Q \subseteq P$. Then $a \in Q$ is a **maximal element of** Q if $a \leq x$, and $x \in Q$ imply $a = x$.*

We denote the set of all maximal elements of Q by Max Q. If Q has a top element, $\top_Q$, then Max $Q = \top_Q$ and $\top_Q$ is called the **maximum** element of Q.

A minimal element and minimum are defined dualy. A minimum element of Q is a maximum element of Q^δ and minimum of Q is maximum of Q^δ. The set of all minimal elements of Q is denoted Min$Q$. If Q has a bottom element, $\perp_Q$, then Min$Q = \perp_Q$ and $\perp_Q$ is called the **minimum** element of Q.

Figure 2.4: Maximal Elements and Maximum

In the Figure 2.4 P_1 has maximal elements a_1, a_2, a_3 and minimal elements a_4 and a_5 but no maximum or minimum. P_2 has a_1 as maximum and a_2 as minimum.

2.10 Sums and Products of Ordered Sets

Two ordered sets are join together in several different ways. In each of these constructions we require that sets being joined are disjoint.

Definition 2.7. *Let P and Q be disjoint ordered sets. The **linear sum** $P \oplus Q$ is defined by taking the following order relation on $P \cup Q$: $x \leq y$ if and only if*

$$x, y \in P \text{ and } x \leq y \text{ in } P,$$
$$\text{or } x, y \in Q \text{ and } x \leq y \text{ in } Q,$$
$$\text{or } x \in P \text{ and } y \in Q.$$

A diagram for $P \oplus Q$ is obtained by placing a diagram for P directly below a diagram of Q and adding a line segment from each maximal element of P to each minimal element of Q.

Definition 2.8. *Let $P_1, ..., P_n$ be ordered sets: The* **Cartesian product** $P_1 \times ... \times P_n$ *is an ordered set with the coordinatewise order defined by*

$$(x_1, ..., x_n) \leq (y_1, ..., y_n) \text{ if and only if } (\forall i) x_i \leq y_i \text{ in } P_i$$

A product $P \times Q$ is drawn by replacing each point of a diagram of P by a copy of a diagram of Q, and connecting corresponding points. In particular, 2^n is the cartesian product of the chain 2, n times.

Lemma 2.9. *Let $X = \{1, 2, 3, ..., n\}$ and define $\varphi : \mathcal{P}(X) \longrightarrow 2^n$ by $\varphi(A) = (\epsilon_1, ..., \epsilon_n)$ where*

$$\epsilon_i = \begin{cases} 1 & (i \in A) \\ 0 & (i \notin A) \end{cases}$$

Then φ is an order-isomorphism.

Proof. To show that φ is an isomorphism, we have to show :

(i) $A \leq B$ implies $\varphi(A) \leq \varphi(B)$

(ii) φ is one-to-one

(iii) φ is onto

Proof of (i)

Let $A, B \in \mathcal{P}(X)$ be two subsets of $\mathcal{X}$, with $A \subseteq B$ and let $\varphi(A) = (\epsilon_1, ..., \epsilon_n)$ and $\varphi(B) = (\delta_1, ..., \delta_n)$. We need to show that $\varphi(A) \leq \varphi(B)$. $A \subseteq B \Longleftrightarrow (\forall i)\ i \in A$ implies $i \in B$. This is equivalent to $(\forall i)\epsilon_i = 1$ implies $\delta_i = 1$, which is equivalent to $(\forall i)\epsilon_i \leq \delta_i$. This later statement gives us $\varphi(A) \leq \varphi(B)$ in 2^n.

Proof of (ii)

Given $\varphi : \mathcal{P}(X) \longrightarrow 2^n$, $\varphi(A) = (\epsilon_1, ..., \epsilon_n)$ and $\varphi(B) = (\delta_1, ..., \delta_n)$ and $\varphi(A) = \varphi(B)$. Then, since $\varphi(A) = \varphi(B)$, we have $(\epsilon_1, \epsilon_2, ..., \epsilon_n) = (\delta_1, \delta_2, ..., \delta_n)$. This implies that $\epsilon_i = \delta_i$, $(\forall i)\ i = 1, 2, ...n$.

We can have either $i \in A$ or $i \notin A$. If $i \in A$, then $\epsilon_i = 1$, which implies $\delta_i = 1$. Thus, $i \in B$ and $A \subseteq B$. If $i \notin A$, then $\epsilon_i = 0 = \delta_i$. Hence $i \notin B$. Thus by contrapositive $B \subseteq A$. Therefore φ is one-to-one.

Proof of (iii)

Let $x = (\epsilon_1, ..., \epsilon_n)$ and $x \in 2^n$. Then $x = \varphi(A)$ with $A = \{i \mid \epsilon_i = 1\}$ and φ is onto. $\square$

2.11 Down-Sets and Up-Sets

Definition 2.10. *Let P be an ordered set and $Q \subseteq P$.*

*(i) Q is a **down-set** if, whenever $x \in Q$, $y \in P$ and $y \leq x$, we have $y \in Q$.*

*(ii) Dually, Q is an **up-set** if, whenever $x \in Q$, $y \in P$ and $y \geq x$, we have $y \in Q$.*

We can think about a down-set as one which is 'closed under going down' and about an up-set as one which is 'closed under going up'.

Given an arbitrary subset Q of P, we define:

'down Q' : $\downarrow Q := \{y \in P \mid (\exists x \in Q)\ y \leq x\}$

'down x' : $\downarrow x := \{y \in P \mid (y \leq x)\}$

'up Q' : $\uparrow Q := \{y \in P \mid (\exists x \in Q)\ y \geq x\}$

'up x' : $\uparrow x := \{y \in P \mid (y \geq x)\}$

Up-sets (down-sets) of the form $\uparrow x$ ($\downarrow x$) are called **principal**.

The family of all down-sets of P is denoted by $\mathcal{O}$ (P) and is itself an ordered set, under the inclusion. When P is finite, every non-empty down-set Q of P is of the form $\bigcup_{i=1}^{k} \downarrow x$, as the reader may easy verify.

Example 2.11. *In Figure 2.1(iii) the sets $\{c\}$, $\{a, b, c, d, e\}$ and $\{a, b, d, f\}$ are all down-sets, but the set $\{b, d, e\}$ is not a down set because a, b and $a \notin \{b, d, e\}$. The set $\{e, f, g\}$ is an up-set, but $\{a, b, d, f\}$ is not.*

Example 2.12. *If P is an antichain, then $\mathcal{O}(P)$=$\mathcal{P}(P)$.*

Example 2.13. *If P is the chain $\mathbf{n}$, then $\mathcal{O}(P)$ consists of all the sets $\downarrow x$ for $x \in P$, together with the empty set.*

Example 2.14. *If P is the chain $\mathbb{Q}$ of rational numbers, then $\mathcal{O}(P)$ contains the empty set, $\mathbb{Q}$ itself and all sets $\downarrow x$ (for $x \in \mathbb{Q}$). We have also other sets in $\mathcal{O}(P)$, like $\downarrow x \backslash \{x\}$ (for $x \in \mathbb{Q}$) and $\{y \in \mathbb{Q} \mid y < a\}$ (for $a \in \mathbb{R} \backslash \mathbb{Q}$).*

2.12 Maps Between Ordered Sets

Definition 2.15. *Let P and Q be ordered sets. A map $\varphi : P \to Q$ is said to be* *(i)* ***order-preserving*** *if $x \leq y$ in P implies $\varphi(x) \leq \varphi(y)$ in Q;*

(ii) an ***order-embedding*** *(and we write $\varphi : P \hookrightarrow Q$) if $x \leq y$ in P if and only if $\varphi(x) \leq \varphi(y)$ in Q;*

(iii) an ***order-isomorphism*** *if it is an order-embedding which maps P onto Q*

In Figure 2.5 φ_1 is not order-preserving, since $a < v$ but $\varphi_1(a) > \varphi_1(b)$. On the other hand φ_2 is order-preserving but is not order-embedding. Note that $\varphi_2(b) < \varphi_2(c)$ but $b \not< c$. The function φ_3 is order-embedding but not order-isomorphism, since is not one-to-one map.

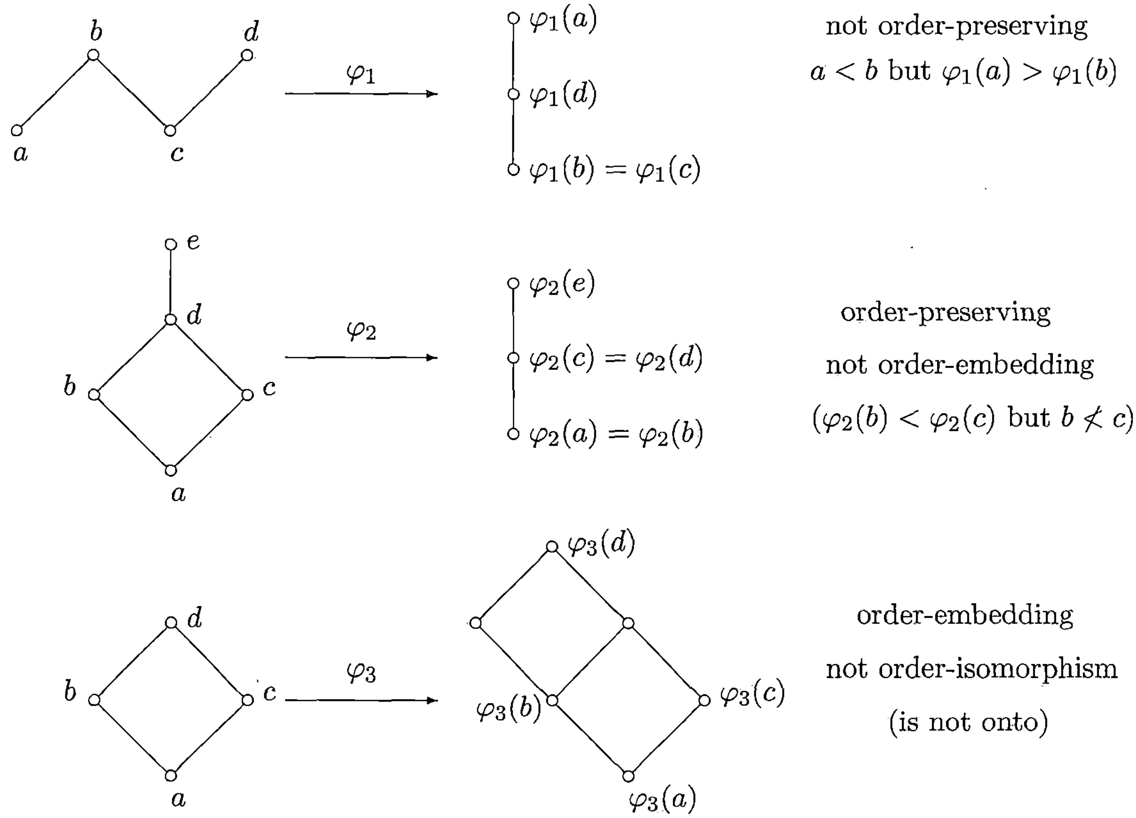

Figure 2.5: Maps Between Ordered Sets

Chapter 3

Lattices and Complete Lattices

Two of the most important classes of ordered sets are lattices and complete lattices. In this chapter we present the basic theory of such ordered sets.

3.1 Lattices as Ordered Sets

Definition 3.1. *Let P be an ordered set and let $S \subseteq P$. An element $x \in P$ is an* ***upper-bound*** *of S if $s \leq x$ for all $s \in S$. Dually an element $x \in P$ is an* ***lower-bound*** *of S if $s \geq x$ for all $s \in S$.*

The set of all upper-bounds of S is denoted by S^u and the set of all lower bounds by S^l:

$$S^u := \{x \in P | (\forall s \in S)\ s \leq x\} \ \ \textit{and} \ \ S^l := \{x \in P | (\forall s \in S)\ s \geq x\}.$$

Since $\leq$ is transitive, S^u is an up-set and S^l is a down-set. If S^u has a least element x, then x is called the **least upper bound** of S. The least upper bound of S is also called the **supremum** of S and is denoted by $\sup S$. If S^l has a greates element x, then x is called the **greatest lower bound** of S. The greatest lower bound of S is also called the **infimum** of S and is denoted by $\inf S$. The supremum of S exists if and only if there exists $x \in P$ such that

$$(\forall y \in P)[((\forall s \in S)\ s \leq y) \ \Leftrightarrow \ x \leq y]$$

In ordered set (i) from Figure 3.1 $S = \{7, 8, 11, 12\}$. The elements 15, 16 are the only upper bounds and 2 and 3 are the only lower bounds. The set $S^u = \{15, 16\}$ is the set of all upper bounds and 15 is the sup S, since 15 is the least element of $\{15, 16\}$. The set $S^l = \{2, 3\}$ is the set of all lower bounds and we have no infimum.

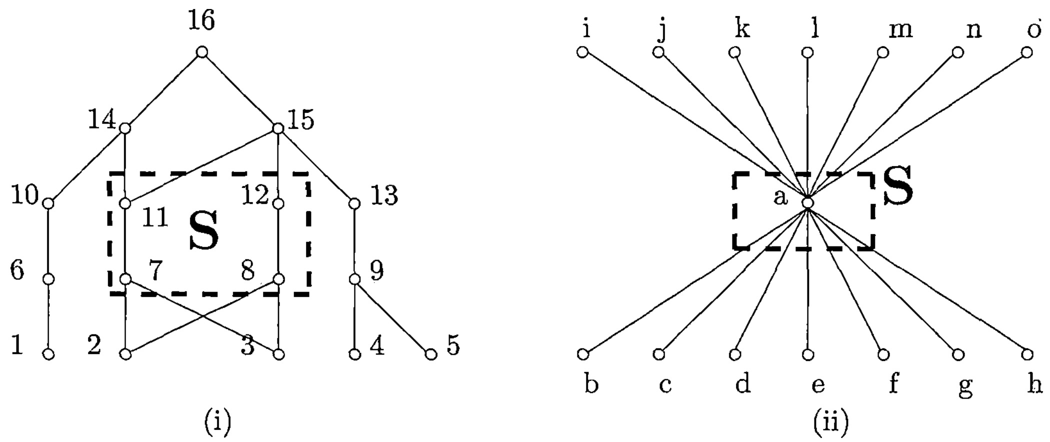

Figure 3.1: Upper and Lower Bounds

In ordered set (ii), $S = \{a\}$, $S^u = \{$ i,j,k,l,m,n,o$\}$, $S^l = \{$ a,b,c,d,e,f,g,h$\}$. Here we do not have any supremum or infimum.

Remark 3.2. *There are two extreme cases for S as subset of P: when S is empty or S is P itself. If S is the empty set , then every element $x \in P$ satisfies $s \leq x$ for all $s \in S$. Thus $\phi^u = P$ and hence sup ϕ exists if and only if P has a bottom element, in which case sup $\phi = \bot$. Dually, inf $\phi = \top$ whenever P has a top element.*

Notation. We write $x \vee y$ (read as 'x **join** y') in place of sup $\{x, y\}$ when it exists and $x \wedge y$ (read as 'x **meet** y') in place of inf $\{x, y\}$ when it exists. From this observe that the commutative law holds. That is, $x \vee y = y \vee x$ and similarly for meet. Similary we write $\bigvee S$ (the **join of** S) and $\bigwedge S$ (the **meet of** S) instead of sup S and inf S when these exist.

Remarks 3.3.

(1) Let P be an ordered set. If $x, y \in P$ and $x \leq y$, then $\{x, y\}^u = \uparrow y$ and $\{x, y\}^l = \downarrow x$. Since the least element of $\uparrow y$ is y and the greatest element of $\downarrow x$ is x, we have

$x \vee y = y$ and $x \wedge y = x$ whenever $x \leq y$. In particular, since $\leq$ is reflexive, we have $x \vee x = x$ and $x \wedge x = x$.

(2) In an ordered set P, $x \vee y$ may fail to exist for two different reasons:

(a) because x and y have no common upper bound, as in Figure 3.2 (i)

(b) because they have no least upper bound, as in in Figure 3.2 (ii) where we find that $\{a, b\}^u = \{c, d\}$ and thus $a \vee b$ does not exists as $\{a, b\}^u$ has no least element.

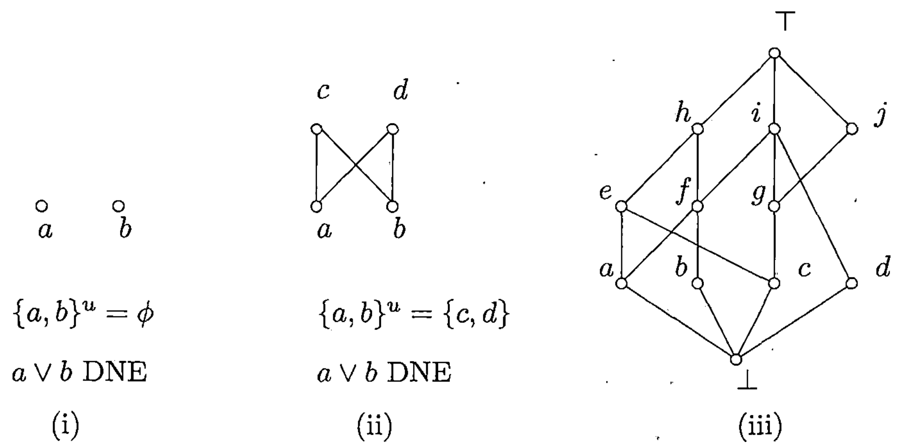

Figure 3.2: Join and Meet

(3) In Figure 3.2 (iii) $\{b, c\}^u = \{\top, h, i\}$ and since $\{b, c\}^u$ has distinct minimal elements h and i, $b \vee c$ does not exists. On the other hand, $\{a, b\}^u = \{\top, h, i, f\}$ has a least element f and thus $a \vee b = f$. Also, $\{h, i\}^l = \{f, a, b, c, \perp\}$ and f is the greatest element, therefore $h \wedge i = f$.

