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# INVESTIGATION OF 4-CUTWIDTH CRITICAL GRAPHS

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A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

---

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

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by

Dolores Chavez

June 2006

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June 2006

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## ABSTRACT

The 2004 article, “On 3-cutwidth critical graphs”, by Yixun Lin and Aifeng Yang, characterized the set of 3-cutwidth critical graphs by five specified elements. In this project we extend this idea to 4-cutwidth critical graphs. We will demonstrate the methods used in finding these graphs and also illustrate some of the things they have in common. Unlike the 3-cutwidth critical case we verify that there are infinitely many 4-cutwidth critical graphs. We also provide a dictionary of the 4-cutwidth critical graphs.

## ACKNOWLEDGEMENTS

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Another asset in this process has been my committee, to whom I owe many thanks for their support. I have also been very fortunate to work with the faculty here. Each and every one of them has been kind and patient, including those that were not my professors. I want to thank them all for their help and insight. Last, but not least, I must extend my deepest gratitude to everyone else in the Mathematics department. The staff and students have been very encouraging. It has been my pleasure working with them throughout these years.

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# Chapter 1

## Introduction

### 1.1 Background

Graphs are useful tools for analyzing situations involving a set of elements in which various pairs of elements are related by some property. They are used to solve problems in many fields. We can distinguish between two chemical compounds with the same molecular formula but different structures using graphs. Examples of graphs that have physical links are electrical networks, where the electrical components are the vertices and connecting wires are the edges. Graphs can also be seen as sets with logical sequencing, such as computer flow charts, where the instructions are the vertices and the logical flow from one instruction to other successor instruction(s) defines the edges. Other examples can be evolutionary trees in biology, computer data structures, street maps, telephone networks and the scheduling of tasks in a complex project.[ROS91]

In a theoretical aspect, graph labeling plays a major role in understanding and interpreting graphs. The bandwidth, pathwidth, treewidth and cutwidth problems are problems in which one must number, or label, the vertices of a given graph in order to optimize some parameter. This paper will concentrate on the cutwidth of graphs, which involves finding a labeling of the vertices of a graph so that the maximum number of edges

between consecutive vertices is minimized.

## 1.2 General Graph Theory Definitions

The following set of definitions will be used throughout this paper. The actual picture we draw of a graph is called a graph diagram. A graph is called connected if we can get from any vertex to any other vertex by traveling along the edges of the graph.[TUC95]

**Definition 1.1.** *A graph,  $G$ , is a structure with a set of vertices,  $V$ , and a set of edges,  $E$ , which join pairs of distinct vertices.*

Let  $G = (V, E)$  be denote the graph with vertex set  $V$ ,  $|V| = n$ , and edge set  $E$ . For example, let  $A_1$  be a graph with 11 vertices and 12 edges, see Figure 1.1.

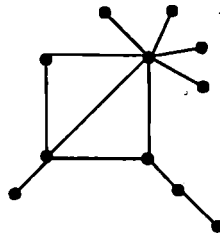


Figure 1.1:  $A_1$

This paper will be restricted to simple graphs. That is, we do not allow our graphs to have more than one edge connecting two vertices, ie, multiple edges. Further, an edge cannot “loop” so that both ends terminate at the same vertex. All graphs in this project are simple graphs. For example, the graph shown in Figure 1.2 is not part of this project because it has a loop and multiple edges.[FS96]

**Definition 1.2.** *A path is a sequence of edges where successive edges share a vertex.*

**Definition 1.3.** *A labeling of a graph,  $G = (V, E)$  with  $|V| = n$ , is a bijection  $f : V \rightarrow \{1, \dots, n\}$ , which can be regarded as an embedding of  $G$  onto a path  $P_n$ .*

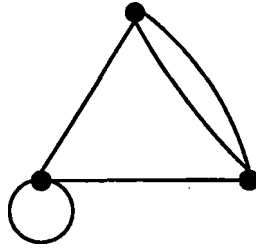


Figure 1.2: Graph with loop and multiple edges

A possible labeling for  $A_1$  is shown below in Figure 1.3:

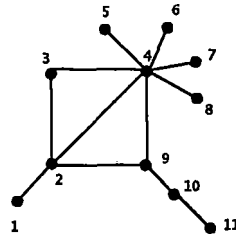


Figure 1.3:  $A_1$  with a labeling

**Definition 1.4.** *The degree of vertex  $x$ , denoted  $\deg(x)$ , is the number of edges incident to the vertex.*

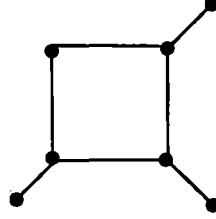
For example, in Figure 1.3, the vertex of highest degree is  $d$  because it has 7 incident edges. So, in  $A_1$ ,  $\deg(d) = 7$ .

**Definition 1.5.** *A subgraph is a graph formed by a subset of vertices and edges of a graph.*

In Figure 1.4 we have a subgraph of  $A_1$ , call it  $A'_1$ .

### 1.3 Cutwidth

The cutwidth problem is one of several types of graph labeling problems studied in combinatorics. In general, a graph labeling problem is concerned with finding an optimal

Figure 1.4:  $A'_1$ 

way to label the vertices of a graph. [TUC95]

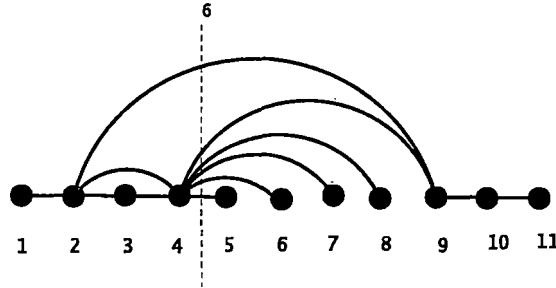
**Definition 1.6.** For a given labeling  $f$  of  $G$ , the cutwidth of  $G$  with respect to  $f$  is

$$c(G, f) = \max_{1 \leq i < n} |uv \in E : f(u) \leq i < f(v)|,$$

which represents the congestion of the linear embedding.

Since a labeling  $f$  of a graph  $G$  is a bijection, then  $f$  is a one-to-one and onto function. So, each vertex on the domain graph  $G$  has a corresponding vertex on the range graph, or linear embedding of  $G$ . Similarly, each edge has a corresponding path.

For example, in the linear embedding of  $A_1$  the  $c(A_1, f) = 6$ , where  $f$  is the labeling from Figure 1.3. The linear embedding can be seen in Figure 1.5.

Figure 1.5: Linear Embedding of  $A_1$ 

To get a better understanding of cutwidth we introduce the following. A vertex 1 is adjacent to 2 when there is an edge from 1 to 2. We also say that neighboring vertices are

vertices that are listed or drawn next to each other on the linear embedding of a graph  $G$ , whether they are connected by an edge or not. For example, in Figure 1.5, vertices 2 and 3 are both adjacent and neighbor vertices. This is because they are both connected by an edge and are also drawn next to each other on the linear embedding. Vertices 2 and 9 are adjacent vertices because of the edge that connects them, but they are not neighbor vertices since they are not drawn next to each other on the linear embedding of  $A_1$ . Vertices 8 and 9 are neighbor vertices on the linear embedding, but they are not adjacent because there does not exist an edge from vertex 8 to vertex 9. The cut of a graph is given between any neighboring vertices. The cutwidth of a graph with respect to a particular labeling,  $f$ , is the maximum cut. The cutwidth of a graph is the minimum of the cutwidths over all numberings. More precisely the cutwidth of a graph is defined below.

**Definition 1.7.** *The cutwidth of  $G$ ,  $c(G)$ , is defined by*

$$c(G) = \min_f c(G, f),$$

*where the minimum is taken over all labelings  $f$ .*

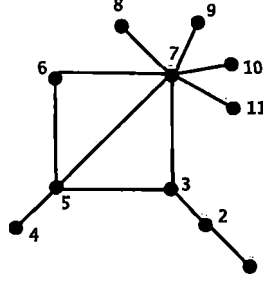
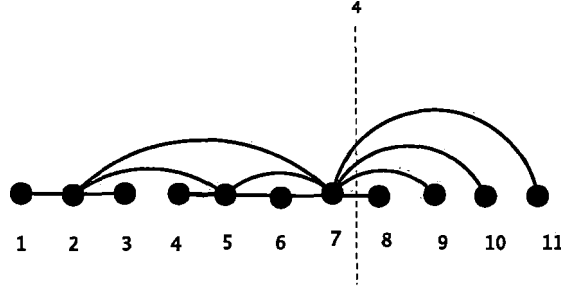
A labeling  $f$  attaining the above minimum value is called an optimal labeling.

The labeling of  $A_1$ , or any graph, is not unique. By changing the labeling of the vertices of a graph, we may change the cutwidth. We can relabel  $A_1$ , as shown in Figure 1.6, to obtain a minimum value for the cutwidth. Let  $g$  represent the new labeling.

Figure 1.7 shows the linear embedding of  $A_1$  under the new relabeling,  $g$ . Using this figure we clearly see that  $c(A_1, g) = 4$ .

## 1.4 Cutwidth Critical

In this project we will be looking for graphs that are 4-cutwidth critical. Before we look at those properties, let's look at the general case of what it means for a graph to

Figure 1.6:  $A_1$  relabeledFigure 1.7: Linear embedding of  $A_1$  after relabeling

be  $k$  – cutwidth critical. First, the following definitions are required.

**Definition 1.8.** A graph  $G'$  is a subdivision of  $G$  if they can both be obtained from the same graph by inserting new vertices of degree 2.

For example, in Figure 1.8 we have a graph  $G$  and two subdivisions, say  $G'$  and  $G''$ .

**Definition 1.9.** A graph  $G$  is said to be  $k$  – cutwidth critical if:

1.  $c(G) = k$ ;
2. for every proper subgraph  $G'$  of  $G$ ,  $c(G') < k$ ;
3.  $G$  is homeomorphically minimal, that is,  $G$  is not a subdivision of any simple graph.

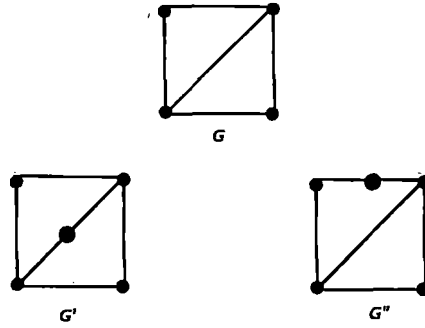


Figure 1.8:  $G$  and two subdivisions

When we show that a graph is  $k$  – *cutwidth critical*, we need to satisfy all three properties. For the first property, we will need to show that a graph has a cutwidth of  $k$ . For the second property we will need to show that any proper subgraph has a cut strictly less than  $k$ . Recall, that a subgraph is formed by a subset of vertices and edges of a given graph. For example, Figure 1.9 shows  $A_1$  and a proper subgraph, say  $A'_1$ .

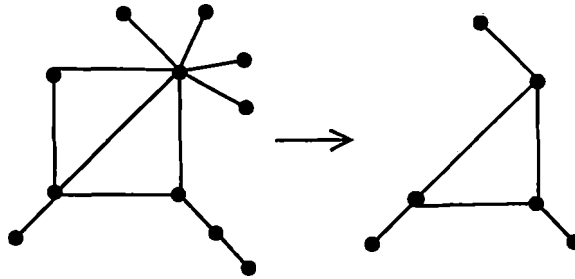


Figure 1.9: Graphs  $A_1$  and a proper subgraph  $A'_1$

It is understood that a proper subgraph is just a proper subset of vertices or edges of a larger graph. So, we can remove several edges and vertices to obtain any proper subgraph. When verifying the second property of  $k$  – *cutwidth critical* it suffices to show the case of removing just one edge because removing multiple edges is a subgraph of removing

one edge.

