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# The evolution of equation-solving: Linear, quadratic, and cubic

Annabelle Louise Porter

# THE EVOLUTION OF EQUATION-SOLVING:

# LINEAR, QUADRATIC, AND CUBIC

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A Project

Presented to the

Faculty of

California State University,

San Bernardino

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In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Teaching:

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ABSTRACT

Algebra and algebraic thinking have been cornerstones of problem solving in many different cultures over time. Since ancient times, algebra has been used and developed in cultures around the world, and has undergone quite a bit of transformation. This paper is intended as a professional developmental tool to help secondary algebra teachers understand the concepts underlying the algorithms we use, how these algorithms developed, and why they work. It uses a historical perspective to highlight many of the concepts underlying modern equation solving. The paper includes suggestions of some ways to use historical approaches to not only enhance an algebra course, but to help students improve algebraic thinking and understand the deep-rooted connections between algebra and geometry. In addition, it will provide resources and references for those teachers wishing to explore the topic further.

ACKNOWLEDGEMENTS

I would like to thank my family for their encouragement and help in completing this program. In particular, I would like to thank my husband Scot, who provided me with the support I needed to succeed. The camaraderie I experienced with my fellow master's candidates made this experience more than I could have hoped for. I would like to thank Shawnee McMurran, my advisor, for the time and effort she put in to making my project possible.

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# INTRODUCTION

## Project Overview

Humans developed computations for several reasons. The need to track business transactions and the need to keep track of time were two primary purposes for the evolution of calculations. Humans used mathematical calculations for a variety of practical applications. The need to be able to keep count of animals in a herd, exchange money, deal with property issues, and keep a calendar in order to know when to plant crops are just a few such applications.

Over time humans began to develop increasingly sophisticated methods for calculations. Ancient Egyptians employed the method of false position to solve applied problems in which the variables vary directly. They used a simple technique involving proportional reasoning to find the unknown variable. Evidence of this can be found in Egyptian scrolls such as the *Rhind Papyrus* and the *Moscow Papyrus*, dated as far back as 1650 BC. This technique for equation-solving marks one of the earliest records of algebra.

Algebra got its name from the Arabic word al-jabr employed in the title of a book written in Baghdad in 825 AD by Mohammed ibn-Mūsa Al-Khwārizmī. Loosely translated, the title of his book *Hisab al-jabr w'al-mugabalah* means "the science of transposition and cancellation." Early algebra focused on equations and solution techniques. It developed over a period of approximately 3500 years. Beginning in 1700 BC, there is evidence of the development of symbolic notation and methodical equation solving. Modern algebra has expanded into abstract topics including groups, rings, and fields. The history of mathematics allows us to trace the evolution of algebra and provides a means by which we can make connections between concrete and abstract algebraic ideas.

Although much of the ancient history of mathematics has been lost, the earliest evidence of algebraic thinking appears to come from Babylonia. Cuneiform clay tablets dating back to King Hammurabi in 1700 BC show evidence of calculations with the area and perimeter of a rectangle. The tablets demonstrate that it is possible to calculate the length and width of a rectangle given its area and perimeter. The method used here involves a parameter which is used to describe each of the unknowns. This method of introducing a third unknown as a parameter differs significantly from the methods of substitution or elimination that are taught in contemporary algebra classes.

Much of our knowledge regarding equation-solving begins with the ancient Babylonians. The method of "false-position" is one of the earliest means by which people solved linear equations (Berlinghoff & Gouvea, 2004). This method bears similarities to the "guess-and-check" method taught in many beginning algebra classes. The method of "false position" employs a clever application of proportional reasoning. However, this method can be used only on equations involving variables that vary directly with each other. The method of false position was extended to that of "double false position" (Berlinghoff & Gouvea, 2004). This method also relied on proportional reasoning, but could be applied to systems of two linear equations with two unknowns. Variations of this method existed in other cultures over time. For example, the Chinese method of surplus and deficit is essentially the same as double false position. The Babylonians also employed a variation of the method.

Mohammed ibn-Mūsa Al-Khwārizmī moved beyond linear equations and into solving quadratic equations. He divided quadratic equations into six types. He then devised a method for solving each type, including familiar ideas like completing the square. Girolamo Cardano (~1545 AD) spent many years of his life investigating cubic equations. After many years of struggle and family turmoil, Cardano established a method to solve the general cubic

$$0 = ax^3 + bx^2 + cx + d$$

and the many variations of depressed cubics in his book *Ars Magna*.

Although solution algorithms for linear, quadratic, and cubic equations have evolved over time, the underlying concepts are similar to what they were in the beginning. One of the most interesting and beneficial applications of having a historical perspective to mathematics is to help both students and teachers understand why certain algorithms work, discover how those algorithms might be derived, and identify their underlying concepts.

## Literature Review

Numerous texts trace the history of mathematics. Many of these texts present an overview of mathematical developments. A few will select a main idea, and investigate it thoroughly. However, it is difficult to find a source that specifically traces the development of equation solving and its applications to the secondary classroom.

*Math through the Ages* (Berlinghoff & Gouvea, 2004) is an excellent book from which to learn the history of some key mathematical ideas. The text focuses on a few main ideas, and expands upon them. Specifically, it provides interesting stories and histories on people. However, it does not show most of the actual work that was needed to derive the formulae and ideas presented. On the other hand, *Journey through Genius* (Dunham, 1990) provides many of the proofs and derivations of formulae in addition to interesting background information. However in this book, the focus of each chapter is a specific theorem, rather than the evolution of a mathematical idea. Swetz's (1994) *From Five Fingers to Infinity* provides a broader range of topics of historical mathematics. Swetz de-emphasizes individuals, and presents the materials by geographic location and time. For instance, one chapter is specific to the evolution of mathematics in ancient China. This presentation style is quite valuable in getting such a large amount of information across. However, it lacks the interesting personal stories present in books such as *Math through the Ages* and *Journey through Genius* that can motivate the reader to investigate a topic further. *The Historical Roots of Elementary Mathematics* (Bunt, Jones, & Bedient, 1976) is very similar in style and information to *Math through the Ages*. Both books present information in short chapters specific to a main idea (e.g. Greek numeration systems). In addition, both books cover a wide range of topics that are broken down by date. However, *The Historical Roots of Elementary Mathematics* does not delve into the stories describing the people behind the discoveries. The four volume collection *The World of Mathematics* (Newman, 1956) consists of individual articles compiled together in an effort to convey the "…diversity, the utility and the beauty of mathematics" (Newman, iii). Newman attempted to show the richness and range of mathematics. This collection spans ideas from the Rhind Papyrus to the "Statistics of Deadly Quarrels" (Newman, p 1254). *The World of Mathematics* presents an amazingly broad view of the many applications of mathematics to the sciences. *An Introduction to the History of Math* (Eves, 1956) covers the same topics as several of the other books, in much the same manner. It traces the development of mathematics from numeration systems through to the development of calculus. It includes specific information of the individuals that developed many of the critical ideas in the history of mathematics. Boyer's (1968) *A History of Mathematics* is almost entirely about Greek mathematics. It covers ancient Greek mathematics to a degree that none of the other mentioned texts do.

Perhaps one of the most valuable tools for a secondary teacher available is *Historical Topics for the Mathematics Classroom* (National Council for Teachers of Mathematics, 1989). This text consists of a series of "capsules" (short chapters). Each capsule gives a brief historical overview of a particular topic (e.g. Napier's Rods). The capsules are grouped by general topic (algebra, geometry, trigonometry, etc.). Specifically, this text provides a historical context to graphical approaches to equation-solving. In addition, it provides a concise overview of the methods employed to solve quadratics and cubics. The people that developed these methods are named, though little is said about their personal history.

The many texts available on the history of mathematics all attempt to convey an enormous amount of information in different ways. Some briefly describe many of the contributions that people have made to mathematics. Others describe the contributions of a culture, paying less attention to individuals (thereby allowing more time for the derivations of formulae). Although many texts include the evolution of equation solving in their exposition, such material is often spread throughout the text. In addition, most texts are not geared specifically for secondary teachers. I believe my project will complement this body of knowledge. As opposed to covering the breadth of mathematics, I will focus on equation solving in a way that will connect to the secondary classroom. My intention is to provide a source that can help secondary teachers understand where their textbook formulae came from and to familiarize teachers with some of the people and stories that contributed to the development of the mathematics we use today. I would like to demonstrate that the problem solving techniques for equations that are used in the secondary high school curriculum today did not simply fall from the sky! Rather, modern methods for solving equations took time, dedication and effort to evolve.

It is critical for teachers to understand the history behind their topic in order for them to make educated decisions regarding the manner and order of the presentation of material. These decisions impact students' motivation and curiosity, which in turn impacts the amount of information they absorb in a meaningful manner. Understanding the history of the evolution of equation solving will help teachers make decisions on how to present information to their students in the secondary school.

# CHAPTER TWO

# LINEAR EQUATION-SOLVING

## Historical Overview

Humans have been solving linear equations for centuries. Linear equations arise naturally when applying mathematics to the real world (Berlinghoff & Gouvea, 2004). The *Rhind Papyrus* was written by the scribe Ahmes in approximately 1650 BC. This document gives evidence of Ancient Egyptian linear word problems and their solutions. The solutions to these problems are not derived in a manner that most mathematics students would recognize today. The following algorithm is often taught in a one year algebra course: 1) label a variable 2) write an equation 3) perform the "order of operations" in reverse in order to isolate the variable. However, in Ancient Egypt, scribes used the method of "false position." First, the scribe would "posit" (guess) a possible solution to the word problem. This guess was usually some convenient value to work with and need not be anywhere near the correct solution. He would then determine the result yielded by his guess. If he did not guess the correct solution, he would calculate the ratio he would need to multiply his incorrect result by in order to attain the correct result. He would then multiply the original guess by that ratio.

Proportional reasoning played a key role in the method of false position. Problem 26 from the *Rhind Papyrus* illustrates this idea well. "Find a quantity such that when it is added to one quarter of itself, the result is 15." The solution using typical modern algorithms would begin by defining a variable to represent the unknown quantity. A common choice for this variable is $x$. Then, the problem may be represented algebraically by the equation

$$x+\frac{1}{4}x=15\,.$$

So, by combining like terms, the equation becomes

$$\frac{5}{4}x=15\,.$$

Multiplying both sides of the equation by four fifths yields

$$x=15\cdot\frac{4}{5}\,.$$

Thus, the unknown quantity is 12.

Compare this with the solution using the method of "false position." Make a convenient guess. A convenient guess for this example would be some multiple of 4. Let the guess, G, be 16. Calculate the result using this guess in the problem statement: When the quantity of 16 is added to one quarter of itself (i.e. 4) the result is 20. The symbolic representation of this statement is

$$16+\frac{1}{4}(16)=16+4=20$$

The proportion by which this result should be multiplied in order to get the correct solution of 15 is fifteen-twentieths:

$$20\cdot\frac{15}{20}=15 .$$

Now multiply the original guess by this proportion:

$$16\cdot\frac{15}{20}=12 .$$

Thus, the unknown quantity is 12.

In general, the algorithm for the solution to a linear equation using false position can be demonstrated as follows. Note that this method works only if the variable is directly related to the result. Therefore, for illustrative purposes, first let the word problem be represented by the linear equation

$$Mx=N .$$

Remember that this algebraic shorthand would not have been employed at the time.

1. Make a guess, $G$. Typically, $M$ could be represented by a ratio and this guess would be a multiple of the denominator of $M$. However, any guess will do.
2. Calculate the result with the guess: $M \cdot G$
3. If $MG$ is not equal to the desired result $N$, then $G$ is not the correct solution. The proportion by which $MG$ should be multiplied to achieve the desired result $N$ is given by $\frac{N}{MG}$. Indeed, we see

$$MG \cdot \frac{N}{MG} = N .$$

4. Multiply the original guess by this proportion to find the correct solution:

$$x = G \cdot \frac{N}{MG} = \frac{N}{M} .$$

Using modern notation, it is clear that this is the correct solution to the equation

$$Mx = N .$$

The calculations provided above should indicate why this method will work for all linear equations where there is a direct proportional relationship between the input and the output.

We can illustrate the underlying concept of proportions geometrically with similar triangles.

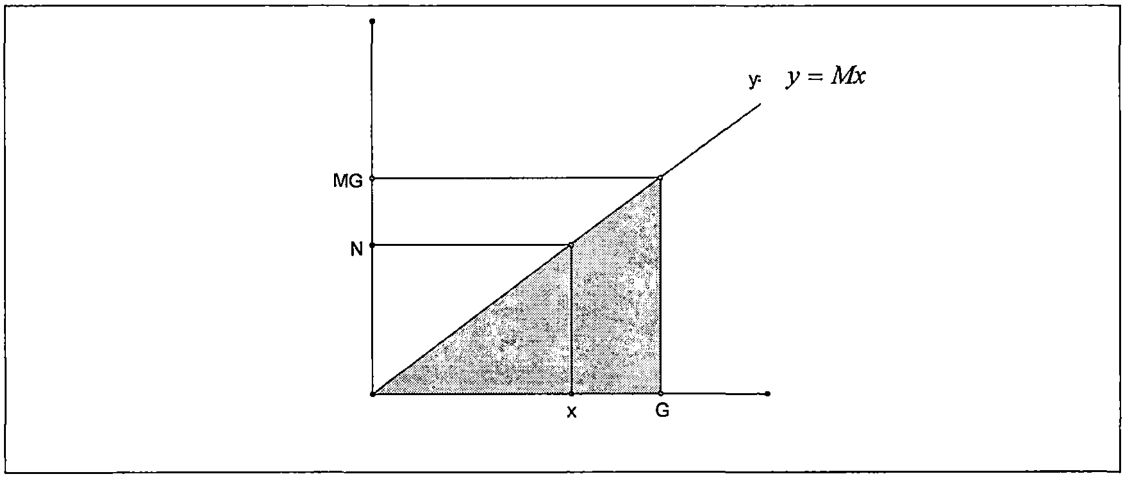



Figure 1. False Position Using Similar Triangles

Consider the line $y = Mx$. We are looking for the $x$-coordinate corresponding to the $y$-coordinate $N$. We guess an $x$-coordinate and find the corresponding y coordinate on the line. The point $(G, MG)$ lies on the line $y = Mx$. Thus, it becomes possible to create similar right triangles, using the proportionality of corresponding legs to obtain

$$\frac{G}{MG} = \frac{x}{N}.$$

So $x$ must be given by

$$G \cdot \frac{N}{MG}.$$

The idea of applying ratios to solve mathematical problems was not unique to the ancient Egyptians. Chinese scholars produced the text *The Nine Chapters on the Mathematical Art*. This text was edited by Liu Hui in 236 A.D., though the time of its origin is still in question (Berlinghoff & Gouvea, 2004). It seems to have originated sometime between 1100 B.C. and 100 B.C. "Proportionality seems to have been a central idea for these early Chinese mathematicians, both in geometry (e.g. similar triangles) and in algebra (e.g. solving problems by using proportions)" (Berlinghoff & Gouvea, 2004). The original text contains problems and solutions, as did the *Rhind Papyrus*, but Liu Hui added commentary and justifications for the solutions.

Proportionality was the main concept applied by ancient scholars when solving linear equations, particularly in the method of false position. Variations on the method of false position were employed for more complex linear equations. One such method is that of "double false position." This method will give solutions for problems that can be represented by the (modern) equation $Mx + B = N$, and solutions for problems that can be represented by systems of two equations and two unknowns. This method was so effective, that mathematicians continued to use it even after the advent of the algebraic notation that provided the means to efficiently write equations (Berlinghoff & Gouvea, 2004). In addition, one of the current benefits to using the method of "double false position" is that many students have trouble writing an algebraic equation from a word problem. However, most can substitute values in to see if they work. This method ties in to what is currently called "guess and check," which can be an intermediate step in going from a word problem to an equation.

The method of double false positions follows a format similar to that of false position, however, the method requires two guesses. Given a problem that can be represented in the form $Mx + B = N$. Begin by making a guess,

$G_1$, for the solution. Calculate the result and compare it to $N$. If it is not the correct solution, calculate the magnitude of the difference, $E_1$, between the result and $N$.[1] Now, make a second guess, $G_2$. If it is not the correct solution, calculate the magnitude of difference, $E_2$, between the result and N. In order to find the correct solution, use both guesses and errors in the following way:

If both guesses yield either underestimates (less than the result desired) or overestimates (greater than the result desired), then the formula to find the solution is given by:

$$x = \frac{E_1 \cdot G_2 - E_2 \cdot G_1}{E_1 - E_2}.$$

If one guess yields an underestimate (less than the result desired) and the other guess yields an overestimate, (greater than the result desired), then the formula to find the solution is given by:

$$x = \frac{E_1 \cdot G_2 + E_2 \cdot G_1}{E_1 + E_2}.$$

[1] Mathematicians did not generally acknowledge the use of negative numbers until the $17^{th}$ century AD, hence only positive errors would be considered.

The latter formula is used as a means to avoid dealing with negative numbers (Berlinghoff, 2005, p 123).

It is possible to display these general solutions geometrically with similar triangles. In the figure below, the correct solution, x, yields the correct result in the linear relationship y=mx+b. Both guesses are overestimates (and could similarly have been underestimates).