Definition 3.4. *Let P be a non-empty ordered set.*

*(i) If $x \vee y$ and $x \wedge y$ exist for all $x, y \in P$, then P is called a **lattice***

*(ii) If $\bigvee S$ and $\bigwedge S$ exist for all $S \subseteq P$, then P is called a **complete lattice***

Example 3.5. *Each of $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{N}$ is a lattice under its usual order. By Remark 3.2(1), if $x \leq y$ then $x \vee y = y$ and $x \wedge y = x$. Hence, every chain is a lattice in which $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$. Therefore $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{N}$ are lattices.*

Example 3.6. *None of the lattices* $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, *and* $\mathbb{N}$ *is complete; every one lacks a top element, and a complete lattice must have top and bottom.*

Example 3.7. *For any set* X, *the ordered set* $\langle \mathcal{P}(X); \subseteq \rangle$ *is a complete lattice in which*

$$\bigvee\{A_i \mid i \in I\} = \bigcup\{A_i \mid i \in I\},$$

$$\bigwedge\{A_i \mid i \in I\} = \bigcap\{A_i \mid i \in I\}.$$

Proof. First, we note that we shall indicate the index set by subscripting it. Thus, instead of $\bigcup\{A_i \mid i \in I\}$ we write $\bigcup_{i\in I} A_i$, and instead of $\bigcap\{A_i \mid i \in I\}$ we write $\bigcap_{i\in I} A_i$. Let $\{A_i\}_{i\in I}$ be a family of elements of $\mathcal{P}(X)$. Since $\bigcup_{i\in I} A_i \supseteq A_j$ for all $j \in I$, it follows that $\bigcup_{i\in I} A_i$ is an upper bound for $\{A_i\}_{i\in I}$. Now, let $B \in \mathcal{P}(X)$ be another upper bound of $\{A_i\}_{i\in I}$. Then $B \supseteq A_i$ for all $i \in I$ and hence $B \supseteq \bigcup_{i\in I} A_i$. Thus $\bigcup_{i\in I} A_i$ is indeed the lowest upperbound of $\{A_i\}_{i\in I}$ in $\mathcal{P}(X)$. The assertion about meets is proved dually. □

Example 3.8. *With* $\mathcal{N}_0 = \{0, 1, 2, 3, \ldots\}$, $a \leq b$ *if and only if* $a|b$, $a \vee b = \mathrm{lcd}(a, b)$ *and* $a \wedge b = \gcd(a, b)$, $\langle \mathcal{N}_0; \mathrm{lcd}; \gcd \rangle$ *is a lattice.*

Proof. We'll first prove that the relation $a \leq b$ iff $a|b$ is an order relation. Since every $x \in \mathbb{N}_0$ is divisible by itself the relation is reflexive. If $x, y \in \mathbb{N}_0$, $x|y$ and $y|x$ imply $x = y$ and thus the relation is antisymmetric. Now, let $x, y, z \in \mathbb{N}_0$, $x|y$ and $y|z$. Then, there exist $p, q \in \mathbb{N}_0$ such that $y = x \cdot p$ and $z = y \cdot q$. That implies $z = y \cdot q = (x \cdot p) \cdot q = x \cdot (p \cdot q)$ and $x|z$. Thus the relation is transitive. Therefore $\langle \mathbb{N}_0; \mathrm{lcd}; \gcd \rangle$ is an ordered set. Second we have to prove that the $\mathrm{lcm}(a, b)$ and the $\gcd(a, b)$ are precisely $a \vee b$ and $a \wedge b$, respectively. Let $\mathrm{lcm}(a, b) = c$. That implies $a|c$ and $b|c$ and by the definition of relation in our ordered set, $a \leq c$ and $b \leq c$. Since by definition c divides any other multiple of a and b, c is the $\sup(a, b)$ and therefore $\mathrm{lcm}(a, b) = a \vee b$. Dually we can show that $\gcd(a, b) = a \wedge b$. Therefore $\langle \mathbb{N}_0; \mathrm{lcd}; \gcd \rangle$ is a lattice. □

Lemma 3.9. *Let* P *be a lattice. Then for all* $a, b, c, d \in P$,

(i) $a \leq b$ *implies* $a \vee c \leq b \vee c$ *and* $a \wedge c \leq b \wedge c$

(ii) $a \leq b$ *and* $c \leq d$ *imply* $a \vee c \leq b \vee d$ *and* $a \wedge c \leq b \wedge d$.

Proof. For the (i) part let P be a lattice and $a, b, c, d \in P$ and let $a \leq b$. Then by Remark (1) above $a \vee b = b$. Consider the element $b \vee c = (a \vee b) \vee c$ (since $a \vee b = b$). By the associative and commutative laws (see next theorem) $b \vee c = (a \vee b) \vee c = b \vee (a \vee c)$. Using the definition of join, $b \vee c \geq b$ and $b \vee c \geq a \vee c$. Therefore we have the result $a \vee c \leq b \vee c$ and the proof is complete. For the second part by Remark (1) above $a \wedge b = a$. Consider the element $a \wedge c = (a \wedge b) \wedge c$ (since $a \wedge b = a$). By associativity $a \wedge c = a \wedge (b \wedge c)$ and by the definition of meet $a \wedge c \leq b \wedge c$.

For the (ii) part $a \leq b$ implies $a \vee b = b$, and $c \leq d$ implies $c \vee d = d$. Then the element $b \vee d$ in lattice P is equal to $(a \vee b) \vee (c \vee d)$ and using the associative and commutative laws is equal to $(a \vee c) \vee (b \vee d)$. Thus we have $b \vee d = (a \vee c) \vee (b \vee d)$ and therefore, $a \vee c \leq b \vee d$. The element $a \wedge c$ in lattice P is equal to $(a \wedge b) \wedge (c \wedge d)$ since $a \wedge c = a$ and $c \wedge d = c$. Using the associative and commutative laws $a \wedge c = (b \wedge d) \wedge (a \wedge c)$. Thus we have $a \wedge c \leq b \wedge d$. $\square$

3.2 Lattices as Algebraic Structure

Given a lattice L, we define binary operations **join** and **meet** on the non-empty set L by

$$a \vee b := \sup\{a, b\} \quad \text{and} \quad a \wedge b := \inf\{a, b\} \quad (a, b \in L)$$

In this section we view a lattice as an algebraic structure $\langle L; \vee, \wedge \rangle$.

Lemma 3.10. Connecting Lemma.

Let L be a lattice and let $a, b \in L$. Then the following are equivalent:

(i) $a \leq b$;

(ii) $a \vee b = b$;

(iii) $a \wedge b = a$.

Proof. We already showed in Remark (1) above that, (i) implies (ii) and (i) implies (iii). Now, assume (ii) $a \vee b = b$. We know from the definition of join that $a \leq a \vee b$. Thus, $a \leq a \vee b = b$ implies $a \leq b$ which is (i). To show that (iii) implies (i), we take $a \wedge b = a$ which implies that a is a lower bound for $\{a, b\}$, hence $a \leq b$. $\square$

Theorem 3.11. *Let L be a lattice. Then $\vee$ and $\wedge$ satisfy, for all $a, b, c \in L$,*

(L1)	$(a \vee b) \vee c = a \vee (b \vee c)$	*and* $(L1)^\delta$	$(a \wedge b) \wedge c = a \wedge (b \wedge c)$	*(associative laws)*
(L2)	$a \vee b = b \vee a$	*and* $(L2)^\delta$	$a \wedge b = b \wedge a$	*(commutative laws)*
(L3)	$a \vee a = a$	*and* $(L3)^\delta$	$a \wedge a = a$	*(idempotency laws)*
(L4)	$a \vee (a \wedge b) = a$	*and* $(L4)^\delta$	$a \wedge (a \vee b) = a$	*(absorption laws)*

Proof. Let $a, b \in P$ and $a \leq b$. Then, $\{a, b\}^u = b \uparrow$ and $\{a, b\}^l = a \downarrow$, since the least element of $b \uparrow$ is b and the greatest element of $a \uparrow$ is a. Thus, $a \vee b = b$ and $a \wedge b = a$ whenever $a \leq b$. In particular if in the set $\{a, b\}$ we let $b = a$, we get $a \vee a = a$ and $a \wedge a = a$, which proofs $L(4)$.

To prove $L(2)$ we remember that $a \vee b = \sup\{a, b\}$ and that is equal to $\sup\{b, a\} = b \vee a$.

To prove $L(3)$ we'll use the above lemma. If we have two elements belong to the lattice L, both elements are less or equal to their join. In particular if our elements are a and $a \wedge b$, then $a \leq a \vee (a \wedge b)$. By Lemma 3.4 $a \wedge a \vee (a \wedge b) = a$ and since $a \vee a = a$ (by $L(3)$), we have $a \vee (a \wedge b) = a$.

To prove $L(1)$ let $a, b, c \in L$ and $d \in \{a, b, c\}^u$ be any upper bound of the set $\{a, b, c\}$. If d is an upper bound of $\{a, b, c\}$ then, by definition, $d \geq a$, $d \geq b$ and $d \geq c$, and moreover the upper bound d is an upper bound for any subset of $\{a, b, c\}$. Thus, we have $d \in \{a, b\}^u$ and $d \geq c$, which implies $d \geq a \vee b$ and $d \geq c$, which implies $d \in \{a \vee b, c\}^u$. In the same time we can say that $d \in \{b, c\}^u$ and $d \geq a$, which implies $d \in \{b \vee c, a\}^u$ and thus $d \in \{b \vee c, a\}^u$. Thus the set of $\{b \vee c, a\}^u = \{b \vee c, a\}^u$. That implies that the two sets have the same least element and therefore $(a \vee b) \vee c = a \vee (b \vee c)$.

Note that the dual statements $L(1)^\delta - L(4)^\delta$ are obtained simply by interchanging the $\vee$ and $\wedge$. $\square$

Theorem 3.12. *Let $\langle L; \vee, \wedge \rangle$ be a non-empty set equiped with two binary operations which satisfy* $(L1) - (L4)$ *and* $(L1)^\delta - (L4)^\delta$ *from 3.11.*

(i) For all $a, b \in L$, we have $a \vee b = b$ if and only if $a \wedge b = a$

(ii) Define $\leq$ on L by $a \leq b$ if and only if $a \vee b = b$. Then $\leq$ is an order relation.

(iii) With $\leq$ as in (ii), $\langle L; \leq \rangle$ is a lattice in which the original operations agree with the induced operations, that is, for all $a, b \in L$,

$$a \vee b = \sup\{a, b\} \quad \text{and} \quad a \wedge b = \inf\{a, b\}.$$

Proof. Assume $a \vee b = b$. From $(L4)^{\delta}$ we have $a = a \wedge (a \vee b)$ and replacing $a \vee b$ by b, we have $a = a \wedge b$. Conversely, assume $a \wedge b = a$. Replacing in ?? $(L4)$, $(b \wedge a)$ by a, we get $b = b \vee a$. Thus the proof of (i) is complete.

Now define $\leq$ as in (ii). By $L3$, $a \vee a = a$ and since by assumption $a \leq b$ if $a \vee b = b$ we have $a \leq a$ which shows reflexivity. To verify antisymmetry suppose first that $a \leq b$. That implies $a \vee b = b$. Second, suppose $b \leq a$, which implies $b \vee a = a$. Since by $L2$, $a \vee b = b \vee a$ we have $a = b$. To show that $\leq$ is transitive let $a, b, c \in L$, $a \leq b$ and $b \leq c$. If $a \leq b$, then $a \vee b = b$ and if $b \leq c$, then $b \vee c = c$. Replacing these two joins in $L1$, $a \vee (b \vee c) = (a \vee b) \vee c$ we find that $a \vee c = b \vee c$ and since $b \vee c = c$ we have $a \vee c = c$ which implies that $a \leq c$. Thus $\leq$ is reflexive, antisymmetric and transitive, therefore is an order relation.

With $\leq$ and $\vee$ defined as in (ii), to show that $a \vee b = \sup\{a, b\}$ in the ordered set $\langle L; \leq \rangle$ we need to show that $a \vee b \in \{a, b\}^u$ and that $d \in \{a, b\}^u$ implies $d \geq a \vee b$. The following equality $(a \vee b) \vee a = (a \vee a) \vee b$ holds by the associativity and commutativity laws of $\vee$. Since $a \vee a = a$, by ?? $(L3)$, we have $a \vee (a \vee b) = a \vee b$. Using Lemma ??, $a \vee b \geq a$. On the other hand $(a \vee b) \vee b = a \vee (b \vee b) = a \vee b$, which by (ii) implies that $a \vee b \geq b$. These two results, $a \vee b \geq a$ and $a \vee b \geq b$ allow us to conclude that $a \vee b \in \{a, b\}^u$. Now let $d \in \{a, b\}^u$. We need to prove that $d \geq a \vee b$. If d is an element of the set of upper bounds of the set $\{a, b\}$, then $d \geq a$ and $d \geq b$, which imply that $d \vee a = d$ and $d \vee b = d$. Looking at the element $d \vee (a \vee b)$ we see that by associativity this is equal to $(d \vee a) \vee b$ which equals $d \vee b$, which equals d. Thus we have $d \vee (a \vee b) = d$ and therefore, by (ii), $d \geq a \vee b$. $\square$

3.3 Sublattices, Products and Homomorphism

Definition 3.13. *Let L be a lattice and $\phi \neq M \subseteq L$. Then M is a* ***sublattice*** *of L if*

$$a, b \in M \text{ implies } a \vee b \in M \text{ and } a \wedge b \in M$$

Example 3.14. *Any one element subset of a lattice is a sublattice. More generally a non-empty chain in a lattice is a sublattice.*

Example 3.15. *In the diagrams in Figure* **??** *the shaded elements in lattice* (i) *and* (ii) *form sublattices. In* (iii)*, since* $M = \{b, e, g, d\}$, $e, g \in M$ *but* $e \vee g = h$ *and* $h \notin M$*, the shaded elements do not form a sublattice. The shaded elements in* (iv) *do not form a sublattice, since* $e \wedge g = c$ *and* c *is not shaded. In Figure* **??** (v) *the shaded elements also do not form a sublattice since* $b \vee d = f$ *and* f *is not in the subset of shaded elements.*

Example 3.16. *The reader can easily check that in Figure* **??** (iv) *the subset of shaded elements forms a lattice in its own, without being a sublattice of* L.

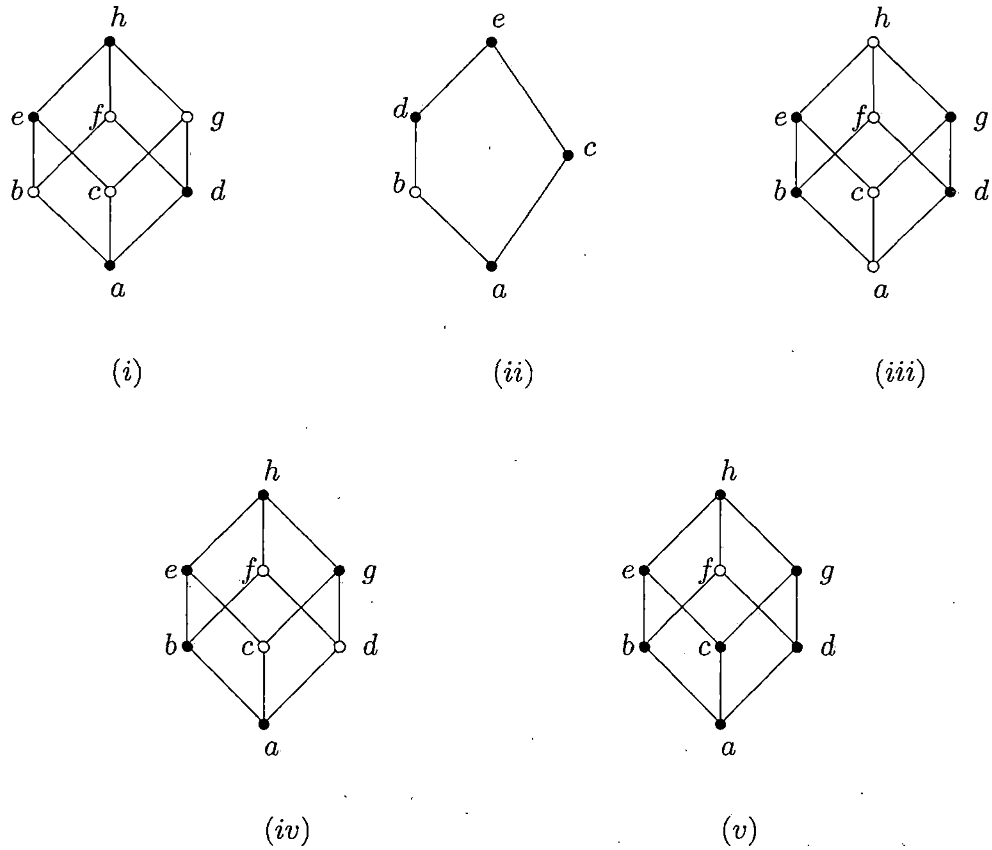

Figure 3.3: Sublattices

Definition 3.17. *Let L and K be lattices with the ordered set $L \times K$. We define $\vee$ and $\wedge$ coordinatewise on $L \times K$, as follows:*

$$(l_1, k_1) \vee (l_2, k_2) = (l_1 \vee l_2, k_1 \vee k_2)$$

$$(l_1, k_1) \wedge (l_2, k_2) = (l_1 \wedge l_2, k_1 \wedge k_2)$$

It is routine to check that $L \times K$ satisfies the identities ?? $(L1 - (L4)^{\delta}$ and therefore is a lattice.

Definition 3.18. *Let L and K be lattices. A map $f : L \to K$ is said to be a* ***homomorphism*** *if f is* ***join-preserving*** *and* ***meet-preserving****, that is, for all $a, b \in L$,*

$$f(a \vee b) = f(a) \vee f(b) \text{ and } f(a \wedge b) = f(a) \wedge f(b).$$

A bijective homomorphism is an **isomorphism**. If $f : L \to K$ is a one-to-one homomorphism, then $f(L)$ is a sublattice of K isomorphic to L and we refer to f as an **embedding** map.

Examples 3.19. *In Figure 2.5, recall that φ_1 is not order-preserving since $a \| c$ and $\varphi(a) \geq \varphi(c)$ but each of $\varphi_2, \varphi_3, \varphi_4$ is an order preserving map. The map φ_2 is an homomorphism, the remainder are not. The map φ_4 preserves joins but does not preserve all meets. The map φ_3 is meet preserving but does not preserve all joins.*

Proposition 3.20. *Let L and K be lattices and $f : L \to K$ a map.*

(i) The following are equivalent:

(a) f is order-preserving;

(b) $(\forall a, b \in L)$ $f(a \vee b) \geq f(a) \vee f(b)$;

(c) $(\forall a, b \in L)$ $f(a \wedge b) \leq f(a) \wedge f(b)$;

(ii) f is a lattice isomorphism if and only if it is an order-isomorphism.

Proof.

Part (i) (a implies b).

Let L and K be lattices and $f : L \to K$ an order-preserving map, we need to prove that

$(\forall a, b \in L) \quad f(a \vee b) \geq f(a) \vee f(b)$.

Given $a, b \in L$ and $a \leq b$ we have:

	$a \vee b = b$	(by Lemma ??),
which implies	$f(a \vee b) = f(b)$	
and	$f(a) \leq f(b)$	(since the map f is order preserving)
which implies	$f(a) \vee f(b) = f(b)$	(by Lemma ??).

Using $(L3)$ from Theorem 3.5, $b \vee b = b$. That implies $f(b) \vee f(b) = f(b)$. Replacing here the values for $f(b)$ found above, we have:

$$(f(a) \vee f(b)) \vee f(a \vee b) = f(a \vee b)$$

and therefore, by Lemma ??,

$$f(a \vee b) \geq f(a) \vee f(b).$$

Part (i) (b implies a).

Given $(\forall a, b \in L)$, $a \leq b$ and $f(a \vee b) \geq f(a) \vee f(b)$, we need to show that $f(a) \leq f(b)$. Given $a, b \in L$ and $a \leq b$ we have $a \vee b = b$. Replacing this in $f(a \vee b) \geq f(a) \vee f(b)$ and using the fact $f(a) \vee f(b) \geq f(a)$ we obtain

$$f(b) \geq f(a) \vee f(b) \geq f(a)$$

and therefore $f(a) \leq f(b)$ what we needed to show.

Part (i) (a implies c). Let L and K be lattices and $f : L \to K$ an order-preserving map, we need to prove that $(\forall a, b \in L) \quad f(a \vee b) \geq f(a) \vee f(b)$.

Given $a, b \in L$ and $a \leq b$ we have:

	$a \wedge b = a$	(by Lemma ??),
which implies	$f(a \wedge b) = f(a)$	
and	$f(a) \leq f(b)$	(since the map f is order preserving)
which implies	$f(a) \wedge f(b) = f(a)$	(by Lemma ??).