Note that it is possible to remove an edge without removing a vertex, as shown in Figure 1.10. This will still result in a proper subgraph, but this type of subgraph does not effect the cutwidth.

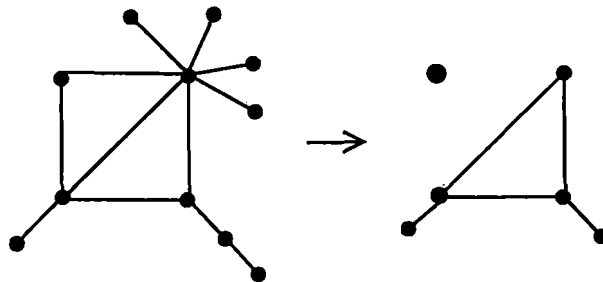


Figure 1.10: Removing edges but vertex remains

And last, for the third property we will need to show that a graph is homeomorphically minimal. Using Definition 1.8, a graph is homeomorphically minimal if it is not a subdivision of any simple graph.

So, a graph is homeomorphically minimal if it does not contain an unnecessary vertex of degree 2. For example,  $A_1$  is not homeomorphically minimal, since  $\deg(10) = 2$ , see Figure 1.11. Removing the vertex does not change the structure of the graph.

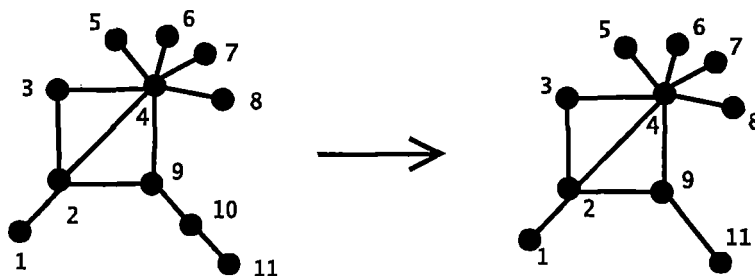


Figure 1.11: Removing vertex 10 from  $A_1$

One thing we must be careful with is to make sure we do not change the structure



of the graph when removing the degree 2 vertex. For example, in  $A_1$ ,  $\deg(3)=2$ . However, if we remove vertex 3, then we get multiple edges, as shown in Figure 1.12.

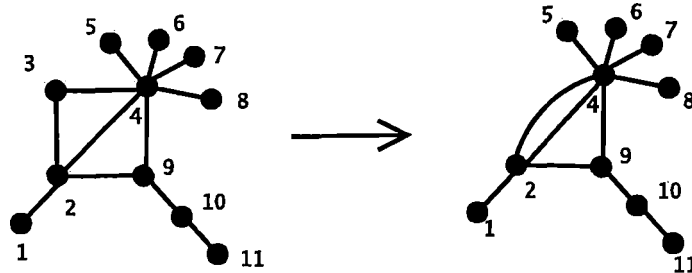


Figure 1.12: Removing vertex 3 from  $A_1$

We cannot get multiple edges because we want our graphs to remain simple. So, vertex 3 is necessary and vertex 10 is not.

## 1.5 Summary of 3-cutwidth critical graphs

This is a summary of the 3-cutwidth critical graphs that were characterized in the journal article “On 3-cutwidth critical graphs” by Yixun Lin and Aifeng Yang [LY04]. In the article, they concluded that the set of five graphs, denoted as  $H_1$ ,  $H_2$ ,  $H_3$ ,  $H_4$ , and  $H_5$  in Figure 1.12, are the only 3-cutwidth critical graphs. They did so by first proving four lemmas, then summarized their findings in a theorem. For example, the first lemma (Lemma 3.1 pg. 343) states, “A tree  $T$  is 3-cutwidth critical if and only if  $T$  is either  $H_1$  or  $H_2$ ”. The diagrams of all 3-cutwidth critical graphs are shown below in Figure 1.13, the numbers on the vertices of each graph represent an optimal labeling.

From this article we have several basic properties that will be used in the project. One is the following proposition:

**Proposition 1.10.** 1. If  $G'$  is a subgraph of  $G$ , then  $c(G') \leq c(G)$ .

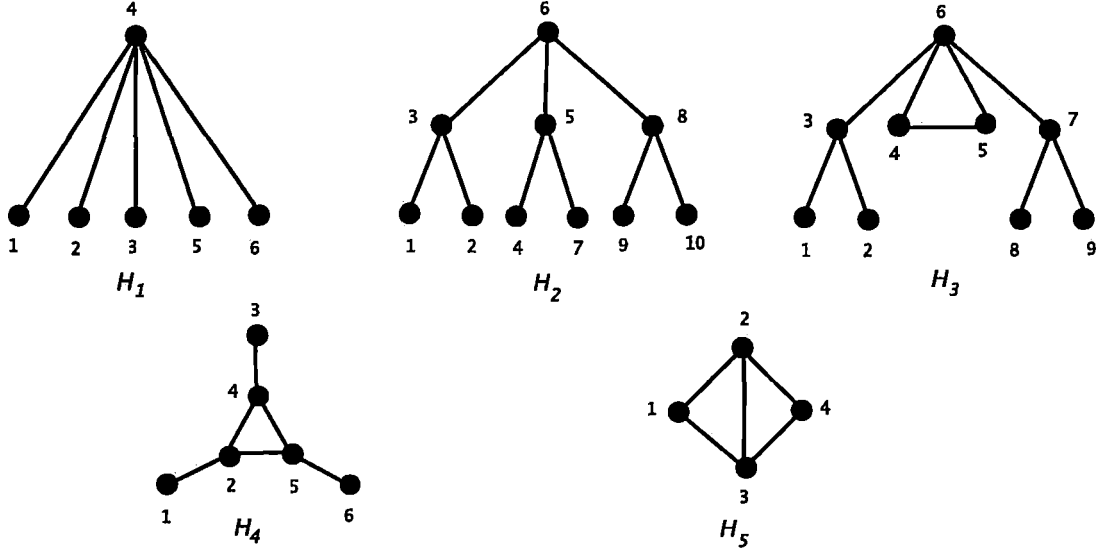


Figure 1.13: 3-cutwidth critical graphs

2. If  $G'$  is homeomorphic to  $G$ , (i.e, they can both be obtained from the same graph by inserting new vertices of degree two into its edges, called a subdivision of the graph), then the  $c(G') = c(G)$

This proposition is useful because we can look at the cutwidths of subgraphs, which can be easier. It is easier because we are in a sense minimizing the the number of graphs in our investigation. The following is another proposition that we used in several proofs.

**Proposition 1.11.** *For any caterpillar  $T$ ,  $c(T) = \lceil \deg(T)/2 \rceil$ . In particular,  $c(K_{1,n}) = \lceil n/2 \rceil$ .*

For example,  $H_1$  is a star of degree 5. It is also denoted as  $K_{1,5}$ , since the maximum degree is 5 and the edges all come from a vertex. By Proposition 1.11,  $c(H_1)$  is the next integer larger than the degree of  $H_1$  divided by two, ie, the  $c(H_1) = \lceil 5/2 \rceil = 3$ . This is because when we put the vertex of highest degree in the center of the linear embedding,

the edges get split two ways, otherwise the cut will not be minimal. We shall use this proposition in the proof of one of our 4-cutwidth critical trees,  $F_1$ .

The end result, appearing on page 345[LY04], which we find the most useful, is stated below.

**Theorem 1.12.** *All 3-cutwidth critical graphs are  $H_1$ ,  $H_2$ ,  $H_3$ ,  $H_4$ , and  $H_5$ .*

This theorem is useful because in the start of this investigation, we used  $H_1$ ,  $H_2$ ,  $H_3$ ,  $H_4$ , and  $H_5$  to find 4-cutwidth critical graphs.

## Chapter 2

# Methods Used To Find 4-Cutwidth Critical Graphs

In this project, we consider graphs that satisfy the following properties.

**Definition 2.1.** *A graph  $G$  is said to be 4 – cutwidth critical if:*

1.  $c(G) = 4$ ;
2. *for every proper subgraph  $G'$  of  $G$ ,  $c(G') < 4$ ;*
3.  *$G$  is homeomorphically minimal, that is,  $G$  is not a subdivision of any simple graph.*

There were five 3-cutwidth critical graphs that were characterized in [LY04]. The five specified elements are the only graphs that are 3-cutwidth critical, by Theorem 1.12. This fact was very useful in finding 4-cutwidth critical graphs in several ways. All diagrams of the graphs mentioned in the remainder of the paper can be found in the appendix. Also, in the following sections the methods are outlined, the proofs show up later in the text.

### 2.1 Method 1

One can naturally assume that we should be able to attach edges to a graph that is 3-cutwidth critical and end up with something that is cutwidth 4. The first graph we

looked at was  $H_1$ , which is also denoted as  $K_{1,5}$ .

We can add two edges at vertex 4 to get  $F_1$  or  $K_{1,7}$ , shown below in Figure 2.1.

Note that  $\deg(4) = 5$ , which is the maximum degree of  $H_1$ .

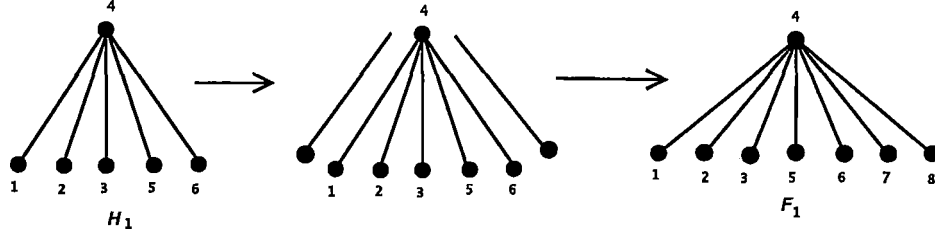


Figure 2.1: Using  $H_1$  to get  $F_1$

As a result  $F_1$  a 4-cutwidth critical graph, which is verified in Chapter 3. We can again apply a similar step to  $H_2$  to get  $F_2$ , the difference is the type of edges we attach, see Figure 2.2. But again, we must be careful of where we attach the edges. The maximum degree for  $H_2$  is 3, but this occurs in 3 places. We must attach the edges at vertex 6, otherwise we would not get a 4-cutwidth critical graph.

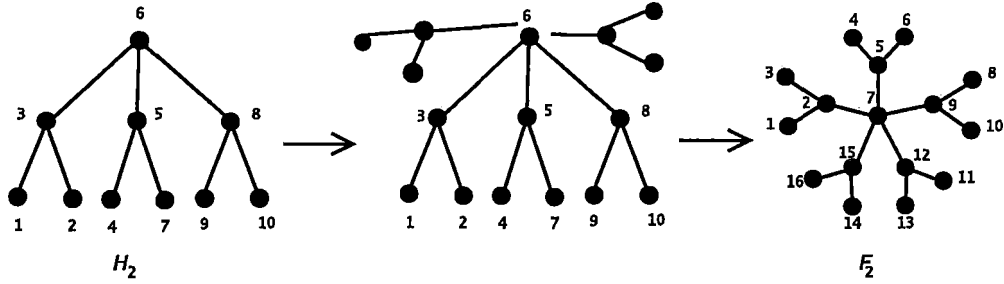


Figure 2.2: Using  $H_2$  to get  $F_2$

Similarly, we use  $H_3$  to get  $F_{10}$ ,  $H_4$  to get  $F_{11}$ , and we use  $H_5$  to get  $F_{15}$  and  $F_{18}$ . (See figures in the Appendix).

