1998

Using symbolic dynamical systems: A search for knot invariants

Russell Clark Wheeler

USING SYMBOLIC DYNAMICAL SYSTEMS: A SEARCH FOR KNOT INVARIANTS

A Project

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

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ABSTRACT

Recent research for knot invariants has shown that the fundamental group of the knot complement contains a great deal of information. This research has led mathematicians to examine representations of the commutator subgroup of such groups as they map into a fixed finite group, called the target group.

The structure of the set of homomorphisms from the commutator subgroup to the target group can be graphed and understood through symbolic dynamics even if the set of homomorphisms is infinite, even uncountable. The representations of these knot complements in a dynamical system appear as special periodic points when their domain is restricted. The system will produce other not so special periodic points and sometimes even nonperiodic points. This information has helped us understand more about the structure of the knot exterior and the variety of its covering spaces. However, work in this area is far from exhausted.

Dynamical systems as applied to fundamental groups of knot exteriors can be summarized more generally as follows. The dynamical system defined by the set of homomorphisms from the commutator subgroup of the fundamental group of a knot exterior to any target group, with the shift map has a structure of a shift of finite type. Such a structure can be completely described symbolically by a finite directed graph. This dynamical system is special in that it has

direct application not only to knots but links as well. Invariants of such a dynamical system, such as the number of periodic points of each given period, directly determine invariants of the associated knot. Also, when an abelian target group is used with the dynamical system, we can glean information about the infinite cyclic cover as well as the branched cyclic covers of the knot.[1]

The following is an expose' designed to highlight the important mathematical developments from Topology, Group Theory and Combinatorics that contributed to the development of these special types of dynamical systems called symbolic dynamical systems. Also, included is the step-by-step development of the symbolic dynamical system for the trefoil knot as an example following each step of the theoretical development of the algorithm for these symbolic dynamical systems.

ACKNOWLEDGMENTS

I would like to take this opportunity to thank Dr. Rolland Trapp for guiding me to the discovery and understanding of symbolic dynamical systems. Working with Dr. Trapp has been a pleasure from start to finish. I would also like to thank Dr. Zahid Hasan for his caring concern for my overall success throughout the graduate program and for his time reviewing and providing constructive advice on this project. Thanks to Dr. Chetan Prakash for his tutelage through the development of this work. A special thanks goes to a special lady who has been more help to me throughout my graduate experience than she knows, Dr. Terry Hallett, Graduate Program Coordinator and my personal academic advisor.

DEDICATION

A Special Thanks To My Wife, LaDonna Marie ... without whose confidence, moral support and patience, a master's education could not have been. Also, in memory of my parents Lawrence Eugene and Norma Lee Wheeler who were my motivation for pursuing a higher education in the first place.

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INTRODUCTION

A Lighthearted Look

Three strings went into a bar and sat down at a table. The first string asked, "Is there a waitress here?"

The second string said, "No, you have to go up to the bar and get your own."

So the first string got up, went over to the bartender and said, "I'll have three Scotches."

The bartender said, "We don't serve your kind in here."

"What kind is that?" said the string.

"Strings, we don't serve strings here," replied the bartender.

So the first string went back to the table and said to the other strings, "they won't serve us here."

The second string said, "Oh yea, we'll see about that." He got up, went over to the bartender, pounded on the bar, and said, "Hey bartender, I want three Scotches, and I want them now!"

The bartender said, "I told your friend, and now I'm telling you, we don't serve strings in here. Now beat it."

The second string went back to the table and shrugged.

The third string stood up and said, "Let me handle this." He tied himself into a nasty tangle and pulled the strands out of his ends, creating a wild mop of a hairdo. Then he walked over to the bar, leaned over close, and said, "Bartender, I would like three Scotches, please."

The bartender turned around and looked at the string, then he looked the string up and down. The bartender replied, "You're not fooling me; you're one of those strings, aren't you?"

The string looked the bartender straight in the eye and said, "Nope, I'm a frayed knot.[2]

In this case, if the bartender had a knot invariant test handy (possibly from a symbolic dynamical system) he could have quickly proven to himself that the tangled mess he was looking at was in fact a string (the unknot) and not a knot.

A Historical Overview

Why knots? What's so important about a tangled-up loop of string and whether or not it can be untangled without cutting or gluing?[3]

In the 1880s, the early days of knot theory, it was believed, among scientists, that all of space was pervaded by a substance called ether. Lord Kelvin (William Thomas, 1824-1907) proposed that atoms were only knots in the fabric of ether. His conclusion was that different elements would then correspond to different kinds of knots. Thus, a Scottish physicist, Peter Guthrie Tait (1831-1901) spent years tabulating knots in the hope that by listing all possible knots, he would eventually create a table of the elements.