Using $(L3)^{\delta}$ from Theorem 3.5, $a \wedge a = a$. That implies $f(a) \wedge f(a) = f(a)$. Replacing here the values for $f(a)$ found above, we have:

$$(f(a) \wedge f(b)) \wedge f(a \wedge b) = f(a \wedge b)$$

and therefore, by Lemma ??,

$$f(a \wedge b) \leq f(a) \wedge f(b).$$

Part (i) (c implies a).

Given $(\forall a, b \in L)$, $a \leq b$ and $f(a \wedge b) \leq f(a) \wedge f(b)$, we need to show that $f(a) \leq f(b)$. Given $a, b \in L$ and $a \leq b$ we have $a \wedge b = a$. Replacing this in $f(a \wedge b) \leq f(a) \wedge f(b)$ and using the fact $f(a) \wedge f(b) \leq f(b)$ we obtain

$$f(a) \leq f(a) \wedge f(b) \leq f(b)$$

and therefore $f(a) \leq f(b)$ what we needed to show.

Part (ii) $(\Rightarrow)$

Given f one -to-one, a, b elements of lattice with $a \leq b$ and $f(a \vee b) = f(a) \vee f(b)$ we need to show that $f(a) \leq f(b)$. In the given $f(a \vee b) = f(a) \vee f(b)$ replace $a \vee b$ by b, and obtain

$$f(b) = f(a) \vee f(b),$$

which implies $f(a) \leq f(b)$ (by Lemma ??).

Part (ii) $(\Leftarrow)$ Given $a, b \in L$, $a \leq b$ implies $f(a) \leq f(b)$, we need to show that f is bijective and preserves join and meet. Since f is order isomorphism, f is one-to-one and onto (see 2.3). Since f is surjective there exists $c \in L$ such that $f(a) \vee f(b) = f(c)$. By Lemma ?? $f(a) \leq f(c)$ and $f(b) \leq f(c)$. Since f is order embedding, $a \leq c$ and $b \leq c$. Hence $a \vee b \leq c$. Since f is order isomorphism, $f(a) \vee b) \leq f(c) = f(a) \vee f(b)$ and thus $f(a) \vee b) \leq f(a) \vee f(b)$. By (i), $f(a \vee b \geq f(a) \vee f(b)$. Thus $f(a \vee b) = f(a) \vee f(b)$ and therefore preserves join. To show that f preserves meet it is enough to change join with meet and $\leq$ with $\geq$. Therefore, since f is surjective there exists $c \in L$ such that $f(a) \wedge f(b) = f(c)$. By Lemma ?? $f(a) \geq f(c)$ and $f(b) \geq f(c)$. Since f is order embedding, $a \geq c$ and $b \geq c$. Hence $a \wedge b \geq c$. Since f is order isomorphism, $f(a \wedge b) \geq f(c) = f(a) \wedge f(b)$ and thus $f(a \wedge b) \geq f(a) \wedge f(b)$. By (i), $f(a \wedge b) \leq f(a) \wedge f(b)$. Thus $f(a \wedge b) = f(a) \vee f(b)$ and therefore preserves meet. Since f preserves join and meet, f is a lattice isomorphism. □

3.4 Ideals and Filters

Definition 3.21. *Let L be a lattice. A non-empty subset J of L is called* ***ideal*** *if*

(i) $a, b \in J$ implies $a \vee b \in J$,

(ii) $a \in L$, $b \in J$ *and* $a \leq b$ *imply* $a \in J$.

More compactly we can say that an ideal is a non-empty down set closed under join.

In Figure 3.4 the shaded elements in (i) form an ideal. The shaded elements in (ii) do not form an ideal, since $b, c \in J$ but $b \vee c \notin J$. In the same way, the shaded elements in (iii) do not form an ideal, since $a \in L$, $b \in J$ and $a \leq b$, but $a \notin J$ (the set of shaded elements is not a down-set).

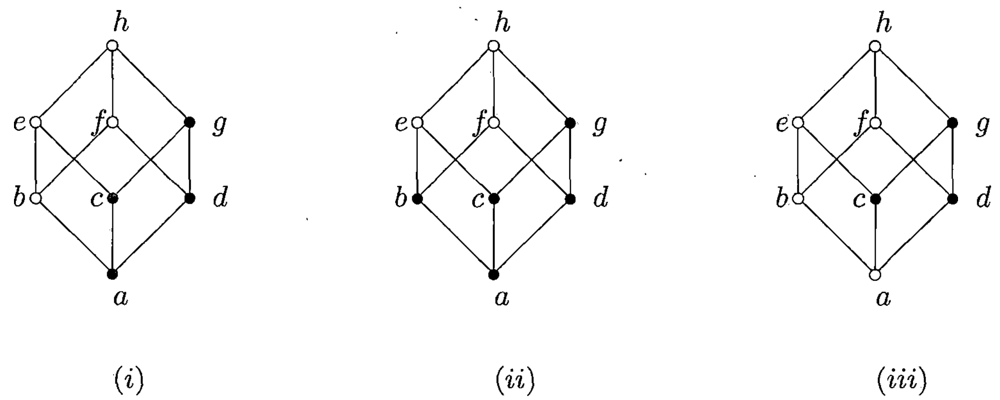

Figure 3.4: Ideals

A dual ideal is called a **filter**. That means a filter is defined as a non-empty subset G of L, such that:

(i) $a, b \in G$ implies $a \wedge b \in G$,

(ii) $a \in L$, $b \in G$ and $a \geq b$ imply $a \in G$.

The set of all ideals (filters) of L is denoted by $\mathcal{I}(L)$ $(\mathcal{F}(L))$, and carries the usual inclusion order.

An ideal or filter is called **proper** if it does not coincide with L.

For each $a \in L$, the set $\downarrow a$ is an ideal and $\uparrow a$ is a filter. $\downarrow a$ is known as **principal ideal** generated by a and $\uparrow a$ as **principal filter** generated by a.

Proposition 3.22. *Let* L *be a lattice and* $J \subseteq L$ *a non-empty subset.* J *is an ideal of* L *if and only if:*

(1) $a, b \in J$ implies $a \vee b \in J$,

(2) $a \in L$, $b \in J$ implies $a \wedge b \in J$

Proof. Let L be a lattice and $J \subseteq L$ a non-empty subset. We need to prove that $J \subseteq L$ satisfies (i) and (ii) from Definition 3.22 if and only if it satisfies (1) and (2).
Proof of ($\Rightarrow$). Given $a, b \in J$ and (1) and (2), we have to show that $a \wedge b \in J$. From the given $a \leq b$ and using Lemma ??, we have $a \wedge b = a$ and since $a \in J$, $a \wedge b \in J$.
Proof of ($\Leftarrow$). Given $a \in L$, $b \in J$, $a \leq b$ and (1) and (2), we have to show that $a \in J$. From the given $a \leq b$ and using Lemma ??, we have $a \wedge b = a$ and since $a \wedge b \in J$, $a \in J$. □

Proposition 3.23. *Let L and K be bounded lattices and $f : L \to K$ a $\{0,1\}$ homomorphism. Then $f^{-1}(0)$ is an ideal and $f^{-1}(1)$ is a filter in L.*

Proof. Given L and K bounded lattices and $f : L \to K$ a $\{0,1\}$ homomorphism prove that $f^{-1}(0)$ is an ideal. Then $f^{-1}(0) = \{x \in L | f(x) = 0\}$. Let $a, b \in f^{-1}(0)$. Now $a \vee b \in L$, since L is a lattice and $f(a \vee b) = f(a) \vee f(b)$ since f is a homomorphism. But $f(a) \vee f(b) = 0 \vee 0 = 0$, since $a, b \in f^{-1}(0)$. Thus $a \vee b \in f^{-1}(0)$. Now let $a \in L$, $b \in f^{-1}(0)$, and $a \leq b$. We need to show that $a \in f^{-1}(0)$.

$$
\begin{aligned}
a \leq b &\Rightarrow a \vee b = b && \text{(by Lemma ??)} \\
&\Rightarrow f(a \vee b) = f(b) \\
&\Rightarrow f(a) \vee f(b) = f(b) && \text{(since } f \text{ is homomorphism)} \\
&\Rightarrow f(a) \leq f(b) = 0 && \text{(by Lemma ?? and since } b \in f^{-1}(0)) \\
&\Rightarrow f(a) = 0 && \text{(since } L \text{ is a bounded lattice)} \\
&\Rightarrow a \in f^{-1}(0) && \text{(since } f \text{ is } 0,1 \text{ homomorphism)}
\end{aligned}
$$

Therefore, $f^{-1}(0)$ is an ideal.

Given L and K bounded lattices and $f : L \to K$ a $\{0,1\}$ homomorphism prove that $f^{-1}(1)$ is a filter. Let $f^{-1}(1) \to 1_L$ the image of 1. Then $f^{-1}(1) = \{x \in L | f(x) = 1\}$. Let $a, b \in f^{-1}(1)$. Now $a \wedge b \in L$, since L is a lattice and $f(a \wedge b) = f(a) \wedge f(b)$ since f is a homomorphism. But $f(a) \vee f(b) = 1 \vee 1 = 1$, since $a, b \in f^{-1}(1)$. Thus $a \wedge b \in f^{-1}(1)$.

Now let $a \in L$, $b \in f^{-1}(1)$, and $a \geq b$. We need to show that $a \in f^{-1}(1)$.

$$\begin{aligned}
a \geq b &\Rightarrow a \wedge b = b && \text{(by Lemma ??)} \\
&\Rightarrow f(a \wedge b) = f(b) && \\
&\Rightarrow f(a) \wedge f(b) && \text{(since } f \text{ is homomorphism)} \\
&\Rightarrow f(a) \wedge f(b) = 1 && \text{(by Lemma ?? and since } b \in f^{-1}(0)) \\
&\Rightarrow f(a) = 1 && \text{(since } L \text{ is a bounded lattice)} \\
&\Rightarrow a \in f^{-1}(1) && \text{(since } f \text{ is } 0,1 \text{ homomorphism)}
\end{aligned}$$

Therefore, $f^{-1}(1)$ is a filter. □

Proposition 3.24. *For any set X the following are ideals in $\mathcal{P}(X)$*

(a) all subsets not containing a fixed element of X;

(b) all finite subsets.

Proof. Let X be a set and let $x_0 \in X$.
Proof of (a). Let G be the set of all subsets of X not containing a fixed element $x_0 \in X$. Then

$$G = \{A \in \mathcal{P}(X) \mid x_0 \notin A\}$$

Let $A, B \in G$. If $x_0 \notin A$ and $x_0 \notin B$, then $x_0 \notin A \cup B$. Thus $A \cup B \in G$. Now, let $A \in G$, $B \in \mathcal{P}(X)$ and $B \subseteq A$. If $x_0 \notin A$ and $B \subseteq A$ we have $x_0 \notin B$ and thus B is an element of G. Therefore by definition, G is an ideal.
Proof of (b). Let G be the set of all finite subsets of X. Then

$$G = \{A \in \mathcal{P}(X) \mid A \text{ is finite}\}$$

Let $A, B \in G$ be finite sets. Since the union of two finite sets is a finite set, $A \cup B \in G$. Now let $A \in G$, $B \in \mathcal{P}(X)$ and $B \subseteq A$. Since B is a subset of a finite set is itself finite and thus $B \in G$. Therefore by definition, G is an ideal. □

Chapter 4

Complete Lattices and $\bigcap$-structure

Recall from Definition 3.4 that a complete lattice is a non-empty set P such that the join (supremum), $\bigvee S$, and the meet (infimum), $\bigwedge S$, exist for all subsets S of P. First we list some immediate consequences of the definitions of least upper bound and greates lower bound.

Proposition 4.1. *Let P be an ordered set, let $S, T \subseteq P$ and assume that $\bigvee S$, $\bigvee T$, $\bigwedge S$ and $\bigwedge T$ exist in P.*

(i) $s \leq \bigvee S$ *and* $s \geq \bigwedge S$ *for all* $s \in S$.

(ii) Let $x \in P$*; then* $x \leq \bigwedge S$ *if and only if* $x \leq s$ *for all* $s \in S$.

(iii) Let $x \in P$*; then* $x \geq \bigvee S$ *if and only if* $x \geq s$ *for all* $s \in S$.

(iv) $\bigvee S \leq \bigwedge T$ *if and only if* $s \leq t$ *for all* $t \in T$.

(v) If $S \subseteq T$*, then* $\bigvee S \leq \bigvee T$ *and* $\bigwedge S \geq \bigwedge T$.

Lemma 4.2. *Let P be a lattice, let $S, T \subseteq P$ and assume that $\bigvee S$, $\bigvee T$, $\bigwedge S$ and $\bigwedge T$ exist in P. Then*

$$\bigvee(S \cup T) = (\bigvee S) \vee (\bigvee T) \quad \text{and} \quad \bigwedge(S \cup T) = (\bigwedge S) \wedge (\bigwedge T)$$

Proof. Since both $\sup(S)$ and $\sup(T)$ exist, to show that the equality holds, it just needs to be shown that the right side, $\sup(S) \vee \sup(T)$, satisfies the definition for least upper bound of the set $S \cup T$.

Upper bound: If x is in $S \cup T$, then $x \in S$ or $x \in T$. Thus, $x \leq \sup(S) \vee \sup(T)$. Hence $\sup(S) \vee \sup(T)$ is an upper bound of $S \cup T$.

Least upper bound: If y is an upper bound for $S \cup T$, then for each z in $S \cup T$, $z \leq y$. This implies in particular that y is an upper bound for S and an upper bound for T. Thus, $\sup(S) \leq y$ and $\sup(T) \leq y$. Therefore, $\sup(S) \vee \sup(T) \leq y$ and $\sup(S) \vee \sup(T)$ is the least upper bound for $S \cup T$. Therefore by definition of join $\bigvee(S \cup T) = (\bigvee S) \vee (\bigvee T)$. Dually we can prove that $\bigwedge(S \cup T) = (\bigwedge S) \wedge (\bigwedge T)$, and the proof is complete. □

Lemma 4.3. *Let P be a lattice. Then $\bigvee F$ and $\bigwedge F$ exist for every finite, non-empty subset F of P.*

Proof. Let $F \subseteq P$, non-empty, consisting of n elements $F = \{a_1, a_2, a_3, \dots a_n\}$. Then $\bigvee F$ can be defined in the following way: define an element $S\{a_1, \dots, a_n\}$ recursively by $S\{a_1, \dots, a_n\} = \sup\{a_1, S\{a_2, \dots, a_n\}$ with $n \geq 2$ and $S\{a\} = a$. For example for $n = 4$, $S\{a_1, \dots, a_n\} = \sup\{a_1, \sup\{a_2, \sup\{a_3, a_4\}\}\}$. First we need to prove $S\{a_1, \dots, a_n\}$ is an upper bound for the set $\{a_1, \dots, a_n\}$. To show $S\ \{a_1, \dots, a_n\} \geq\ a_i$, $\forall i$, $1 \leq i \leq n$, either $a_1 = S\{a_1\} \geq a_i$ or by induction on n $S\{a_2, \dots, a_n\} \geq a_i$. Second we need to prove that $S\ \{a_1, \dots, a_n\}$ is the least upper bound. Suppose $b \geq a_i$, $\forall i$. Then, by induction on n, $b \geq \sup\{a_{n-1}, a_n\} = S\{a_{n-1}, a_n\}$. If $b \geq \sup\{a_k, S\{a_{k+1}, \dots, a_n\} = S\{a_k, \dots, a_n\}$, we have $b \geq \sup\{a_{k-1}, S\{a_k, \dots, a_n\} = S\{a_{k-1}, \dots, a_n\}$. Thus $b \geq S\{a_1, \dots, a_n\}$ and therefore $S\{a_1, \dots, a_n\} = \bigvee\{a_1, \dots, a_n\}$. □

From the last lemma and the discusion about top and bottom in Section 3.1 the following holds.

Corollary 4.4. *Every finite lattice is complete*

By examining the proof of Exemple 3.7 the following holds.

Corollary 4.5. *Let $\mathcal{L}$ be a family of subsets of a set X and let $\{A_i\}_{i \in I}$ be a subset of $\mathcal{L}$.*

(i) If $\bigcup_{i \in I} A_i \in \mathcal{L}$, then $\bigvee_{\mathcal{L}}\{A_i | i \in I\}$ exists and equals $\bigcup_{i \in I} A_i$.

(*ii*) *If* $\bigcap_{i \in I} A_i \in \mathcal{L}$, *then* $\bigwedge_{\mathcal{L}}\{A_i | i \in I\}$ *exists and equals* $\bigcap_{i \in I} A_i$.

Consequently, any (complete) lattice of sets is a (complete) lattice with joins and meets given by union and intersection.

Lemma 4.6. *Let* P *be an ordered set such that* $\bigwedge S$ *exists in* P *for every non-empty subset* S *of* P. *Then* $\bigvee S$ *exists in* P *for every subset* S *of* P *which has an upper bound in* P*; indeed,* $\bigvee S = \bigwedge S^u$.

Proof. Let $S \subseteq P$, with $S \neq \phi$. Assume S has an upper bound in P. Therefore $S^u \neq \phi$. By hypothesis, $\bigwedge S^u \in P$. Let $a = \bigwedge S^u$. We must show that $\bigwedge S = a$. By definition, $\bigwedge S$ is the least upper bound for S. Will now show that $a = \bigwedge S^u$ is also a least upper bound for S. If $s \in S$, we want $s \leq a$. But $\forall t \in S^u$, $s \leq t$ Thus $s \leq \bigwedge S^u$ and a is an upper bound. Now if $b \in P$ is also an upper bound for S we have $b \in S^u$ and thus $\bigwedge S^u \leq b$. This implies $\bigwedge S^u \leq b$ and a is the least upper bound for S. Therefore $\bigwedge S^u = \bigvee S$. □

Theorem 4.7. *Let* P *be a non-empty orederd set. Then the following are equivalent:*

(*i*) P *is a complete lattice;*

(*ii*) $\bigwedge S$ *exists in* P *for every subset* S *of* P*;*

(*iii*) P *has a top element,* $\top$, *and* $\bigwedge S$ *exists in* P *for every non-empty subset* S *of* P.

Proof. Let S be a subset of complete lattice P. Since by Definition of complete lattice, $\bigwedge S$ exist for every subset of P, (*i*) implies (*ii*) automatically. Also, it is easy to prove that (*ii*) implies (*iii*). Since $\bigwedge S$ exists in P for every subset S of P, $\bigwedge S$ exists for empty subset of P. Thus, by Remark 3.2, P has a top element and since by (*ii*), $\bigwedge S$ exists in P for every subset S of P, obvious it exists for every non-empty subset S of P. Now, by Lemma 4.6 $\bigvee S$ exists in P for every subset S of P. Given by (*iii*) that $\bigwedge S$ exists in P for every subset S of P, we have that P is a complete lattice, by definition. Thus we just proved that (*iii*) implies (*i*), and the proof of the theorem is complete. □

This theorem has a simple corollary.

Corollary 4.8. *Let* X *be a set and let* $\mathcal{L}$ *be a family of subsets of* X *ordered by inclusion, such that* *(a)* $\bigcap_{i\in I} A_i \in \mathcal{L}$ *for every non-empty family* $\{A_i\}_{i\in I} \subseteq \mathcal{L}$*, and*

(b) $X \in \mathcal{L}$.