## 2.2 Method 2

The second method we used was to take the 3-cutwidth critical graphs and attach them to the ends of  $K_{1,3}$ , see Figure 2.3.

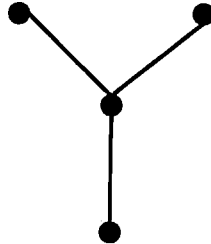


Figure 2.3:  $K_{1,3}$

For example,  $H_1$  is 3-cutwidth critical. We can attach  $H_1$  to the ends of  $K_{1,3}$  to get  $F_3$ . By ends we mean attach at the pendant vertices. The diagrams are shown in Figure 2.4.

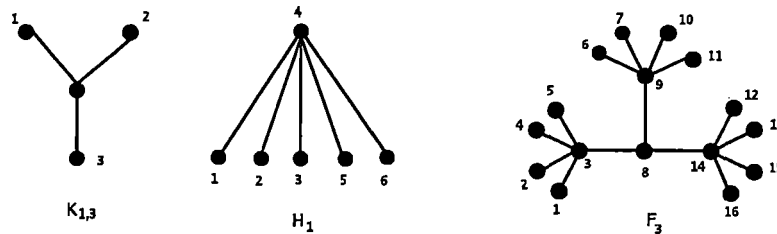
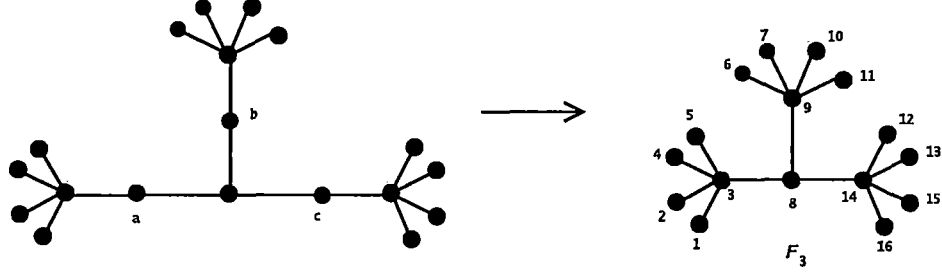


Figure 2.4: Using  $H_1$  to get  $F_3$

This method requires extra precaution. We must be careful when attaching edges to  $K_{1,3}$  because we could end up with a graph that is not homeomorphically minimal. In Figure 2.5 we show  $H_1$  attached to  $K_{1,3}$ . We can see that the vertices labeled  $a$ ,  $b$  and  $c$  are of degree 2. These vertices do not effect the cutwidth of the graph. If we left them then we would not have a graph that is homeomorphically minimal. So, in  $F_3$  the unnecessary vertices are removed, which is also shown in Figure 2.5.

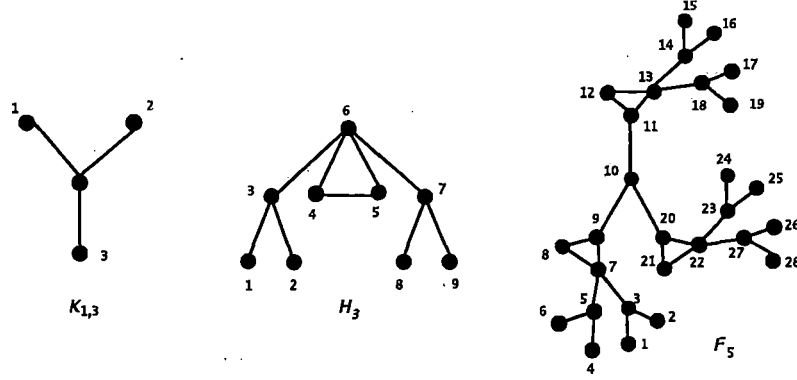
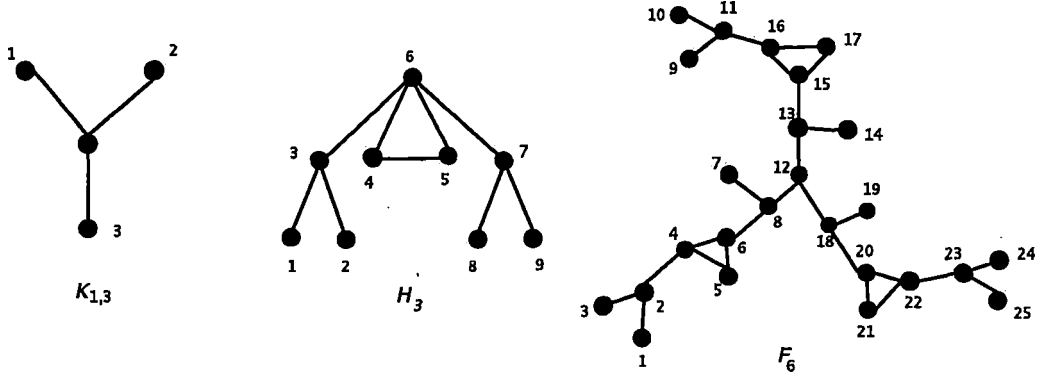
Figure 2.5:  $H_1$  attached to  $K_{1,3}$ 

Not only do we have to be careful when attaching the 3-cutwidth critical graphs to  $K_{1,3}$ , we must also look at the different ways we can attach them. We get  $F_4$  from  $H_2$  and  $F_7$  from  $H_4$  in the same way we got  $F_3$ . However, when we use  $H_3$  and  $H_5$  we get more than one 4-cutwidth critical graph. For example we get  $F_5$  and  $F_6$  from  $H_3$ , and  $F_8$  and  $F_9$ , from  $H_5$ . The difference between  $F_5$  and  $F_6$  is that when we attached them to  $K_{1,3}$  we attached different parts of  $H_3$ . For example, we attached a vertex from one of the cycles to get  $F_5$ , shown in Figure 2.6. For  $F_6$  we attached one of the vertices that is not part of a cycle, shown in Figure 2.7. Similar to the case of getting  $F_3$ , we needed to remove unnecessary vertices for  $F_6$ . In both cases we ended up with two 4-cutwidth critical graphs,  $F_5$  and  $F_6$ , from  $H_3$ .

This method of attaching a 3-cutwidth critical graphs more than one way to get a 4-cutwidth critical graph can also be done with  $H_5$ . As shown in Figures 2.8 and 2.9, using  $H_5$  we get  $F_8$  and  $F_9$ , which are both 4-cutwidth critical.

## 2.3 Method 3

In this section we shall discuss a method used to find the rest of our graphs. It is in a sense a combination of the previous two methods. After methods 1 and 2, we had a set of 4-cutwidth critical graphs. We decided to look at the graphs we had and simply checked

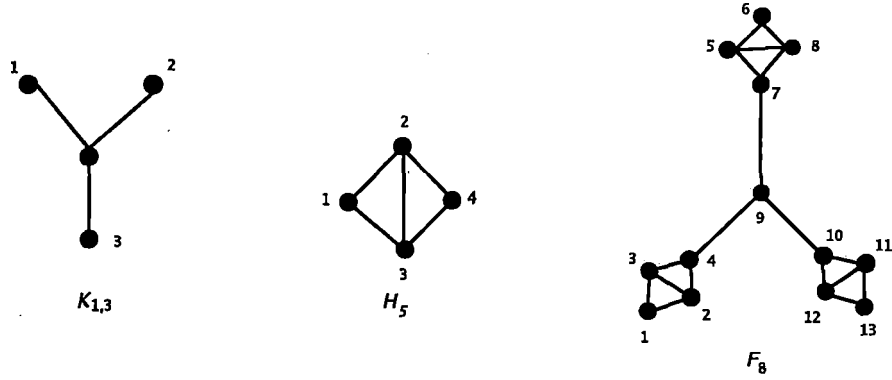
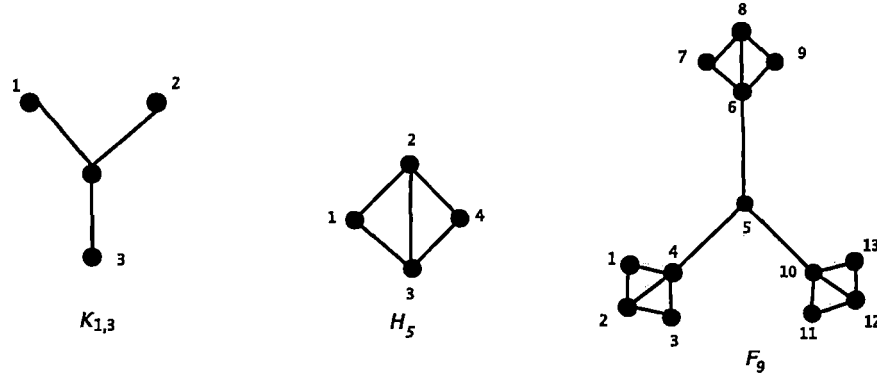
Figure 2.6:  $H_3$  attached to  $K_{1,3}$  to get  $F_5$ Figure 2.7:  $H_3$  attached to  $K_{1,3}$  to get  $F_6$ 

to see if there was something we can change about them that would result in another 4-cutwidth critical graph, which is similar to what we did in Method 1. The first graph we found was  $F_{12}$ , see Figure 2.10.

This graph came from  $F_{11}$ . We looked at the linear embedding of  $F_{11}$  and replaced the edges at the ends of each triangle by a cycle, which can be seen in Figure 2.11.

The next thing we looked at was taking a combination of  $F_{11}$  and  $F_{12}$  to get a 4-cutwidth critical graph. For example, instead of replacing all the edges at the ends of each triangle, we replaced just one. We can also replace just two of the edges. The result

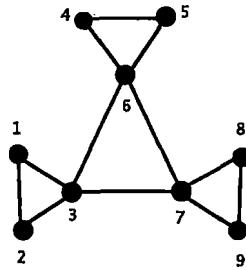
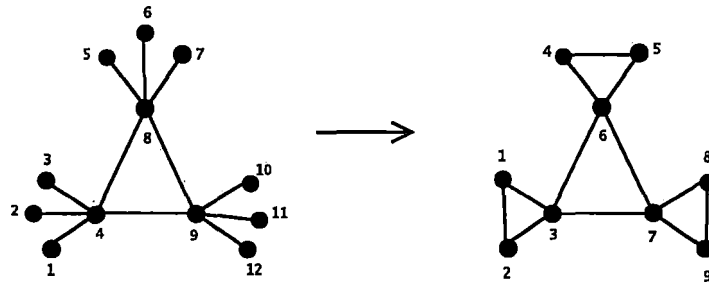


Figure 2.8:  $H_5$  attached to  $K_{1,3}$  to get  $F_8$ Figure 2.9:  $H_5$  attached to  $K_{1,3}$  to get  $F_9$ 

is two more 4-cutwidth critical graphs shown in Figure 2.12.

We then continued taking combinations of graphs, but this time we used the same idea as in Method 2. The graphs shown in Figure 2.13 have a couple of things in common. One is that they are all 4-cutwidth critical and the other is that they all contain  $K_{1,3}$ .