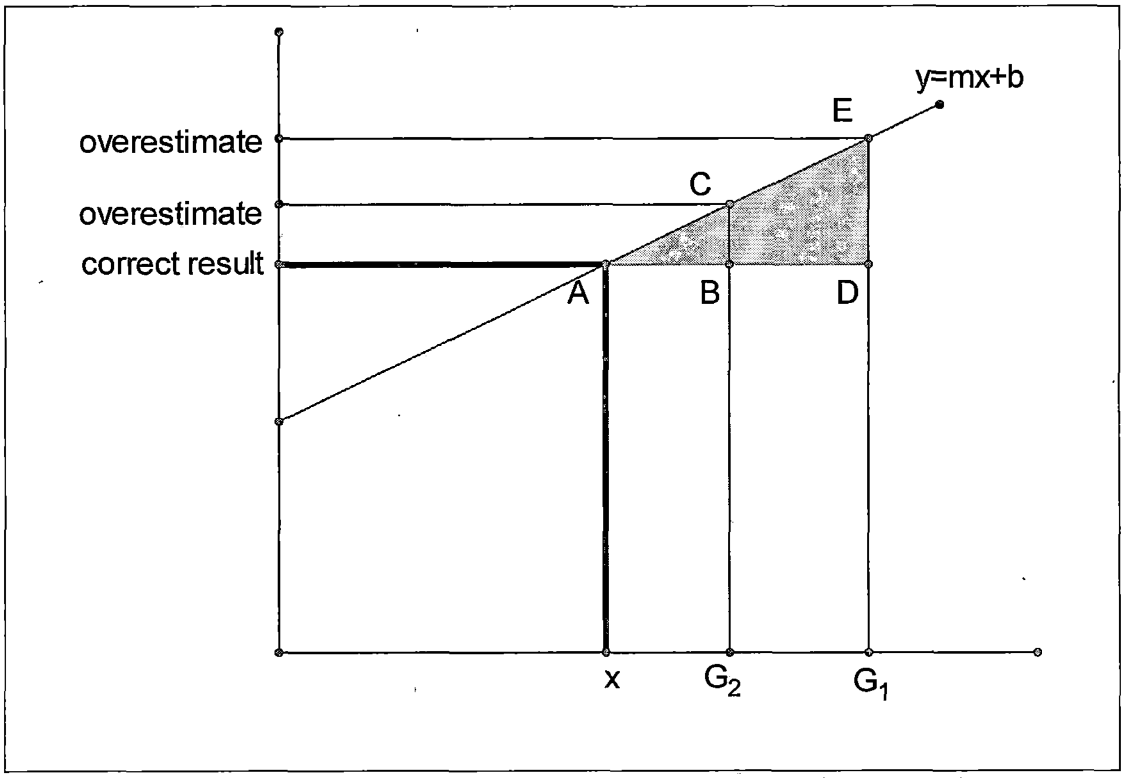



Figure 2. Double False Position Using Similar Triangles (Different Types)

Triangles ABC and ADE are similar because all of their angles are congruent. Each of the following values can be derived from this diagram:

DE = the difference between the correct result and the first guess = error 1= $E_1$

BC = the difference between the correct result and the second guess = error 2= $E_2$

AD = $G_1$-x

AB = $G_2$-x

Thus, the proportion

$$\frac{E_1}{G_1 - x} = \frac{E_2}{G_2 - x}$$

can be created.

Simplification of this proportion yields the familiar equation

$$x = \frac{E_2 G_1 - E_1 G_2}{E_1 - E_2}.$$

In Figure 3 the correct solution, x, yields the correct result in the linear relationship y=mx+b. The first guess is an overestimate, while the second is an underestimate.

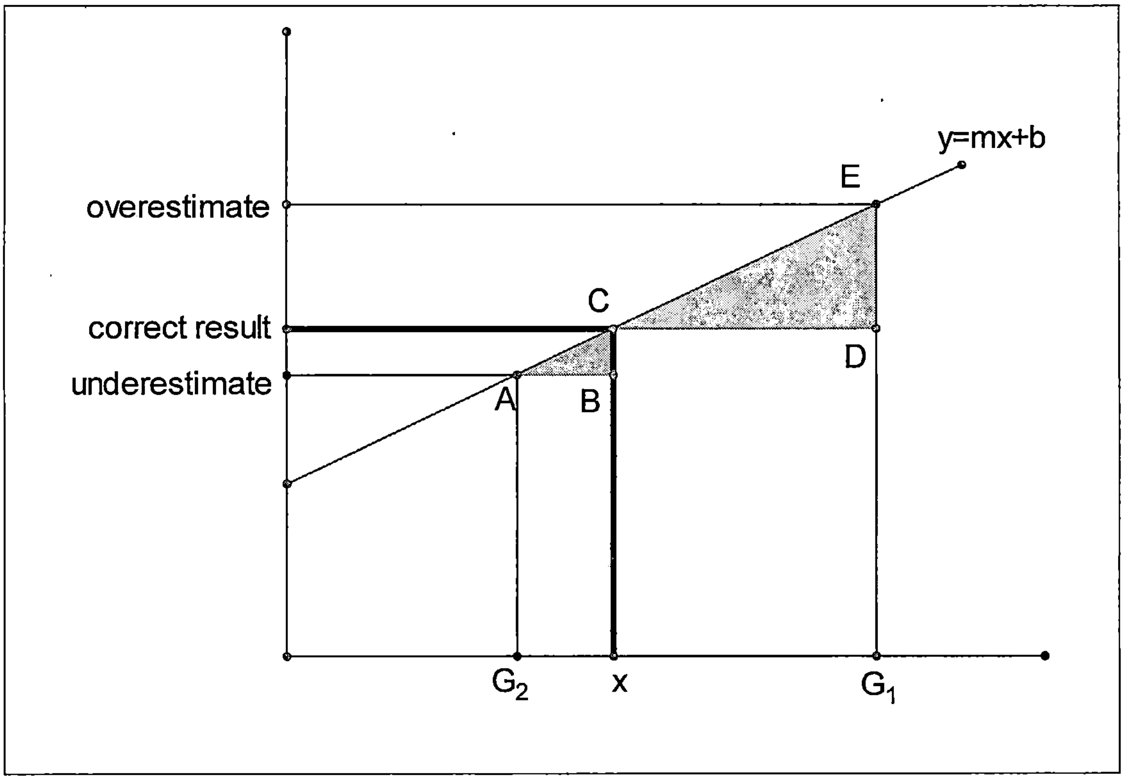



Figure 3. Double False Position Using Similar Triangles (Same Types)

Triangles ABC and CDE are similar because all of their angles are congruent. Each of the following values can be derived from this diagram:

DE = the difference between the correct result and the first guess = error 1 = $E_1$

BC = the difference between the correct result and the second guess = error 2 = $E_2$

BD = $G_1$-x

AB = $x - G_2$

Thus,

$$\frac{E_2}{x-G_2}=\frac{E_1}{G_1-x}.$$

Simplification of this proportion yields

$$x=\frac{E_2G_1+E_1G_2}{E_1+E_2}.$$

Analytic geometry provides another way of interpreting this solution algorithm in modern terms. If these pieces of data were viewed as points, the correct solution could be found using the concept of slope. Let $(x_1, y_1)$ be the first guess and its result. Let $(x_2, y_2)$ be the second guess and its result. Finally, let $(x, y)$ be the correct solution and its result. These three points are collinear, as the solutions are found by performing the same linear operations on each of the guesses. In particular, they each lie on the line with slope $M$ and $y$-intercept $B$. Since the first guess $(x_1, y_1)$ and the correct solution $(x, y)$ lie on the same line,

$$M=\frac{y-y_1}{x-x_1}.$$

Since the second guess $(x_2, y_2)$ and the correct solution $(x, y)$ lie on the same line,

$$M = \frac{y - y_2}{x - x_2}.$$

Since each of these ratios is equal to the same constant, then

$$\frac{y - y_1}{x - x_1} = \frac{y - y_2}{x - x_2}.$$

This equation can be simplified in an effort to solve for $x$, the correct solution.

$$\frac{y - y_1}{x - x_1} = \frac{y - y_2}{x - x_2}$$

Multiplying both sides by $(x - x_2)(x - x_1)$ yields

$$(y - y_1)(x - x_2) = (y - y_2)(x - x_1).$$

Distributing gives

$$x(y - y_1) - x_2(y - y_1) = x(y - y_2) - x_1(y - y_2).$$

By regrouping the terms it follows that

$$x_1(y - y_2) - x_2(y - y_1) = x(y - y_2) - x(y - y_1).$$

Isolating x provides

$$\frac{x_1(y - y_2) - x_2(y - y_1)}{((y - y_2) - (y - y_1))} = x.$$

Finally, rewrite the equation to find

$$\frac{(y-y_1)x_2-(y-y_2)x_1}{((y-y_1)-(y-y_2))}=x\ .$$

In terms of the original guesses and their errors, the final result can be represented as follows

$$x=\frac{E_1\cdot G_2-E_2\cdot G_1}{E_1-E_2}\ .$$

The ancient scholars that developed this method used the concept of proportionality to derive their solutions, since in linear equations the change in the output is proportional to the change in the input (Berlinghoff & Gouvea, 2004, p124). This method is similar to the method of "surplus and deficiency" found in the ancient Chinese texts. However, this Chinese method employed the use of one overestimate (surplus) and one underestimate (deficiency).

Babylonian sources illustrate another variation of false position in the solution of linear equations. In some ways the method provides a connection between "false position" and "double false position." The method involves making one guess (as in false position), but also calculating the result if the guess were increased by one unit (in essence, making a second fixed guess). In this Babylonian variation of false position the solver makes a guess, finds the result, and then calculates the error. Then, the guess is increased by one unit, and the difference in the amount of error is observed. Finally, the proportion by which the change in the error would need to be multiplied in order to decrease the original error to zero is calculated. The solution is now obtained by adding this proportion to the original guess.

Figure 4 depicts a line ($y=mx+b$). The unknown is the correct $x$-coordinate that yields the desired result ($y_0$). Each increase by one on the x axis results in an increase by the amount of the slope on the y axis. Remember that slope is calculated by "rise over run." In this variation of false position, the "run" will always be one. Thus, to reach the desired result, the question is to find out how many "slopes" need to be added to go from the original guess to the correct solution.

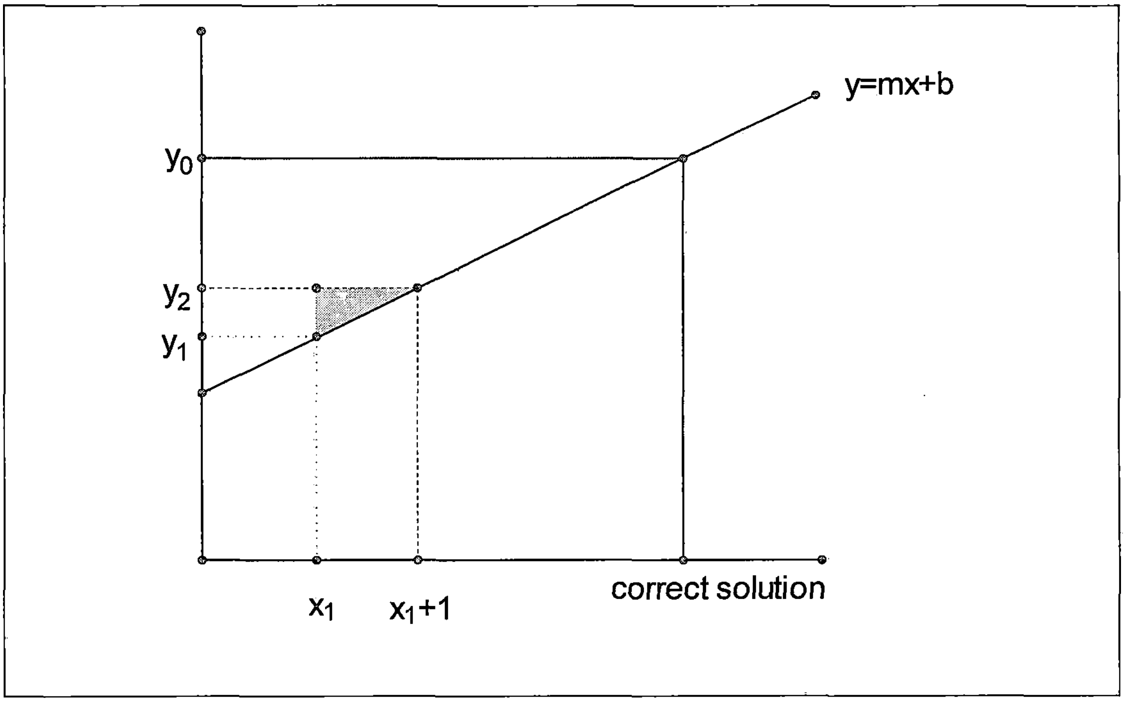


Figure 4. The Babylonian Variation on False Position

These methods of solving linear equations would not be familiar to most secondary school students today. However, with a little time and effort, students would learn to appreciate where the algorithms that are used today came from. These methods would reinforce and give a deeper understanding of proportional reasoning concepts underlying modern algorithms. These methods and their geometrical interpretations might also be introduced into the curriculum for students that are struggling with the modern algorithms, as alternative ways to solve and visualize the solutions of linear equations.

## Applications to the Classroom

Here is an example of a problem that the Ancient Egyptians solved using the method of false position in approximately 1650 BC. Each of the problems below will contain both the historical and modern approaches to the solution.

Problem 1: From *The Rhind Papyrus.*[2]

A quantity; its half and its third are added to it. It becomes 10.

The Solution: Using current algorithms.

Let x be the unknown quantity.

Writing the word problem as an equation using the variable x would yield

$$x+\frac{1}{2}x+\frac{1}{3}x=10.$$

Combining like terms gives

$$\frac{11}{6}x=10.$$

[2] Berlinghoff, W.& Gouvea, F. (2004). Math through the Ages. Farmington, ME: Oxton House Publishers.

Multiplying both sides by the reciprocal of $\frac{11}{6}$

isolates x in

$$x = 10 \cdot \frac{6}{11}.$$

Simplifying leads to

$$x = \frac{60}{11} = 5\frac{5}{11}.$$

Thus, the unknown quantity is $5\frac{5}{11}$.

<u>The Solution</u>: Using "False Position."

Make a guess (using a number that will work easily with the denominators 2 and 3):

$$G = 6$$

Calculate the result using the guess. Substituting the guess into the problem yields

$$6 + \frac{1}{2}(6) + \frac{1}{3}(6).$$

This simplifies to

$$6 + 3 + 2 = 11.$$

Calculate the ratio by which 11 would be multiplied to get the correct result of 10

$$11 \cdot \frac{10}{11} = 10.$$

Multiplying the original guess by this ratio leads to

$$6 \cdot \frac{10}{11}.$$

Thus, the correct solution is $\frac{60}{11} = 5\frac{5}{11}$.

Here is another example that can be solved using the method of "false position."

Problem 2: From *The Rhind Papyrus (Problem 26)*.[3]

When a quantity is added to one-fourth of itself the result is 15.

The Solution: Using current algorithms.

Let x be the quantity. Then, writing an equation to represent the word problem gives

$$x + \frac{1}{4}x = 15.$$

Combining like terms leads to

$$\frac{5}{4}x = 15.$$

[3] Katz, V. (1998). A History of Mathematics: An Introduction. Reading, MA: Addison-Wesley Educational Publishers, Inc.

Multiplying both sides by the reciprocal of $\frac{5}{4}$ results in

$$x = 15 \cdot \frac{4}{5}.$$

Thus, the unknown quantity is 12.

The Solution: Using "false position."

Make a convenient guess (using a number that is a multiple of the denominator 4):

$$G = 8.$$

Calculate the result using the guess. Substituting the guess into the problem yields a problem statement: When the quantity eight is added to one-fourth of itself (i.e. 2) the result is 10. This statement can be represented as

$$8 + \frac{1}{4}(8).$$

This simplifies to

$$8 + 2 = 10.$$

Calculate the ratio by which 10 would be multiplied to get the correct result of 15

$$10 \cdot \frac{15}{10} = 15.$$

Multiplying the original guess by this ratio leads to

$$8 \cdot \frac{15}{10}.$$

Thus, the correct solution is $\frac{120}{10} = 12$.

Here is an example of "double false position" from the early 1800's.

Problem 3: From *Daboll's Schoolmaster's Assistant.*[4]

A purse of 100 dollars is to be divided among four men A,B,C, and D, so that B may have four more dollars than A, and C eight more dollars than B, and D twice as many as C; what is each one's share of the money?

The solution: Using current algorithms.

Let A receive $x$ dollars. Then B receives $x+4$, C receives $x+4+8$, and D receives $2(x+4+8)$. Then, writing an equation to represent the word problem gives

$$x+(x+4)+(x+4+8)+(2(x+4+8))=100.$$

Combining like terms leads to

$$5x+40=100.$$

[4] Berlinghoff, W.& Gouvea, F. (2004). Math through the Ages. Farmington, ME: Oxton House Publishers.

Subtracting 40 from both sides yields

$$5x = 60 \text{ .}$$

Multiplying both sides by the reciprocal of 5 results in

$$x = 60 \cdot \frac{1}{5} \text{ .}$$

Thus A received $12, B received $16, C received $24, and D received $48.

<u>The Solution</u>: Using "Double False Position."

Make a guess of how much money A receives:

$$G_1 = 6$$

Calculate the result using this guess:

$$6 + (6+4) + (6+4+8) + (2(6+4+8)) = 70 \text{ .}$$

This is an underestimate by 30. So, the error ( $E_1$ ) is 30.

Make a second guess: $G_2 = 8$

Calculate the result using this guess:

$$8 + (8+4) + (8+4+8) + (2(8+4+8)) = 80 \text{ .}$$

This is an underestimate by 20. So, the error ( $E_2$ ) is 20.

Since the two errors are the same type (both underestimates), use the formula for double false position that is appropriate:

$$\text{The solution}=\frac{E_1\cdot G_2-E_2\cdot G_1}{E_1-E_2}.$$

Substitution yields

$$\text{The solution}=\frac{30\cdot 8-20\cdot 6}{30-20}.$$

Simplifying leads to

$$\text{The solution}=\frac{120}{10}.$$

So, the solution is 12.

Thus A received $12, B received $16, C received $24, and D received $48.

An example using the Chinese method of "surplus and deficiency."

Problem 4: From *Jiuzhang (Problem 17)*.[5]

The price of 1 acre of good land is 300 pieces of gold; the price of 7 acres of bad land is 500. One

[5] Katz, V.(1998). A History of Mathematics: An Introduction. Reading, MA: Addison-Wesley Educational Publishers, Inc.

has purchased altogether 100 acres; the price was 10,000. How much good land was bought and how much bad?

<u>The Solution</u>: Using current algorithms.

Let the amount of good land be $x$ acres. Let the amount of bad land be $y$ acres. The price of $x$ acres of good land is:

$$x \text{ acres}\cdot\frac{300}{1}\text{ gold pieces per acre} = 300x \text{ gold pieces.}$$

The price of $y$ acres of bad land is:

$$y \text{ acres}\cdot\frac{500}{7}\text{ gold pieces per acre} = \frac{500}{7}y \text{ gold pieces.}$$

The total cost would then be

$$300x + 500\cdot\frac{y}{7}.$$

Thus, the following system of equations can be developed

$$x + y = 100$$
$$300x + 500\cdot\frac{y}{7} = 10000.$$

Solving the first equation for one variable leads to

$$x = 100 - y.$$

Substituting this equation into the second equation gives

$$300(100-y)+500\cdot\frac{y}{7}=10000 .$$

Distributing achieves

$$30000-300y+\frac{500}{7}y=10000 .$$

Combining like terms leads to

$$20000=\left(300-\frac{500}{7}\right)y .$$

This can be simplified to

$$20000=\frac{1600}{7}\cdot y .$$

Finally, multiplying both sides by the reciprocal of $\frac{1600}{7}$ provides

$$20000\cdot\frac{7}{1600}=y .$$

Thus, the solution for y is

$$y=\frac{140000}{1600}=87.5 .$$

Substituting this value back into the first equation (after having solved it for x) yields

$$x=100-y=100-87.5=12.5 .$$

Thus, the amount of good land is 12.5 acres and the amount of bad land is 87.5 acres.