Then $\mathcal{L}$ *is a complete lattice in which*

$$\bigwedge_{i\in I} A_i = \bigcap_{i\in I} A_i,$$
$$\bigvee_{i\in I} A_i = \bigcap\{B \in \mathcal{L} | \bigcup_{i\in I} A_i \subseteq B\}.$$

Proof. $(\mathcal{L};\subseteq)$ is a complete lattice if $\bigvee S$ and $\bigwedge S$ exist for all $S \in \mathcal{L}$. Thus by theorem 4.7 it suffices to show that $\mathcal{L}$ has a top element and the meet of all subsets of $\mathcal{L}$ exist in $\mathcal{L}$. Since $\mathcal{L}$ is a family of subsets of X, X is the top element of $\mathcal{L}$. Now, let $\{A_i\}_{i\in I}$ be a non-empty subset of $\mathcal{L}$. Then $\bigcap_{i\in I} A_i \in \mathcal{L}$, by given (a). By Corollary 4.5, $\bigwedge_{i\in I} A_i$ exists and is equal to $\bigcap_{i\in I} A_i$. Therefore, since $\mathcal{L}$ has a top element and $\bigwedge_{i\in I} A_i$ exists, by Theorem 4.7, $\mathcal{L}$ is a complete lattice. Now

$$\begin{aligned}\bigvee_{i\in I} A_i &= \bigwedge\{A_i | i \in I\}^u \qquad \text{(since } X \text{ is an upper bound of } \{A_i\} \text{ and Lemma 4.6)}\\ &= \bigcap\{B \in \mathcal{L} | (\forall i \in I)\ A_i \subseteq B\}\\ &= \bigcap\{B \in \mathcal{L} | \bigcup_{i\in I} A_i \subseteq B\}\end{aligned}$$

□

Chapter 5

Join-irreducible Elements

Definition 5.1. *Let L be a lattice. An element $x \in L$ is **join-irreducible** if*

(i) $x \neq 0$ (in case L has zero or bottom $\perp$),

(ii) $x = a \vee b$ implies $x = a$ or $x = b$ for all $a, b \in L$,

Condition (ii) is equivalent to

(ii') $a \leq x$ and $b \leq x$ imply $a \vee b \leq x$ for all $a, b \in L$.

A **meet-irreducible** element is defined dually. We denote the set of all join-irreducible elements of L by $\mathcal{J}(L)$ and the set of all meet-ireducible elements by $\mathcal{M}(L)$.

Example 5.2. *In a chain, every non-zero element is join-irreducible. Thus if L is an n-element chain, then $\mathcal{J}(L)$ is an $(n-1)$-element chain.*

Example 5.3. *In a finite lattice L, an element is join-irreducible if and only if has exactly one lower cover (see Section 2.5). This makes $\mathcal{J}(L)$ extremely easy to identify from a diagram of L. In Figure 5.1 the shaded elements are all join-irreducible.*

Exercise 5.4. *Consider the lattice $\langle \mathbb{N}_0; lcm; gcd \rangle$ defined in Example 3.8. A non-zero element $m \in \mathbb{N}_0$ is join-irreducible if and only if m is of the form p^r, where p is a prime and $r \in \mathbb{N}$.*

Proof. Let

$$L = \langle \mathbb{N}_0;\ lcm;\ gcd \rangle\,,\ \text{where } \mathbb{N}_0 = \{0, 1, 2, \ldots\}.$$

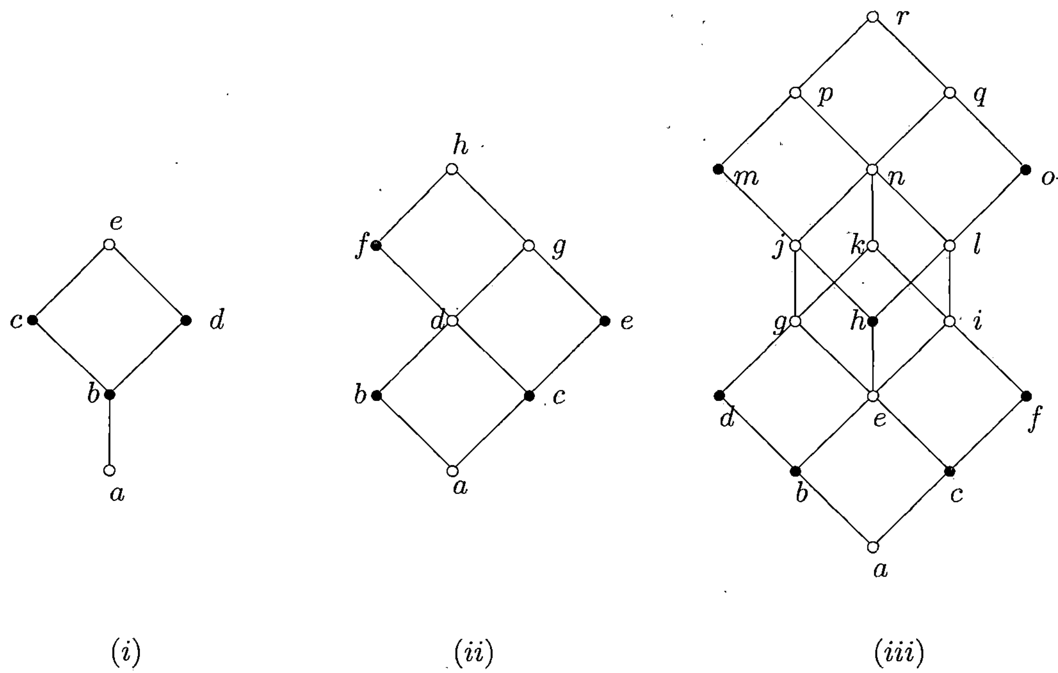

Figure 5.1: Join-irreducible Elements

Define $a \vee b = lcm(a,b)$, $a \wedge b = gcd(a,b)$ and $a \leq b$ if and only if $a|b$.

($\Rightarrow$) Given $m \in \mathbb{N}_0$, $m = {p_1}^{e_1} \cdot {p_2}^{e_2} \cdot \dots \cdot {p_k}^{e_k}$, distinct primes, a join-irreducible element of L, we need to prove that m is of the form p^r. First we'll prove that the powers e^i with $0 \leq i \leq k$ can not be all zeros. For that, we need to show that 1 is the bottom of the lattice L and thus is not join-irreducible. Let $\perp$ be the bottom of lattice L. That implies $\perp \leq x$ for all $x \in L$, or $\perp | x$. Thus, $x = y \cdot \perp$ for some $y \in \mathbb{N}_0$. Now, since x can be any element of L let $x = 2$. That implies the only acceptable values for $\perp$ are $\perp = 1$ or $\perp = 2$, since $\perp | x$. If we let $x = 3$ the only acceptable values for $\perp$ are $\perp = 1$ or $\perp = 3$. Since $\perp | x$ for every $x \in L$, the $\perp$ must be 1. Therefore, since m is join-irreducible in L, m can not be the bottom and the powers e^i with $0 \leq i \leq k$ can not be all zeros.

Suppose one of e_i's is not zero. Choose i with $e_i > 0$ and $e_j = 0$, $\forall j \neq i$. Then $m = p_i^{e_i}$ is of the form p^r. Now, let $m = p_1^{e^1} \cdot p_2^{e^2} \cdot \dots p_n^{e^n}$ with distinct p's and write $m = p_1^{e_1} \cdot k$, where $k = p_2^{e^2} \cdot \dots p_n^{e^n}$, $e_j \geq 0$, $j \in \{2, \dots, n\}$. If $k \neq 1$, lcm$(p_1^{e_1} \cdot k = m$ and by hypothesis m is join-irreducible. This implies that $m = p_1^{e_1}$ or $m = k$. If $m = p_1^{e_1} \cdot k$ we are done. If $m = k$ we repete the above procedure. By induction on n, we have that $m = p_j^{e_j}$ for some $1 \leq j \leq n$. Therefore m is of the form p^r and the proof is complete.

($\Leftarrow$) Given $m \in L$, $m = p^r$, with p prime and $r \in \mathbb{N}$ we need to prove that m is join irreducible. That means we have to prove if $m = a \vee b$, $m = a$ or $m = b$. Let $m = a \vee b$, $a, b \in L$. Then $a \leq m$ and $b \leq m$, by Lemma **??**. Using the definition for join in our lattice $m = lcm(a, b)$, which implies $a|p^r$ and $b|p^r$. Since a divides p^r, a is of the form p^j where $1 \leq j \leq r$ and since b divides p^r, b is of the form p^i where $1 \leq i \leq r$. Thus $m = lcm(p^j, p^i)$, or $m = p^g$ where $g = \max\{i, j\}$. If $j = \max\{i, j\}$, then $m = p^g = p^j = a$. If $i = \max\{i, j\}$, then $m = p^g = p^i = b$ and therefore m is join-irreducible. □

Exercise 5.5. *In the lattice $\mathcal{P}(X)$ the join-irreducible elements are exactly the singleton sets, $\{x\}$, for $x \in X$.*

Proof. Let X be a set. By Example 3.6 $\mathcal{P}(X)$ is a lattice. We need to find the elements $A \in \mathcal{P}(X)$ such that if $A = B \cup C$ then $A = B$ or $A = C$. We claim that the elements A are the singletons $\{x\}$ for some $x \in X$. Suppose $|A| > 1$. Then $A = \{a_1, a_2, \cdots\}$ and we can write the element A as $A = \{a_1\} \cup \{a_2, \cdots\}$. In the last expression A is not $\{a_1\}$ and A is not $\{a_2, \cdots a_n\}$ and thus A is not join-irreducible. On the other hand if $|A| = 1$ then A is join-irreducible and therefore the singletons are the only join-irreducible elements in the lattice $\mathcal{P}(X)$. □

Chapter 6

Modular and Distributive Lattices

Before formally introducing modular and distributive lattices we prove three lemmas wich will put the definitions in 6.4 into perspective.

Lemma 6.1. *Let L be a lattice and let $a, b, c \in L$. Then*

(i) $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$*, and dually,*

(ii) $a \geq c$ *implies* $a \wedge (b \vee c) \geq (a \wedge b) \vee c$*, and dually,*

(iii) $(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) \leq (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$.

Proof. Let L be a lattice and let $a, b, c \in L$
Proof of (i). We need to show that $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$. Now $a \wedge (b \vee c) \geq a \wedge b$ since $b \vee c \geq b$ and $x \geq y \Rightarrow a \wedge x \geq a \wedge y$ by Lemma **??** part (i). Using the same Lemma **??** and the fact that $b \vee c \geq c$, $a \wedge (b \vee c) \geq a \wedge c$. Thus we have

$$a \wedge (b \vee c) \geq a \wedge b \text{ and}$$

$$a \wedge (b \vee c) \geq a \wedge c$$

which implies $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$ (since $x \geq a$ and $x \geq b \Rightarrow x \geq a \vee b$).

Dually, we need to show that $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$. Now $a \vee (b \wedge c) \leq a \vee b$ since $b \wedge c \leq b$ and $x \leq y \Rightarrow a \vee x \leq a \vee y$ by Lemma **??** part (i). Using the same Lemma **??** and the fact that $b \wedge c \leq c$, $a \vee (b \wedge c) \leq a \vee c$. Thus we have

$$a \vee (b \wedge c) \leq a \vee b \text{ and}$$

$$a \vee (b \wedge c) \leq a \vee c,$$

which implies $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$ (since $x \leq a$ and $x \leq b \Rightarrow x \leq a \wedge b$).
Proof of (*ii*). If $a \geq c$ we need to show that $a \wedge (b \vee c) \geq (a \wedge b)$ and $a \wedge (b \vee c) \geq c$. We start with $b \vee c \geq b$ which implies $a \wedge (b \vee c) \geq a \wedge b$ (by Lemma ??). In the same way since $b \vee c \geq c$ we have $a \wedge (b \vee c) \geq a \wedge c$. Since $a \geq c$ is given, then $a \wedge c = c$ and $a \wedge (b \vee c) \geq c$. Therefore $(a \wedge b) \vee (a \wedge c) \geq (a \wedge b) \vee c$. Dually, if $a \leq c$ we obtain $a \vee (b \wedge c) \leq (a \vee b) \wedge c$, just by replacing meet with join and $\leq$ with $\geq$ and viceversa.
Proof of (*iii*). To prove that $(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) \leq (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$ we need to show that

(a) $(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) \leq (a \vee b)$

(b) $(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) \leq (b \vee c)$

(c) $(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) \leq (c \vee a)$

(a). By the definition of meet and join

$$\begin{array}{lcl} a \vee b \geq b \geq a \wedge b & \text{and thus} & a \vee b \geq a \wedge b \\ a \vee b \geq b \geq b \wedge c & \text{and thus} & a \vee b \geq b \wedge c \\ a \vee b \geq a \geq a \wedge c & \text{and thus} & a \vee b \geq c \wedge a \end{array}$$

Therefore $(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) \leq (a \vee b)$. The inequalities (b) and (c) are proved in a similar way. Therefore $(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) \leq (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$ and the proof is complete. □

Lemma 6.2. *Let L be a lattice and let $a, b, c \in L$. Then the following are equivalent:*

(i) $(\forall a, b, c \in L)\ a \geq c \Rightarrow a \wedge (b \vee c) = (a \wedge b) \vee c$;

(ii) $(\forall a, b, c \in L)\ a \geq c \Rightarrow a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$;

(iii) $(\forall p, q, r \in L)\ p \wedge (q \vee (p \wedge r)) = (p \wedge q) \vee (p \wedge r)$.

Proof. Let L be a lattice and $a, b, c, p, q, r \in L$.
((*i*) implies (*ii*)). Given $a \geq c$ and $a \wedge (b \vee c) = (a \wedge b) \vee c$ we need to show that $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$. Given $a \geq c$ by Lemma ?? $a \wedge c = c$ and replacing c in

$(a \wedge b) \vee c$ we get $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

((ii) implies (i)). Given $a \geq c$ and $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, if we replace $a \wedge c$ with c we get $a \wedge (b \vee c) = (a \wedge b) \vee c$.

((ii) implies (iii)). Given $a \geq c$ and $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ we need to show that $p \wedge (q \vee (p \wedge r)) = (p \wedge q) \vee (p \wedge r)$. Let $a = p$, $b = q$ and $c = p \wedge r$. Then $p \geq p \wedge r$ ($a \geq c$). Therefore by (ii) $a \geq c$ and $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ or, replacing as above,

$$\begin{aligned} p \wedge (q \vee (p \wedge r)) &= (p \wedge q) \vee (p \wedge (p \wedge r)) && \text{(by } (ii)\text{)} \\ &= (p \wedge q) \vee ((p \wedge p) \wedge r) && \text{(by ??)} \\ &= (p \wedge q \vee (p \wedge r) && \text{(since } p \wedge p = p\text{)} \end{aligned}$$

((iii) implies (ii)). Given $p \wedge (q \vee (p \wedge r)) = (p \wedge q) \vee (p \wedge r)$ we need to show that whenever $a \geq c$, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$. Let $p = a$, $q = b$ and $r = c$. Then $p \geq r$ ($a \geq c$) and by (iii) $p \wedge (q \vee (p \wedge r)) = (p \wedge q) \vee (p \wedge r)$ or, replacing as above,

$$\begin{aligned} (a \wedge b) \vee (a \wedge c) &= a \wedge (b \vee (a \wedge c)) && \text{(by } (ii)\text{)} \\ &= a \wedge (b \vee c) && \text{(since } a \wedge c = c\text{)} \end{aligned}$$

□

Lemma 6.3. *Let L be a lattice. Then the following are equivalent:*

(D) $(\forall a, b, c \in L)$ $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$;

(D^δ) $(\forall p, q, r \in L)$ $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$.

Proof. Assume D holds. Then for $p, q, r \in L$,

$$\begin{aligned} (p \vee q) \wedge (p \vee r) &= ((p \vee q) \wedge p) \vee ((p \vee q) \wedge r) && \text{(by } (D)\text{)} \\ &= (p \wedge (p \vee q)) \vee ((p \vee q) \wedge r) && \text{(by Lemma ??} (L2)^\delta\text{)} \\ &= p \vee ((p \vee q) \wedge r) && \text{(by Lemma ??} (L4)^\delta\text{)} \\ &= p \vee (r \wedge (p \vee q)) && \text{(by Lemma ??} (L2)^\delta\text{))} \\ &= p \vee ((r \wedge p) \vee (r \wedge q)) && \text{(by } (D)\text{)} \\ &= (p \vee ((r \wedge p)) \vee (r \wedge q)) && \text{(by Lemma ??} (L1)\text{)} \\ &= (p \vee ((p \wedge r)) \vee (r \wedge q)) && \text{(by Lemma ??} (L2)^\delta \end{aligned}$$

$$
\begin{aligned}
&= p \vee (r \vee q) && \text{(by Lemma ??}(L4))\\
&= p \vee (q \vee r) && \text{(by Lemma ??}(L2))
\end{aligned}
$$

Therefore, (D) implies (D^δ). Dually, assume D^δ holds. Then for $a, b, c \in L$,

$$
\begin{aligned}
(a \wedge b) \vee (a \wedge c) &= ((a \wedge b) \vee a) \wedge ((a \wedge b) \wedge c) && (by(D^\delta))\\
&= (a \vee (a \wedge b)) \wedge ((a \wedge b) \vee c) && \text{(by Lemma ??}(L2))\\
&= a \wedge ((a \wedge b) \vee c) && \text{(by Lemma ??}(L4))\\
&= a \wedge (c \vee (a \wedge b)) && \text{(by Lemma ??}(L2)))\\
&= a \wedge ((c \vee a) \wedge (c \vee b)) && \text{(by }(D^\delta))\\
&= (a \wedge ((c \vee a)) \wedge (c \vee b)) && \text{(by Lemma ??}(L1^\delta))\\
&= (a \wedge ((a \vee c)) \wedge (c \vee b)) && \text{(by Lemma ??}(L2))\\
&= a \wedge (c \vee b) && \text{(by Lemma ??}(L4^\delta))\\
&= a \wedge (b \vee c) && \text{(by Lemma ??}(L2))
\end{aligned}
$$

and therefore, (D^δ) implies (D) . □

Definition 6.4. *Let L be a lattice.*

*(i) L is said to be **distributive** if it satisfies the **distributive law,***

$$(\forall a, b, c \in L)\ a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

*(ii) L is said to be **modular** if it satisfies the **modular law,***

$$(\forall a, b, c \in L)\ a \geq c \Rightarrow a \wedge (b \vee c) = (a \wedge b) \vee c.$$

Remarks 6.5.

(1) Lemma 6.1 shows that any lattice is 'half-way' to being both modular and distributive. To establish distributivity or modularity we only need to check a simple inequality.

(2) Lemma 6.2 shows that any distributive lattice is modular.

(3) *Distributivity can be defined either by* (D) *or by* (D^δ) *(from Lemma 6.3). In other words,* L *is distibutive if and only if* L^δ *is. By the Duality Principle* L *is modular if and only if* L^δ *is.*

(4) *The universal quantifiers in Lemmas 6.2 and 6.3 are essential. It is not true that if particular elements* a, b, c *in an arbitrary lattice* L *satisfy* $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, *then they also satisfy* $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

Example 6.6. *Any powerset lattice* $\mathcal{P}(X)$ *is distributive.*

Example 6.7. *Any chain is distributive.*

Proof. Let C be a chain and let $a, b, c \in C$. We need to show that for any $a, b, c \in C$, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$. By Lemma 6.1 $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$. Therefore we only need to prove that $(a \wedge b) \vee (a \wedge c) \geq a \wedge (b \vee c)$. For that we have to investigate a few cases.