Kelvin was proven wrong as an accurate model of atomic structure was published near the end of the 1800's. Conse-

quently chemists, along with the rest of the scientific community, lost interest in knots for nearly a century. However, during this period mathematicians maintained their intrigue of knots and a century of development in knot theory resulted.

By the 1980's, biochemists found knotting in DNA molecules which resulted in a revised interest in knot theory. Synthetic chemists have recently been pursuing the concept that knotted molecules could be created where the properties of the molecules may be determined by the type of knot it replicated. Ironically, a misguided model of atoms fostered a field of mathematics that after a 100 years finally yielded several major applications in the fields of biology and chemistry.[4]

Topology is the study of deformations of geometric figures that preserve their properties and is one of the major areas of research in mathematics today. Knot theory is one of many areas of study in topology and has led to many important advances in other areas of topology.[5]

A major area of focus within knot theory has been and still is the search for knot invariants. An invariant is a means by which one can identify a known knot from any projection of that knot and where applications of this invariant will consistently produce the same identification regardless of the ambient isotopy applied to the knot.

The study of representations of knot groups has yielded

a significant amount of useful information about knots over the past 70 years. The search for knot invariants has been a major motivation behind the efforts to examine knot group representations. What makes the study of mappings of knot group representations into a finite group possible, is the fact that the associated set of homomorphisms is finite and can be tabulated even though the group itself may be infinite or even uncountable.[6]

This paper attempts to expose this new direction of study in representations of knot groups. Let G be any knot group, Σ be a finite group called a target group and let K=[G,G] denote the commutator subgroup of G. Further let Hom(K,Σ) denote the set of homomorphisms from K into Σ as representations of K. The commutator subgroup has a presentation that is referred to as finite Z-dynamic meaning that the knot group G is an infinite cyclic extension of the commutator subgroup K. Such a structure generates a shift of finite type which can be completely described by what is known as a finite directed graph denoted by Γ.[7]

As mentioned earlier a key factor in this approach is that we can gain information about the group, regardless of its complexity, from manageable and relatively uncomplicated target groups. In addition, if we use finite abelian groups we are able to learn more about the homotopy and the homology of branched cyclic covers.[8]

These techniques also apply to links.[9] However, this

paper will explore these concepts only in terms of knots. No special effort to develop like concepts for links will be made nor will we discuss any conceptual difference between the two.

DEVELOPMENT: An Algorithm for Symbolic Dynamical Systems (SDS)

Homotopy, Homologies and Cohomologies

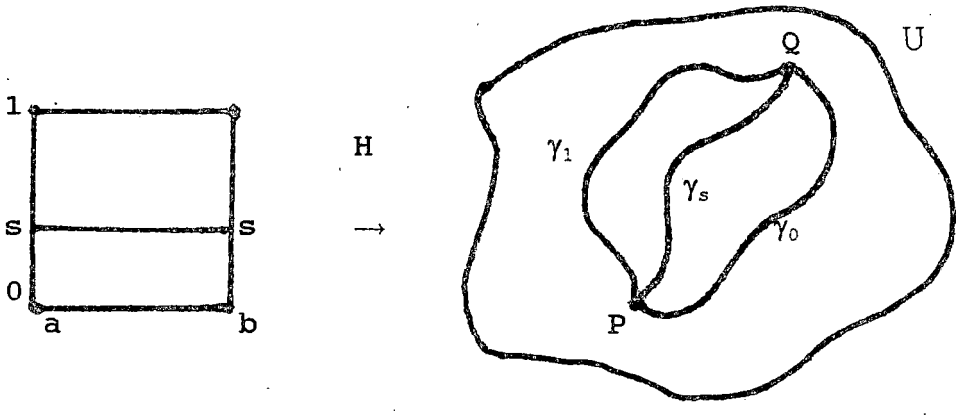
The fundamental group of a knot complement is the central concept in the development of an algorithm for symbolic dynamical systems of knots. However, before we can fully understand and appreciate fundamental groups some knowledge of homotopy, homologies and cohomologies is desirable. Intuitively, a homotopy is the deformation or movement of a path while the end points remain fixed.

Definition: *Let γ_s be a smooth family of paths all defined on the same interval [a,b] with the parameter s varying in the unit interval and the parameter t varying in the interval [a,b]. A homotopy is a map $H:[a,b]x[0,1]\rightarrow U$, where $H(a,s)=P$, $H(b,s)=Q$ for all $0\leq s\leq 1$ and $\gamma_s(t)=H(t,s)$ so each γ_s is a path in U from P to Q.*

For a family of paths to be smooth, we mean that the coordinates of the point $\gamma_s(t)$ are smooth functions of both s and t. The homotopy H is then a C^∞ function meaning that the two coordinate functions can be extended to be infinitely differentiable functions on some open neighborhood of the rectangle.