The Solution: Using the Chinese method of "surplus and deficiency."

Begin by making a guess for the amount of good land:

$$G_1 = 5 .$$

Calculate the amount of bad land:

$$y = 100 - 5 = 95 .$$

Now calculate the yield based on the amounts of land:

$$300(5) + 500 \cdot \frac{95}{7} = \frac{5800}{7} .$$

This is an underestimate by $\frac{12000}{7}$, a "deficiency." Now make a guess that might give an overestimate, a "surplus."

$$G_2 = 20 .$$

Calculate the amount of bad land:

$$y = 100 - 20 = 80 .$$

Now calculate the yield based on the amounts of land

$$300(20) + 500 \cdot \frac{80}{7} = \frac{82000}{7} .$$

This is an overestimate by $\frac{12000}{7}$, a "surplus."

Thus, to solve the problem, use the formula

$$\frac{E_1 G_2 + E_2 G_1}{E_1 + E_2} .$$

Substitution yields

$$\frac{E_1G_2+E_2G_1}{E_1+E_2}=\frac{\left(\frac{1200}{7}\right)(5)+\left(\frac{1200}{7}\right)(20)}{\left(\frac{1200}{7}\right)+\left(\frac{1200}{7}\right)}=12.5\ .$$

Thus, there are 12.5 acres of good land, and 87.5 acres of bad land.

Here is an example of a problem that was solved using the Babylonian variation of "false position."

<u>Problem 5</u>: From the *VAT 8389 (Problem 76)*. [6]

One of two fields yields $\frac{2}{3}$ sila per sar, the second yields $\frac{1}{2}$ sila per sar (sila and sar are measures for capacity and area, respectively). The yield of the first field was 500 sila more than that of the second; the areas of the two fields were together 1800 sar. How large is each field?

<u>The Solution</u>: Using current algorithms.

Let the first field have an area of $x$ sar. Let the second field have an area of $y$ sar. Then, $x$ and $y$ need

[6] Katz, V. (1998). <u>A History of Mathematics: An Introduction.</u> Reading, MA: Addison-Wesley Educational Publishers, Inc.

to satisfy the following system of equations

$$x + y = 1800$$
$$\frac{2}{3}x - \frac{1}{2}y = 500.$$

Solving the first equation for one variable leads to

$$x = 1800 - y\ .$$

Substituting this equation into the second equation gives

$$\frac{2}{3}(1800 - y) - \frac{1}{2}(y) = 500\ .$$

Distributing achieves

$$1200 - \frac{2}{3}y - \frac{1}{2}y = 500\ .$$

Combining like terms and simplifying leads to

$$700 = \frac{7}{6}y\ .$$

Finally, multiplying both sides by the reciprocal of $\frac{7}{6}$ provides

$$700 \cdot \frac{6}{7} = y\ .$$

Thus, the y is 600.

Substituting this value back into the first equation (after having solved it for x) yields

$$x = 1800 - y = 1800 - 600 = 1200\ .$$

Thus, the first field was 1200 sar, and the second field was 600 sar.

The Solution: Using the Babylonian variation on "false position."

Make a guess: Let the first field be 900 sar.

Calculate what the second field must have based on the fixed amount of their sum. Since their sum is 1800 sar, the second field must be 900 sar. Calculating the difference in their yields results in

$$\frac{2}{3}(900)-\frac{1}{2}(900)=150 .$$

Calculate the error in the result:

$$500-150=350 .$$

Increasing the guess for the first field by one unit in turn decreases the guess for the second field by one unit, and calculating the resulting difference in the yield gives

$$\frac{2}{3}(900+1)-\frac{1}{2}(900-1) .$$

Distributing leads to

$$\frac{2}{3}\cdot 900+\frac{2}{3}\cdot 1-\frac{1}{2}\cdot 900+\frac{1}{2}\cdot 1 .$$

Grouping the terms results in

$$\left(\frac{2}{3}\cdot 900-\frac{1}{2}\cdot 900\right)+\left(\frac{2}{3}\cdot 1+\frac{1}{2}\cdot 1\right).$$

Factoring the result gives

$$900\left(\frac{2}{3}-\frac{1}{2}\right)+1\left(\frac{2}{3}+\frac{1}{2}\right).$$

Thus, increasing the guess by one unit has increased the resulting yield by

$$\frac{2}{3}+\frac{1}{2}=\frac{7}{6}.$$

Finally, in order to achieve the correct result, divide the original error by this proportion to find out how many sar the original guess needs to be increased:

$$350\div\frac{7}{6}=350\cdot\frac{6}{7}=300.$$

Thus, the original guess must be increased by 300 sar. The first field was 1200 sar, and the second was 600 sar.

To create problems that can be solved with the methods of false position and double false position, fill in the blanks and change the pertinent information in the following templates.

False Position

An unknown quantity is added to its ____ (half, third, etc.). It becomes ____ (total amount). What is the unknown quantity?

Figure 5. Template for False Position

Double False Position

An unknown quantity ____ (type) is added to another unknown quantity ____ (type), and the result is ____ (constant). ____ (proportion) of the first quantity together with ____ (proportion) of the second quantity results in ____ (constant). What are the two quantities?

Figure 6. Template for Double False Position

# CHAPTER THREE

# QUADRATIC EQUATION-SOLVING

## Historical Overview

In 1766 a successor of Islamic prophet Muhammed, caliph al-Mansūr, founded Baghdad, a new capital of his empire. When the initial impulses of Islamic orthodoxy gave way to a more tolerant atmosphere, Baghdad soon became a commercial and intellectual center (Katz, 1998). Over the next 200 years, the succeeding caliphs set up a world-renowned library, which included manuscripts from Athens and Alexandria, and a research institute, the *Bayt al-Hikma*. By the end of the ninth century, the most influential and famous historical mathematical texts had been translated into Arabic and were being studied in Baghdad. The Islamic scholars studied the works of Euclid, Archimedes, Apollonius, Diophantus, Ptolemy, along with Babylonian and Hindu works.

One of the earliest Islamic algebra texts was written by Mohammed ibn-Mūsa Al-Khwārizmī in 825 AD (Katz, 1998). Within the book itself, and within the title *Al-kitāb al-muhtsar fī hisāb al-jabr wa-l-muqābala*, Al-Khwārizmī used the term "al-jabr", which would evolve into the word algebra. Al-Khwārizmī noted that what people generally wanted was a solution to an equation. His book was a manual for solving equations. Specifically, Al-Khwārizmī dealt with three types of quantities: the square of the unknown, the root of the square (the unknown), and numbers (constants). Since the Islamic mathematicians did not deal with negative numbers, coefficients and roots of equations needed to be positive. The modern term "square root" is the same idea that Al-Khwārizmī refers to as a "root." The term "square root" refers to the side length of a square. The side length is the "root" (the origin) of the square.

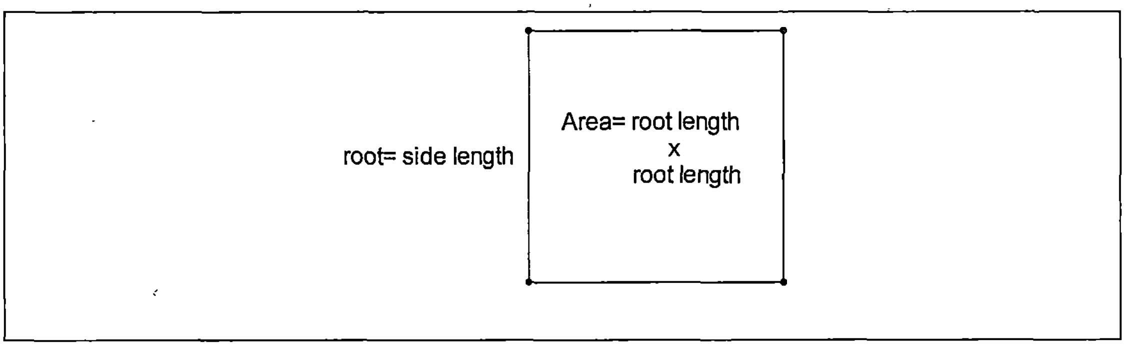



Figure 7. The "Square Root"

Given a square of area A, the "root" of the square would be the length of one of its sides. For example, the root of a square of area 9 would be 3. The root of a square of area 2 would be $\sqrt{2}$ (the "square root" of 2).

One current method of solving quadratic equations involves putting all of the terms on one side of the equation, so that the other side would be zero. This lends itself to factoring and using the zero product property, or using the Quadratic Formula (which is just a method of finding roots so that an equation can be factored). However, these methods require the use of negative coefficients in order to attain real solutions. The Islamic mathematicians did not accept the use of negative coefficients, and thus developed methods to solve quadratic equations while keeping the coefficients, and solutions, positive. This led to six types of equations that can be written using the quantities "squares," "roots," and "numbers."

1) Squares equal to roots: $ax^2 = bx$
2) Squares equal to numbers: $ax^2 = c$
3) Roots equal to numbers: $bx = c$
4) Squares and roots equal to numbers: $ax^2 + bx = c$
5) Squares and numbers equal to roots: $ax^2 + c = bx$
6) Roots and numbers equal to squares: $bx + c = ax^2$

Solutions to the first three types of equations could be achieved via elementary methods. What follows are illustrations of how types 1-3 might be solved using familiar methods and notation.

One current method that could be used to solve Al-Khwārizmī's first type of quadratic equation, $ax^2 = bx$, is fairly brief. First, divide both sides by of the equation $x$. This can be done because of the assumption that $x \neq 0$. The solution to Type 1 yields only one solution, as Al-Khwārizmī did not allow zero as a solution. Dividing provides

$$\frac{ax^2}{x} = \frac{bx}{x}.$$

Simplify, and then divide both sides by $a$,

$$\frac{ax}{a} = \frac{b}{a}.$$

Thus, the solution is

$$x = \frac{b}{a}.$$

The solution method to the second type of Al-Khwārizmī's equations, $ax^2 = c$, is similar. First, divide both sides of the equation by $a$ to yield

$$\frac{ax^2}{a} = \frac{c}{a}.$$

The next step is to find the length of the side of a square with area $\frac{c}{a}$, that is the "root" of the square. In modern terms, this refers to taking the square root.

Take the square root of both sides

$$x^2 = \frac{c}{a}$$

$$\sqrt{x^2} = \sqrt{\frac{c}{a}}.$$

Thus the solution is

$$x = \sqrt{\frac{c}{a}}.$$

(Remember that Al-Khwārizmī would have only acknowledged positive solutions.) This type of equation could also be solved using "false position," as described earlier.

Al-Khwārizmī's third type of equation would not currently be called a quadratic, as the square term is missing. Therefore, the solution can be found using methods that apply to linear equations. The solution to the equation $bx = c$ can be found by dividing both sides of the equation by $b$,

$$\frac{bx}{b}=\frac{c}{b}.$$

Thus, the solution is

$$x=\frac{c}{b}.$$

The process of obtaining the solution to Al-Khwārizmī's fourth type of equation is more complex than the first three types. Fortunately, it has a geometric derivation that aids in understanding the algebraic solution. The equation would take the form $ax^2+bx=c$. Assume $a=1$ for the sake of simplicity. If $a$ were a number other than one, begin by dividing every term by the value of $a$. So we consider the equation $x^2+bx=c$. Represent each term on the left side of this equation geometrically. The term $x^2$ can be represented by a square with side length $x$, and the term $bx$ can be represented by a rectangle with sides of lengths $x$ and $b$. Create a rectangle from these two figures as shown in Figure 8. Then this new rectangle must have area equal to $c$ in order to satisfy the equation.

$$x^2 + bx = c$$

ABCD is a square with side length *x*

ED is a side length representing *b*

ADEF is a rectangle with area *bx*

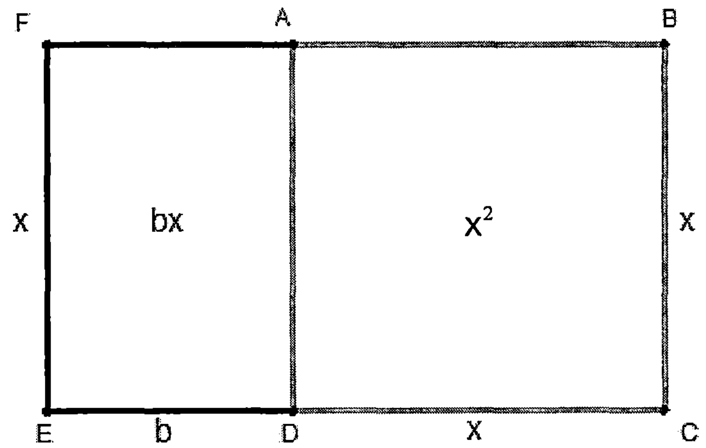



Figure 8. Type 4 Diagram

In an effort to create a square out of this rectangle, so that we can determine the square's "root," first divide the rectangle ADEF in half through the midpoints of AF and DE. Then attach one of these halves to the top of the square, as shown in Figure 9.

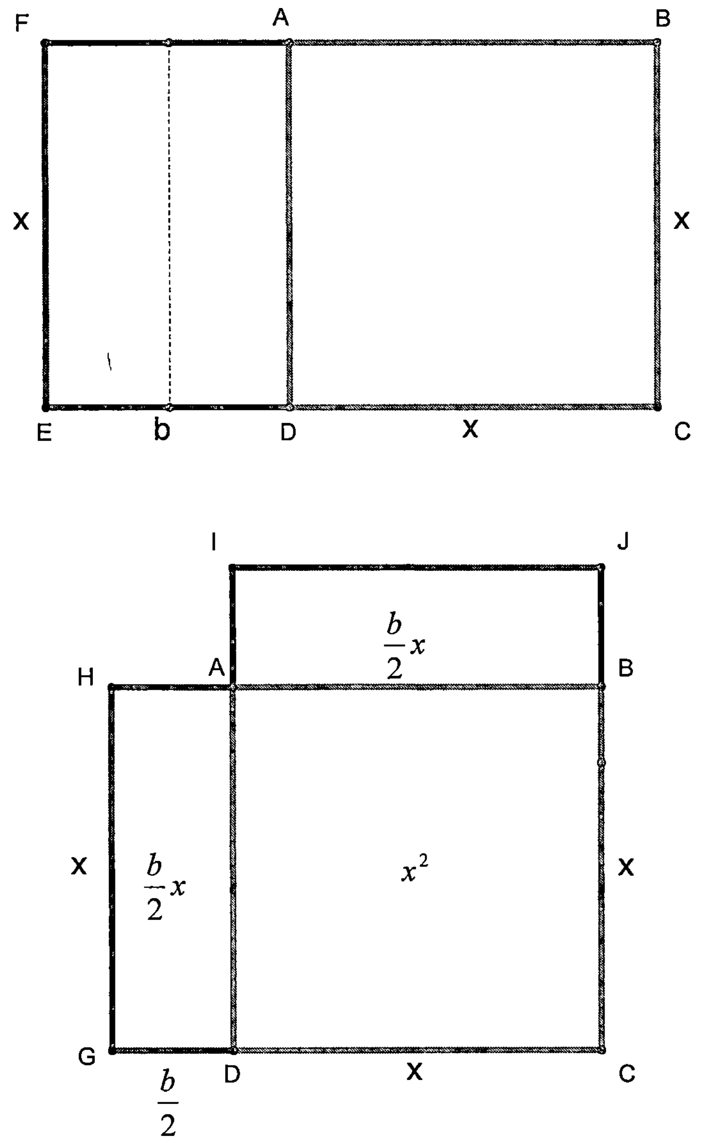



Figure 9. Type 4 Gnomon

This creates a shape which is sometimes called a gnomon (a square missing a square). In order to create a complete square with side length $x+\frac{b}{2}$, it is necessary to "complete the square."

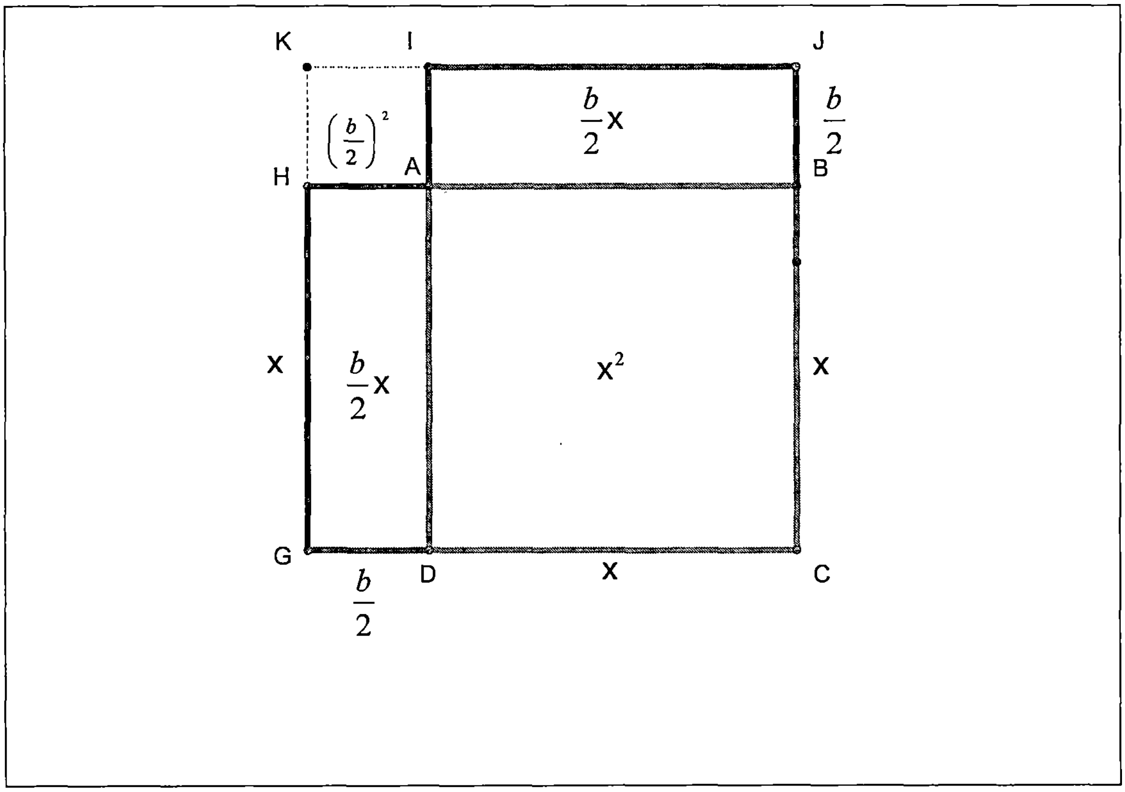



Figure 10. Type 4 Completing the Square

This creates the equation $\left(x+\frac{b}{2}\right)^2 = c+\left(\frac{b}{2}\right)^2$. Note that both equations are different ways to represent the area of the square KJCG. (Remember that the sum of the areas of rectangles HADG and IJBA together with square ABCD was equal to $c$ in the original diagram.) It now becomes possible to view the area as the square with side length $\left(x+\frac{b}{2}\right)^2$. Alternatively, this new figure can be viewed as the sum of the areas of ABCD, IJBA, ADGH, and KHAI. Since figures ABCD, IJBA, and ADGH sum to $c$, the total area is $c+\left(\frac{b}{2}\right)^2$. Geometrically, the solution becomes clear before the algebraic solution appears. The solution, $x$, is the length of the side of the new large square KJCG minus the length of the side of square IAHK, that is $\frac{b}{2}$.