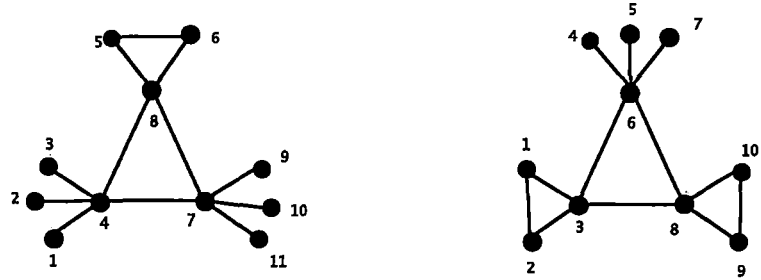
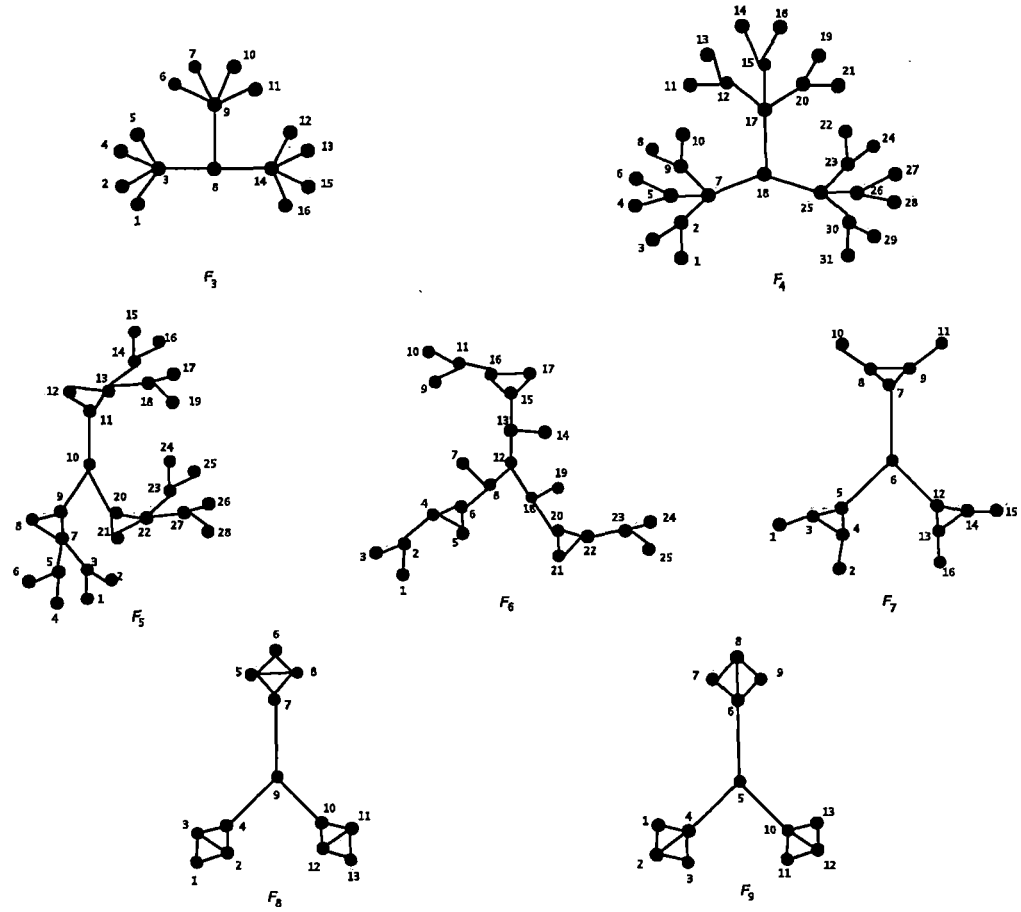
We took combinations of these 4-cutwidth critical graphs to obtain more. Since these graphs all contain  $K_{1,3}$ , we can interchange or exchange the attachments. For example, in Figure 2.14, we interchange the "ends" of  $K_{1,3}$  on  $F_7$  and  $F_3$ . The resulting graphs are 4-cutwidth critical.

Figure 2.10:  $F_{12}$ Figure 2.11: Using  $F_{11}$  to get  $F_{12}$ 

Similarly, we can take combinations of all the graphs shown in Figure 2.13. In the previous example, we took combinations of just two graphs, but we can also take combinations of three graphs. In Figure 2.15, we show several examples of the resulting 4-cutwidth critical graphs.

## 2.4 Method 4

The graphs in this section were not found using any of the previous methods. Although they are the last graphs mentioned, they were not the last ones found. When I started the project, I simply started to play around with graphs so that I can become familiar with finding their cutwidth. These 4-cutwidth critical graphs were discovered by trial and error and are shown in Figure 2.16.

Figure 2.12: Using a combination of  $F_{11}$  and  $F_{12}$ Figure 2.13: All 4-cutwidth critical graphs that contain  $K_{1,3}$

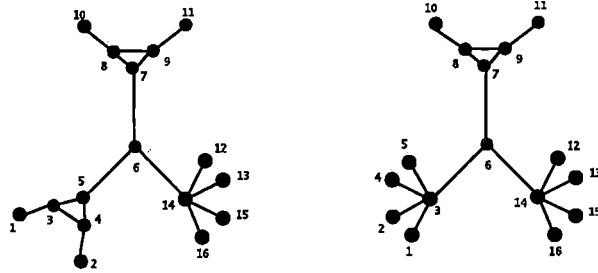


Figure 2.14: 4-cutwidth critical graphs from  $F_7$  and  $F_3$

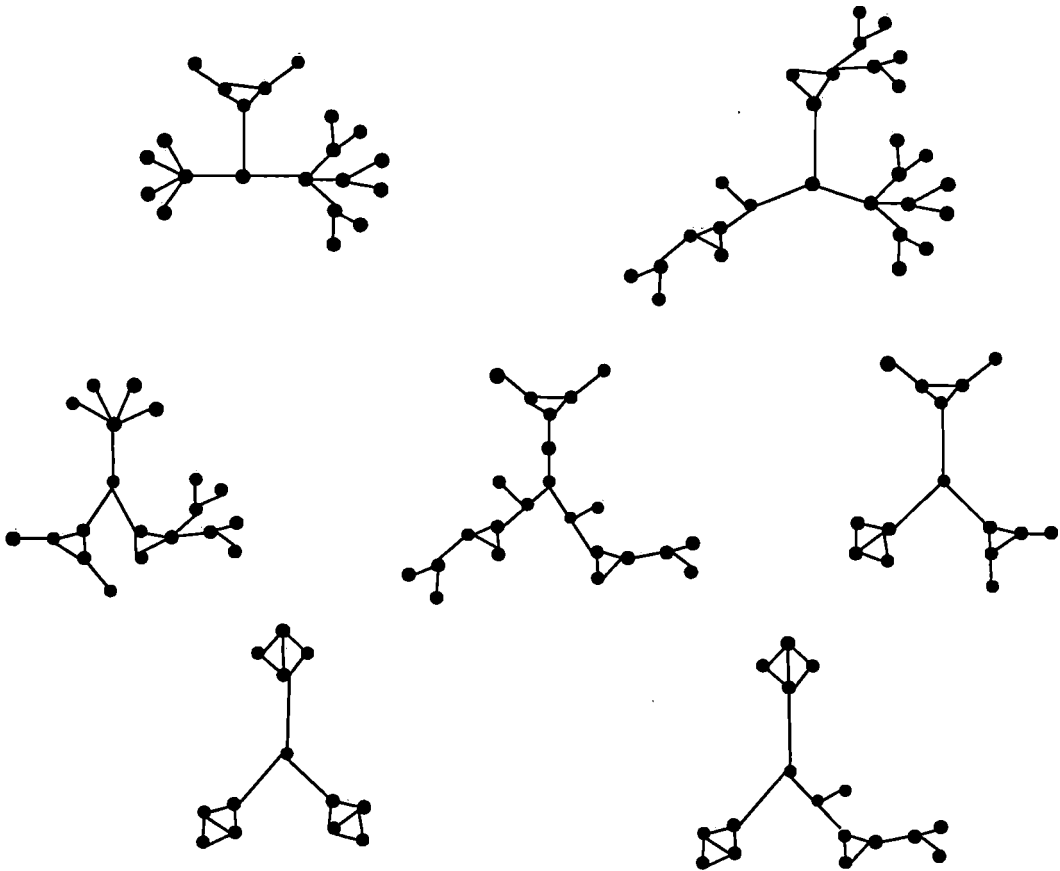


Figure 2.15: Combinations of 4-cutwidth critical graphs

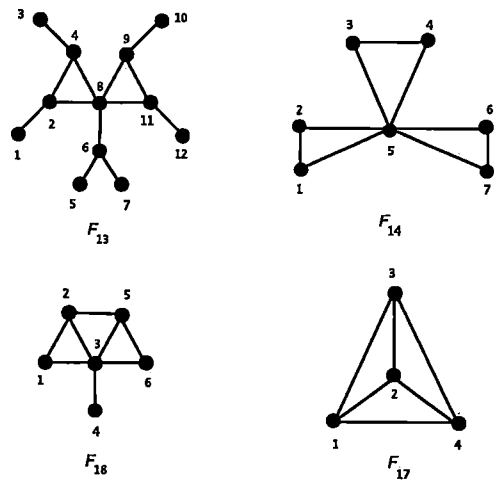


Figure 2.16: More 4-cutwidth critical graphs

## Chapter 3

# 4-Cutwidth Critical Trees

A particular type of graph that we see a lot of in this project is trees.

**Definition 3.1.** *A graph is a tree if it is a connected graph with no circuits, where a circuit is a simple closed path.*

Since a tree cannot have a circuit, a tree cannot contain multiple edges or a loop. So, any tree must be a simple graph. Here we will prove that the trees we have,  $F_1$ ,  $F_2$ ,  $F_3$ , and  $F_4$  shown in Figure 3.1, are indeed 4-cutwidth critical. These graphs can be seen in Figure 3.1. We begin with the proof of  $F_1$ .

**Lemma 3.2.**  *$F_1$  is 4-cutwidth critical.*

*Proof.* We will need to show the  $F_1$  satisfies all three properties of 4-cutwidth critical graphs.

1. The first property states that  $F_1$  must have a cutwidth of 4.  $F_1$  is a star,  $K_{1,7}$ . By Proposition 1.11,  $c(F_1) = c(K_{1,7}) = \lceil 7/2 \rceil = 4$ . Thus the  $c(F_1) = 4$ , as desired.
2. The next property we will satisfy is that every proper subgraph of  $F_2$  has a cutwidth strictly less than four. It suffices to consider the case of removing one edge. Without loss of generality, we may remove  $(4,5)$ , which results in the proper subgraph,  $F'_1$ , shown in Figure 3.2.