1. If $a = b = c$

$$\begin{aligned}(a \wedge b) \vee (a \wedge c) &= a \vee a \\ &= a \\ &\geq a \wedge (b \vee c)\end{aligned}$$

2. If $\quad a = b, \qquad a = c, \qquad b = c$

$$\begin{array}{lll}(a \wedge b) \vee (a \wedge c) = & & \\ \quad = a \vee (a \wedge c) & = (a \wedge b) \vee c & = (a \wedge b) \vee (a \wedge c) \\ \quad = a \vee (b \wedge c) & = a \vee (b \wedge c) & =\geq a \wedge b \\ \quad \geq a \wedge (b \vee c) & \geq a \wedge (b \vee c) & \end{array}$$

3. If a is the least element between a, b, c

$$\begin{aligned}(a \wedge b) \vee (a \wedge c) &= a \vee a \\ &= a \\ &\geq a \wedge (b \vee c)\end{aligned}$$

If b is the least element between a, b, c

$$\begin{aligned}(a \wedge b) \vee (a \wedge c) &= b \vee (a \wedge c)\\ &= (a \wedge c) \vee b\\ &\geq a \wedge (b \vee c)\end{aligned}$$

If c is the least element between a, b, c

$$\begin{aligned}(a \wedge b) \vee (a \wedge c) &= (a \wedge b) \vee c\\ &= (a \wedge b)\\ &\geq a \wedge (b \vee c)\end{aligned}$$

Therefore any chain is distributive. □

Remarks 6.8.

(1) The following theorem whose proof can be found in [1] implies that is possible to determine whether or not a finite lattice is modular or distributive for its diagram.

(2) We write $K \rightarrowtail L$ to indicate that the lattice L has a sublattice isomorphic to the lattice K.

Theorem 6.9. *(*The $\mathrm{M}_3 - \mathrm{N}_5$ Theorem*) Let L be a lattice.*

(i) L is non-modular if and only if $N_5 \rightarrowtail L$.

(ii) L is non-distributive if and only if $N_5 \rightarrowtail L$ or $M_3 \rightarrowtail L$.

As an application of Theorem 6.9 consider the lattices $\mathbf{N}_5$ (the **pentagon**) and $\mathbf{M}_3$ (the **diamond**) shown in Figure 6.1. The lattice M_3 is modular, but is not distributive. To see this, note that

$$u \wedge (v \vee w) = u \wedge q = u \neq p = p \vee p = (u \wedge v) \vee (u \wedge w).$$

The lattice $\mathbf{N}_5$ is not modular and so also not distributive. In the diagram we have

$$v \geq u \text{ and } v \wedge (e \vee v) = v \wedge q = v > u = p \vee u = (v \wedge e) \vee v.$$

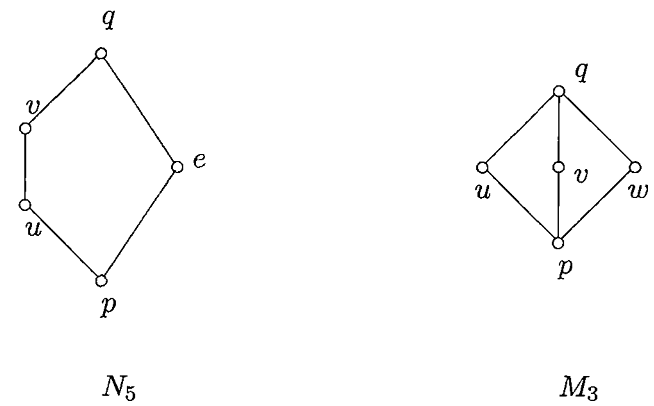

Figure 6.1: $M_3 - N_5$ Lattices

Consider the four lattices in Figure 6.2. The lattice L_1 and L_2 have sublattices isomorphic with $N5$ (the shaded elements). The $M_3 - N_5$ Theorem implies imediately that they are non-modular. The lattice L_3 contains M_3 so, is not distributive. It is apparent from the diagrams that N_5 does not embed in L_3 and that neither N_5 or M_3 embeds in L_4. However, to justify this fully requires an enumeration of cases.

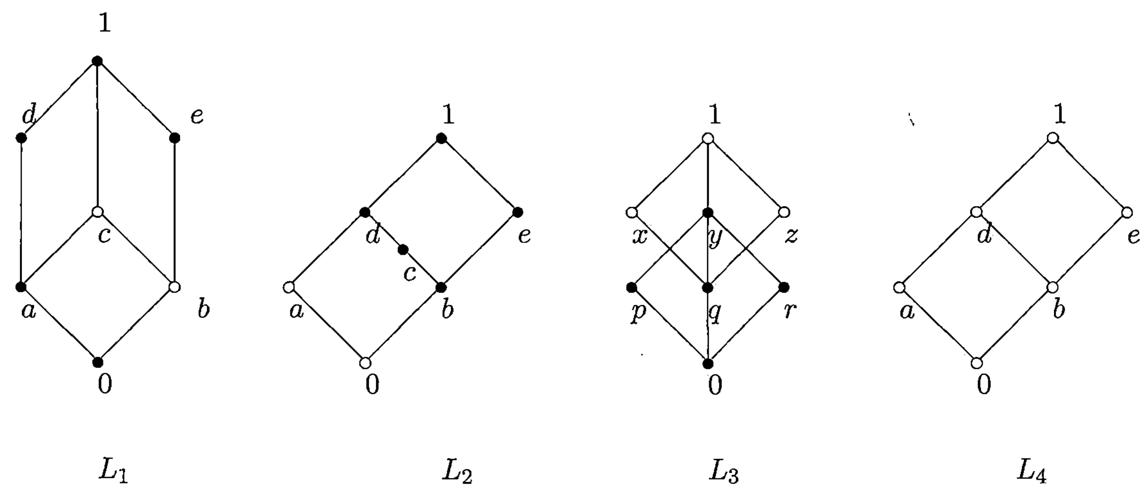

Figure 6.2: $M_3 - N_5$ Theorem

To decide wheter a given lattice L is non-modular, modular but non-distributive, or distributive, we therefore proceed as follows. If a sublattice of L isomorphic to N_5 (M_3) can be exibited, then L is non-modular (non-distributive), by the $M_3 - N_5$ Theorem. If a search for a copy of N_5 or M_3 fails, we conjecture that L is modular (distributive).

To substantiate this claim we have to inspect every pair of elements of the lattice and see if the relations for distributivity or modularity hold.

It should be emphasized that the statement of the $M_3 - N_5$ Theorem refers to the occurance of the pentagon or diamond as a sublattice of L. This means that the meets and the joins in a candidate copy of N_5 or M_3 must be the same as those in the lattice L. In L_4 for example the pentagon $K = \{0, a, d, 1, e\}$ is not a sublattice, since $d \wedge e = b \notin K$.

Exercise 6.10.

(i) Let L be a distributive lattice and let $a, b, c \in L$. Prove that

$$(a \vee b = c \vee b \text{ and } a \wedge b = c \wedge b) \Rightarrow a = c.$$

(ii) Find elements a, b, c in M_3 and N_5 violating (i).

(iii) Deduce that a lattice L is distributive if and only if (i) holds for all $a, b, c \in L$.

Proof.

Proof of (i). Let L be a distributive lattice, $a, b, c \in L$, $a \vee b = c \vee b$ and $a \wedge b = b \wedge c$. We need to show that $a = c$. By definition, in a distributive lattice

$$(\forall a, b, c \in L) \quad a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

Now

$$\begin{aligned} a \wedge (b \vee c) &= a \wedge (c \vee b) && \text{(by Lemma ?? } L2) \\ &= a \wedge (a \vee b) && \text{(since } \ b \vee c = a \vee b) \\ &= a && \text{(by Lemma ?? } (L4)) \end{aligned}$$

and

$$\begin{aligned} (a \wedge b) \vee (a \wedge c) &= (c \wedge b) \vee (a \wedge c) && \text{(since } a \wedge b = c \wedge b) \\ &= (c \wedge b) \vee (c \wedge a) && \text{(by Lemma ?? } (L2)) \\ &= c \wedge (b \vee c) && \text{(by Lemma 6.3 } \ D) \\ &= c && \text{(by Lemma ?? } (L4^{\delta})) \end{aligned}$$

Therefore $a = c$.

Proof of (ii). In M_3 (see Figure 4.1),

$$u \vee v = q = v \vee w \qquad \text{and} \qquad u \wedge v = p = v \wedge w \qquad \text{but} \qquad u \neq w$$

In N_5 (see Figure 4.1),

$$v \vee e = q = u \vee e \qquad \text{and} \qquad v \wedge e = p = u \wedge e \qquad \text{but} \qquad u \neq v$$

Proof of (iii). The ($\Rightarrow$) was proved in part (i). Conversely, let L be a lattice, $a, b, c \in L$. Assume that for all $a, b, c \in L$, $a \vee b = c \vee b$ and $a \wedge b = \wedge c$ implies $a = c$. We have to show that $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$. Consider the element $a \wedge (b \vee c)$

$$\begin{aligned}
a \wedge (b \vee c) &= a \wedge (c \vee b) && \text{(by Lemma ?? } (L2)) \\
&= a \wedge (a \vee b) && \text{(since } a \vee b = c \vee b) \\
&= a && \text{(by Lemma ?? } (L4^{\delta})) \\
&= c && \text{(since } a = c) \\
&= c \vee (c \wedge b) && \text{(by Lemma ?? } (L4)) \\
&= c \vee (a \wedge b) && \text{(since } a \vee b = c \vee b) \\
&= (a \wedge c) \vee (a \wedge b) && \text{(since } a = c \text{ means } a \wedge c = c)
\end{aligned}$$

Therefore, by definition the lattice L is distributive. □

Exercise 6.11. *The lattice $\langle \mathbb{N}_0; lcm; gcd \rangle$ is distributive.*

Proof. Let L be the lattice $\langle \mathbb{N}_0; lcm; gcd \rangle$, where $N_0 = \{0, 1, 2, 3, 4, \cdots\}$. Let $a, b, c \in L$ with $a \vee b = \mathrm{lcm}(a, b)$ and $a \wedge b = \gcd(a, b)$. By Exercise 6.10, assuming that $a \vee b = c \vee b$ and $a \wedge b = c \wedge b$ we need to show that $a = c$.

Let

$$a = p_1^{e_1} \cdot p_2^{e_2} \cdot \cdots \cdot p_m^{e_m}$$

$$b = p_1^{f_1} \cdot p_2^{f_2} \cdot \cdots \cdot p_q^{e_q}$$

$$c = p_1^{g_1} \cdot p_2^{g_2} \cdot \cdots \cdot p_r^{e_r}$$

where, p_i's are prime numbers and $e, f, g, m, q, r \in \mathbb{N}$. Let $n = \max\{m, q, r\}$. Then

$$a = p_1^{e_1} \cdot p_2^{e_2} \cdot \dots \cdot p_n^{e_n}$$

$$b = p_1^{f_1} \cdot p_2^{f_2} \cdot \dots \cdot p_n^{e_n}$$

$$c = p_1^{g_1} \cdot p_2^{g_2} \cdot \dots \cdot p_n^{e_n}$$

where some of the powers e_i's might be zeros.

Considering the join and meet as lcm and gcd we have

$$\begin{aligned} a \vee b &= \operatorname{lcm}(a, b) \\ &= \operatorname{lcm}(p_1^{e_1} \cdot p_2^{e_2} \cdot \dots \cdot p_n^{e_n}, p_1^{f_1} \cdot p_2^{f_2} \cdot \dots \cdot p_q^{e_q}) \\ &= p_1^{\max\{e_1, f_1\}} \cdot p_2^{\max\{e_2, f_2\}} \cdot \dots \cdot p_n^{\max\{e_n, f_n\}} \end{aligned}$$

$$\begin{aligned} c \vee b &= \operatorname{lcm}(c, b) \\ &= \operatorname{lcm}(p_1^{g_1} \cdot p_2^{g_2} \cdot \dots \cdot p_n^{g_n}, p_1^{f_1} \cdot p_2^{f_2} \cdot \dots \cdot p_q^{e_q}) \\ &= p_1^{\max\{g_1, f_1\}} \cdot p_2^{\max\{g_2, f_2\}} \cdot \dots \cdot p_n^{\max\{g_n, f_n\}} \end{aligned}$$

$$\begin{aligned} a \wedge b &= \gcd(a, b) \\ &= \gcd(p_1^{e_1} \cdot p_2^{e_2} \cdot \dots \cdot p_n^{e_n}, p_1^{f_1} \cdot p_2^{f_2} \cdot \dots \cdot p_q^{e_q}) \\ &= p_1^{\min\{e_1, f_1\}} \cdot p_2^{\min\{e_2, f_2\}} \cdot \dots \cdot p_n^{\min\{e_n, f_n\}} \end{aligned}$$

$$\begin{aligned} c \wedge b &= \gcd(c, b) \\ &= \gcd(p_1^{g_1} \cdot p_2^{g_2} \cdot \dots \cdot p_n^{g_n}, p_1^{f_1} \cdot p_2^{f_2} \cdot \dots \cdot p_q^{e_q}) \\ &= p_1^{\min\{g_1, f_1\}} \cdot p_2^{\min\{g_2, f_2\}} \cdot \dots \cdot p_n^{\min\{g_n, f_n\}} \end{aligned}$$

By assumption $a \vee b = b \vee c$ and $a \wedge b = b \wedge c$. Then

$$p_1^{\max\{e_1, f_1\}} \cdot p_2^{\max\{e_2, f_2\}} \cdot \dots \cdot p_n^{\max\{e_n, f_n\}} = p_1^{\max\{g_1, f_1\}} \cdot p_2^{\max\{g_2, f_2\}} \cdot \dots \cdot p_n^{\max\{g_n, f_n\}}$$

and

$$p_1^{\min\{e_1, f_1\}} \cdot p_2^{\min\{e_2, f_2\}} \cdot \dots \cdot p_n^{\min\{e_n, f_n\}} = p_1^{\min\{g_1, f_1\}} \cdot p_2^{\min\{g_2, f_2\}} \cdot \dots \cdot p_n^{\min\{g_n, f_n\}}.$$

That implies

$$\max\{e_1, f_1\} = \max\{g_1, f_1\},$$
$$\max\{e_2, f_2\} = \max\{g_2, f_2\},$$
$$\dots,$$
$$\max\{e_n, f_n\} = \max\{g_n, f_n\}$$

and

$$\min\{e_1, f_1\} = \min\{g_1, f_1\},$$
$$\min\{e_2, f_2\} = \min\{g_2, f_2\},$$
$$\dots$$
$$\min\{e_n, f_n\} = \min\{g_n, f_n\}.$$

Given i, either $e_i \leq f_i$ or $f_i \leq e_i$. If $e_i \leq f_i$, then by the maximum equalities, $g_i \leq f_i$ and thus by the minimum equalities, $e_i = g_i$. If $f_i \leq e_i$, then by minimum equalities $f_i \leq g_i$. Thus by the maximum equalities, $e_i = g_i$. In either case, $e_i = g_i$ and therefore $a = c$. Therefore by Exercise 4.10 the lattice $\langle \mathbb{N}_0; lcm; gcd \rangle$, is distributive. $\square$

Chapter 7

Representation Theorem: the Finite Case

We start the detailed investigation of the structure of distributive lattices with the finite case.

Definition 7.1. *For a distributive lattice L, let $J(L)$ denote the set of all non-zero join-irreducible elements regarded as a poset under the partial ordering of L. For $a \in L$, set*

$$r(a) = \{x \mid x \leq a, \ x \in J(L)\} = \downarrow a \cap J(L).$$

Definition 7.2. *For a poset P, define a subset $A \subseteq P$ to be **hereditary** if and only if $x \in A$ and $y \leq x$ imply that $y \in A$.*

Remark 7.3. *Note, a hereditary subset A satisfies $A = \downarrow A$. Therefore hereditary subsets are down sets (see 2.11). Let $H(P)$ be the set of all hereditary subsets partially ordered by set inclusion. The following check will show that $H(P)$ is a lattice in which join and meet are intersection and union, respectively, and thus $H(P)$ is distributive.*

To show that $H(P)$ is a lattice in which join and meet are intersection and union, respectively, let X and Y two elements in $(H(P)$. We need to show that $X \cup Y \in H(P)$ and $X \cap Y \in H(P)$. Let $x \in X \cap Y$ and $y \leq x$. That implies $x \in X$ and $x \in Y$. Since $y \leq x$ and $X, Y \in H(P)$, by definition of hereditary set, $y \in X$ and $y \in Y$. Hence, $y \in X \cap Y$ and $X \cap Y \in H(P)$. Now let $x \in X \cup Y$ and $y \leq x$. That implies $x \in X$

or $x \in Y$. Since $y \leq x$ and $X, Y \in H(P)$, $y \in X$ or $y \in Y$. Hence, $y \in X \cup Y$ and $X \cup Y \in H(P)$. Therefore, by definition, $H(P)$ is a lattice.

Now to show that $H(P)$ is distributive, let $A, B, C \in H(P)$. and let $x \in A \cap (B \cup C)$. Then, by definition of intersection, $x \in A$ and $x \in B \cup C$. That implies $x \in A$ and $x \in B$ or $x \in C$. Thus $x \in A$ and $x \in B$ or $x \in A$ and $x \in C$, which implies $x \in (A \cap B) \cup (A \cap C)$ and therefore $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$. Now let $x \in (A \cap B) \cup (A \cap C)$. That implies $x \in A \cap B$ or $x \in A \cap C$ which we can rewrite as $x \in A$ and *B or $x \in A$* and *C. Thus $x \in A$ and $x \in B$* or *C which implies $x \in A \cap (B \cup C)$. Therefore $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, wich imply that the lattice $H(P)$ is distributive.*

Theorem 7.4. Birkhoff's representation theorem for finite distributive lattices. (Grätzer, [2])
Let L be a finite distributive lattice. Then the map

$$\varphi : L \to \mathcal{P}(L), \quad \varphi : a \mapsto r(a), \; r(a) = \{x | \; x \leq a, \; x \in J(L)\}$$

has image the sublattice $H(J(L))$ of $\mathcal{P}(L)$ and is an isomorphism from L to $H(J(L))$.

Proof. We need to prove that $\varphi(L) \subseteq H(J(L))$ and

(1) φ is onto,

(2) φ is one-to-one,

(3) $\varphi(a \vee b) = \varphi(a) \vee \varphi(b)$, and $\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b)$, $\forall a, b \in L$

Proof of (1). By the definition of φ, φ has its image in $H(J(L))$. Let $A \in H(J(L))$ and let a be the upper bound of A (A is finite). Then $a = \bigvee A$ and we need to show that $A = \{x | x \leq a, x \in J(L)\}$ or, $A = r(a)$. Let $x \in A$. Then $x \in J(L)$ and $x \leq a$. This implies that $x \in r(a)$ and thus $A \subseteq r(a)$. Now, let $x \in r(a)$. Since $x \leq a$, then $x = x \wedge a$ and since $a = \bigvee A$, $x = x \wedge (\bigvee A)$. Because the lattice L is finite, we can rewrite the last equality as $x = \bigvee (x \wedge y \; | y \in A)$. Since x is join-irreducible $x = x \wedge y$ for some $y \in A$, which implies $x \leq y$. Since A is a downset and $y \in A$, $x \in A$ and hence $r(a) \subseteq A$. Therefore $A = r(a)$

Proof of (2). Let $\varphi(a) = \varphi(b)$ and $a, b \in L$. Since $\varphi(a) = r(a)$ and $\varphi(b) = r(b)$, we have $r(a) = r(b)$, which implies $\bigvee r(a) = \bigvee r(b)$. Since L is finite, every element is a sup of the join-irreducible elements below it, thus $\bigvee r(a) = a$ and $\bigvee r(b) = b$. Hence $a = b$. Therefore φ is one-to-one function.