Algebraically, after obtaining the equation $\left(x+\frac{b}{2}\right)^2=c+\left(\frac{b}{2}\right)^2$, the next step would be to take the square root of both sides in an attempt to isolate $x$. This leads to

$$\sqrt{\left(x+\frac{b}{2}\right)^2}=\sqrt{c+\left(\frac{b}{2}\right)^2}.$$

Considering only positive roots gives

$$x+\frac{b}{2}=\sqrt{c+\left(\frac{b}{2}\right)^2}.$$

Hence

$$x = -\frac{b}{2} + \sqrt{c + \left(\frac{b}{2}\right)^2} \ .$$

This expression represents the length of the side of the new square KJCG minus the square IAHK. Negative solutions were disregarded in this derivation since Al-Khwārizmī derived his solutions geometrically, so the solution was a length represented in a diagram.

The solution to Al-Khwārizmī's fifth type of equation begins in a similar fashion to the solution of the fourth type. The equation would take the form

$$x^2 + c = bx$$

where the leading coefficient has again been given the value of one for the sake of simplicity. Take the case $c > x^2$, then it becomes possible to represent this equation with the diagram shown in Figure 11.

$$ax^2 + c = bx$$

ABCD be a square with side length $x$

HADN be a rectangle with area $c$

NC is a side length representing $b$

HBCN is a rectangle with area $bx$

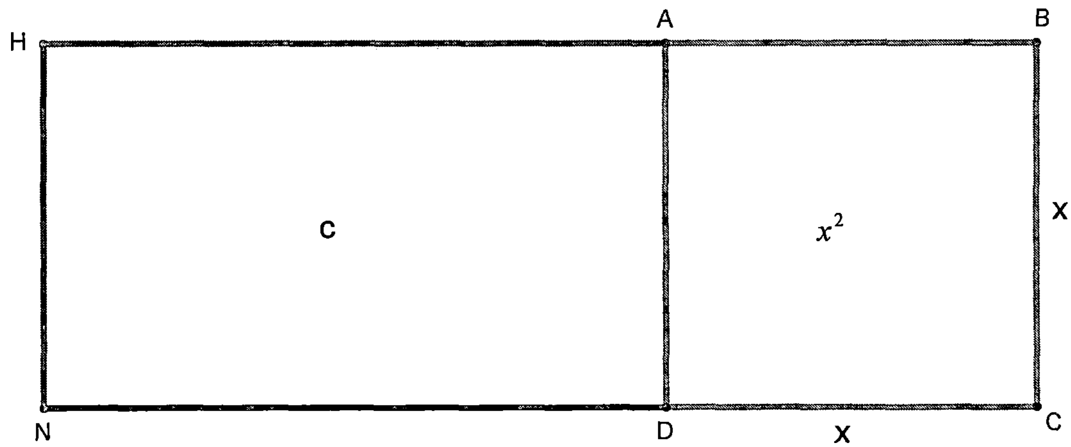



Figure 11. Type 5 Diagram

To create the desired figure, first construct the perpendicular bisector of BH and let its intersection with BH be called G. Then construct a circle with radius AG centered at G (see Figure 12).

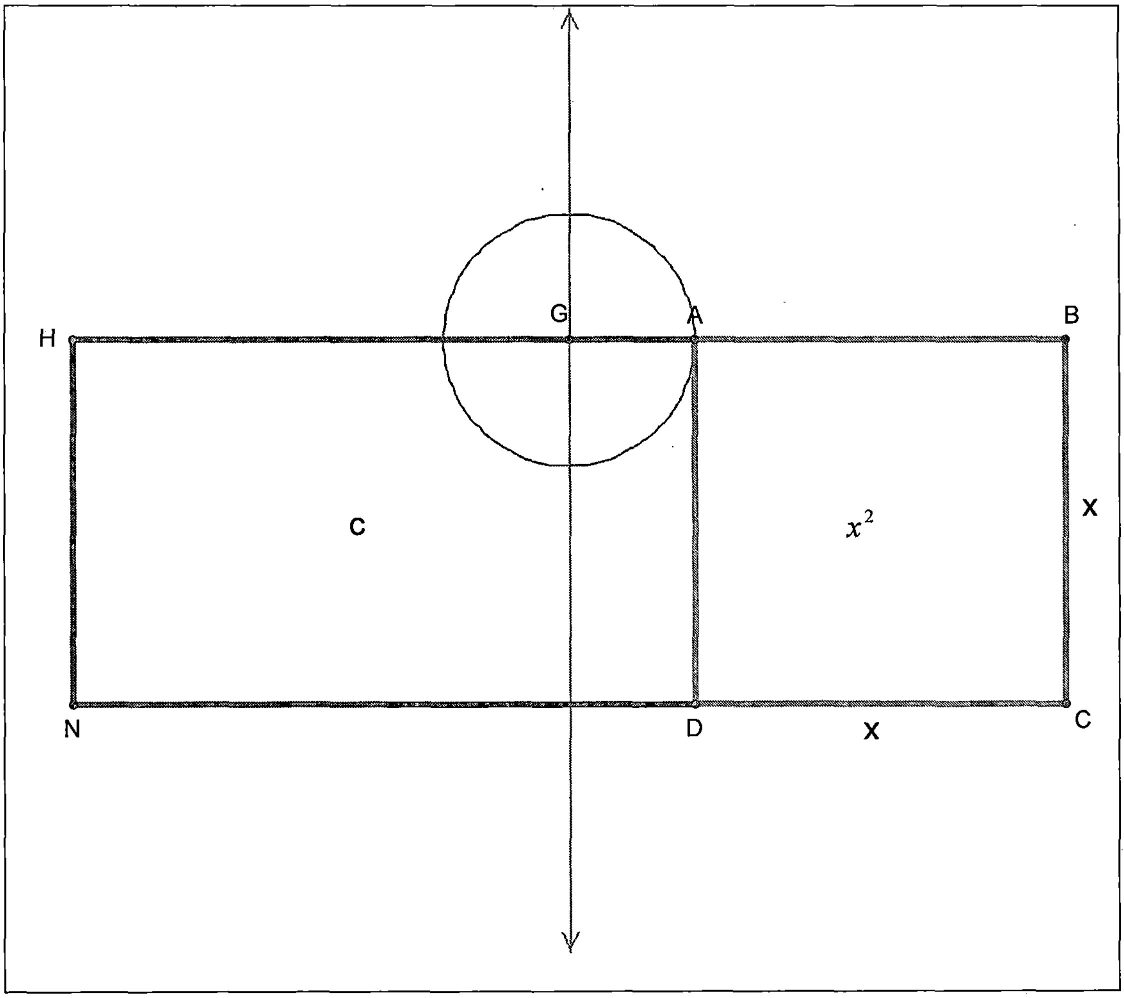


Figure 12. Type 5 Bisection

Next, construct the intersection K between the circle and the perpendicular bisector of BH on the side outside of AHND. Use this segment to construct rectangle RLMH on the side HG (see Figure 13).

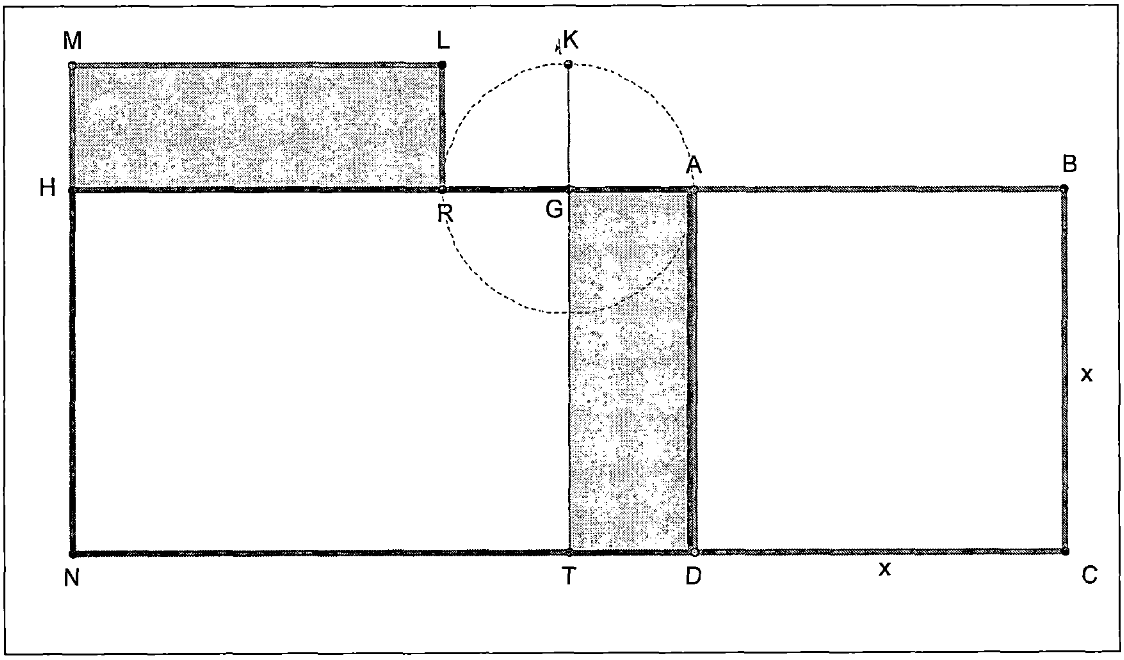


Figure 13. Type 5 Creating Congruent Rectangles

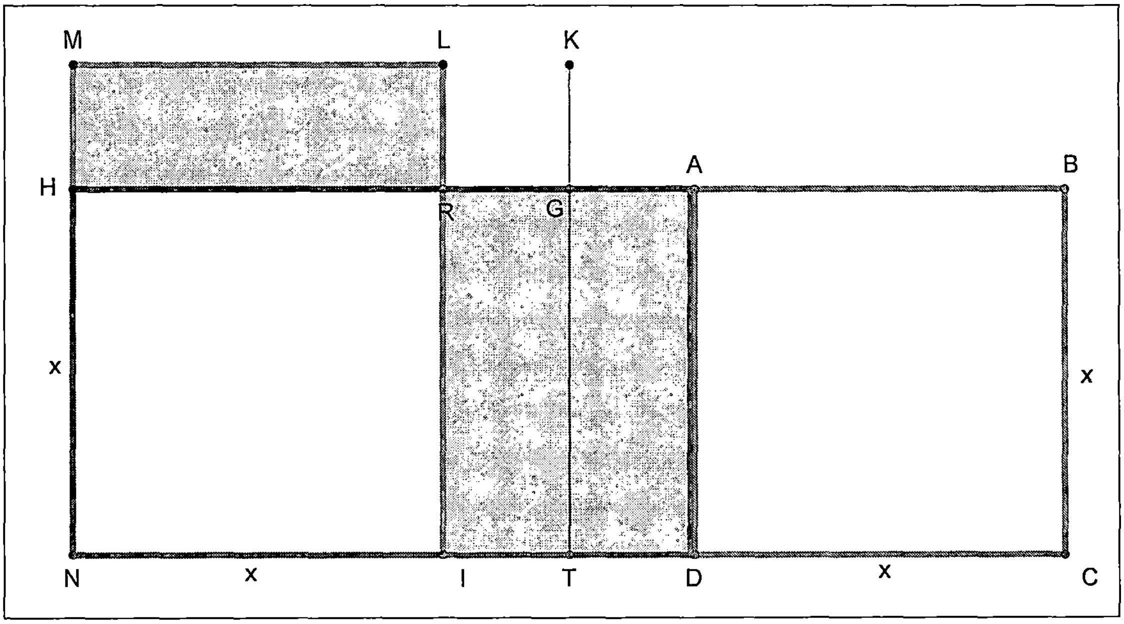


Figure 14. Type 5 Gnomon

Notice that RHNI is congruent to ABCD since RI = AD and RH = BH-AB-AR. Observe that the concave hexagon LRGTNM is a gnomon with area equal to $c$. Thus, the solution $x$ is equal to the side length $\frac{b}{2}$ minus the length MH. To calculate the length of MH, begin by completing the square MKTN.

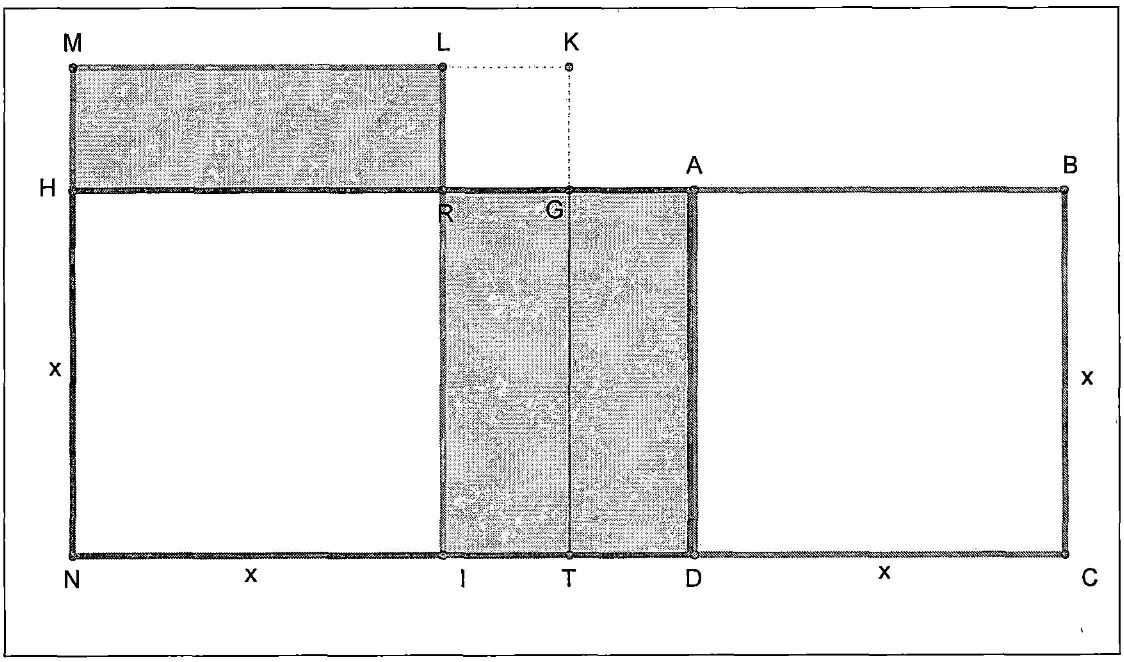



Figure 15. Type 5 Completing the Square

Now, MKTN has area $c+(RG)^2$. Now MH is congruent to LR. The length of LR can be obtained by taking the square root of the square LKGR. The area of LKGR can be found by subtracting $c$ (the area of AHNT) from the area of square MKTN, a square with side length $\frac{b}{2}$. So,

$$MH = LR = \sqrt{\left(\frac{b}{2}\right)^2 - c}$$

Thus the solution, $x$, is

$$x = \frac{b}{2} - \sqrt{\left(\frac{b}{2}\right)^2 - c} .$$

The case $0 < c < x^2$ yields a different geometric interpretation of Al-Khwārizmī's fifth type of equation. It yields a second positive solution. Figure 16 is similar to Figure 11, however the area, *c*, of AHND is assumed to be smaller than that of square ABCD.

ABCD is a square with side length $x$

HADN is a rectangle with area $c$

NC is a side length representing $b$

HBCN is a rectangle with area $bx$

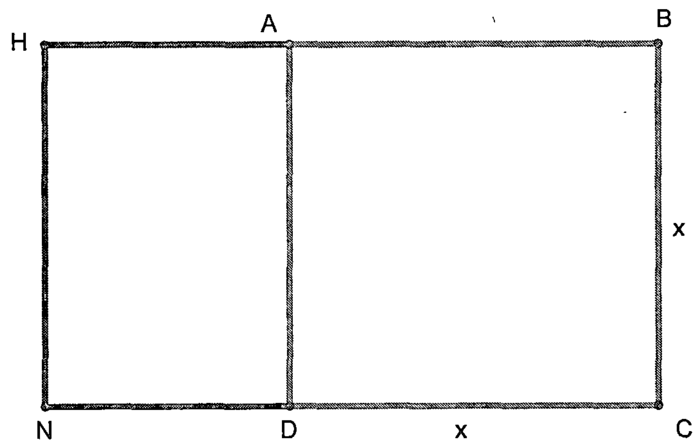



Figure 16. Type 6 Diagram

In an effort to create a gnomon, from which it will be possible to derive the value of $x$, proceed in a manner very similar to the previous case. The first step is to bisect segment NC at G and to construct a circle with radius length NG centered at N.

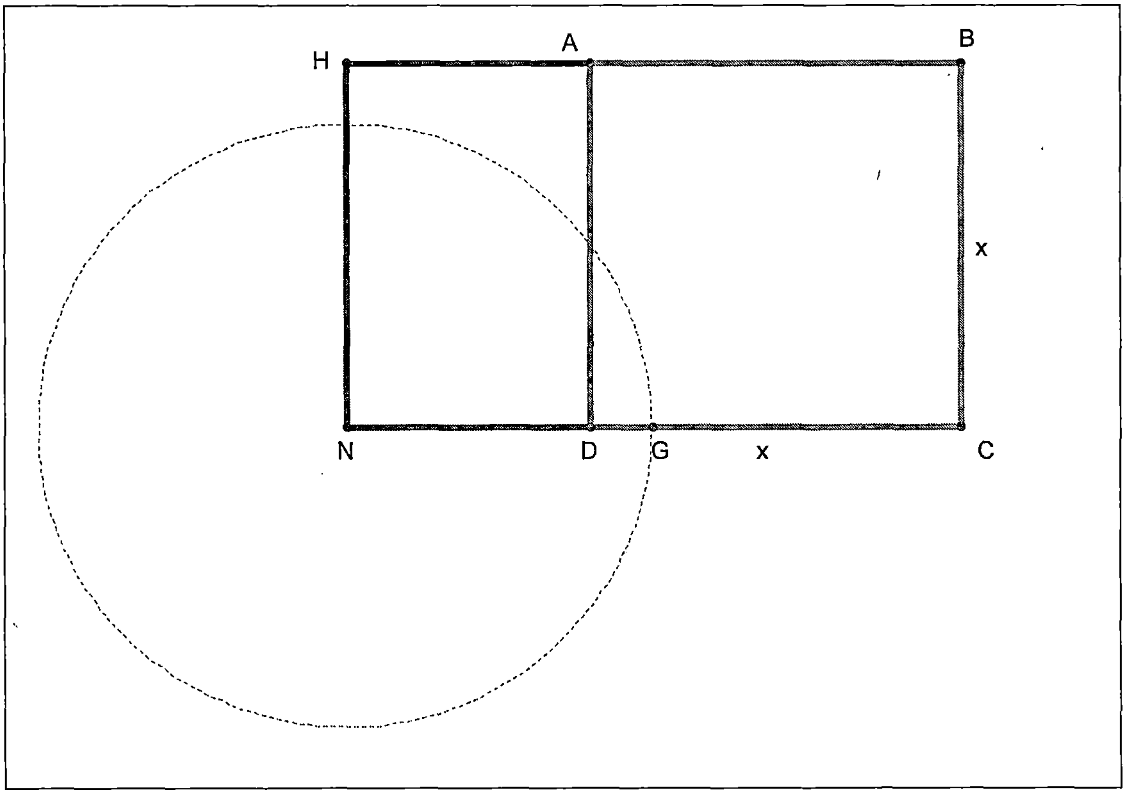


Figure 17. Type 5 Bisection

Next, construct perpendicular lines from where the circle intersects HN, and point G. Let I be the intersection of the circle with HN and construct the square GNIJ.