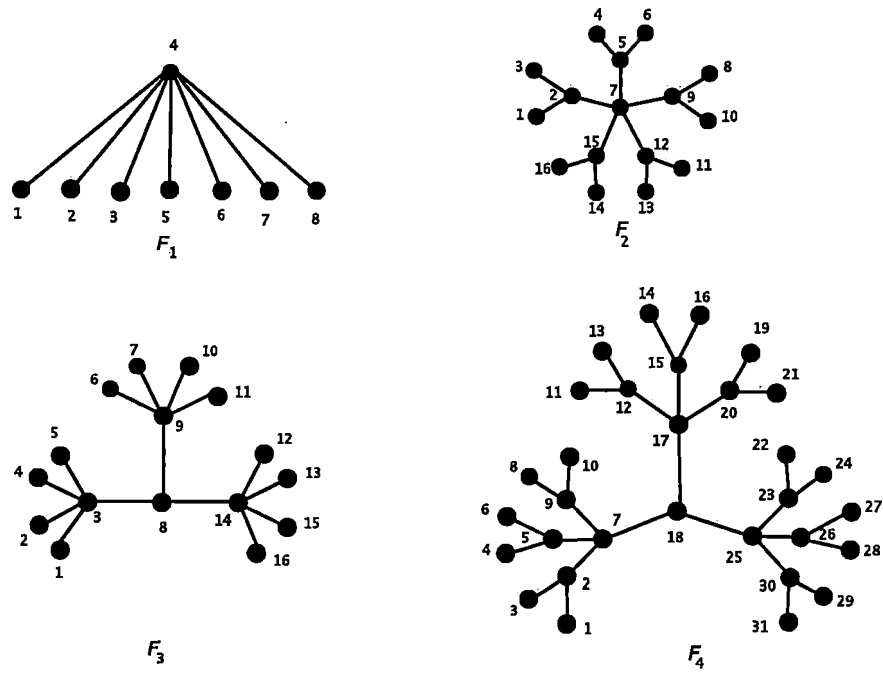


Figure 3.1: 4-cutwidth critical trees

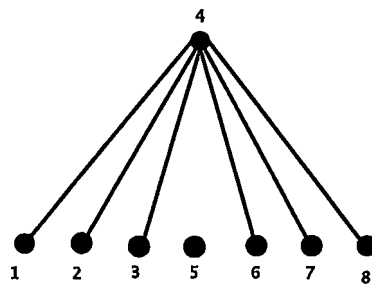


Figure 3.2:  $F'_1$

$F'_1$  is homeomorphic to a star of degree 6. So, by Proposition 1.11,  $c(F'_1) = c(K_{1,6}) = \lceil 6/2 \rceil = 3$ . Therefore, any proper subgraph of  $F_1$  has cutwidth strictly less than 4.

3. Now, we know that  $F_1$  satisfies the third property of being homeomorphically minimal because it does not contain a vertex of degree 2.

All three properties are satisfied, therefore,  $F_1$  is 4-cutwidth critical.  $\square$

The format of the proofs for  $F_2$ ,  $F_3$ , and  $F_4$  are similar to each other, so we will show the proof of one tree,  $F_2$ . This proof is more interesting than the proof of  $F_1$  because we can not use Proposition 1.11.

**Lemma 3.3.**  *$F_2$  is 4-cutwidth critical.*

*Proof.* We will show  $F_2$  satisfies all three properties of a 4-cutwidth critical graph.

1. The first property we will verify is that  $c(F_2) = 4$ . Denote the vertex of degree 5 in  $F_2$  by  $x$ , and denote its neighbors by  $a$ ,  $b$ ,  $c$ ,  $d$  and  $e$ . In addition, let  $a_1$  and  $a_2$  be adjacent to  $a$ ,  $b_1$  and  $b_2$  be adjacent to  $b$ ; similarly for  $c$ ,  $d$  and  $e$ . Now,  $F_2$  has the following labeling, shown in Figure 3.3:

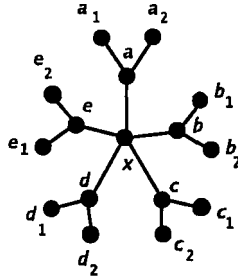


Figure 3.3:  $F_2$  relabeled

Recall that  $f$  is a labeling of  $G$  is a bijection  $f : V \rightarrow \{1, 2, \dots, n\}$ , so the following is a set of natural numbers. Let  $x$  be the median of  $\{f(x), f(a), f(b), f(c), f(d), f(e)\}$



such that the maximum  $\{f(a), f(b)\} < f(x) < \text{minimum } \{f(c), f(d), f(e)\}$ . Let  $f(c)$  be the vertex adjacent to  $f(x)$ . Now, there are two cases to consider:

Case (i): If  $f(c_1) < f(c) < f(c_2)$  then the cut of  $F_2$  at  $[c_1, c]$  is given by  $\{cc_1, xc, xd, xe\}$  as shown in Figure 3.4:

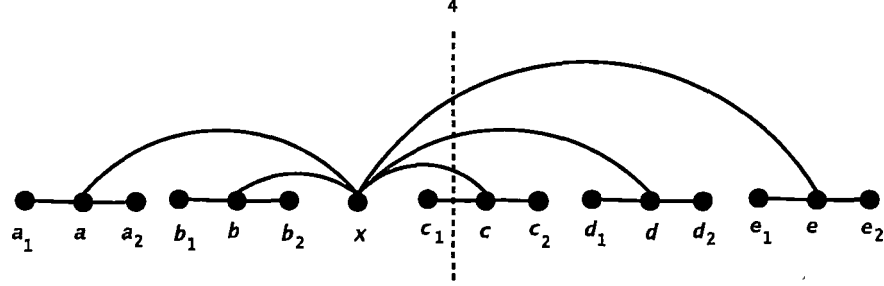


Figure 3.4: Linear embedding of  $F_2$  (case (i))

This shows that the maximum size of all the cuts is 4, when  $f(c_1) < f(c) < f(c_2)$ .

Case (ii): If  $f(c) < f(c_1) < f(c_2)$ , then the cut of  $F_2$  at  $[c_1, c]$  is given by  $\{cc_1, xc, xd, xe\}$  as shown in Figure 3.5:

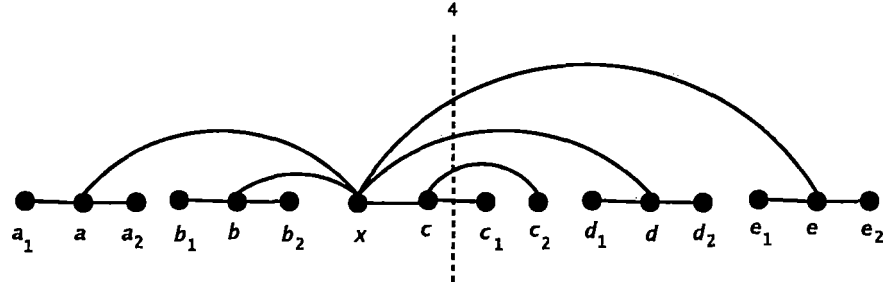


Figure 3.5: Linear Embedding of  $F_2$  (case (ii))

Again we have that the cut of  $F_2$  is 4. So, in either case, when  $x$  is the median, the cut cannot be  $< 4$ .

Now, if  $x$  is not the median of  $F_2$ , then the cut of  $F_2 \geq 4$ . Therefore,  $c(F_2)=4$ .

2. Next we will show that any proper subgraph of  $F_2$  has a cutwidth  $< 4$ . There are also two cases we need to consider, since any other subgraphs of  $F_2$  will be subgraphs of the following:

Case (i): We can remove an edge coming from the center, ie,  $(2, 7)$ ,  $(5, 7)$ ,  $(9, 7)$ ,  $(12, 7)$  or  $(15, 7)$  to obtain a proper subgraph of  $F_2$ . Without loss of generality, we can remove  $(12, 7)$ . We now have a subgraph,  $F'_2$ , shown in Figure 3.6.

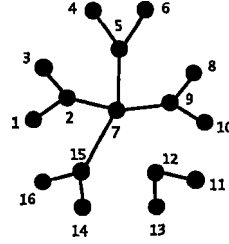


Figure 3.6:  $F'_2$

Then the  $c(F'_2)$  is given by the following linear embedding, shown in Figure 3.7.

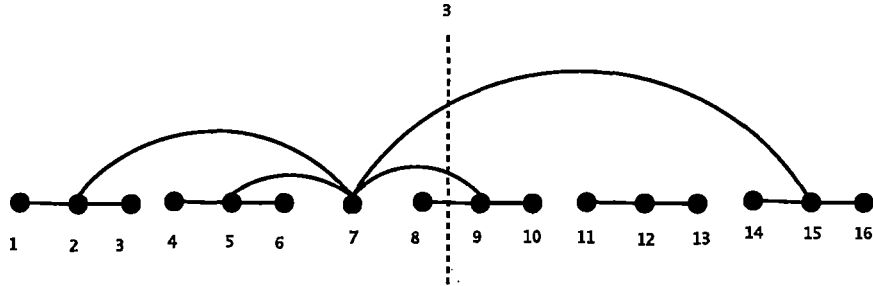


Figure 3.7: Linear Embedding of  $F'_2$

This shows  $c(F'_2) \leq 3$ .

Case (ii): We can also remove a “dangling edge” to get a proper subgraph. A “dangling edge” is an edge connected to a pendant vertex. So removing one of the following edges,  $(1, 2)$ ,  $(2, 3)$ ,  $(4, 5)$ ,  $(5, 6)$ ,  $(8, 9)$ ,  $(9, 10)$ ,  $(11, 12)$ ,  $(12, 13)$ ,  $(14, 15)$  or  $(15, 16)$ ,

results in a proper subgraph of  $F_2$ . Without loss of generality we may remove (8, 9).

We now have the following subgraph,  $F_2''$  in Figure 3.8.

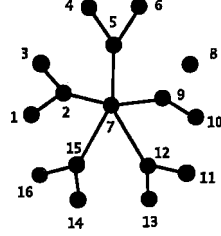


Figure 3.8:  $F_2''$

The  $c(F_2'')$  is given by the linear embedding shown in Figure 3.9.

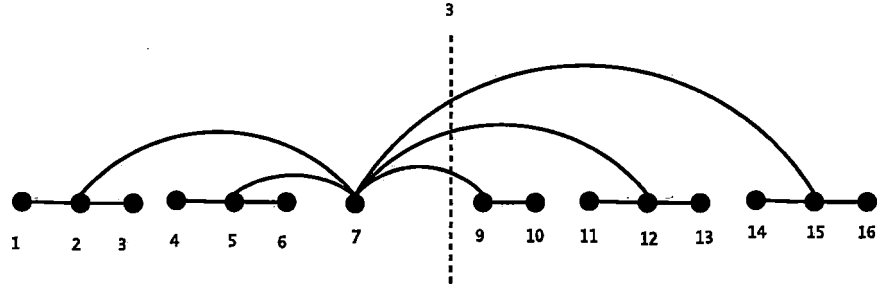


Figure 3.9: Linear Embedding of  $F_2''$

This shows  $c(F_2'') \leq 3$ . So, any proper subgraph of  $F_2$  has a cutwidth of at most 3, which is strictly less than 4.

3. We know that  $F_2$  is homeomorphically minimal because it does not contain a vertex of degree 2.

The three properties are satisfied, therefore,  $F_2$  is 4-cutwidth critical.

□

## Chapter 4

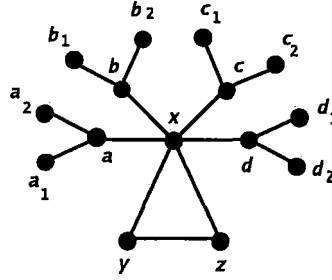
# Proving graphs are 4-cutwidth critical

In this chapter we will show a few proofs of our other 4-cutwidth critical graphs. The proofs are similar in that we must show that the graphs satisfy all three conditions of a 4-cutwidth critical graph. The first proof we will show is of  $F_{10}$ . This proof is slightly different than the ones previously mentioned because  $F_{10}$  contains one cycle. We will also show a proof of  $F_{18}$ . We decided to show the proof of  $F_{18}$  because it contains more than one cycle with shared edges. It is also different because we verify the second property of cutwidth critical by using a 3-cutwidth critical graph. (See Appendix for graphs)

**Lemma 4.1.**  *$F_{10}$  is 4-cutwidth critical.*

*Proof.* We will show that  $F_{10}$  satisfies all three conditions of 4-cutwidth critical graphs.