Proof of (3). To show that $\varphi(a \vee b) = \varphi(a) \vee \varphi(b)$, we need to show that $r(a \vee b) = r(a) \cup r(b)$. Let $x \in r(a \vee b)$. Then

$$\begin{aligned} x &= x \wedge (a \vee b) && \text{(by Theorem ??, } (L4)^{\delta}) \\ &= (x \wedge a) \vee (x \wedge b) && \text{(since } L \text{ is distributive)} \end{aligned}$$

Since x is join-irreducible, $x = x \wedge a$ or $x = x \wedge b$. This implies that $x \in r(a)$ or $x \in r(b)$. Thus,

$$x \in r(a) \cup r(b) \quad \text{and} \quad r(a \vee b) \subseteq r(a) \cup r(b). \tag{7.1}$$

Now let $x \in r(a) \cup r(b)$. Then, $x \in r(a)$, or $x \in r(b)$. This implies that $x \leq a$, or $x \leq b$, which implies that $x \leq a \vee b$, which implies $x \in r(a \vee b)$. Thus

$$r(a) \cup r(b) \subseteq r(a \vee b). \tag{7.2}$$

Therefore by (7.1) and (7.2), $r(a \vee b) = r(a) \cup r(b)$.

Now if $\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b)$, we need to show that $r(a \wedge b) = r(a) \cap r(b)$. Let $x \in r(a \wedge b)$. Then $x \leq a \wedge b$, which implies $x \leq a$ and $x \leq b$. This implies that $x \in r(a)$ and $x \in r(b)$. Thus,

$$x \in r(a) \cap r(b) \quad \text{and} \quad r(a \wedge b) \subseteq r(a) \cap r(b). \tag{7.3}$$

Now let $x \in r(a) \cap r(b)$. Then, $x \in r(a)$ and $x \in r(b)$. This implies that $x \leq a$, and $x \leq b$, which implies that $x \leq a \wedge b$. That means $x \in r(a \wedge b)$ and

$$r(a) \cap r(b) \subseteq r(a \wedge b). \tag{7.4}$$

Therefore by (7.1) and (7.2), $r(a \wedge b) = r(a) \cap r(b)$. □

In a lattice a ring of sets is any subset of $\mathcal{P}(X)$ closed under union and intersection, that is, a sublattice of $\mathcal{P}(X)$.

Corollary 7.5. (Grätzer, [2]) *A finite lattice is distributive if and only if it is isomorphic to a ring of sets.*

Proof.

($\Rightarrow$). Let L be a finite distributive lattice and let $P = (J(L))$ be the ordered set of join-irreducible elements. By Remark 7.3 $H(J(L))$ is closed under union and intersection and therefore $H(J(L))$ is a ring of sets. Hence, by Theorem 7.4 the finite distributive lattice L is isomorphic to a ring of sets.

($\Leftarrow$). Let L be a finite lattice isomorphic to a ring of sets. That is L is isomorphic to a sublattice of $\mathcal{P}(X)$ for some set X. Since $\mathcal{P}(X)$ is distributive, L is too. $\square$

Example 7.6. *In Figure 7.1, L is a lattice, $J(L)$ are the join-irreducible elements of L and $H(J(L))$ is the set of all hereditary subsets of $J(L)$. In L_1, $J(L_1) = \{a, b, c\}$ and the set of all hereditary subsets of $J(L_1)$ is $H(J(L)) = \{\phi, \{a\}, \{b\}, \{a, c\}\}$.*

The map

$$\varphi : L \to H(J(L)), \quad z \mapsto r(z) = \{z \in L | x \leq z, x \in J(L)\}$$

is an isomorphism:

$$\varphi(\bot) = r(\bot) = \{x \in L | x \leq \bot, x \in J(L)\} = \phi$$
$$\varphi(a) = r(a) = \{x \in L | x \leq a, x \in J(L)\} = \{a\}$$
$$\varphi(b) = r(b) = \{x \in L | x \leq b, x \in J(L)\} = \{a, b\}$$
$$\varphi(c) = r(c) = \{x \in L | x \leq c, x \in J(L)\} = \{b, c\}$$
$$\varphi(\top) = r(\top) = \{x \in L | x \leq \top, x \in J(L)\} = \{a, b, c\}$$
$$\varphi(\bot) = r(\bot) = \{x \in L | x \leq \bot, x \in J(L)\} = \phi$$
$$\varphi(a) = r(a) = \{x \in L | x \leq a, x \in J(L)\} = \{a\}$$

Therefore, by Theorem 7.3 the lattice L_1 is distributive.

In L_2, $J(L_2) = a, b, d, e$ and the set of all hereditary subsets of $J(L_1)$ is

$$H(J(L)) = \{\phi, \{a\}, \{b\}, \{a, b\}, \{b, d\}, \{e, a, b\}, \{e, b, d\}, \{a, b, d, e\}\}.$$

The map $\varphi : L \to H(J(L)$, $z \mapsto r(z) = \{z \in L | x \leq z, x \in J(L)\}$ is an isomorphism:

$$\varphi(b) = r(b) = \{x \in L | x \leq b, x \in J(L)\} = \{b\}$$
$$\varphi(c) = r(c) = \{x \in L | x \leq c, x \in J(L)\} = \{a, b\}$$
$$\varphi(d) = r(d) = \{x \in L | x \leq d, x \in J(L)\} = \{b, d\}$$

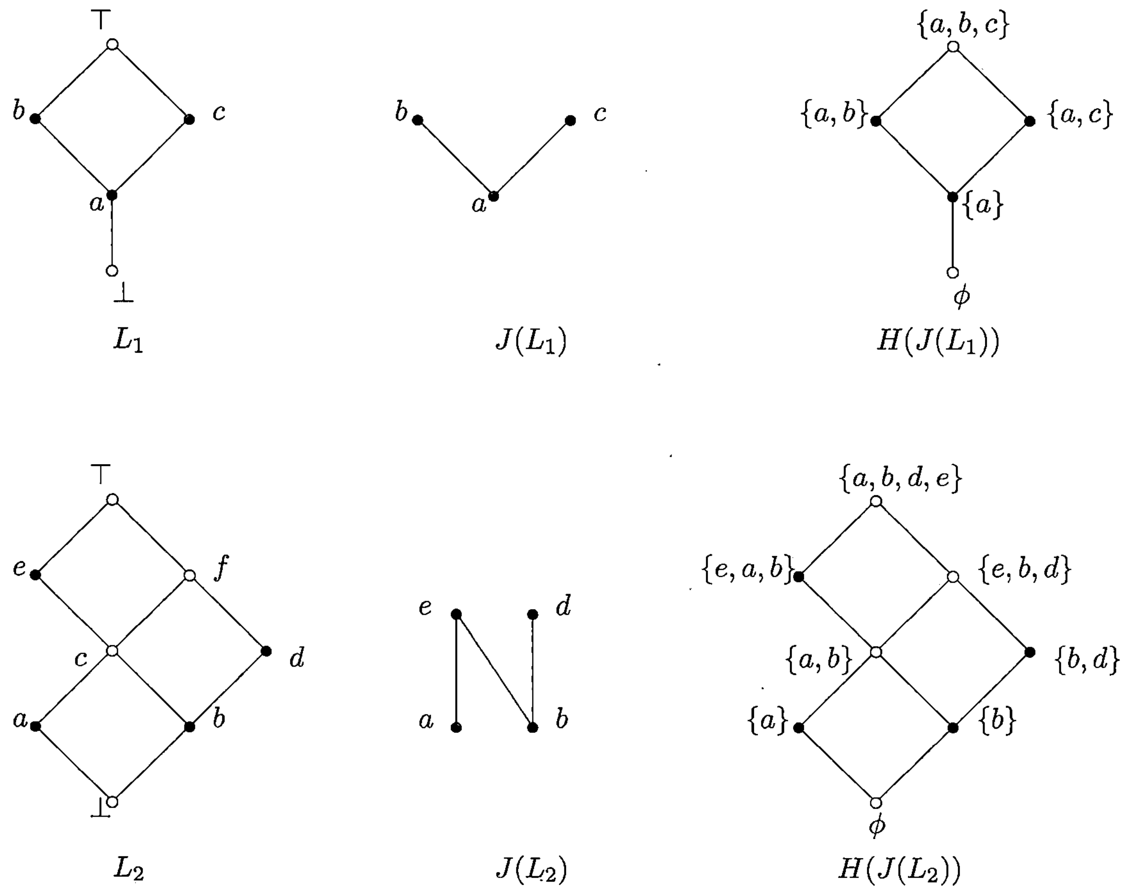

Figure 7.1: Distributive Lattices

$$\varphi(e) = r(e) = \{x \in L | x \leq e, x \in J(L)\} = \{e, a, b\}$$
$$\varphi(f) = r(f) = \{x \in L | x \leq f, x \in J(L)\} = \{e, b, d\}$$
$$\varphi(\top) = r(\top) = \{x \in L | x \leq \top, x \in J(L)\} = \{a, b, d, e\}$$

Therefore, by Theorem 7.3 the lattice L_2 is distributive.

Example 7.7. *The lattice in Figure 7.2 is not distributive.*

In L_3, $J(L_3) = \{a, b, c\}$ and the set $H(J(L_3))$ is

$$H(J(L_3)) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

Since L has 5 elements and $H(J(L))$ has 8 elements they can not be isomorphic and therefore, by Theorem 7.4 L is not distributive.

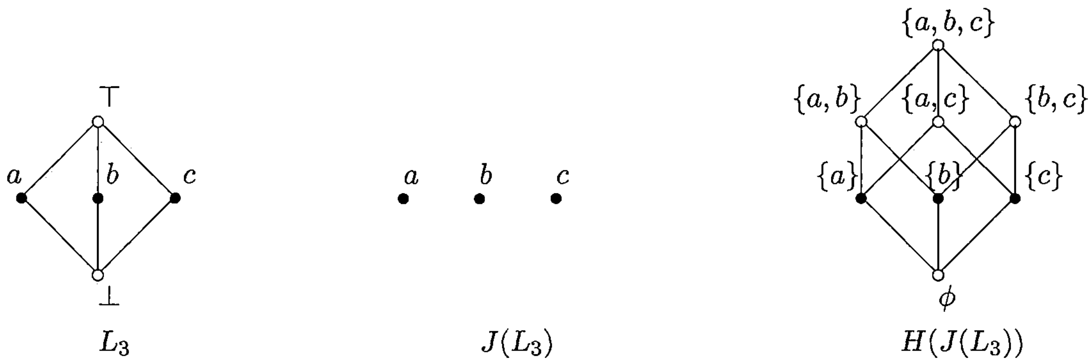

Figure 7.2: Non-distributive Lattices

Exercise 7.8. *Using Theorem 7.4, prove that the lattice from Figure 7.3 is distributive.*

Proof. Let L be a lattice. The join-irreducible elements of L are $J(L) = \{a, b, g, h, i\}$ and the corresponding hereditary subsets $H(J(L))$ are

$$\{\phi, \{g\}, \{h\}, \{i\}, \{g, h\}, \{g, i\}, \{h, i\}, \{g, h, i\}, \{a, g, h, i\}, \{b, g, h, i\}, \{a, b, g, h, i\}\}$$

Let $\varphi : L \longrightarrow H(J(L))$, $z \mapsto \varphi(z) = \{x \in L | x \leq z, x \in J(L)\}$.

$$\varphi(\perp) = r(\perp) = \{x \in L | x \leq \perp, x \in J(L)\} = \phi$$
$$\varphi(g) = r(g) = \{x \in L | x \leq g, x \in J(L)\} = \{g\}$$
$$\varphi(h) = r(h) = \{x \in L | x \leq h, x \in J(L)\} = \{h\}$$
$$\varphi(i) = r(i) = \{x \in L | x \leq i, x \in J(L)\} = \{i\}$$
$$\varphi(d) = r(d) = \{x \in L | x \leq d, x \in J(L)\} = \{g, h\}$$
$$\varphi(e) = r(e) = \{x \in L | x \leq e, x \in J(L)\} = \{g, i\}$$
$$\varphi(f) = r(f) = \{x \in L | x \leq f, x \in J(L)\} = \{h, i\}$$
$$\varphi(c) = r(c) = \{x \in L | x \leq c, x \in J(L)\} = \{g, h, i\}$$
$$\varphi(a) = r(a) = \{x \in L | x \leq a, x \in J(L)\} = \{a, g, h, i\}$$
$$\varphi(b) = r(b) = \{x \in L | x \leq b, x \in J(L)\} = \{b, g, h, i\}$$
$$\varphi(\top) = r(\top) = \{x \in L | x \leq \top, x \in J(L)\} = \{a, b, g, h, i\}$$

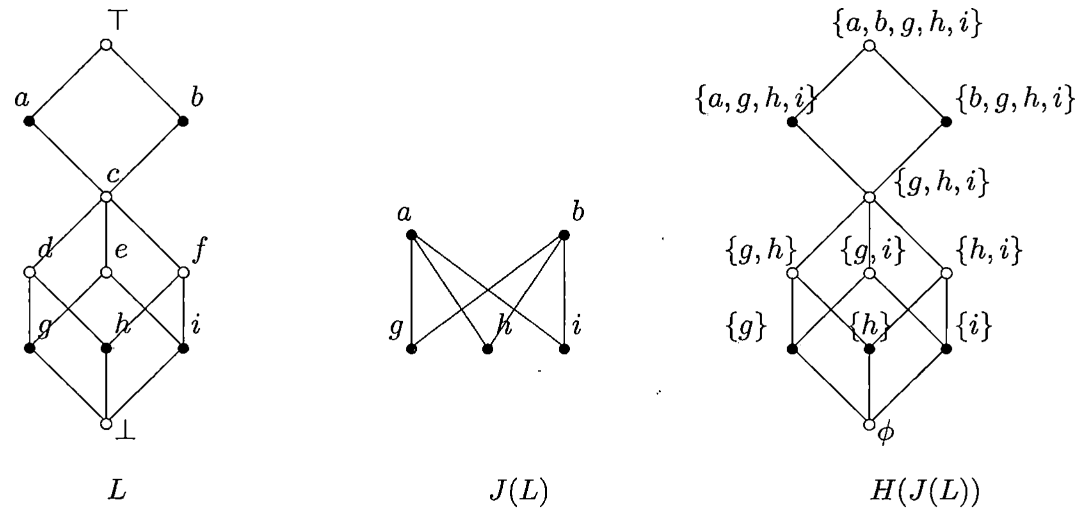

Figure 7.3: Distributive Lattice

Now, we need to check if φ preserves meet and join

$$\begin{aligned}
\varphi(\perp \vee g) =& \varphi(g) = \{g\} = \phi \cup \{g\} = \varphi(\perp) \vee \varphi(g)\\
\varphi(\perp \wedge g) =& \varphi(\perp) = \phi = \{\perp\} \cap \{g\} = \varphi(\perp) \wedge \varphi(g)\\
\varphi(\perp \vee h) =& \varphi(h) = \{h\} = \phi \cup \{h\} = \varphi(\perp) \vee \varphi(h)\\
\varphi(\perp \wedge h) =& \varphi(\perp) = \phi = \{\perp\} \cap \{h\} = \varphi(\perp) \wedge \varphi(h)\\
\varphi(\perp \vee i) =& \varphi(i) = \{i\} = \phi \cup \{i\} = \varphi(\perp) \vee \varphi(i)\\
\varphi(\perp \wedge i) =& \varphi(\perp) = \phi = \{\perp\} \cap \{i\} = \varphi(\perp) \wedge \varphi(i)\\
\varphi(g \vee h) =& \varphi(d) = \{g, h\} = \{g\} \cup \{h\} = \varphi(g) \vee \varphi(h)\\
\varphi(g \wedge h) =& \varphi(\perp) = \phi = \{g\} \cap \{h\} = \varphi(g) \wedge \varphi(h)\\
\varphi(g \vee i) =& \varphi(e) = \{g, i\} = \{g\} \cup \{i\} = \varphi(g) \vee \varphi(i)\\
\varphi(g \wedge i) =& \varphi(\perp) = \phi = \{\perp\} \cap \{h\} = \varphi(g) \wedge \varphi(i)\\
\varphi(i \vee h) =& \varphi(f) = \{i, h\} = \{i\} \cup \{h\} = \varphi(i) \vee \varphi(h)\\
\varphi(i \wedge h) =& \varphi(\perp) = \phi = \{i\} \cap \{h\} = \varphi(i) \wedge \varphi(h)\\
\varphi(\perp \vee d) =& \varphi(d) = \{g, h\} = \{g, h\} \cup \phi = \varphi(d) \vee \varphi(\perp)\\
\varphi(\perp \wedge d) =& \varphi(\perp) = \phi = \{\perp\} \cap \{g\} = \varphi(\perp) \wedge \varphi(g)
\end{aligned}$$

$$\varphi(\perp \vee e) = \varphi(e) = \{g,i\} = \{g,i\} \cup \phi = \varphi(\perp) \vee \varphi(e)$$

$$\varphi(\perp \wedge e) = \varphi(\perp) = \phi \quad = \{\perp\} \cap \{g,i\} = \varphi(\perp) \wedge \varphi(e)$$

$$\varphi(\perp \vee f) = \varphi(f) = \{h,i\} \quad = \{h,i\} \cup \phi = \varphi(\perp) \vee \varphi(i)$$

$$\varphi(\perp \wedge f) = \varphi(\perp) = \phi \quad = \{\perp\} \cap \{h,i\} = \varphi(\perp) \wedge \varphi(f)$$

$$\varphi(\perp \vee c) = \varphi(c) = \{g,h,i\} \quad = \{g,h,i\} \cup \phi = \varphi(\perp) \vee \varphi(c)$$

$$\varphi(\perp \wedge c) = \varphi(\perp) = \phi \quad = \{\perp\} \cap \{g,h,i\} = \varphi(\perp) \wedge \varphi(c)$$

$$\varphi(\perp \vee a) = \varphi(a) = \{a,g,h,i\} = \phi \cup \{a,g,h,i\} = \varphi(\perp) \vee \varphi(a)$$

$$\varphi(\perp \wedge a) = \varphi(\perp) = \phi \quad = \{\perp\} \cap \{a,gh,i\} = \varphi(\perp) \wedge \varphi(a)$$

$$\varphi(\perp \vee b) = \varphi(b) = \{b,g,h,i\} = \phi \cup \{b,g,h,i\} = \varphi(\perp) \vee \varphi(b)$$

$$\varphi(\perp \wedge b) = \varphi(\perp) = \phi \quad = \{\perp\} \cap \{b,g,h,i\} = \varphi(\perp) \wedge \varphi(b)$$

$$\varphi(\perp \vee \top) = \varphi(\top) = \{a,b,g,h,i\} \quad = \phi \cup \{a,b,g,h,i\} = \varphi(\perp) \vee \varphi(\top)$$

$$\varphi(\perp \wedge \top) = \varphi(\perp) = \phi \quad = \{\perp\} \cap \{a,b,g,h,i\} = \varphi(\perp) \wedge \varphi(\top)$$