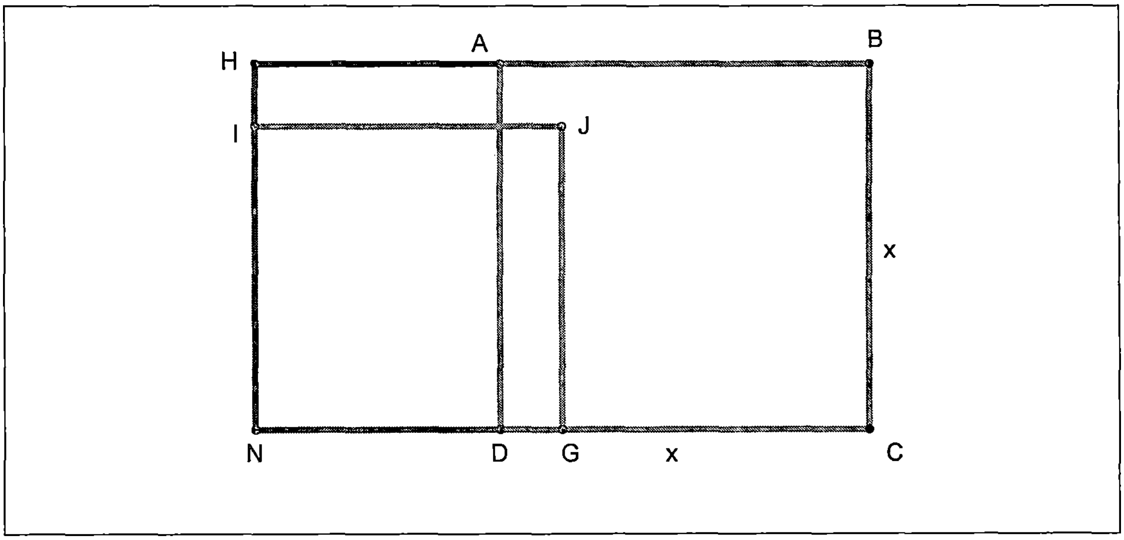



Figure 18. Type 5 Gnomon

Construct a square with length DG off of DG.

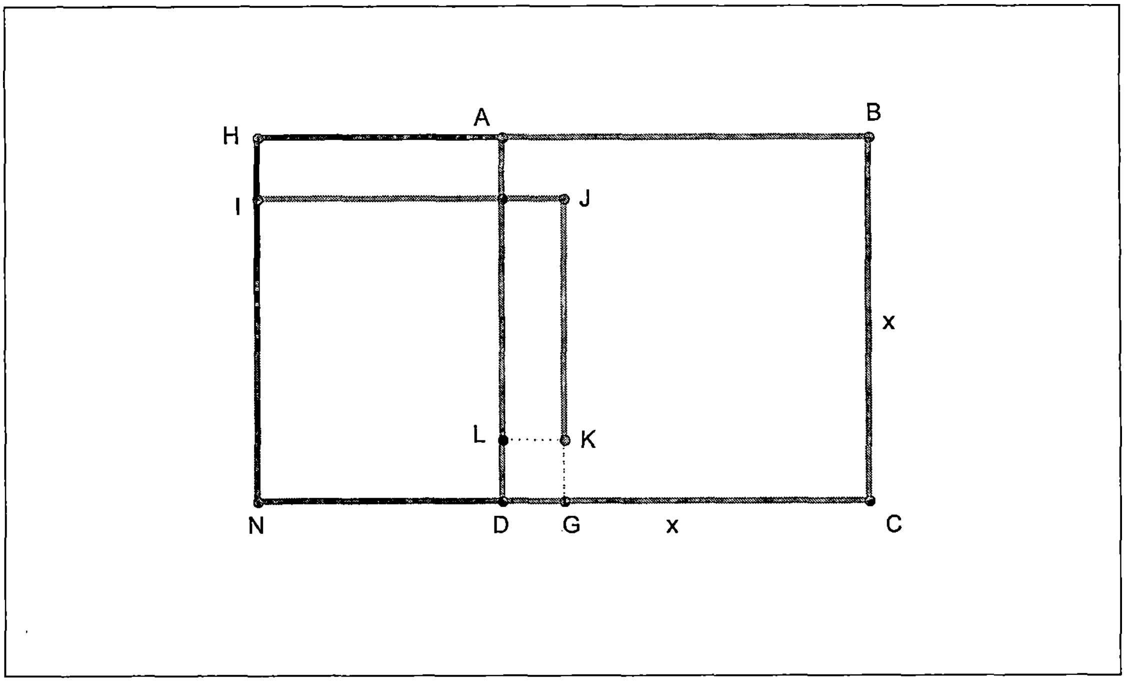



Figure 19. Type 5 Completing the Square

Now the diagram displays gnomon NDLKJI. Square IJGN is missing square LKGD. The solution, x, is equal to $\frac{b}{2}$ plus the side length DG (recall that the length NC is equal to b, and point G bisected NC). To calculate the area of square LKGD, begin by finding the area of square IJGN. Then subtract the area of HADN.

$$\text{Area of LKGD}=\left(\frac{b}{2}\right)^2-c$$

Thus the side length DG is given by

$$DG=\sqrt{\left(\frac{b}{2}\right)^2-c}\ .$$

Finally, the solution, $x$, is

$$x=\frac{b}{2}+\sqrt{\left(\frac{b}{2}\right)^2-c}\ .$$

Al-Khwārizmī found means to solve all the possible quadratic equations that could be written with positive coefficients and constants. An illustration of the solution method for the sixth and final type of Al-Khwārizmī's equations can be found in the "Applications to the Classroom" section. At this point, Al-Khwārizmī still did not acknowledge the solutions to his equations to be roots of equations. Thomas Harriot developed the idea that the solutions to quadratic equations were zeros of the equations. This non-geometric interpretation made it possible to consider negative solutions. It then became possible to create a formula that allowed the calculation of solutions to all quadratic equations, The Quadratic Formula.

## Applications to the Classroom

There is a classic example that Al-Khwārizmī used to demonstrate his method of "completing the square."

Problem 1: *The Classic "Completing the Square" Problem.*[7]

> What must be the square which, when increased by ten of its own roots, amounts to thirty-nine?

The Solution: Using "Completing the Square" pictorially together with Al-Khwārizmī's verbal description.

> You halve the number of roots, which in the present instance yields five.
>
> This you multiply by itself; the product is twenty-five.

[7] Katz, Victor (1998). A History of Mathematics: An Introduction. Reading, MA: Addison-Wesley Educational Publisher, Inc.

Add this to thirty-nine; the sum is sixty four.
Now take the root of this which is eight, and subtract from it half the number of roots, which is five; the remainder is three.
This is the root of the square which you sought for.
(Katz, 1998, p355)

Students often run straight to the Quadratic Formula to solve quadratic equations, even when other alternatives are available. For instance, factoring and using the zero product property is often a more efficient method to find the solution(s). The decision to use the Quadratic Formula or another technique requires students to use critical thinking and analyzing skills. Like the Quadratic Formula, Al-Khwārizmī's method of Completing the Square also works on all quadratics, and when encouraged to use the method, teachers and students may find that it is often less complicated than the Quadratic Formula. Practice in comparing and employing various solution methods can help students develop such skills.

Problem 2: An Example Comparing the Quadratic Formula to Completing the Square.

Find the root(s) of the equation

$$y = x^2 + 8x + 2 .$$

The Solution: Using the Quadratic Formula.

For an equation of the form

$$ax^2 + bx + c = 0 ,$$

the Quadratic Formula is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} .$$

In order to find the solution(s) to the equation when y=0, substitute the values $a = 1$, $b = 8$, and $c = -2$ into the Quadratic Formula to obtain

$$x = \frac{-8 \pm \sqrt{8^2 - 4 \cdot 1 \cdot -2}}{2 \cdot 1} .$$

Simplifying yields

$$x = \frac{-8 \pm \sqrt{64 + 8}}{2} .$$

Combining terms beneath the square root gives

$$x = \frac{-8 \pm \sqrt{72}}{2} .$$

Which can be simplified to

$$x = \frac{-8 \pm 6\sqrt{2}}{2} .$$

Factor the 2 out of the numerator to obtain

$$\frac{2\left(-4\pm 3\sqrt{2}\right)}{2}.$$

After dividing by the common factor, the roots are

$$x=-4\pm 3\sqrt{2}\ .$$

(Teachers may recognize this final step as one which students frequently make mistakes on.)

The Solution: Using "Completing the Square."

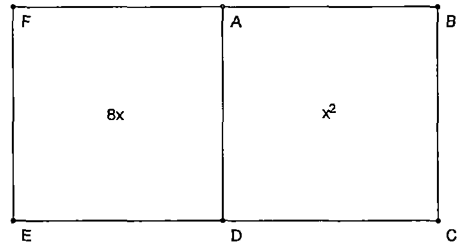
In order to find the roots, substitute zero for $y$ to obtain

$$0=x^2+8x-2\ .$$

Completing the square interprets the terms as areas, so make all of the terms positive yields

$$x^2+8x=2\ .$$

Here is the step-by-step process to find the roots of the equation.

<table>
<tr>
<td>Step 1:<br><br>Diagram the areas.</td>
<td>

<br>(pictures not drawn to scale)</td>
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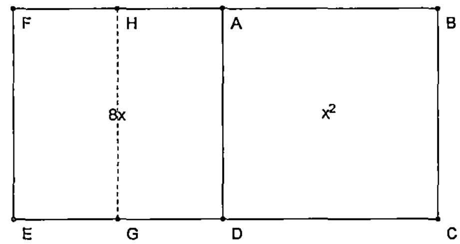
<tr>
<td>Step 2:<br><br>Divide rectangle FADE in half through midpoints H and G.</td>
<td>
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Figure 20a. The Steps to Solving an Example of Type 4

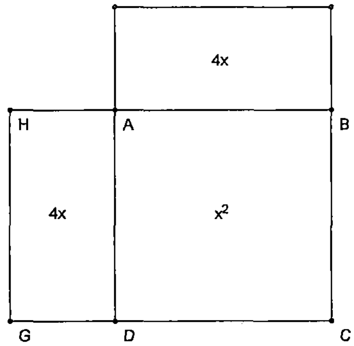
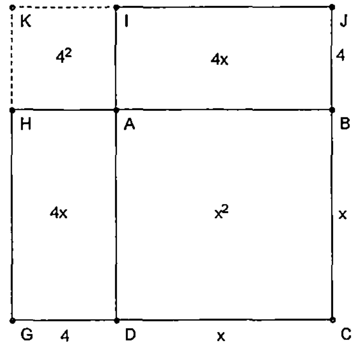
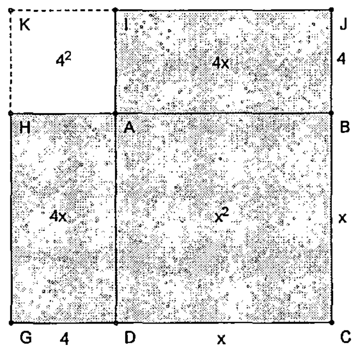
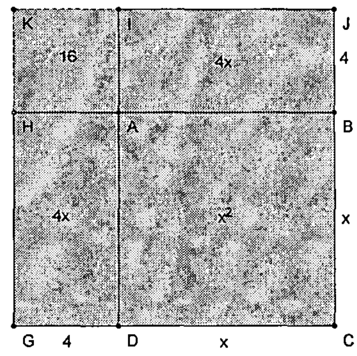
| | |
|---|---|
| Step 3:<br>Append half of the rectangle to the top of square ABCD. |  |
| Step 4:<br>Calculate what size square would need to be added to gnomon HGCJIA to create a complete square. |  |
| Step 5:<br>"Complete the Square" by adding in the missing square KIAH. | <br>Remember that the yellow area totaled 2 |

Figure 20b. The Steps to Solving an Example of Type 4

Step 6:

The new square has sides $x+4$ and area 18. Thus, $(x+4)^2 = 18$. Taking the square root of each side yields $x+4 = \pm\sqrt{18}$, which can be simplified to $x+4 = \pm 3\sqrt{2}$. Thus, $x = -4 \pm 3\sqrt{2}$.




The total area is 2+16

Figure 20c. The Steps to Solving an Example of Type 4

Once students become familiar with the diagramming and geometric concepts underlying the "completing the square" method, they will be able to apply the method to problems that do not lend themselves to geometric solutions. This includes problems that have negative coefficients, and those that have no real solution (i.e. complex solutions). Here is one such example.

Problem 3: Find the root(s) of the equation

$$y = x^2 - 10x + 45.$$

Solution: Using Completing the Square.

Substituting zero for $y$ yields

$$0 = x^2 - 10x + 45 .$$

Rewriting the equation so that the variables are on one side, and the constant is on the other gives

$$-45 = x^2 - 10x .$$

In order to create a perfect square from the terms $x^2 - 10x$, it is necessary to divide the coefficient of x by two, and rewrite the problem as

$$-45 + ? = x^2 - 5x - 5x + ? .$$

The equation, rewritten as a perfect square after adding 25 to both sides, becomes

$$-45 + 25 = (x - 5)^2 .$$

Simplifying leads to

$$-20 = (x - 5)^2 .$$

Taking the square root of each side, and simplifying when possible, yields

$$\pm 2i\sqrt{5} = x - 5 .$$

Thus, the roots are

$$x = 5 \pm 2i\sqrt{5} .$$

One application to the classroom would be to ask students to find the geometric solution to the sixth type of Al-Khwārizmī's equations. Begin by showing students the solutions to the first five types of equations, particularly Types 4 and 5. Ask the students to use a similar diagram to construct the geometric diagram that Al-Khwārizmī used to derive his solution to the Type 6 equation. This problem is quite challenging, and may require the use of manipulatives.

Problem 4: Find the geometric representation to display and derive Al-Khwārizmī's solution to his sixth type of equation, $bx + c = ax^2$.

The Solution:

Begin by representing each of the parts of the problem geometrically. Rectangle ABHR represents area $bx$. Rectangle ABHR represents area $c$. Square ABCD represents $x^2$ (allow the coefficient of $x^2$ to be one for the sake of simplicity).

ABHR is a rectangle with area c

RD is a side length representing b

RHCD is a rectangle with area bx

ABCD is a square with side length x

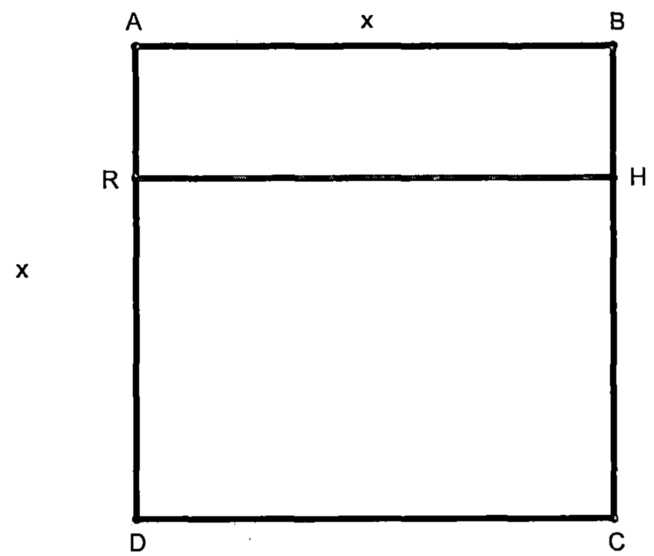



Figure 21. The Steps to Deriving Type 6

Begin by bisecting length HC at G and constructing a square off of HC.

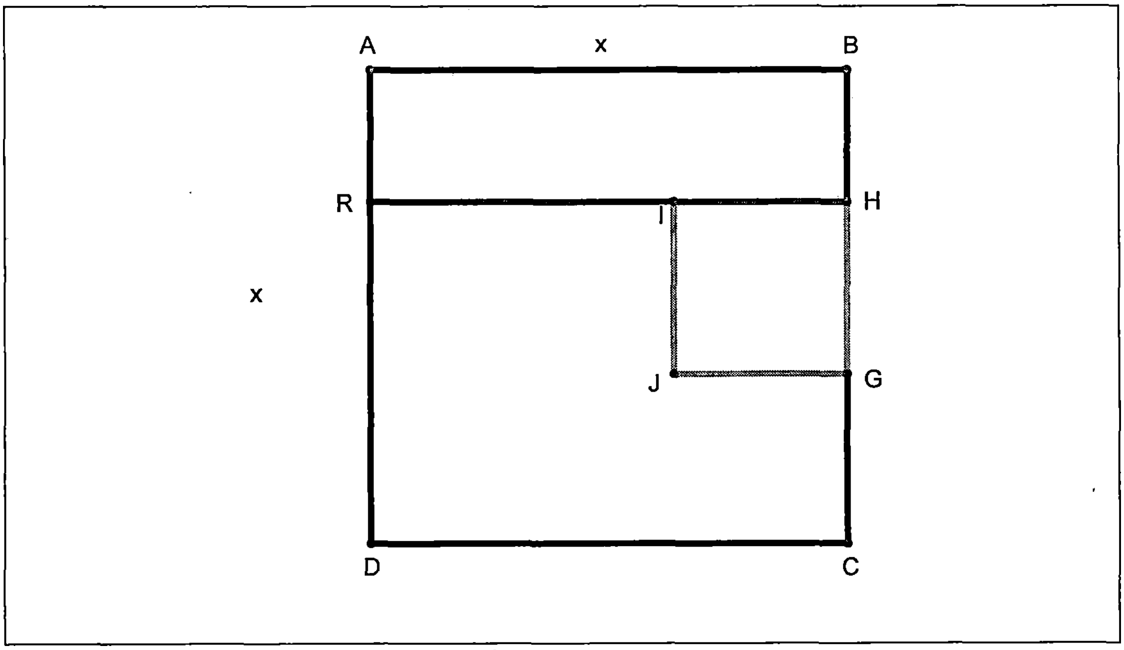


Figure 22. Type 6 Bisection

Now, the diagram shows a square IHGJ with area $\left(\frac{b}{2}\right)^2$.