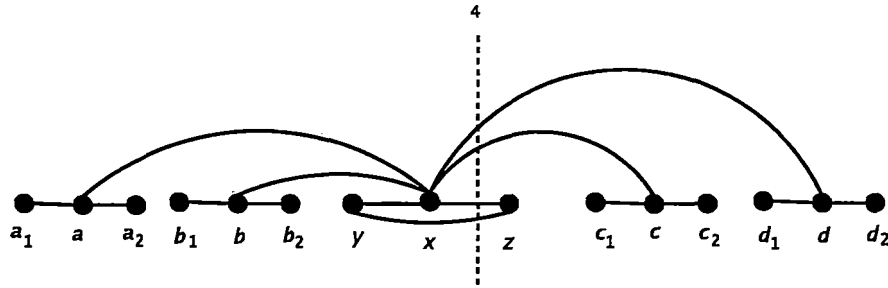
1. The first property we will verify is that the  $c(F_{10}) = 4$ . The labeling of  $F_{10}$  asserts  $c(F_{10}) \leq 4$ . We will show that we cannot get a cut less than 4. Denote the vertex of degree six by  $x$  and denote its neighbors by  $a, b, c, d, y, z$ , where  $x, y, z$ , forms the cycle (triangle). In addition, let  $a_1$  and  $a_2$  be adjacent to  $a$ ; let  $b_1$  and  $b_2$  be adjacent to  $b$  and similarly for  $c$  and  $d$  (See Figure 4.1).

Figure 4.1:  $F_{10}$  relabeled

For a labeling  $f$  of  $F_{10}$ , if  $f(x)$  is not the median of

$\{f(x), f(a), f(b), f(y), f(z), f(c), f(d)\}$ , then it is clear that  $c(F_{10}, f) \geq 4$ . Let  $x$  be the median of  $\{f(x), f(a), f(b), f(y), f(z), f(c), f(d)\}$ , then there are two cases to consider.

Case (i): Let  $\max \{f(a), f(b), f(y)\} < f(x) < \min \{f(z), f(c), f(d)\}$ . A linear embedding is shown in Figure 4.2. In this case the cutwidth is given by  $\{xz, xc, xd, yz\}$ . So,  $c(F_{10}, f) \geq 4$ .

Figure 4.2: Linear Embedding of  $F_{10}$ , (case (i))

Case (ii): Let  $f(a) < f(b) < f(c) < f(x) < f(y) < f(z) < f(d)$ . Let  $i = f(c)$ . If  $\max\{f(c_1), f(c_2)\} > i$ , say  $f(c_1) > i$ , then  $\{ax, bx, cx, c, c_2\}$  give the cutwidth of  $F_{10}$ . Otherwise,  $f(c_1)$  and  $f(c_2) < i$  then  $\{ax, bx, cc_1, cc_2\}$  give the cutwidth of  $F_{10}$ . Both linear embeddings are shown in Figure 4.3. So,  $c(F_{10}, f) \geq 4$ .

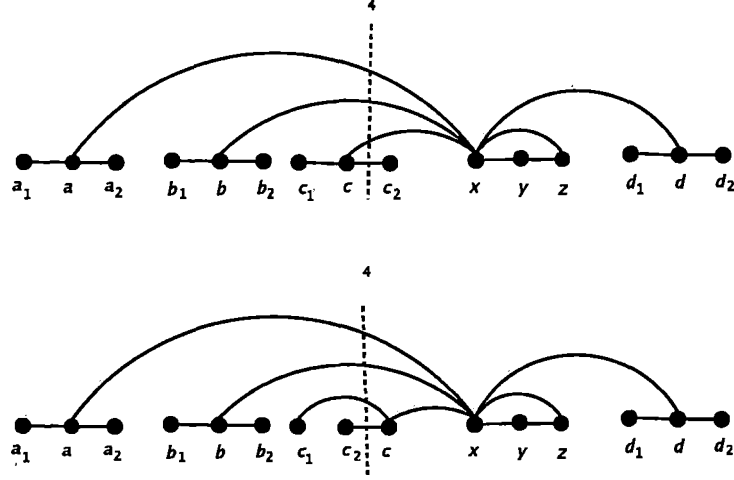


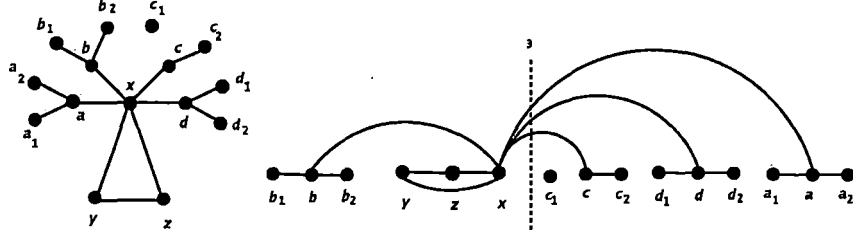
Figure 4.3: Linear Embedding of  $F_{10}$ , (case (ii))

We have shown that the  $c(F_{10}) \geq 4$  when  $x$  is the median. Therefore,  $c(F_{10}) = 4$ .

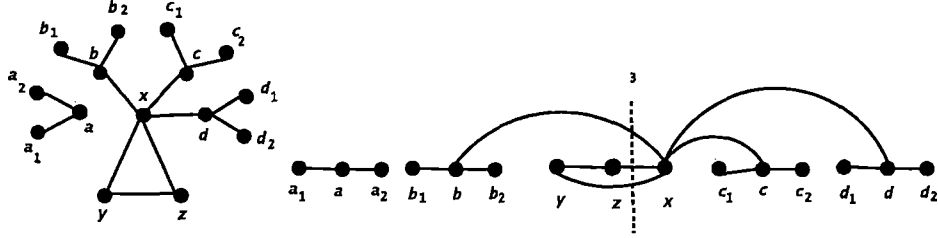
2. Now that we have shown the  $c(F_{10}) = 4$ , the next property we will verify is that every proper subgraph has cutwidth strictly less than 4. There are four cases to consider, since any other subgraphs of  $F_{10}$  will be subgraphs of the following:

Case (i): First we consider removing a “dangling edge”, where a dangling edge is an edge connected to a pendant vertex. So, removing one of the following edges:  $(a_1, a)$ ,  $(a_2, a)$ ,  $(b_1, b)$ ,  $(b_2, b)$ ,  $(c_1, c)$ ,  $(c_2, c)$ ,  $(d_1, d)$  and  $(d_2, d)$  results in a proper subgraph of  $F_{10}$ . Without loss of generality, we can remove  $(c_1, c)$ . We now have the subgraph,  $F_{10}^1$ , shown in Figure 4.4. Now,  $c(F_{10}^1)$  is given by the linear embedding, also shown in Figure 4.4.

Case (ii): Next we consider the case of removing an edge from the center, but that is not part of the cycle, to obtain a proper subgraph. We can remove either  $(a, x)$ ,  $(b, x)$ ,  $(c, x)$ , or  $(d, x)$ . Without loss of generality, we may remove  $(a, x)$ . We now have the following subgraph,  $F_{10}^2$ , shown in Figure 4.5. Then  $c(F_{10}^2)$  is given by the linear

Figure 4.4:  $F_{10}^1$  and Linear Embedding, (case (i))

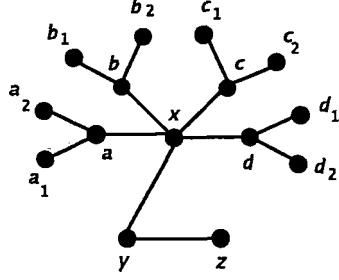
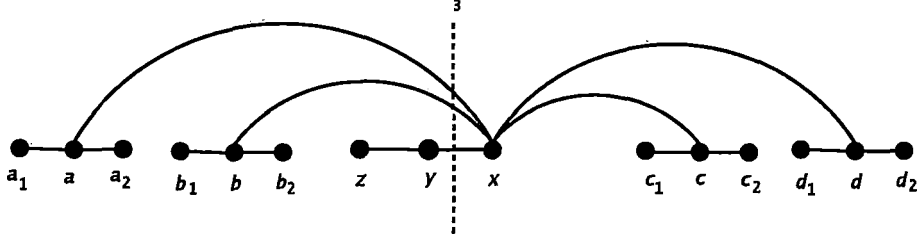
embedding, also in Figure 4.5. This shows  $c(F_{10}^2) \leq 3$ .

Figure 4.5:  $F_{10}^2$  and Linear Embedding, (case (ii))

Case (iii): Here we remove an edge coming from the center, but is part of the cycle. So, we can remove  $(y, x)$  or  $(z, x)$ . Say we remove  $(z, x)$ . Then we have the following proper subgraph,  $F_{10}^3$ , shown in Figure 4.6. Then  $c(F_{10}^3)$  is given by the following linear embedding, shown in Figure 4.7. Note that we again have to rearrange the labeling to obtain the cutwidth. This shows  $c(F_{10}^3) \leq 3$ .

Case (iv): In the final case, we only need to consider removing the edge of the cycle that is not from the center, ie, remove  $(y, z)$ . Which results in the subgraph,  $F_{10}^4$ , shown in Figure 4.8. The linear embedding of  $F_{10}^4$  is shown in Figure 4.9. This shows  $c(F_{10}^4) \leq 3$ .

In all four cases, every proper subgraph has a cutwidth strictly less than 4. Therefore, the second condition is satisfied.

Figure 4.6:  $F_{10}^3$  (case (iii))Figure 4.7: Linear Embedding of  $F_{10}^3$ , (case (iii))

3. We know that  $F_{10}$  is homeomorphically minimal because it does not contain an unnecessary vertex of degree 2.

The three condtions are satisfied, therefore,  $F_{10}$  is 4-cutwidth critical.

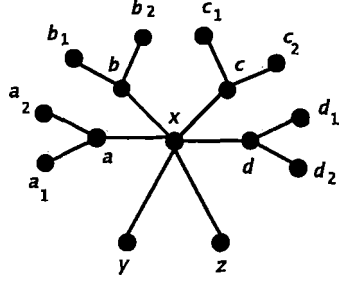
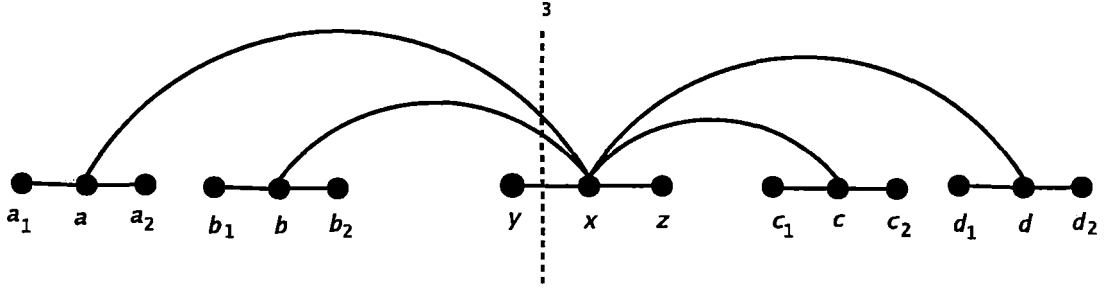
□

**Lemma 4.2.**  $F_{18}$  is 4-cutwidth critical.

*Proof.* For the proof we need to show  $F_{18}$  satisfies all three properties of a 4-cutwidth critical graph.

1. The first condition we will satisfy is that  $c(F_{18}) = 4$ . The cut of four is given because there are four paths between vertices  $a$  and  $b$ . Each path will contribute to cutwidth on the linear embedding. So,  $c(F_{18}) = 4$ .