$$\varphi(g \vee d) = \varphi(d) = \{g,h\} \quad = \{g\} \cup \{g,h\} = \varphi(g) \vee \varphi(d)$$

$$\varphi(g \wedge d) = \varphi(g) = \{g\} \quad = \{g\} \cap \{g,h\} = \varphi(g) \wedge \varphi(h)$$

$$\varphi(g \vee e) = \varphi(e) = \{g,i\} \quad = \{g\} \cup \{g,i\} = \varphi(g) \vee \varphi(e)$$

$$\varphi(g \wedge e) = \varphi(g) = \{g\} \quad = \{g\} \cap \{g,i\} = \varphi(g) \wedge \varphi(e)$$

$$\varphi(g \vee f) = \varphi(c) = \{g,h,i\} \quad = \{g\} \cup \{h,i\} = \varphi(g) \vee \varphi(f)$$

$$\varphi(g \wedge f) = \varphi(\perp) = \{\phi\} \quad = \{g\} \cap \{h,i\} = \varphi(g) \wedge \varphi(h)$$

$$\varphi(g \vee c) = \varphi(c) = \{g,h,i\} \quad = \{g\} \cup \{h,i\} = \varphi(g) \vee \varphi(c)$$

$$\varphi(g \wedge c) = \varphi(g) = \{g\} \quad = \{g\} \cap \{g,h,i\} = \varphi(g) \wedge \varphi(c)$$

$$\varphi(g \vee a) = \varphi(a) = \{a,g,h,i\} \quad = \{g\} \cup \{a,g,h,i\} = \varphi(g) \vee \varphi(a)$$

$$\varphi(g \wedge a) = \varphi(g) = \{g\} \quad = \{g\} \cap \{a,g,h,i\} = \varphi(g) \wedge \varphi(a)$$

$$\varphi(g \vee b) = \varphi(b) = \{b,g,h,i\} \quad = \{g\} \cup \{b,g,h,i\} = \varphi(g) \vee \varphi(b)$$

$$\varphi(g \wedge b) = \varphi(g) = \{g\} \quad = \{g\} \cap \{b,g,h,i\} = \varphi(g) \wedge \varphi(b)$$

$$\varphi(g \vee \top) = \varphi(\top) = \{a,b,g,h,i\} \quad = \{g\} \cup \{a,b,g,h,i\} = \varphi(g) \vee \varphi(\top)$$

$$\varphi(g \wedge \top) = \varphi(g) = \{g\} \quad = \{g\} \cap \{a,b,g,h,i\} = \varphi(b) \wedge \varphi(\top)$$

$$\varphi(h \vee d) = \varphi(d) = \{g,h\} \quad = \{h\} \cup \{g,h\} = \varphi(h) \vee \varphi(d)$$

$$\varphi(h \wedge d) = \varphi(h) = \{h\} \quad = \{h\} \cap \{g, h\} = \varphi(h) \wedge \varphi(d)$$
$$\varphi(h \vee e) = \varphi(c) = \{g, h, i\} \quad = \{h\} \cup \{g, i\} = \varphi(h) \vee \varphi(e)$$
$$\varphi(h \wedge e) = \varphi(\perp) = \{\phi\} \quad = \{h\} \cap \{g, i\} = \varphi(h) \wedge \varphi(e)$$
$$\varphi(h \vee f) = \varphi(f) = \{h, i\} \quad = \{h\} \cup \{h, i\} = \varphi(h) \vee \varphi(f)$$
$$\varphi(h \wedge f) = \varphi(h) = \{h\} \quad = \{h\} \cap \{h, i\} = \varphi(h) \wedge \varphi(f)$$
$$\varphi(h \vee c) = \varphi(c) = \{g, h, i\} \quad = \{h\} \cup \{g, h, i\} = \varphi(h) \vee \varphi(c)$$
$$\varphi(h \wedge c) = \varphi(h) = \{h\} \quad = \{h\} \cap \{g, h, i\} = \varphi(h) \wedge \varphi(c)$$
$$\varphi(h \vee b) = \varphi(b) = \{b, g, h, i\} \quad = \{h\} \cup \{b, g, h, i\} = \varphi(h) \vee \varphi(b)$$
$$\varphi(h \wedge b) = \varphi(h) = \{h\} \quad = \{h\} \cap \{b, g, h, i\} = \varphi(h) \wedge \varphi(b)$$
$$\varphi(h \vee \top) = \varphi(\top) = \{a, b, g, h, i\} \quad = \{h\} \cup \{a, b, g, h, i\} = \varphi(h) \vee \varphi(\top)$$
$$\varphi(h \wedge \top) = \varphi(h) = \{h\} \quad = \{h\} \cap \{a, b, g, h, i\} = \varphi(h) \wedge \varphi(\top)$$
$$\varphi(i \vee d) = \varphi(c) = \{g, h, i\} \quad = \{i\} \cup \{g, h\} = \varphi(i) \vee \varphi(d)$$
$$\varphi(i \wedge d) = \varphi(\perp) = \{\phi\} \quad = \{i\} \cap \{g, h\} = \varphi(i) \wedge \varphi(d)$$
$$\varphi(i \vee e) = \varphi(e) = \{g, i\} \quad = \{i\} \cup \{g, i\} = \varphi(i) \vee \varphi(e)$$
$$\varphi(i \wedge e) = \varphi(i) = \{i\} \quad = \{i\} \cap \{g, i\} = \varphi(i) \wedge \varphi(e)$$
$$\varphi(i \vee f) = \varphi(f) = \{h, i\} \quad = \{i\} \cup \{h, i\} = \varphi(i) \vee \varphi(f)$$
$$\varphi(i \wedge f) = \varphi(i) = \{i\} \quad = \{i\} \cap \{h, i\} = \varphi(i) \wedge \varphi(f)$$
$$\varphi(i \vee c) = \varphi(c) = \{g, h, i\} \quad = \{i\} \cup \{g, h, i\} = \varphi(i) \vee \varphi(c)$$
$$\varphi(i \wedge c) = \varphi(i) = \{i\} \quad = \{i\} \cap \{g, h, i\} = \varphi(i) \wedge \varphi(c)$$
$$\varphi(i \vee a) = \varphi(a) = \{a, g, h, i\} \quad = \{i\} \cup \{a, g, h, i\} = \varphi(i) \vee \varphi(a)$$
$$\varphi(i \wedge a) = \varphi(i) = \{i\} \quad = \{i\} \cap \{a, g, h, i\} = \varphi(i) \wedge \varphi(a)$$
$$\varphi(i \vee b) = \varphi(b) = \{b, g, h, i\} \quad = \{i\} \cup \{b, g, h, i\} = \varphi(i) \vee \varphi(b)$$
$$\varphi(i \wedge b) = \varphi(i) = \{i\} \quad = \{i\} \cap \{b, g, h, i\} = \varphi(i) \wedge \varphi(b)$$
$$\varphi(i \vee \top) = \varphi(\top) = \{a, b, g, h, i\} \quad = \{i\} \cup \{a, b, g, h, i\} = \varphi(i) \vee \varphi(\top)$$
$$\varphi(i \wedge \top) = \varphi(h) = \{i\} \quad = \{i\} \cap \{a, b, g, h, i\} = \varphi(i) \wedge \varphi(\top)$$
$$\varphi(d \vee e) = \varphi(c) = \{g, h, i\} \quad = \{g, h\} \cup \{g, i\} = \varphi(d) \vee \varphi(e)$$
$$\varphi(d \wedge e) = \varphi(g) = \{g\} \quad = \{g, h\} \cap \{g, i\} = \varphi(d) \wedge \varphi(e)$$

$$\varphi(d \vee f) = \varphi(c) = \{g, h, i\} \quad = \{g, h\} \cup \{h, i\} = \varphi(d) \vee \varphi(f)$$
$$\varphi(d \wedge f) = \varphi(h) = \{h\} \quad = \{g, h\} \cap \{h, i\} = \varphi(d) \wedge \varphi(f)$$
$$\varphi(d \vee c) = \varphi(c) = \{g, h, i\} \quad = \{g, h\} \cup \{g, h, i\} = \varphi(d) \vee \varphi(c)$$
$$\varphi(d \wedge c) = \varphi(d) = \{g, h\} \quad = \{g, h\} \cap \{g, h, i\} = \varphi(d) \wedge \varphi(c)$$
$$\varphi(d \vee a) = \varphi(a) = \{a, g, h, i\} \quad = \{g, h\} \cup \{a, g, h, i\} = \varphi(d) \vee \varphi(a)$$
$$\varphi(d \wedge a) = \varphi(d) = \{g, h\} \quad = \{g, h\} \cap \{a, g, h, i\} = \varphi(d) \wedge \varphi(a)$$
$$\varphi(d \vee b) = \varphi(b) = \{b, g, h, i\} \quad = \{g, h\} \cup \{b, g, h, i\} = \varphi(d) \vee \varphi(b)$$
$$\varphi(d \wedge b) = \varphi(d) = \{g, h\} \quad = \{g, h\} \cap \{b, g, h, i\} = \varphi(d) \wedge \varphi(b)$$
$$\varphi(d \vee \top) = \varphi(\top) = \{a, b, g, h, i\} \quad = \{g, h\} \cup \{a, b, g, h, i\} = \varphi(d) \vee \varphi(\top)$$
$$\varphi(d \wedge \top = \varphi(d) = \{g, h\} \quad = \{g, h\} \cap \{a, b, g, h, i\} = \varphi(d) \wedge \varphi(\top)$$
$$\varphi(e \vee f) = \varphi(c) = \{g, h, i\} \quad = \{g, i\} \cup \{h, i\} = \varphi(e) \vee \varphi(f)$$
$$\varphi(e \wedge f) = \varphi(i) = \{i\} \quad = \{g, i\} \cap \{h, i\} = \varphi(e) \wedge \varphi(f)$$
$$\varphi(e \vee c) = \varphi(c) = \{g, h, i\} \quad = \{g, i\} \cup \{g, h, i\} = \varphi(e) \vee \varphi(c)$$
$$\varphi(e \wedge c) = \varphi(e) = \{g, i\} \quad = \{g, i\} \cap \{g, h, i\} = \varphi(e) \wedge \varphi(c)$$
$$\varphi(e \vee a) = \varphi(a) = \{a, g, h, i\} \quad = \{g, i\} \cup \{a, g, h, i\} = \varphi(e) \vee \varphi(a)$$
$$\varphi(e \wedge a) = \varphi(e) = \{g, i\} \quad = \{g, i\} \cap \{a, g, h, i\} = \varphi(e) \wedge \varphi(a)$$
$$\varphi(e \vee b) = \varphi(b) = \{b, g, h, i\} \quad = \{g, i\} \cup \{b, g, h, i\} = \varphi(e) \vee \varphi(b)$$
$$\varphi(e \wedge b) = \varphi(e) = \{g, i\} \quad = \{g, i\} \cap \{b, g, h, i\} = \varphi(e) \wedge \varphi(b)$$
$$\varphi(e \vee \top) = \varphi(\top) = \{a, b, g, h, i\} \quad = \{g, i\} \cup \{a, b, g, h, i\} = \varphi(e) \vee \varphi(\top)$$
$$\varphi(e \wedge \top) = \varphi(e) = \{g, i\} \quad = \{g, i\} \cap \{a, b, g, h, i\} = \varphi(e) \wedge \varphi(\top)$$
$$\varphi(f \vee c) = \varphi(c) = \{g, h, i\} \quad = \{h, i\} \cup \{g, h, i\} = \varphi(f) \vee \varphi(c)$$
$$\varphi(f \wedge c) = \varphi(f) = \{h, i\} \quad = \{h, i\} \cap \{g, h, i\} = \varphi(f) \wedge \varphi(c)$$
$$\varphi(f \vee a) = \varphi(a) = \{a, g, h, i\} \quad = \{h, i\} \cup \{a, g, h, i\} = \varphi(f) \vee \varphi(a)$$
$$\varphi(f \wedge a) = \varphi(f) = \{h, i\} \quad = \{h, i\} \cap \{a, g, h, i\} = \varphi(f) \wedge \varphi(a)$$
$$\varphi(f \vee b) = \varphi(b) = \{b, g, h, i\} \quad = \{h, i\} \cup \{b, g, h, i\} = \varphi(f) \vee \varphi(b)$$
$$\varphi(f \wedge b) = \varphi(f) = \{h, i\} \quad = \{h, i\} \cap \{b, g, h, i\} = \varphi(f) \wedge \varphi(b)$$

$$\begin{aligned}
\varphi(f \vee \top) = \varphi(\top) = \{a,b,g,h,i\} &= \{h,i\} \cup \{a,b,g,h,i\} = \varphi(f) \vee \varphi(\top) \\
\varphi(f \wedge \top) = \varphi(f) = \{h,i\} &= \{h,i\} \cap \{a,b,g,h,i\} = \varphi(f) \wedge \varphi(\top) \\
\varphi(c \vee a) = \varphi(a) = \{a,g,h,i\} &= \{g,h,i\} \cup \{a,g,h,i\} = \varphi(c) \vee \varphi(a) \\
\varphi(c \wedge a) = \varphi(c) = \{g,h,i\} &= \{g,h,i\} \cap \{a,g,h,i\} = \varphi(c) \wedge \varphi(a) \\
\varphi(c \vee b) = \varphi(b) = \{b,g,h,i\} &= \{h,i\} \cup \{b,g,h,i\} = \varphi(c) \vee \varphi(b) \\
\varphi(c \wedge b) = \varphi(c) = \{g,h,i\} &= \{g,h,i\} \cap \{b,g,h,i\} = \varphi(c) \wedge \varphi(b) \\
\varphi(c \vee \top) = \varphi(\top) = \{a,b,g,h,i\} &= \{h,i\} \cup \{a,b,g,h,i\} = \varphi(c) \vee \varphi(\top) \\
\varphi(c \wedge \top) = \varphi(c) = \{g,h,i\} &= \{g,h,i\} \cap \{a,b,g,h,i\} = \varphi(c) \wedge \varphi(\top) \\
\varphi(a \vee b) = \varphi(\top) = \{a,b,g,h,i\} &= \{a,g,h,i\} \cup \{b,g,h,i\} = \varphi(a) \vee \varphi(b) \\
\varphi(a \wedge b) = \varphi(c) = \{g,h,i\} &= \{a,g,h,i\} \cap \{b,g,h,i\} = \varphi(a) \wedge \varphi(b) \\
\varphi(a \vee \top) = \varphi(\top) = \{a,b,g,h,i\} &= \{a,g,h,i\} \cup \{a,b,g,h,i\} = \varphi(a) \vee \varphi(\top) \\
\varphi(a \wedge \top) = \varphi(a) = \{a,g,h,i\} &= \{a,g,h,i\} \cap \{a,b,g,h,i\} = \varphi(a) \wedge \varphi(\top) \\
\varphi(b \vee \top) = \varphi(\top) = \{a,b,g,h,i\} &= \{b,g,h,i\} \cup \{a,b,g,h,i\} = \varphi(b) \vee \varphi(\top) \\
\varphi(b \wedge \top) = \varphi(b) = \{b,g,h,i\} &= \{b,g,h,i\} \cap \{a,b,g,h,i\} = \varphi(b) \wedge \varphi(\top)
\end{aligned}$$

Therefore the map $\varphi : L \to P(L)$ is an isomorphism and by Theorem 7.4, the lattice L is distributive. $\square$

Chapter 8

Representation Theorem: the General Case

The crucial Theorem 7.4 and its most important consequence, Corollary 7.5 depend of the existence of sufficiently many join-irreducible elements. In an infinite distributive lattice, there may be no join-irreducible element.

Example 8.1. *Let L be a lattice whose elements are finite unions of (a, b)'s where $a, b \in \mathbb{R}$ and $(a, a) = \phi$. The relation "$\leq$" is inclusion and meet and join are union and intersection. Clearly this lattice is closed under $\cup$ and $\cap$, since union of finite sets is again a finite set and intersection of finite sets is a finite set or empty set both in our lattice L. The lattice is distributive, since meet and join are union and intersection. But in this lattice there are no join-irreducible elements. For that consider the element (a, b). Since $(a, b) = (a, c) \cup (c', b)$ where $a < c' < c < b$ and neither $(a, b) = (a, c)$ nor $(a, b) = (c', b)$, no element of the form $(a, b) \in L$ is join-irreducible. By the same argument no finite union $(a_1, a_2) \cup (a_3, a_4) \cup \ldots \cup (a_{n-1}, a_n)$ is join-irreducible either.*

In the infinite case, the role of join-irreducible elements is taken over by prime ideals. The crucial result is the existence of sufficiently many prime ideals.

To prove the next theorem some form of the Axiom of Choice is needed. Recall the Axiom of Choice says that given an arbitrary collection of nonempty sets, we may choose an element from each of them, without any sort of rule, method, or scheme to do so. The most convenient form for this proof is:

Lemma 8.2. ***Zorn's Lemma.*** (Grätzer, [2]) *Let A be a set and let $\mathcal{X}$ be a non-empty subset of $\mathcal{P}(A)$. Let us assume that $\mathcal{X}$ has the following property: If C is a chain in $\langle \mathcal{X}; \subseteq \rangle$, then $\bigcup C \in \mathcal{X}$. Then $\mathcal{X}$ has a maximal member.*

Theorem 8.3. (M. H. Stone [1936]) (Grätzer, [2])

Let L be a distributive lattice, let I be an ideal, let D be a dual ideal of L, and let $I \cap D = \phi$. Then there exists a prime ideal P of L such that $P \supseteq I$ and $P \cap D = \phi$.

Proof. Let $\mathcal{X}$ be the set of all ideals of L that contain I and are disjoint of D. Since $I \subseteq \mathcal{X}$, $\mathcal{X}$ is not empty. Given chain C in $\mathcal{X}$, we need to show $\bigcup C \in \mathcal{X}$. If we write $M = \bigcup C$, must show:

(1) M is an ideal of L,

(2) M contains I,

(3) M disjoint from D.