Next, construct a square off of side BG with length BG.

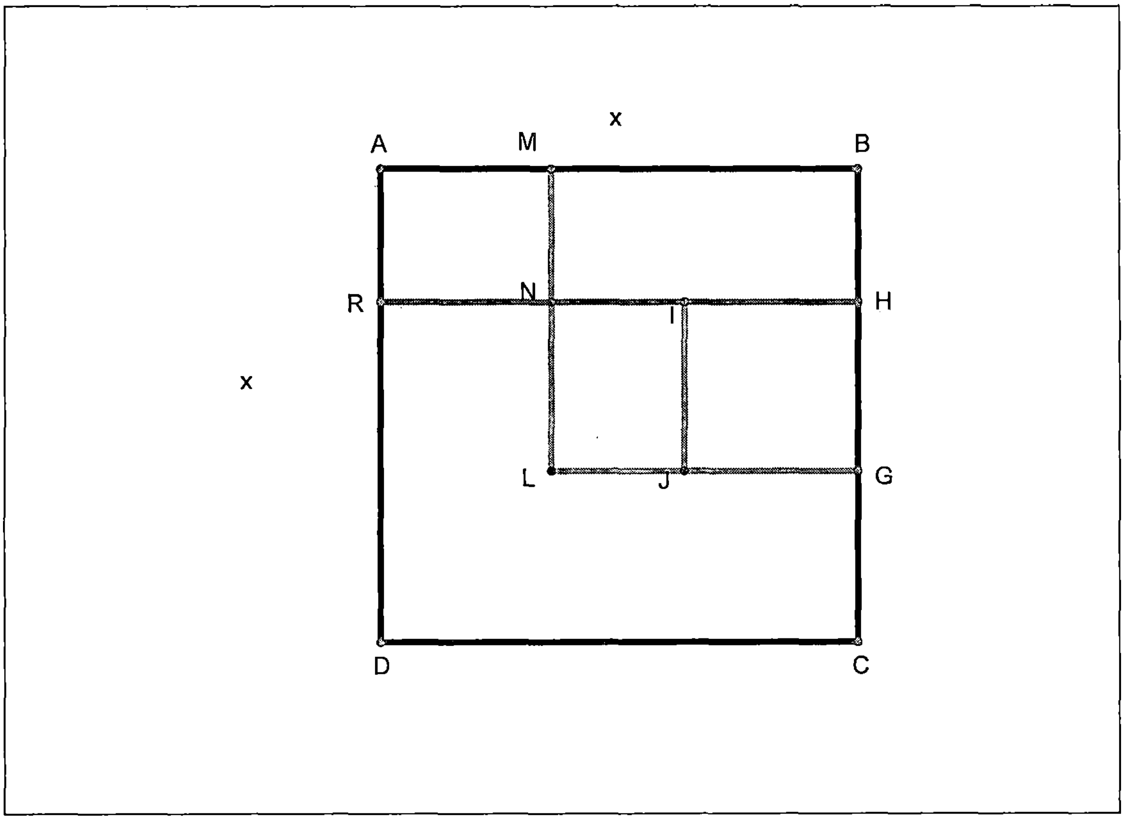


Figure 23. Type 6 Gnomon

Now it becomes possible to see that rectangles AMNR and NIJL are congruent. Therefore the solution, x, is equal to the sum of length CG and GB. CG was constructed to be $\frac{b}{2}$. To calculate BG, first find the area of square BGLM.

The area of BGLM is $c+\left(\frac{b}{2}\right)^2$. Thus BG is

$$\sqrt{c+\left(\frac{b}{2}\right)^2}.$$

The solution, x, is then

$$\frac{b}{2}+\sqrt{c+\left(\frac{b}{2}\right)^2}.$$

There are questions of varying difficulty levels that teachers can pose to their students. Students could be asked to identify the gnomon in one of the figures of the derivation. In addition, students could be asked to justify why particular steps of the derivation are allowed. This includes verifying that pieces of figures are congruent or similar, and justifying what type of shape a figure might be (e.g. square).

CHAPTER FOUR

# CUBIC EQUATION-SOLVING

## Historical Overview

Mathematician Fra Luca Pacioli noted that there was not yet a solution to the general cubic equation in his book *Summa de Arithmetica* in the year 1494 (Dunham, 1990). Specifically, Pacioli was of the opinion that finding a solution to the general cubic was as likely as squaring the circle (Dunham, 1990)[8]. However, many mathematicians were working on this problem during the fifteenth and sixteenth centuries. Sciopine del Ferro took up the challenge to find the solution to the general cubic while teaching at the University of Bologna between 1500 and 1515 (Katz, 1998). In fact, del Ferro did find a method for solving the cubic, but not to the general form. The general form of a cubic would now be written as

$$ax^3 + bx^2 + cx + d = 0 .$$

---

[8] Recall that squaring a circle was once considered an extremely difficult task, and was later proven to be impossible. (For more information, see: Dunham, W. (1990). Journey through Genius. New York, NY: Penguin Books.)

The solution method del Ferro found solved depressed cubics, those without a square term:

$$ax^3 + cx + d = 0 .$$

Curiously, his method of finding a solution was not publicized, but rather, was kept a secret. This secrecy was a function of the academic attitude at the time. The current trend in academia is for professors to publish new results as quickly and often as possible. However, in the sixteenth century, university professors were expected to challenge others, and to meet the challenges of others. Their professorial worth was on the line every time they took up a challenge, as was the security of their jobs. For this reason, del Ferro did not publish his results. Rather, he kept his breakthrough a secret shared with no one but his student Antonio Maria Fior, and his successor Annibale della Nave, whilst on his deathbed. Fior and Nave did not publicize the solution, but word spread that the solution to the cubic was known. Soon another mathematician named Niccolo Fontana (best known as "Tartaglia") boasted that he too knew the solution to the cubic.

Fior publicly challenged Fontana in 1535. Each mathematician provided problems for the other to solve. For example, "A man sells a sapphire for 500 ducats, making a profit of the cube root of his capital. How much is that profit?" This problem could be written algebraically as

$$x^3 + x = 500 .$$

Tartaglia discovered the solution to this cubic problem, while Fior was unable to solve many of Tartaglia's non-cubic mathematical questions (Katz, 1998). For this reason, Tartaglia was declared the winner of the mathematical duel. His prize was 30 banquets prepared by Fior, which Tartaglia declined in favor of simply having the honor of being the victor (Katz, 1998).

Gerolamo Cardano, a mathematician giving public lectures on mathematics in Milan, heard about Tartaglia's solution to the cubic. After many entreaties, Tartaglia agreed to share his method with Cardano, provided that Cardano would not publish these methods. This is how Tartaglia solved the cubic $x^3 + cx = d$ .

When the cube and its things near
Add to a new number, discrete,
Determine two new numbers different

$_{4}$By that one; this feat

$_{5}$Will be kept as a rule

$_{6}$Their product always equal, the same,

$_{7}$To the cube of a third

$_{8}$Of the number of things named.

$_{9}$Then, generally speaking,

$_{10}$The remaining amount

$_{11}$Of the cube roots subtracted

$_{12}$Will be your desired count.

(Katz, 1998, p359)

Line 1 refers to $x^3$ (the cube) and $cx$ (its things). In Line 2, Tartaglia is referring to creating the term $x^3+cx=d$. Let $v$ and $w$ be the two new numbers in Line 3. Let their difference be represented by $v-w=d$ in this line. Lines 6 and 7 refers to the term $v\cdot w=\left(\frac{c}{3}\right)^3$. Thus, Lines 9 through 12 say that $\sqrt[3]{v}-\sqrt[3]{w}$ is the solution to the problem. This can be checked by substituting this solution into the original cubic $x^3+cx=d$. This would initially give the expression

$$\left(\sqrt[3]{v}-\sqrt[3]{w}\right)^3+c\left(\sqrt[3]{v}-\sqrt[3]{w}\right),$$

which should be equal to "*d*." Expanding this expression leads to

$$v-w-3\sqrt[3]{v}\sqrt[3]{vw}+3\sqrt[3]{w}\sqrt[3]{vw}+c\left(\sqrt[3]{v}-\sqrt[3]{w}\right).$$

Substituting $v-w=d$ and $v\cdot w=\left(\frac{c}{3}\right)^3$ gives

$$d-3\sqrt[3]{v}\sqrt[3]{\left(\frac{c}{3}\right)^3}+3\sqrt[3]{w}\sqrt[3]{\left(\frac{c}{3}\right)^3}+c\sqrt[3]{v}-c\sqrt[3]{w}.$$

This can be simplified to

$$d-c\sqrt[3]{v}+c\sqrt[3]{w}+c\sqrt[3]{v}-c\sqrt[3]{w},$$

which becomes "*d*." Thus, $\sqrt[3]{v}-\sqrt[3]{w}$ is a solution to the equation $x^3+cx=d$. The question then becomes how to find the values for $v$ and $w$. This will be discussed in the geometric derivation that follows. Although this method is not the traditional method used to solve cubic equations, it has a stronger appeal in terms of applications to the classroom.

Cardano and his student, Lodovico Ferrari, continued working on solutions to the various forms of the cubic (where a variation of the 4 terms would be missing).

Cardano investigated del Ferro's papers and found that he had discovered the solution to the cubic before Tartaglia. Anxious to publish the solutions to the cubic, and not wanting to betray Tartaglia, Cardano used the fact that del Ferro had discovered the solution first to evade his promise of secrecy. Cardano published *Ars Magna, sive de Regulis Algebraicis* (*The Great Art, On the Rules of Algebra*) (Katz, 1998). Cardano's Formula (in modern notation) to solve the cubic $x^3 + cx = d$ is

$$x = \sqrt[3]{\frac{d}{2} + \sqrt{\frac{d^2}{4} + \frac{c^3}{27}}} - \sqrt[3]{-\frac{d}{2} + \sqrt{\frac{d^2}{4} + \frac{c^3}{27}}}.$$

Before examining Cardano's method, it is interesting to note that both Niccolo Fontana and Gerolamo Cardano both led fascinating lives. Fontana was disfigured as a boy when a soldier slashed his face with a sword. The legend tells that he survived only because a dog licked his wound, causing it to heal (Dunham, 1990). Due to the serious injury to his face, Fontana had a speech impediment. His nickname became Tartaglia (The Stammerer), and he is best known by that nickname today (Dunham, 1990). Gerolamo Cardano was plagued by infirmities throughout his life (Dunham, 1990). He kept track of his many afflictions, and left a detailed accounting of them in his autobiography. Cardano had a vision of a woman in white in a dream. When he later met a woman that he felt resembled that of his dream, he married her. When his wife died, leaving him with two sons and a daughter, Cardano was left to raise his children alone. He writes that disaster struck in the form of a "wild woman," whom his eldest son Giambattista married (Dunham, 1990). The couple soon produced a male child named Fazio. Unfortunately, the wife boasted that none of the children were Giambattista's. Giambattista prepared a cake laced with arsenic that killed his wife. He was subsequently convicted and beheaded. Cardano raised Fazio as a son, and the relationship thrived. Near the end of his life, Cardano was jailed for heresy against the Church of Italy for several issues, including writing a book titled *In Praise of Nero*. These two mathematicians of the 16th century led fascinating lives, and their stories serve as a reminder that mathematicians are humans too.

Tartaglia and Cardano both played important roles in deriving the solution to cubic equations. Cubics of the form $x^3 + cx = d$ are considered "depressed" because the square term is missing. Cubics that begin in the form $x^3 + bx^2 + cx = d$

can all be rewritten as depressed cubics. In fact, substituting

$$x = m - \frac{b}{3}$$

(where b is the coefficient of the second degree term) results in a cubic in ***m*** with no square term. Substituting this value for *x* yields

$$\left(m - \frac{a}{3}\right)^3 + b\left(m - \frac{a}{3}\right)^2 + c\left(m - \frac{a}{3}\right) = d\ .$$

Simplifying leads to

$$m^3 - bm^2 + \frac{b^2}{3}m - \frac{b^3}{27} + bm^2 - \frac{2b^2}{3}m + \frac{b^3}{9} + cm - \frac{cb}{3} = d\ .$$

Finally, by grouping the terms, the equation becomes

$$m^3 + (-b + b)m^2 + \left(\frac{b^2}{3} - \frac{2b^2}{3} + c\right)m = d - \left(-\frac{cb}{3}\right).$$

The coefficients of the square term become zero, thus creating a cubic with no square term. This is a depressed cubic, for which it is possible to use Cardano's Formula. Hence, it is possible to find a solution to all cubic equations using Cardano's Formula. Keep in mind that all cubic graphs have at least one real root, as they are odd functions. Quantitatively this means that the range of the graph will be all real numbers, guaranteeing hence that the graph will have at least one real root. The Intermediate Value Theorem (commonly covered in calculus) guarantees that there exists at least one real root. The Intermediate Value Theorem states that if $f(x)$ is continuous on the interval $[a,b]$, and k lies between $f(a)$ and $f(b)$, then $f(x)$ will have value $k$ for some value of $x$ on the interval $[a,b]$. Essentially, the function cannot get from one point to the other without crossing horizontal line $y=k$.

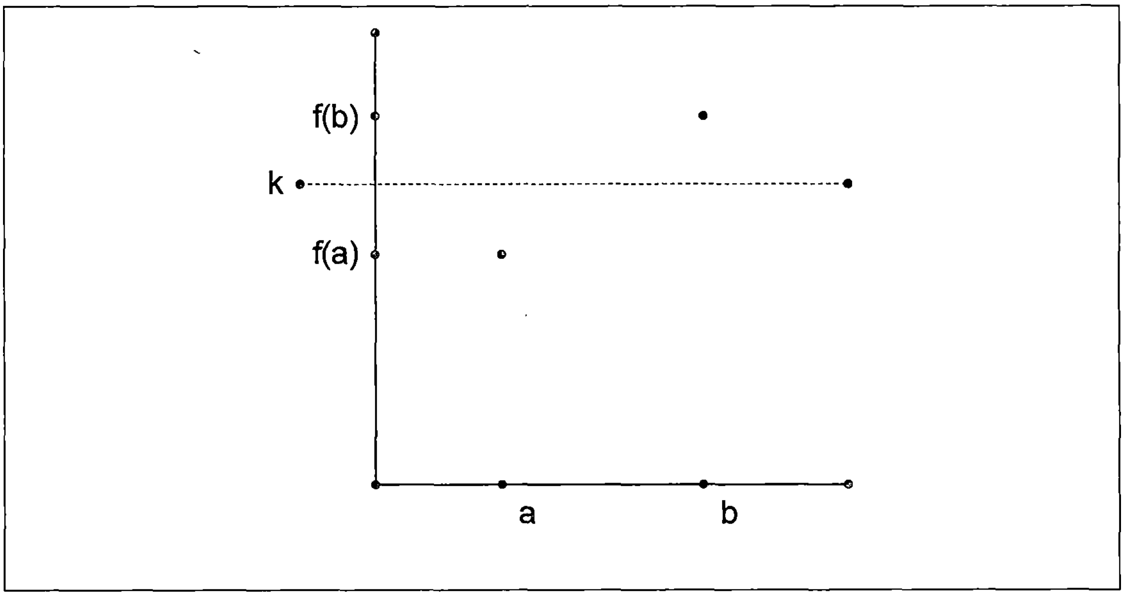



Figure 24. The Intermediate Value Theorem

A geometric understanding for Cardano's solution to the cubic equation requires some three-dimensional diagrams. The following two-dimensional nets can be used to create the shapes necessary to see Cardano's solution.