Figure 4.8:  $F_{10}^4$  (case (iv))Figure 4.9:  $F_{10}^4$  and the Linear Embedding of  $F_{10}^4$ , (case (iv))

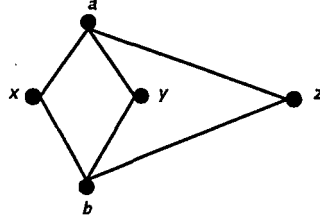
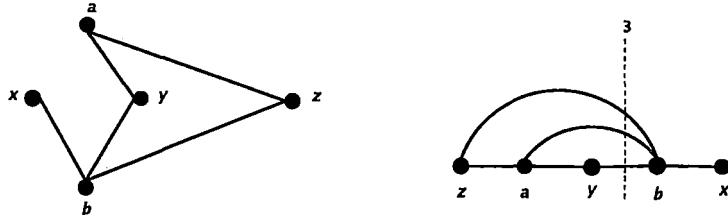
2. The next property we will verify is that every proper subgraph has cutwidth strictly less than 4. There are only two cases to consider, since any other subgraphs of  $F_{18}$  will be subgraphs of the following:

Case(i): We can remove  $(a, b)$  to obtain  $F'_{18}$  shown in Figure 4.11.  $F'_{18}$  is homeomorphic to  $H_5$ . By Theorem 1.12 and Proposition 1.10,  $c(F'_{18}) = c(H_5) = 3$ .

Case (ii): In this case we can remove any of the other edges to obtain a proper subgraph. Without loss of generality, remove  $(a, x)$ , to obtain  $F''_{18}$ , shown in Figure 4.12 along with the corresponding linear embedding.

So, the cutwidth of every proper subgraph of  $F_{18}$  is strictly less than 4, as desired.

3. We know  $F_{18}$  is homeomorphically minimal because if the vertex of degree two is

Figure 4.10:  $F'_{18}$ Figure 4.11:  $F''_{18}$  and the Linear Embedding of  $F''_{18}$ 

removed, then we will have overlapping edges and we are not allowed overlapping edges between the same pair of vertices.

The three conditions are satisfied, therefore,  $F_{18}$  is 4-cutwidth critical.  $\square$

## Chapter 5

# Family of 4-cutwidth critical graphs

This chapter brings us to a special type of 4-cutwidth critical graphs. Here we show that there exist an infinite family. First, the following definitions and lemma are required.

**Definition 5.1.** *A cycle is a sequence of consecutively linked edges whose starting vertex is the ending vertex and in which no edge can appear more than once.*

**Definition 5.2.** *A graph is connected if we can get from any vertex to any other vertex by traveling along the edges of the graph. The opposite of connected is disconnected.*

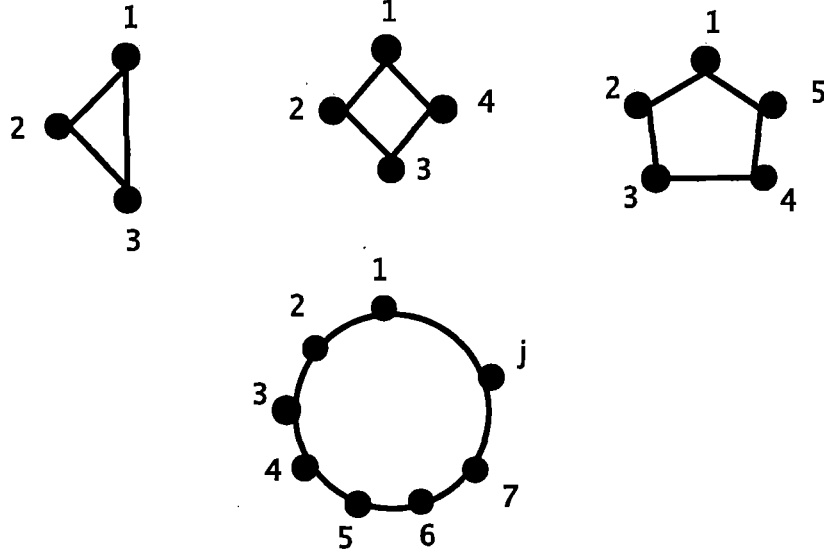
The following lemma asserts that the cut of any cycle is 2.

Note: A cycle is a connected graph.

**Definition 5.3.** *Let  $C_i$  denote the cyclic graph with  $i$  vetices, where  $3 \leq i \leq \infty$ .*

Figure 5.1 contains figures with examples of  $C_i$ , where  $i=3,4,5$  and  $C_j$ , where  $3 \leq j, j+1 \leq i-1$ .

Using the above definition, we can prove the following lemma.

Figure 5.1: Examples of  $C_i$ 

**Lemma 5.4.**  $c(C_i) = 2$ , for all  $j, j + 1$ , where  $1 \leq j, j + 1 \leq i - 1$ .

*Proof.* We need to show that the  $c(C_i) = 2$  between all neighbors on the linear embedding.

The labeling of  $C_i$  asserts that  $C_i \leq 2$ . Now we need to show that  $c(C_i)$  can not be less than 2.

Assume, by way of contradiction,  $c(C_i) = 1$ .

If the  $c(C_i) = 1$ , removing an edge, ie, breaking the cycle, results in a disconnected graph. It does not matter whether we remove the edge from the ends of the linear embedding or anywhere in the center. It will still result in a disconnected graph. This contradicts our assumption because we started with a cycle. In a cycle we can remove an edge and still have a connected graph. So,  $c(C_i) \neq 1$ . Which clearly asserts that the  $c(C_i) = 2$  everywhere.

Therefore,  $c(C_i) = 2, \forall i$  where,  $3 \leq i \leq \infty$ . □

The following is a definition of our 4-cutwidth critical family.

**Definition 5.5.** Let  $Z_i$ , where  $1 \leq i \leq \infty$ , denote a graph with two edge disjoint cycles, ie,

they share vertices but do not share edges. The “small” cycle is denoted by black vertices, the “big” cycle contains black and white vertices.

Example of graphs that belong to this family are shown in Figure 5.2. We shall use  $Z_2$  as an example in the context of the proof.

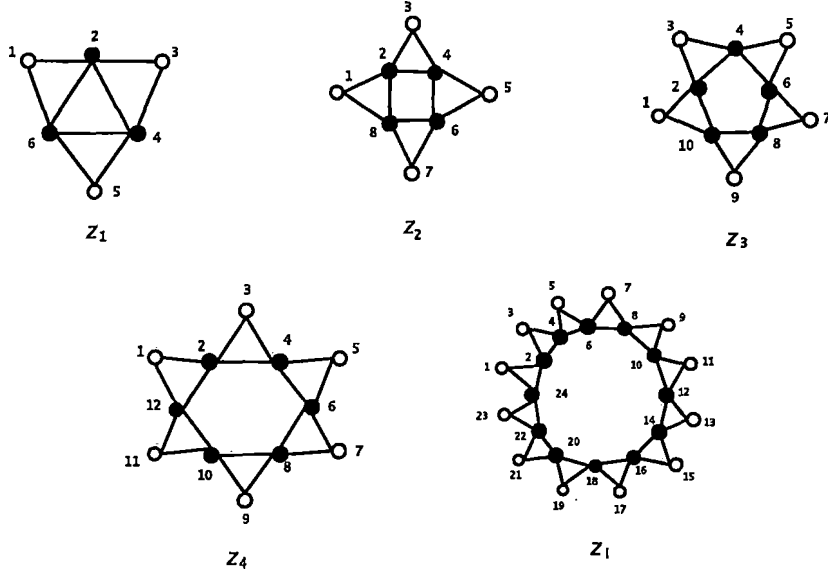


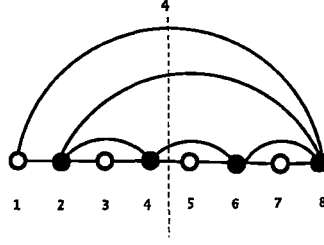
Figure 5.2: Family of 4-cutwidth critical graphs

**Theorem 5.6.**  $Z_i$  is 4-cutwidth critical.

*Proof.* We need to show  $Z_i$  satisfies the three conditions of 4-cutwidth critical.

1. First we will show  $c(Z_i) = 4, \forall i$ , where  $1 \leq i \leq \infty$ .

$Z_i$  is defined to have two edge independent cycles. Any linear embedding of  $Z_i$  will have a region containing edges from both cycles, which is forced because they share vertices, but do not share an edge. For example in the linear embedding of  $Z_2$ , is shown in Figure 5.3. Each of these edge independent cycles contribute a cut of 2, from Lemma 5.4.

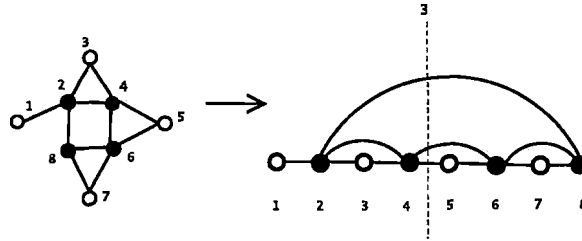
Figure 5.3: Linear Embedding of  $Z_2$ 

Therefore,  $c(Z_i) = 4$ ,  $\forall i$ , where  $1 \leq i \leq \infty$ .

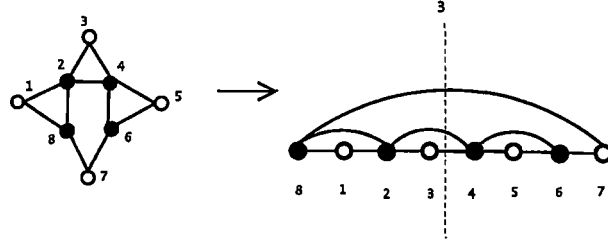
2. Next we will show that any proper subgraph has cutwidth  $< 4$ .

Let  $l, k$  be any vertices of  $Z_i$ , where edge  $(l, k) \in Z_i$ ,  $\forall i$ , where  $1 \leq i \leq \infty$ . Any proper subgraph of  $Z_i$ , say  $Z'_i$ , is obtained by removing at least one edge, ie,  $(l, k) \notin Z_i$ . Removing an edge will break one of the cycles. Consider a linear embedding of  $Z'_i$ , where we put  $l, k$  at ends, then  $c(Z'_i) = 3$ .

For example, in  $Z_2$  we can remove an edge from the "big" cycle, say  $(1, 8)$ , to get a subgraph,  $Z'_2$ . In the linear embedding of  $Z'_2$  we get a cutwidth of 3. This can be seen in Figure 5.4.

Figure 5.4:  $Z'_2$  and the linear embedding of  $Z'_2$ 

We can also remove an edge from the "small" cycle, say  $(6, 8)$  to get a subgraph,  $Z''_2$ . In the linear embedding of  $Z''_2$  we also get a cut of 3. But we had to put vertices 6 and 8 at the ends, as shown in Figure 5.5.

Figure 5.5:  $Z_2''$  and the linear embedding of  $Z_2''$ 

Thus,  $c(Z_i') < 4$ ,  $\forall i$ , where  $\forall i$ , where  $1 \leq i \leq \infty$ .

3. Show  $Z_i$  is homeomorphically minimal,  $\forall i$ , where  $1 \leq i \leq \infty$ .

Assume  $Z_i$  is not homeomorphically minimal. If a graph is not homeomorphically minimal then we should be able to remove a vertex of degree 2 and not change the graph.  $Z_i$  contains vertices of degree 2. Removing a degree 2 vertex from  $Z_i$  results in multiple edges, which is not allowed since we are investigating simple graphs. In  $Z_2$ , we can remove vertex 3, but then we have two edges connecting vertices 2 and 4, as shown in Figure 5.6.