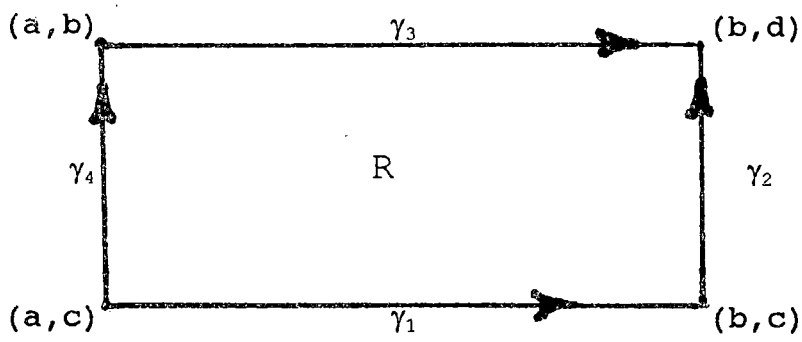
Two paths, γ_0 and γ_1, from an interval $[a,b]$ to a topological space U, with the same endpoints are said to be smoothly homotopic in U if there is a homotopy, H, that

smoothly deforms one path onto the other. That is,



where $H(a,s)=P$ and $H(b,s)=Q$ are constant functions on the interval $[a,b]$ for all $0\leq s\leq 1$. Note that $\gamma_s(t)=H(t,s)$ so each γ_s is a path in U from P to Q. A comparable definition exists for γ_0 and γ_1 being closed paths on U where the only difference is the requirement that $H(a,s)=H(b,s)$ for all $0\leq s\leq 1$ in the second part of the above definition.[10]

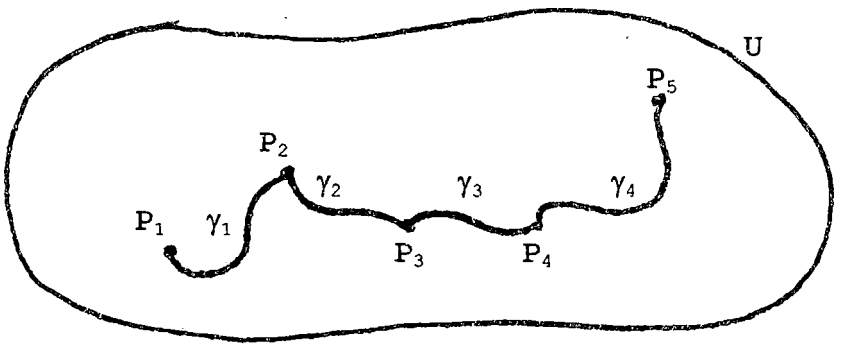
Before we discuss homology and cohomology groups let's first define and discuss 1-chains and 0-chains. Given a topological space X, a 1-chain in X is expressed as $\gamma=n_1\gamma_1+n_2\gamma_2+\cdots+n_i\gamma_i$, where each γ_i is defined to be a continuous path in X, each n_i is an integer and all paths are defined on the unit interval. Thus all 1-chains are formal linear combinations of paths. The boundary of a bounded rectangle R, denoted by ∂R with sides parallel to the axes is a 1-chain expressed as $\gamma=\gamma_1+\gamma_2-\gamma_3-\gamma_4$, where each γ_i is a straight path as shown:



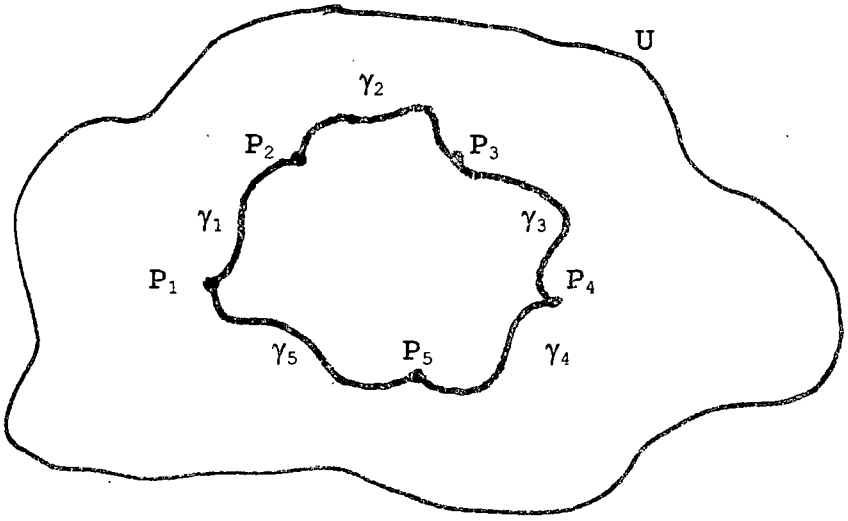
Such a boundary is referred to as a 1-boundary. Next a 0-chain in X is a function from X to the set of integers that is zero outside a finite set. In practice it means that a finite set of points are selected and each is assigned either a positive or negative multiplicity. A 0-boundary is a 0-chain where the sum of the multiplicities is zero.