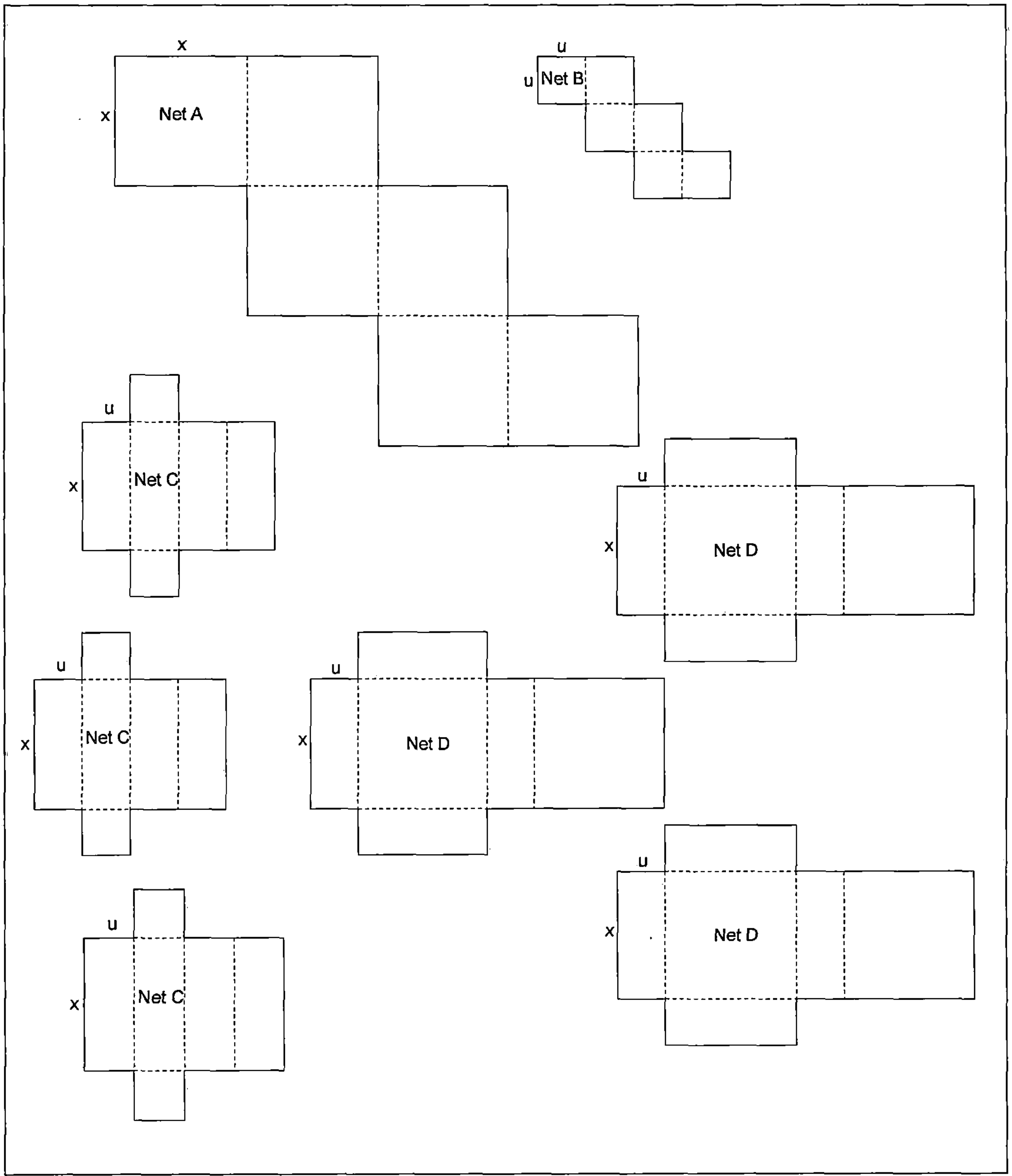


Figure 25. The Nets for the Cubic Derivation

Each of these nets, when cut and folded along the lines provided, creates a three dimensional shape that will be used to construct the geometric derivation of Cardano's solution to the equation

$$x^3 + cx = d .$$

Note that this geometric derivation ties together with Tartaglia's poem to find the solution of a cubic (discussed earlier). In fact, $u^3$ (the small cube) corresponds to Tartalia's $w^3$. In addition, $x^3$ corresponds to Tartaglia's $v^3$. Geometrically, this would refer to a cube and a rectangle added together to create "$d$." Let Figure 1 (blue cube) represent $x^3$. Then Figure 2 (red cube) is the cube that is missing after the pieces are put together represent $u^3$ (the piece will not be put together with the rest). Two of Figure 3, the pink rectangular prisms representing $ux(u+x)$, are necessary to complete the picture. Figure 4, the yellow prism represents $x^2u$. Figure 5, the green rectangular prism represents $xu^2$. When all the figures are cut out and attached to each other (excluding Net B), they comprise a large cube with dimensions $(x+u)^3 - u^3$. Thus, the three dimensional shape implies that the large cube (which is missing a small cube) is equal to the sum of its parts. This shape is analogous to the quadratic "gnomon," and might therefore be named a "cubic gnomon."

$$(x+u)^3 - u^3 = x^3 + 3x^2u + 3xu^2 .$$

Factoring produces

$$(x+u)^3 - u^3 = x^3 + xu[3(x+u)] .$$

Regrouping the terms leads to

$$(x+u)^3 - u^3 = x^3 + x[3u(x+u)] .$$

This sum represents a cube and six rectangular prisms added together. In other words, this sum is the "d" in the original equation $x^3 + cx = d$. Thus, $3u(x+u)$, the rectangles, is the value "$c$" in the equation. Consequently, it is possible to write "$d$" as the original form from which $x^3 + x[3u(x+u)]$ was derived,

$$(x+u)^3 - u^3 .$$

Hence,

$$d = (x+u)^3 - u^3 \quad \text{and} \quad c = 3u(x+u)$$

This implies that

$$x + u = \frac{c}{3u} .$$

With substitution, it is possible to derive that

$$d = \left(\frac{c}{3u}\right)^3 - u^3 .$$

Cubing each term leads to

$$d = \frac{c^3}{27u^3} - u^3 .$$

Multiplying both sides by $u^3$ generates

$$d \cdot u^3 = \frac{c^3}{27} - u^6 .$$

Writing an equation in terms of $u$ leads to

$$\frac{c^3}{27} = u^6 + du^3.$$

In an effort to write a quadratic in $u^3$, substitute $y = u^3$ into the previous equation. This leads to

$$y^2 + dy = \frac{c^3}{27} .$$

The Quadratic Formula provides the solution

$$y = -\frac{d}{2} + \sqrt{\frac{d^2}{4} + \frac{c^3}{27}} .$$

Notice that the negative solution is not included, as negative solutions were disregarded at the time. From which is follows that

$$u^3 = -\frac{d}{2} + \sqrt{\frac{d^2}{4} + \frac{c^3}{27}} .$$

Taking the cube root of both sides leads to

$$u=\sqrt[3]{-\frac{d}{2}+\sqrt{\frac{d^2}{4}+\frac{c^3}{27}}}\ .$$

Recall that originally $d=(x+u)^3-u^3$. Solving the equation for x gives $x=\sqrt[3]{d+u^3}-u$. Substituting the derived values for "$u$" and "$u^3$" provides

$$x=\sqrt[3]{d+\left(-\frac{d}{2}+\sqrt{\frac{d^2}{4}+\frac{c^3}{27}}\right)}-\sqrt[3]{-\frac{d}{2}+\sqrt{\frac{d^2}{4}+\frac{c^3}{27}}}\ .$$

This leads to Cardano's solution for $x$

$$x=\sqrt[3]{\frac{d}{2}+\sqrt{\frac{d^2}{4}+\frac{c^3}{27}}}-\sqrt[3]{-\frac{d}{2}+\sqrt{\frac{d^2}{4}+\frac{c^3}{27}}}\ .$$

## Applications to the Classroom

The cube root is analogous to the square root, which is described in Figure 1. The square root refers to the length of a side of a square with a desired area, while the cube root refers to the length of a side of a cube with a desired volume.

Although Cardano expanded on his Islamic predecessors by including the possibility of negative solutions, he was still not able to find all the possible solutions to cubic equations. Rafael Bombelli continued to expand on, and refine, the work of Cardano. Bombelli wrote a book that contained a logical progression from linear to quadratic to cubic equations. He also included the idea that it seemed possible to take the square root of a negative number, something that is encountered when using Cardano's formula for some cubic equations. He used notation that was a stepping stone toward the notation currently used in algebra. For example, Bombelli began writing R.Sq. to represent the square root. Bombelli also noted that it was possible to take the cube root of numbers that were negatives. He noted that this required a different set of rules for these new numbers, which he called "plus of plus" and "minus of minus" (Katz, 1998). In modern terms, Bombelli had begun working with imaginary numbers. Bombelli's promotion of the existence of imaginary numbers allowed the use of Cardano's Formula even when the sum beneath the root would be negative.

Problem 1: Use Cardano's Formula to solve a cubic equation.

Use Cardano's Formula to get one real solution of the equation $x^3 + 63x = 316$. Then, use this solution to find the remaining solutions to the equation.

The Solution:

Cardano's Formula gives

$$x=\sqrt[3]{\frac{d}{2}+\sqrt{\frac{d^2}{4}+\frac{c^3}{27}}}-\sqrt[3]{-\frac{d}{2}+\sqrt{\frac{d^2}{4}+\frac{c^3}{27}}}$$

as the solution to equations of the form

$$x^3+cx=d\ .$$

Thus, substituting the values of $c$ and $d$ into Cardano's formula gives

$$x=\sqrt[3]{\frac{316}{2}+\sqrt{\frac{316^2}{4}+\frac{63^3}{27}}}-\sqrt[3]{-\frac{316}{2}+\sqrt{\frac{316^2}{4}+\frac{63^3}{27}}}\ .$$

Simplifying this equation results in

$$x=\sqrt[3]{343}-\sqrt[3]{27}=7-3=4\ .$$

And so it follows that $x-4$ is a factor of the original cubic. It is now possible to use long division or synthetic division to determine the quadratic by which $x-4$ would be multiplied in order to obtain the original cubic. The division of $x^3+63x-316$ by $x-4$ results in $x^2+4x+79$. And so it follows (by the Quadratic Formula) that the remaining solutions of the original cubic are $-2\pm 5i\sqrt{3}$.

In some problems, students will find themselves faced with the issue of taking the cube root of a value they are unfamiliar with. Here is an example that would be the first step in solving this type of problem.

Problem 2: An Example Involving the Cube Root of an Imaginary Number.

a) Verify that $(5+i)^3 = 110+74i$.

b) Conclude that $\sqrt[3]{110-74i} = 5+i$.

c) Similarly show that $\sqrt[3]{110-74i} = 5-i$

d) Use Cardano's Formula to find one real solution to $x^3 - 78x = 220$.

The Solution:

a) Using the binomial theorem, expand

$$(5+i)^3 = (5)^3 + 3(5)^2(i) + 3(5)(i)^2 + (i)^3.$$

Simplifying this expression gives the desired result

$$(5+i)^3 = 110+74i$$

b) To show that

$$\sqrt[3]{110-74i} = 5-i,$$

first cube both sides

$$110-74i = (5-i)^3.$$

Then use the binomial theorem to expand the expression

$$(5-i)^3 = (5)^3 + 3(5)^2(-i) + 3(5)(-i)^2 + (-i)^3 .$$

Simplifying this expression gives the desired result

$$(5-i)^3 = 110 - 74i .$$

c) Substitute $c = -78$ and $d = 220$ into Cardano's Formula to obtain the real solution

$$x = \sqrt[3]{\frac{220}{2} + \sqrt{\frac{220^2}{4} + \frac{(-78)^3}{27}}} - \sqrt[3]{-\frac{220}{2} + \sqrt{\frac{220^2}{4} + \frac{(-78)^3}{27}}} .$$

Simplifying the terms under the cube root leaves

$$x = \sqrt[3]{110 + \sqrt{-5476}} - \sqrt[3]{-110 + \sqrt{-5476}} .$$

Rewriting in complex form results in

$$x = \sqrt[3]{110 + 74i} - \sqrt[3]{-110 + 74i} .$$

Factoring a negative one out of the second cube root gives

$$x = \sqrt[3]{110 + 74i} + \sqrt[3]{110 - 74i} .$$

Note that $\sqrt[3]{-110 + 74i} = \sqrt[3]{(-1)(110 - 74i)} = -\sqrt[3]{110 - 74i}$ .

Substituting the known quantities from parts a) and

b) leaves

$$x=(5+i)+(5-i).$$

Hence, one of the solutions to the cubic is

$$x=10.$$

Problem 3: Large templates for building the geometric representation for the equation $x^3+6x=20$ are provided. Create the three dimensional figure that represents the equation. Note that the figures are drawn to the scale necessary to create the geometric representation of this equation. Students could actually measure the side length of the appropriate pieces to find the desired solution.

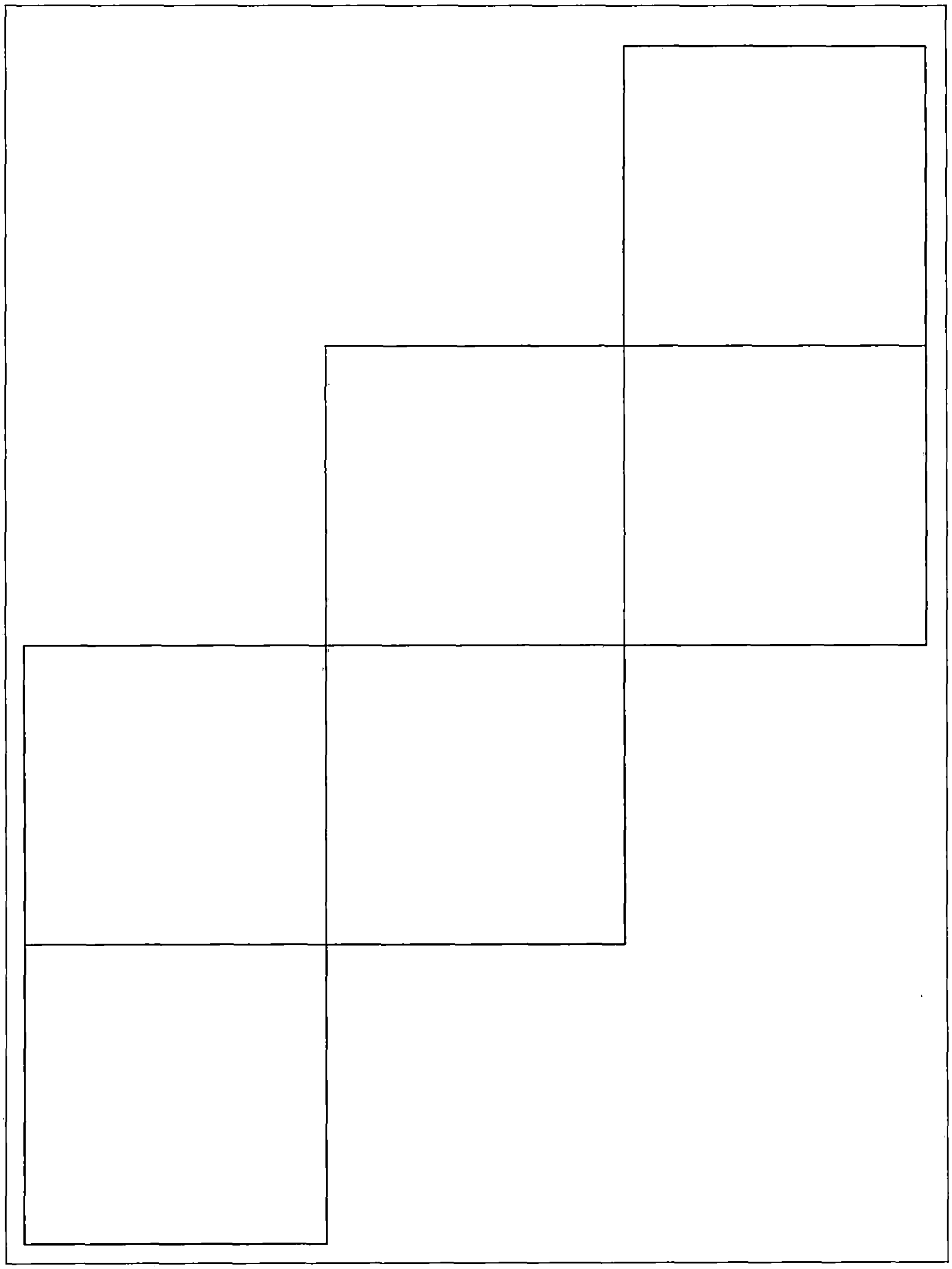


Figure 26. Cubic Net A

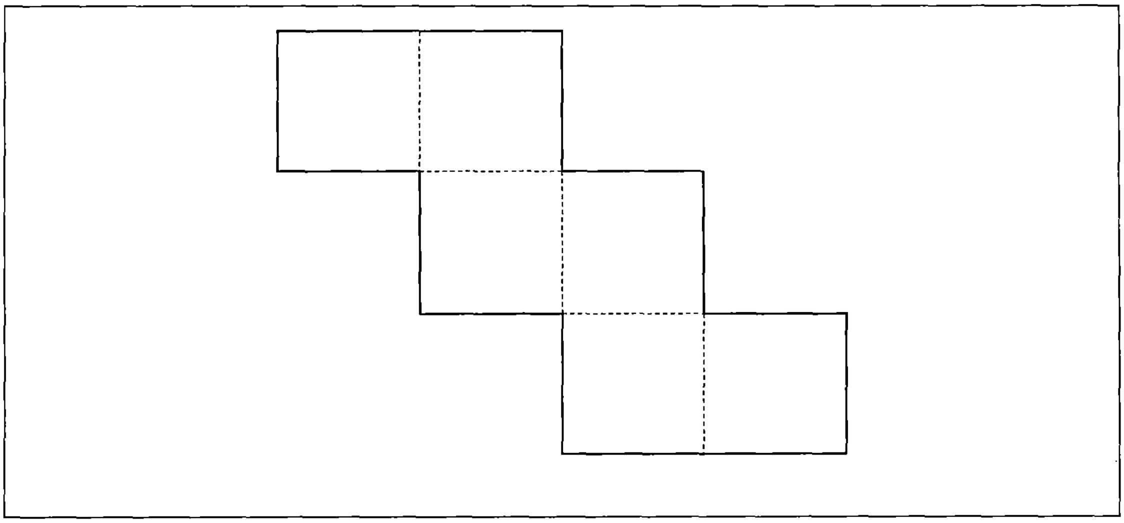
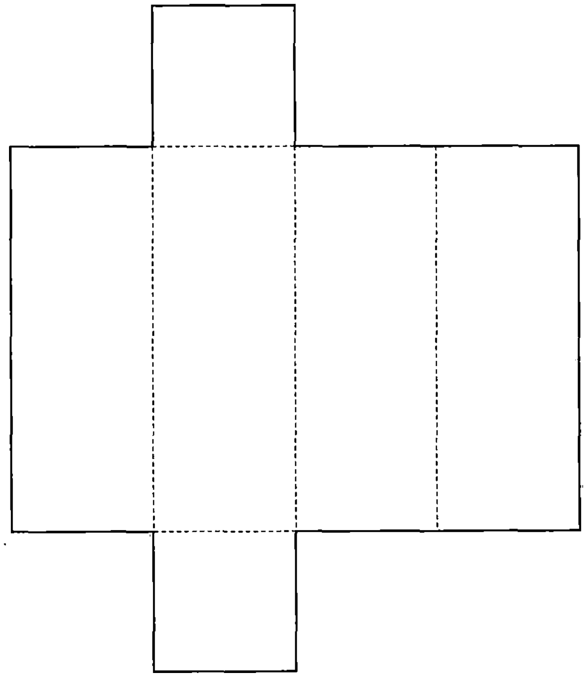
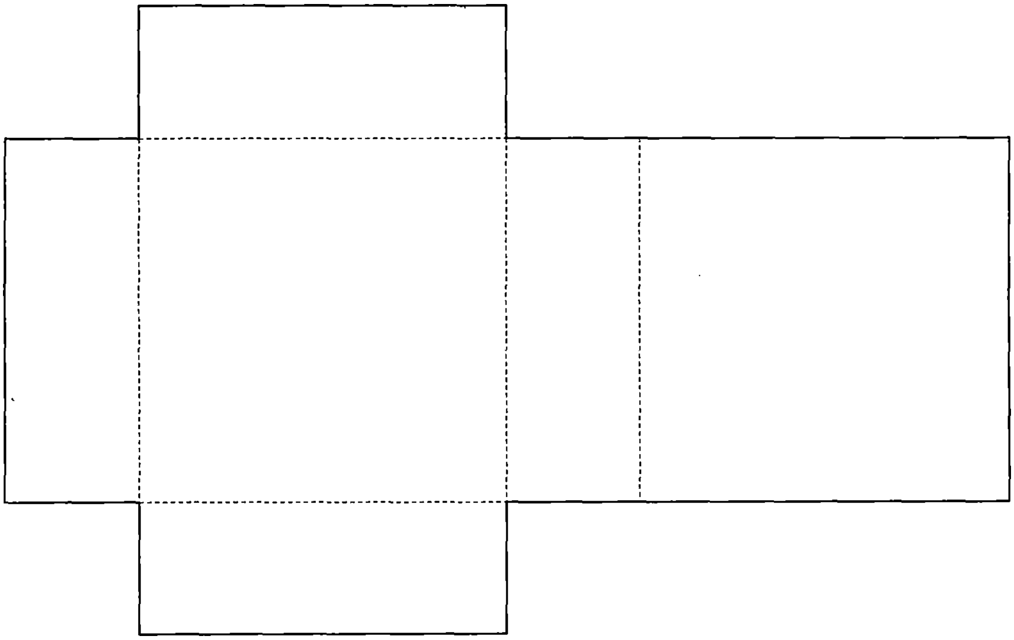


Figure 27. Cubic Net B



(Remember that the figure requires three of these nets.)