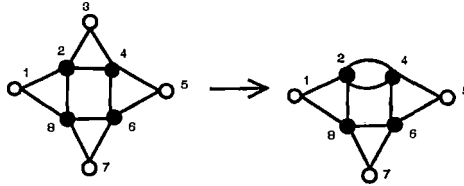


Figure 5.6: Removing a vertex of degree 2

Thus,  $Z_i$  is homeomorphically minimal,  $\forall i$ , where  $1 \leq i \leq \infty$ .

$Z_i$  satisfies the three conditions of 4-cutwidth critical. Therefore,  $Z_i$  is 4-cutwidth critical,  $\forall i$ , where  $1 \leq i \leq \infty$ .

□

## Chapter 6

# Conclusion

In conclusion, we have found infinitely many 4-cutwidth critical graphs. However, there are several that are finite in their type of classification. For example, there are only four 4-cutwidth critical trees. These can be found in Appendix A and are denoted as  $F_1$ ,  $F_2$ ,  $F_3$  and  $F_4$ .

We also have 4-cutwidth critical graphs that only contain one cycles, but the cycles do not share edges. These graphs are denoted  $F_5$ ,  $F_6$ ,  $F_7$ ,  $F_{10}$  and  $F_{11}$ .

$F_8$ ,  $F_9$ ,  $F_{15}$ , and  $F_{18}$  are all 4-cutwidth critical graphs that contain two cycle, but the cycles share one edge.  $F_{13}$  also contains two cycles, but they only share a vertex. There were very few graphs that contained three cycles.  $F_{14}$ , has three cycles that share a vertex.  $F_{17}$  and  $F_{13}$  also have three cycles, but the have a common vertex and edges.

It still remains to be shown that these are indeed the *only* 4-cutwidth critical graphs. But if someone decides to further investigate k-cutwidth critical graphs, the methods previously mentioned can be used. For example, if we were to look for a 5-cutwidth critical graph, we can use method 1 and simply attach two edges at the vertex of highest degree. This would result in  $K_{1,9}$ . By Proposition 1.11, this is a 5-cutwidth critical graph.



## Appendix A

### 4-cutwidth critical graphs

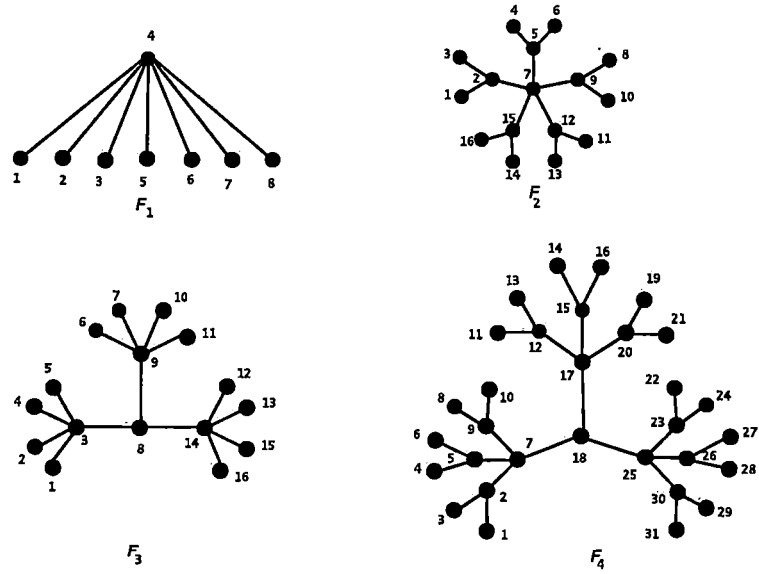


Figure A.1: 4-cutwidth critical trees

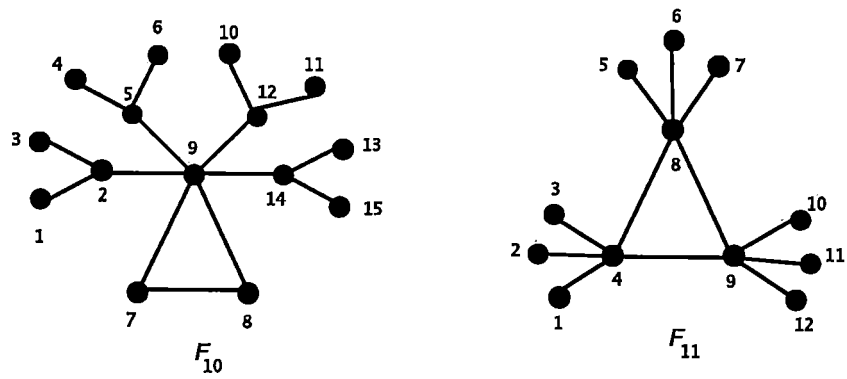


Figure A.2: 4-cutwidth critical graphs with one cycle

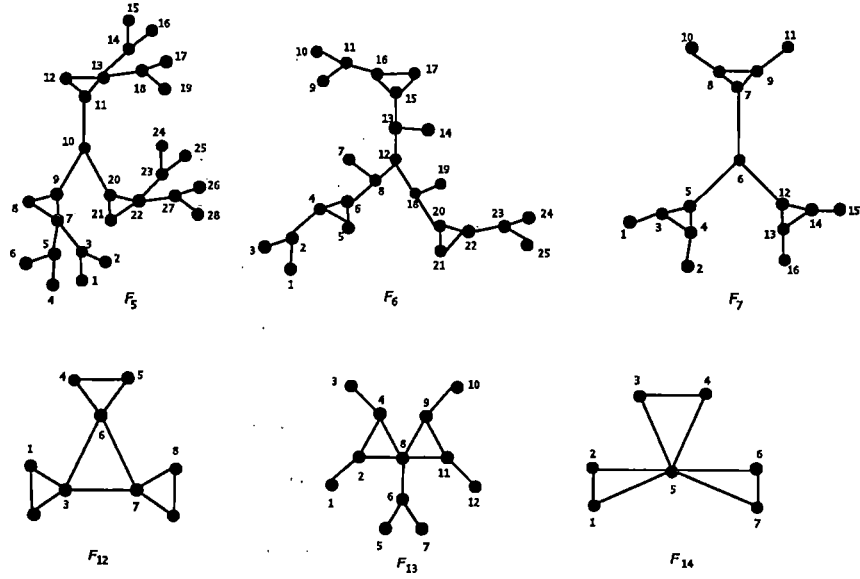


Figure A.3: 4-cutwidth critical graphs with multiple cycles that do not share edges

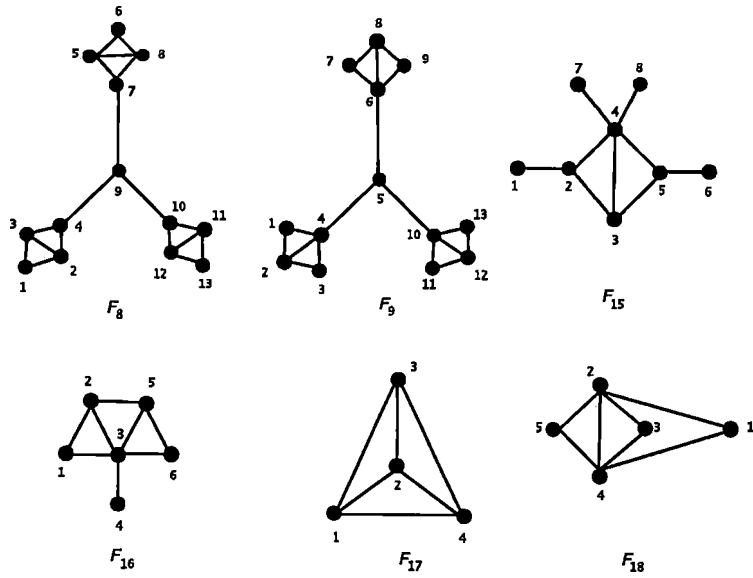


Figure A.4: 4-cutwidth critical graphs with multiple cycles that share edges

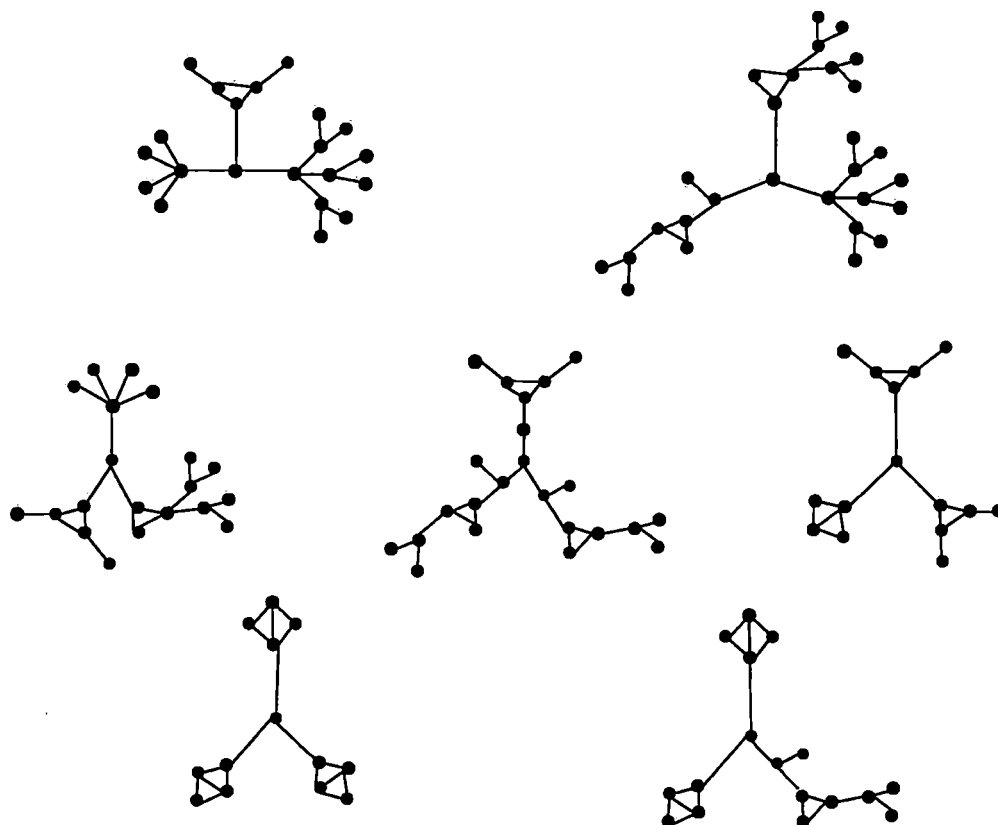


Figure A.5: Combinations of 4-cutwidth critical graphs

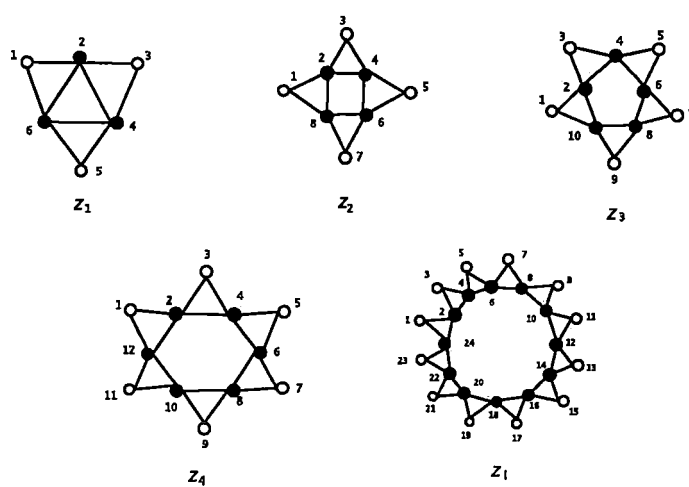


Figure A.6: Family of 4-cutwidth critical graphs

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