Let $a, b \in M$. That implies $a \in X$, $b \in Y$ for some $X, Y \in C$. But C is a chain, so $X \subseteq Y$ or $Y \subseteq X$. Assume $X \subseteq Y$, then $a \in Y$ and $b \in Y$. Since Y is an ideal, $a \vee b \in Y$. That implies $a \vee b \in M$. If $b \leq a$, $a \in M$ $b \in L$ then $a \in X$ implies $b \in X \subseteq M$. Thus, $b \in M$ and by definition M is an ideal. Since members of C contain I, M contains I. To show that M is disjoint from D, suppose $\exists y \in M \cap D$. That means $\exists j$ such that $y \in C_j$. Since $y \in D$, we have $y \in C_j \cap D$ and that contradicts the hypothesis. Thus $M \cap D = \phi$. Therefore by Zorn's lemma, $\mathcal{X}$ has a maximal element P. We claim that P is a prime ideal. First we check if P is an ideal. Since $P \in \mathcal{X}$ and $\mathcal{X}$ is a subset of all ideals of L, P is an ideal. Next suppose that P is not a prime ideal. That means there exist $a, b \notin P$ such that $a \wedge b \in P$. Consider $P \vee (a]$ where $(a] = \downarrow a$. $P \vee (a]$ is not disjoint from D since if so, $P \subsetneqq P \vee (a]$ and that contradicts the maximality of P. Thus $(P \vee (a]) \cap D \neq \phi$ and $(P \vee (b]) \cap D \neq \phi$. That implies there exist $p, q \in P$ such that $p \vee a \in D$ and $q \vee b \in D$. Since D is a dual ideal, $x = (p \vee a) \wedge (q \vee b) \in D$. But

$$\begin{aligned}
x =& ((p \vee a) \wedge q) \vee ((p \vee a) \wedge b) && \text{(since } L \text{ is a distributive lattice)} \\
=& (p \wedge q) \vee (a \wedge q) \vee (p \wedge b) \vee (a \wedge b) && \text{(since } L \text{ is a distributive lattice)} \\
=& (p \wedge q) \vee (p \wedge b) \vee (a \wedge q) \vee (a \wedge b) && \text{(associative and commutative laws)}
\end{aligned}$$

and since $p \wedge q \in P$, $p \wedge b \in P$, $a \wedge q \in P$ and $a \wedge b \in P$, we have that $x \in P$. Thus $P \cap D \neq \phi$ and that is a contradiction. Therefore P is a prime ideal. Now since $I \in X$ for all $X \in \mathcal{X}$ and P is the maximal element of $\mathcal{X}$, $I \in P$. All elements of $\mathcal{X}$ are disjont from D and thus, P is also disjoint from D. $\square$

Corollary 8.4. *Let L be a distributive lattice, let I be an ideal of L, and let $a \in L$ and $a \notin I$. Then there is a prime ideal P such that $P \supseteq I$ and $a \notin P$.*

Proof. Let $a \in L \backslash I$ and let $D = \{x \in L | x \geq a\} = \uparrow a$. Show $I \cap D = \phi$. Let $\hat{x} \in D$ which implies $\hat{x} \geq a$. Suppose $\hat{x} \in I$. Then since I is an ideal $a \in I$ which contradicts the hypothesis. Thus $\hat{x} \notin I$ and $I \cap D = \phi$. So, $D = \uparrow a$ is a dual ideal of L, I is an ideal of L, L is a distributive lattice and $I \cap D = \phi$. Therefore by Theorem 8.3 there exists an ideal P of L such that $P \supseteq I$ and $P \cap D = \phi$. But $a \in D$ and that implies $a \notin P$. $\square$

Corollary 8.5. *Let L be a distributive lattice, $a, b \in L$ and $a \neq b$. Then there is a prime ideal containing exactly one of a and b.*

Proof. Suppose $a \nleq b$. Let $D = \uparrow a$ be dual ideal of L with $a \in D$ and therefore, $b \notin D$. Let $I = \downarrow b$ be an ideal of L, and note $a \notin I$ and $b \in I$. By Theorem 8.3, $I \cap D = \phi$ and $\exists P$ prime ideal of L such that $P \supseteq I$ and $P \cap D = \phi$. Since $P \supseteq I$ and $b \in I$ we have $b \in P$. Thus, since $a \in D$ and $P \cap D = \phi$, $a \notin P$. A similar argument holds if $b \nleq a$. Therefore there exists prime ideal of L containing exactly one of a and b. $\square$

Theorem 8.6. (G. Birkhoff [1933] and M. H. Stone [1936]) (Grätzer, [2])
A lattice is distributive if and only if it is isomorphic to a ring of sets.

Proof. Let $I_P(L)$ denote the set of prime ideals of L.

($\Rightarrow$) Let L be a lattice and let

$$\varphi : L \to \mathcal{P}(I_P(L)), \quad a \mapsto \{P \mid a \notin P,\ P \in I_P(L)\}$$

We need to show that φ is one-to-one and preserves meet and join. If $a \neq b$ in L, by Corollary 8.5 there exists $Q \in I_P(L)$ for which we may assume $a \in Q$ and $b \notin Q$. Therefore, $Q \in \varphi(b)$ but $Q \notin \varphi(a)$, which implies $\varphi(a) \neq \varphi(b)$. Therefore φ is a one-to-one function.

Let $a, b \in L$. To show that φ preserves join we need to show that $\varphi(a \vee b) = \varphi(a) \vee \varphi(b)$. Let $P \in I_P(L)$. We first need to show that $a \vee b \notin P \leftrightarrow a \notin P$ or $b \notin P$. The contrapositive of this is: $a \vee b \in P \leftrightarrow a \in P$ and $b \in P$. Now if $a \vee b \in P$ and $a \leq a \vee b$ and $b \leq a \vee b$, since P is an ideal and is closed going down, $a \in P$ and $b \in P$. Now assume that $a, b \in P$. Then, since P is closed under join $a \vee b \in P$. Therefore $a \vee b \notin P$ if and only if $a \notin P$ or $b \notin P$. Now:

$$\begin{aligned}\varphi(a \vee b) =&\{P \mid (a \vee b) \notin P, P \in I_P(L)\}\\ =&\{Q \mid a \notin Q, Q \in I_P(L)\} \cup \{R \mid b \notin R, R \in I_P(L)\}\\ =&\varphi(a) \vee \varphi(b)\end{aligned}$$

To show that φ preserves meet we need to show that $\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b)$. We first need to show that $a \wedge b \notin P \leftrightarrow a \notin P$ and $b \notin P$. The contrapositive of this is: $a \wedge b \in P \leftrightarrow a \in P$ or $b \in P$. Now if $a \wedge b \in P$, since P is a prime ideal, $a \in P$ or $b \in P$. Now assume that a or b is in P. Then, since P is closed going down $a \wedge b \in P$. Therefore $a \wedge b \notin P$ if and only if $a \notin P$ and $b \notin P$. Now:

$$\begin{aligned}\varphi(a \wedge b) =&\{P \mid (a \wedge b) \notin P, P \in I_P(L)\}\\ =&\{Q \mid a \notin Q, Q \in I_P(L)\} \cap \{R \mid b \notin R, R \in I_P(L)\}\\ =&\varphi(a) \wedge \varphi(b)\end{aligned}$$

Since φ is an injective lattice homomorphism, whose image is a sublattice of $\mathcal{P}(I_P(L))$, L is isomorphic to a ring of sets.

($\Leftarrow$) By the Remark 7.3 any ring of sets is distributive and therefore, any lattice isomorphic to a ring of sets is itself distributive. □

In the next two examples I will illustrate the isomorphism from Theorem 8.6

Example 8.7. *The chain $\mathbb{Z}$ is a distributive lattice.*

Proof. By Example 3.5 every chain is a lattice and by Example 6.7 every chain is distributive. Since $\mathbb{Z}$ is a chain, it follows that $\mathbb{Z}$ is a distributive lattice. Hence by Theorem 8.6 $\mathbb{Z}$ is isomorphic to a ring of sets.

Ones can check that all proper ideals in the chain $\mathbb{Z}$ are of the form $\downarrow x$, $\forall x \in \mathbb{Z}$. In fact if I is a proper ideal of chain $\mathbb{Z}$, let $w \notin I$. Take x the greatest element of I so that $x < w$. Then $I = \downarrow x$. The set $\downarrow x$ is a prime ideal since, if $a, b \in \downarrow x$ then

$a \vee b = \max\{a, b\} \in \downarrow x$. If $a \in \downarrow x$ and $b \in \mathbb{Z}$, then $a \wedge b = \min\{a, b\} \in \downarrow x$. Let $a \wedge b \in \downarrow x$. If $a < b$ then $a \in \downarrow x$ and if $b < a$ then $b \in \downarrow x$. Therefore all proper prime ideals are of the form $\downarrow x$, $x \in \mathbb{Z}$. Let X be the set of all prime ideals of $\mathbb{Z}$. For $a \in \mathbb{Z}$ define $\varphi \langle \mathbb{Z}; \leq \rangle \rightarrow \mathcal{P}(X)$ such that $\varphi(a) = \{P \mid a \notin P, P \in X\}$. For example:

$$\begin{aligned} 2 \mapsto & \{\downarrow a \mid 2 \notin \downarrow a, a \in \mathbb{Z}\} \\ = & \{\downarrow 1, \downarrow 0, \downarrow -1, \dots,\} \\ 5 \mapsto & \{\downarrow a \mid 5 \notin \downarrow a, a \in \mathbb{Z}\} \\ = & \{\downarrow 4, \downarrow 3, \downarrow 2, \dots,\} \end{aligned}$$

Observe that φ is one-to-one and preserves meet and join. Let $\varphi(a) = \varphi(b)$. Then

$$\{P \mid a \notin P, P \in X\} = \{P \mid b \notin P, P \in X\}$$
$$\{\downarrow (a-1), \downarrow (a-2), \downarrow (a-3), \dots,\} = \{\downarrow (b-1), \downarrow (b-2), \downarrow (b-3), \dots,\}$$

Thus we must have $a - 1 = b - 1$. Hence $a = b$ and the function is one-to-one. Now observe that φ preserves join and meet. For that let $a, b \in \mathbb{Z}$. Since $\mathbb{Z}$ is a chain we have three cases: $a = b$, $a > b$ and $b > a$. In the first case, if $a = b$,

$$\begin{aligned} \varphi(a \vee b) = & \varphi(a) \\ = & \{\downarrow (a-1), \downarrow (a-2), \downarrow (a-3), \dots,\} \\ = & \{\downarrow (a-1), \downarrow (a-2), \downarrow (a-3), \dots,\} \cup \{\downarrow (a-1), \downarrow (a-2), \downarrow (a-3), \dots,\} \\ = & \varphi(a) \vee \varphi(a) \\ = & \varphi(a) \vee \varphi(b) \end{aligned}$$

and

$$\begin{aligned} \varphi(a \wedge b) = & \varphi(a) \\ = & \{\downarrow (a-1), \downarrow (a-2), \downarrow (a-3), \dots,\} \\ = & \{\downarrow (a-1), \downarrow (a-2), \downarrow (a-3), \dots,\} \cap \{\downarrow (a-1), \downarrow (a-2), \downarrow (a-3), \dots,\} \\ = & \varphi(a) \wedge \varphi(a) \\ = & \varphi(a) \wedge \varphi(b) \end{aligned}$$

If $a > b$

$$
\begin{aligned}
\varphi(a \vee b) =& \varphi(a) \\
=& \{\downarrow (a-1), \downarrow (a-2), \downarrow (a-3), \ldots, \} \\
=& \{\downarrow (a-1), \downarrow (a-2), \downarrow (a-3), \ldots, \} \cup \{\downarrow (b-1), \downarrow (b-2), \downarrow (b-3), \ldots, \} \\
=& \varphi(a) \vee \varphi(b)
\end{aligned}
$$

and

$$
\begin{aligned}
\varphi(a \wedge b) =& \varphi(b) \\
=& \{\downarrow (b-1), \downarrow (b-2), \downarrow (b-3), \ldots, \} \\
=& \{\downarrow (a-1), \downarrow (a-2), \downarrow (a-3), \ldots, \} \cap \{\downarrow (b-1), \downarrow (b-2), \downarrow (b-3), \ldots, \} \\
=& \varphi(a) \wedge \varphi(b)
\end{aligned}
$$

A similar argument holds for $b > a$. □

Example 8.8. *The lattice $L = \langle \mathbb{N}_0;\ lcm;\ gcd \rangle$ is distributive.*

Proof. By example 3.8, L is a lattice and by example 6.11, L is distributive. Hence by Theorem 8.6, L is isomorphic to a ring of sets.

To illustrate Theorem 8.6, first we need to determine prime ideals I_P and prime filters F_P of L. Recall that a prime ideal (filter) must satisfy:

(i) closed with respect to $\vee$ $(\wedge)$,

(ii) closed with respect to down (up),

(iii) If $a \wedge b \in I$ $(a \vee b \in F)$ then $a \in I$ or $b \in I$ $(a \in F$ or $b \in F)$.

Also recall that if $I_P \subseteq L$ is a prime ideal then $L \setminus I_P$ is a prime filter and if $F \subseteq L$ is a prime filter then $L \setminus F$ is a prime ideal. Let $F \subseteq L$ be a prime filter. If $a, b \in F$ and $\gcd(a, b) = 1$, then $1 \in F$ and F is not proper filter. Assume F is prime and proper. We claim that F consists of mp^k, for a fixed, p prime, $k \geq 1$ and all $m \in \mathbb{N}$. For exemple $\uparrow 3$ and $\uparrow 3^2$ are prime filters. To prove that $\uparrow p^k$ is a filter let $a, b \in \uparrow p^k$. Then $a \geq p^k$ and $b \geq p^k$ which implies $p|a$ and $p|b$. Thus p is a common divisor of a and b and therefore p must divide the greatest common divisor of a and b. That means $p|(a \wedge b)$ and $a \wedge b \in \uparrow p^k$.

If $a \in \uparrow p^k$, $b \in L$ and $a \leq b$ we have that $p^k|a$, $a|b$ and thus $p^k|b$ which implies $b \in \uparrow p^k$. Therefore $\uparrow p^k$ is a filter. To prove that $\uparrow p^k$ is a prime filter, let $a \vee b \in \uparrow p^k$. That implies $p|\text{lcm}(a, b$. Since p divides the lowest common multiple of a and b, then $p|a$ or $p|b$ which implies $a \in \uparrow p^k$ or $b \in \uparrow p^k$ and therefore $\uparrow p^k$ is a prime filter.

Therefore every proper prime filter F is of the form $F = \uparrow p^k$ for some $k \geq 1$.

For example: the proper prime filters that contain p^5 are

$$\uparrow p^5, \uparrow p^4, \uparrow p^3, \uparrow p^2, \uparrow p.$$

the proper prime filters that contain p^5q^3, p, q primes, are

$$\uparrow p^5, \uparrow p^4, \uparrow p^3, \uparrow p^2, \uparrow p, \uparrow q^3, \uparrow q^2, \uparrow q.$$

Note that $\uparrow p^iq^j \subseteq \uparrow p^i$ and $\uparrow p^iq^j \subseteq \uparrow q^j$. Also $p^iq^j = p^i \vee q^j$ since $\text{lcm}(p^i, q^j) = p^iq^j$. Thus $p^i \vee q^j \in \uparrow p^iq^j$ and if $\uparrow p^iq^j$ is prime, either $p^i \in \uparrow p^iq^j$ or $q^j \in \uparrow p^iq^j$. Therefore either $\uparrow p^i = \uparrow p^iq^j$ or $\uparrow q^j = \uparrow p^iq^j$. If $a = p_1^{e_1} \cdot \ldots \cdot p_n^{e_n}$ then the proper prime filters containing a are $\uparrow p_i^{e_i}, \ldots, \uparrow p_i$ for $i = 1, 2, \ldots n$.

By the comment above and duality principle, $\mathbb{N}_0 \backslash \uparrow p$ is a prime ideal and by analogy all proper prime ideals not containing a, where $a = p_1^{e_1} \cdot \ldots \cdot p_n^{e_n}$, are

$$\mathbb{N}_0 \backslash \uparrow p_i^{e_i}, \ldots, \mathbb{N}_0 \backslash \uparrow p_i \quad (i = 1, \ldots, n)$$

For example all proper prime ideals not containing p^5 are

$$\mathbb{N}_0 \backslash \uparrow p^5,\ \mathbb{N}_0 \backslash \uparrow p^4,\ \mathbb{N}_0 \backslash \uparrow p^3,\ \mathbb{N}_0 \backslash \uparrow p^2,\ \mathbb{N}_0 \backslash \uparrow p$$

Therefore the Stone map will be

$$a \mapsto \varphi(a) = \{\mathbb{N}_0 \backslash \uparrow p_i^{e_i}, \ldots, \mathbb{N}_0 \backslash \uparrow p_i, \quad i = 1, \ldots, n\}$$

$$1 \mapsto \{\ \}$$

For example

$$\begin{aligned}
2 \mapsto &\{\mathbb{N}_0 \backslash \uparrow 2\} \\
3 \mapsto &\{\mathbb{N}_0 \backslash \uparrow 3\} \\
4 \mapsto &\{\mathbb{N}_0 \backslash \uparrow 2^2,\ \mathbb{N}_0 \backslash \uparrow 2\} \\
6 \mapsto &\{\mathbb{N}_0 \backslash \uparrow 2,\ \mathbb{N}_0 \backslash \uparrow 3\}
\end{aligned}$$

$$12 \mapsto \{\mathbb{N}_0 \backslash \uparrow 2,\ \mathbb{N}_0 \backslash \uparrow 2^2,\ \mathbb{N}_0 \backslash \uparrow 3\}$$

□

We see that join = lcm and meet = gcd in $\mathbb{N}_0$, correspond to the simpler distributive operations of union and intersection in $\mathcal{P}(I_P(\mathbb{N}_0))$. For example:

$$\begin{aligned}\varphi(2 \vee 3) &= \varphi(6)\\ &= \{\mathbb{N}_0 \backslash \uparrow 2,\ \mathbb{N}_0 \backslash \uparrow 3\}\\ &= \{\mathbb{N}_0 \backslash \uparrow 2\} \cup \{\mathbb{N}_0 \backslash \uparrow 3\}\\ &= \varphi(2) \vee \varphi(3)\end{aligned}$$

and

$$\begin{aligned}\varphi(2 \wedge 3) &= \varphi(1)\\ &= \{\}\\ &= \{\mathbb{N}_0 \backslash \uparrow 2\} \cap \{\mathbb{N}_0 \backslash \uparrow 3\}\\ &= \varphi(2) \wedge \varphi(3)\end{aligned}$$

or

$$\begin{aligned}\varphi(4 \vee 6) &= \varphi(12)\\ &= \{\mathbb{N}_0 \backslash \uparrow 2,\ \mathbb{N}_0 \backslash \uparrow 2^2,\ \mathbb{N}_0 \backslash \uparrow 3\}\\ &= \{\mathbb{N}_0 \backslash \uparrow 2,\ \mathbb{N}_0 \backslash \uparrow 2^2\} \cup \{\mathbb{N}_0 \backslash \uparrow 3,\ \mathbb{N}_0 \backslash \uparrow 2\}\\ &= \varphi(4) \vee \varphi(6)\end{aligned}$$

and

$$\begin{aligned}\varphi(4 \wedge 6) &= \varphi(2)\\ &= \{\mathbb{N}_0 \backslash \uparrow 2\}\\ &= \{\mathbb{N}_0 \backslash \uparrow 2,\ \mathbb{N}_0 \backslash \uparrow 2^2\} \cap \{\mathbb{N}_0 \backslash \uparrow 2,\ \mathbb{N}_0 \backslash \uparrow 3\}\\ &= \varphi(4) \wedge \varphi(6)\end{aligned}$$

Chapter 9

Conclusion

I focused my thesis on the work of Birkhoff and Stone about distributive lattices and their representations. Distributive lattices have played a very important role in the development of lattice theory. For long time mathematicians believed that every lattice is distributive. This was finally cleared when Garrett Birkhoff, in the early thirties proved that every finite distributive lattice is isomorphic to the lattice of order ideals of a partially ordered set. This representation theorem becomes a powerful asset in the study of finite distributive lattices. Just a few years later, both, Birkhoff and Stone proved that, in fact, every distributive lattice is isomorphic to a ring of sets. This was a fundamental step forward in mathematics. The result provides a systematic and useful translation of the combinatorics of partially ordered sets into the algebra of distributive lattice. Lattice theory started with distributive lattices. Many great results in general lattice theory are provided by the work on distributive lattices. Many conditions on lattices and on elements and ideals of lattices are weakened forms of distributivity. Therefore, a good knowledge of distributive lattices is indispensable for work in lattice theory. Finally, in many applications the condition of distributivity is imposed on lattices arising in various areas of mathematics, especially algebras.

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