Figure 28. Cubic Net C



(Remember that the figure requires three of these nets.)

Figure 29. Cubic Net D

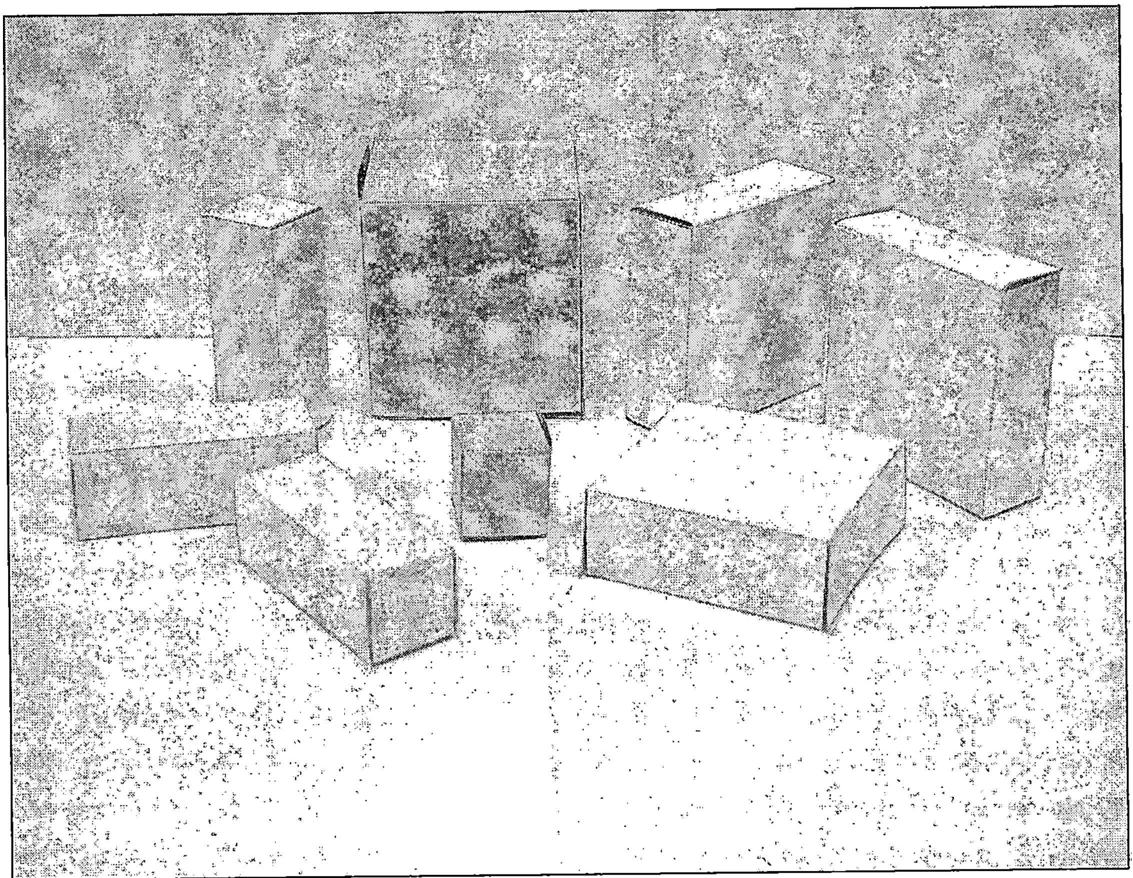


Figure 30. Disassembled Cubic Pieces

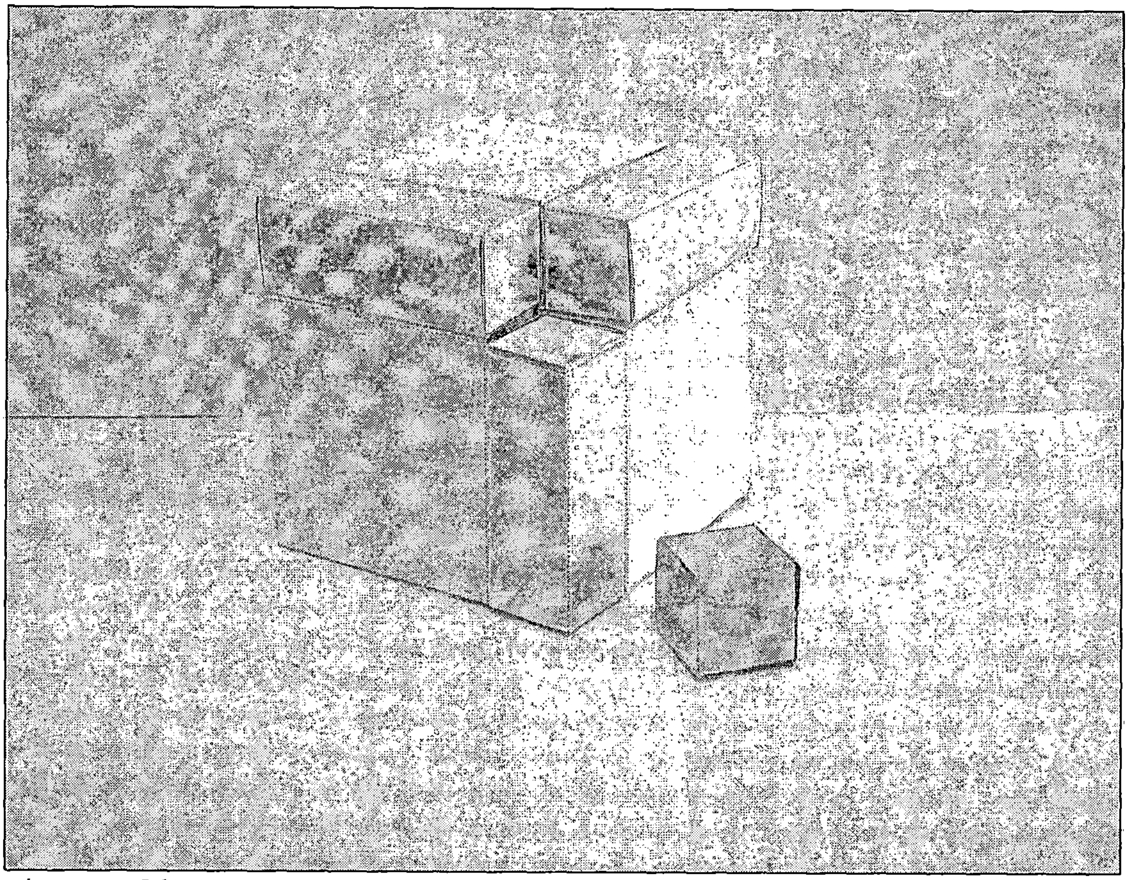


Figure 31. Assembled Cubic Pieces (Front View)

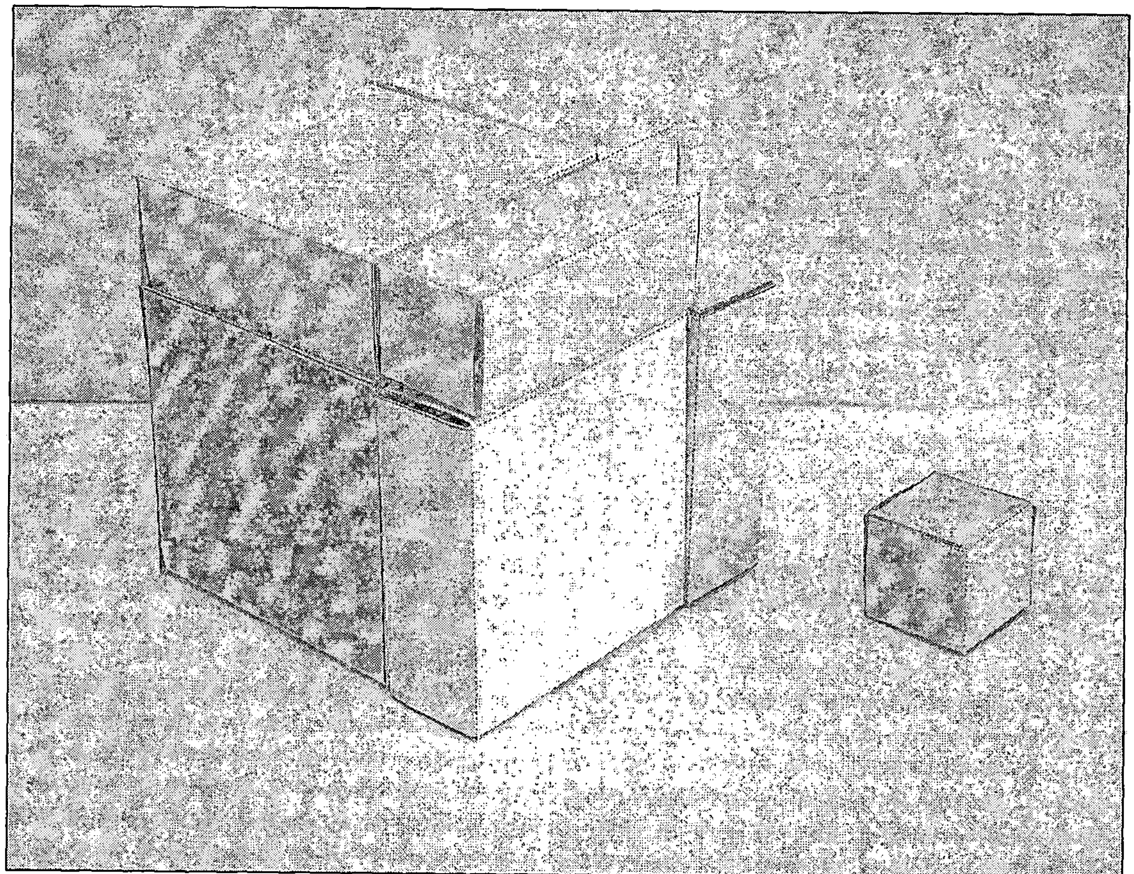


Figure 32. Assembled Cubic Pieces (Back View)

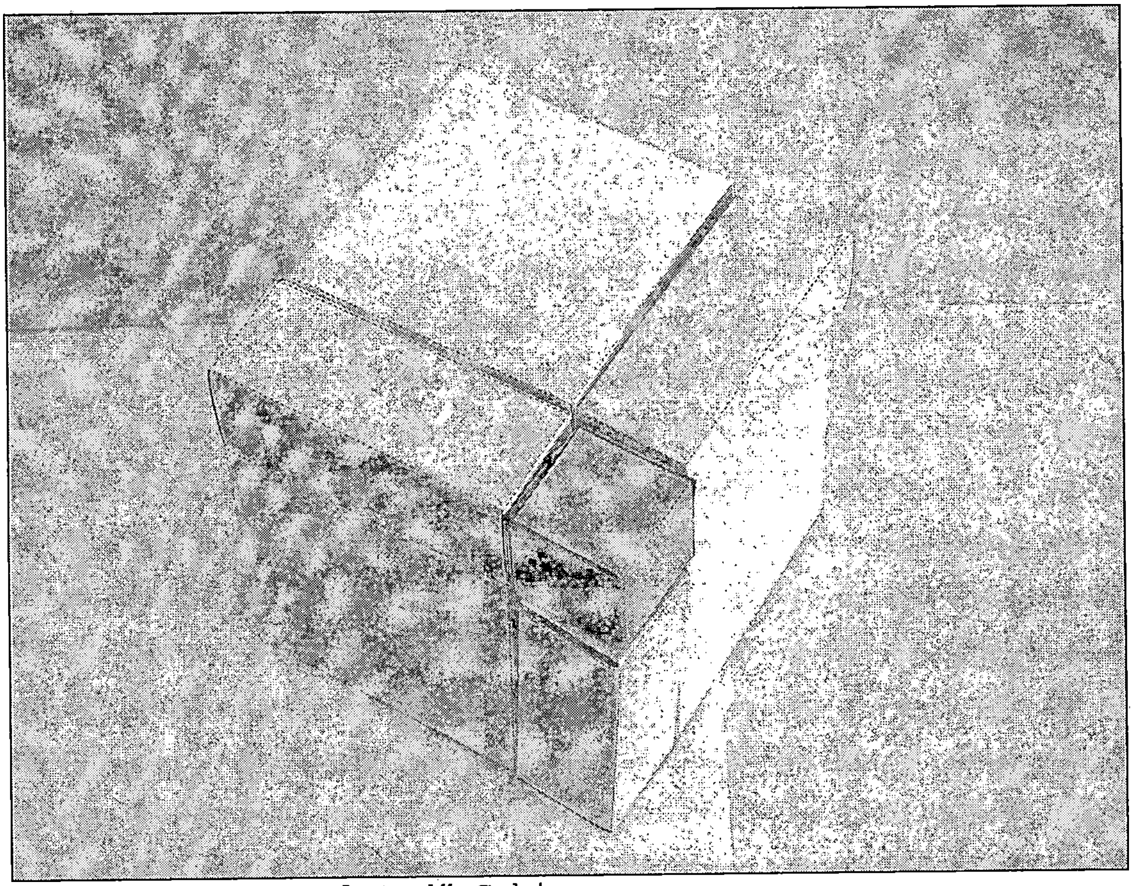


Figure 33. "Completed" Cubic

# CHAPTER FIVE

# CONCLUSIONS

Many would agree that the teaching of mathematics at all levels can be enriched by historical reflection. This enrichment may be most acute at the secondary school level. At this point, students begin to grasp the wide scope of mathematics. Discovering the power of mathematics may cause students to become overwhelmed. Knowledge of the historical context behind the concepts may alleviate the sense of confusion on the part of the student. It may also present feasible responses to the famous "When will I ever use this?" This question is one which secondary mathematics teachers face on a daily basis. Presenting the history behind topics allows students to see how the algorithms and ideas they are currently learning evolved over the course of hundreds of years.

Atival (1995) found that most mathematics taught in the secondary schools today is devoid of historical context. Essentially, the students view the topic as coming only from the classroom teacher. The teacher shows the students how to do the problem, and then decides whether they have done it correctly or not. This situation is harmful to the teaching of mathematics, a cumulative topic, as it presents the information as isolated and unrelated to the many other concepts that aided in its development (Atival, 1995). When students learn that the topics presented are products of a human endeavor that spans hundreds of thousands of years ago through the hard work or many people, it creates a sense of interest in the material, making it more personal and relevant.

The use of manipulatives and models has been shown to be extremely valuable in increasing students' understanding of a subject (National Council of Teachers of Mathematics, 1989). If the manipulatives and models are well-chosen, then the operations on them help to draw general conclusions about what happens in the more abstract cases. In addition, the use of manipulatives motivates those students who are more kinesthetic than auditory or visual. The act of moving pieces around will help the student understand and remember the material.

The examples and explanations in this paper offer both visual and kinesthetic learners an opportunity to excel. In addition, the derivations of the steps to equation-solving connects geometry with algebra. In doing so, it strengthens the skills and understanding in both areas.

The material within this paper addresses several of the content standards that California teachers are expected to address. Specifically, the material on linear equation solving addresses all or part of the following Algebra 1 standards[9]:

2.0 Students understand and use such operations as taking the opposite, finding the reciprocal, taking a root, and raising to a fractional power. They understand and use the rules of exponents.

5.0 Students solve multistep problems, including word problems, involving linear equations and linear inequalities in one variable and provide justification for each step.

9.0 Students solve a system of two linear equations in two variables algebraically and are able to interpret the answer graphically. Students are able to solve a system of two linear inequalities in two variables and to sketch the solution sets.

11.0 Students apply basic factoring techniques to second- and simple third-degree polynomials. These techniques include finding a common factor for all terms in a

[9] Standards taken from the California Department of Education website: www.cde.ca.gov.

polynomial, recognizing the difference of two squares, and recognizing perfect squares of binomials.

19.0 Students know the quadratic formula and are familiar with its proof by completing the square.

20.0 Students use the quadratic formula to find the roots of a second-degree polynomial and to solve quadratic equations.

21.0 Students graph quadratic functions and know that their roots are the x- intercepts.

22.0 Students use the quadratic formula or factoring techniques or both to determine whether the graph of a quadratic function will intersect the x-axis in zero, one, or two points.

In addition, the quadratic equation-solving addresses the following Geometry standards entirely or in part:

2.0 Students write geometric proofs, including proofs by contradiction.

4.0 Students prove basic theorems involving congruence and similarity.

5.0 Students prove that triangles are congruent or similar, and they are able to use the concept of corresponding parts of congruent triangles.

8.0 Students know, derive, and solve problems involving the perimeter, circumference, area, volume, lateral area, and surface area of common geometric figures.

9.0 Students compute the volumes and surface areas of prisms, pyramids, cylinders, cones, and spheres; and students commit to memory the formulas for prisms, pyramids, and cylinders.

11.0 Students determine how changes in dimensions affect the perimeter, area, and volume of common geometric figures and solids.

16.0 Students perform basic constructions with a straightedge and compass, such as angle bisectors, perpendicular bisectors, and the line parallel to a given line through a point off the line.

17.0 Students prove theorems by using coordinate geometry, including the midpoint of a line segment, the distance formula, and various forms of equations of lines and circles.

Although the cubic equation and its solutions are not addressed in the secondary mathematics curriculum, it is a valuable way to tie together several of the standards from the Advanced Algebra curriculum:

3.0 Students are adept at operations on polynomials, including long division.

4.0 Students factor polynomials representing the difference of squares, perfect square trinomials, and the sum and difference of two cubes.

5.0 Students demonstrate knowledge of how real and complex numbers are related both arithmetically and graphically. In particular, they can plot complex numbers as points in the plane.

6.0 Students add, subtract, multiply, and divide complex numbers.

7.0 Students add, subtract, multiply, divide, reduce, and evaluate rational expressions with monomial and polynomial denominators and simplify complicated rational expressions, including those with negative exponents in the denominator.

20.0 Students know the binomial theorem and use it to expand binomial expressions that are raised to positive integer powers.

## Additional References

If the reader wishes to pursue the topic further, here are some additional references that may be helpful.

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