For any 1-chain where $\gamma=n_1\gamma_1+\cdots+n_r\gamma_r$ the boundary of the path denoted by $\partial\gamma$ is defined to be the 0-chain expressed as $\partial\gamma=n_1[\gamma_1(1)-\gamma_1(0)]+\cdots+n_r[\gamma_r(1)-\gamma_r(0)]$. A closed 1-chain is defined to be a 1-chain whose boundary is zero and is also referred to as a 1-cycle.

The following are pictorial examples of a 1-chain, a 1-cycle, a homologous 1-cycle and their related boundaries as they are defined in H_1U.



The above depicts a 1-chain in U, where γ is a path from P_1 to P_5 defined by $\gamma=n_1\gamma_1+n_2\gamma_2+n_3\gamma_3+n_4\gamma_4+n_5\gamma_5$ and where each n_i integer represents the number of times each subpath γ_i is completely traversed. Recall that all 1-chains are formal linear combinations of paths.



This second example depicts a closed 1-chain called a 1-cycle where each point occurs as many times as a final point of a path as it does on an initial point of a path.

We can now define two homology groups of U, an open subset of X where Z_oU is a group of 0-chains on U and B_oU is a subgroup of 0-boundaries of Z_oU. One homology group is

called the zeroth homology group and defined as the quotient group $H_oU=Z_oU/B_oU$. The key here for using such a factor group is that even if the groups Z_oU and B_oU are infinite and uncountable we know the factor group is always finite, small and easy to handle and gives us valuable information about the larger group. The homology group H_oU actually measures the number of connected components in the topological space. The following proposition sums it up.

Proposition: *The homology group H_oU is canonically isomorphic to the free abelian group on the set of path connected components of U.*

The following definition gives a connection between homologies, 1-cycles and boundaries.

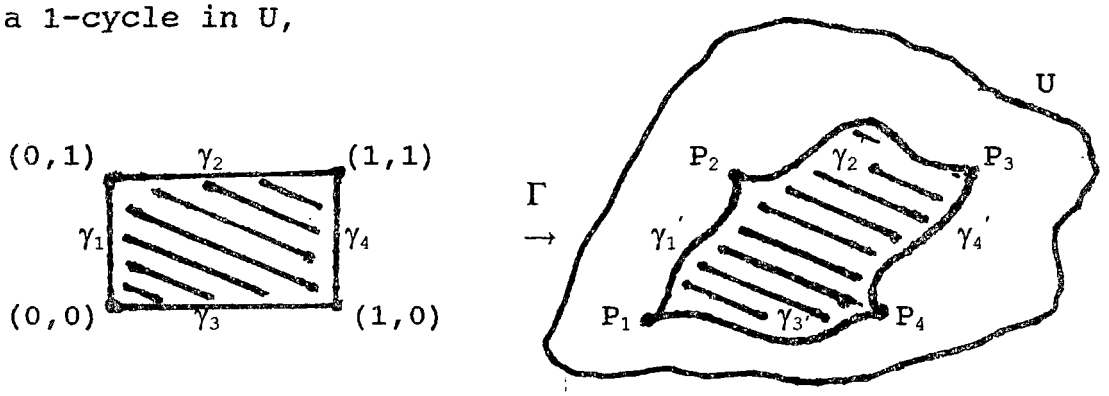
Definition: *Two closed 1-chains are said to be homolgous if the difference between them is a boundary in U.*

The next homology group is called the first homology group of U, denoted and defined by the quotient group as $H_1U=Z_1U/B_1U$. In a similar way Z_1U is defined as the group of closed 1-chains on U and B_1U, a subgroup of Z_1U, *is defined* as a group of 1-boundaries on U. Two closed 1-chains are homologous exactly when they have the same image in H_1U, in which case we say that they define the same homology class.[11]

Definition: *A 1-chain path γ is called a boundary (or a 1-boundary) in U if it can be written as a finite linear combination of boundaries of such maps with integer coeffi-*

cients.

Our goal here is to define 1-boundaries but first some preliminaries are necessary. The following is an example of a 1-cycle in U,



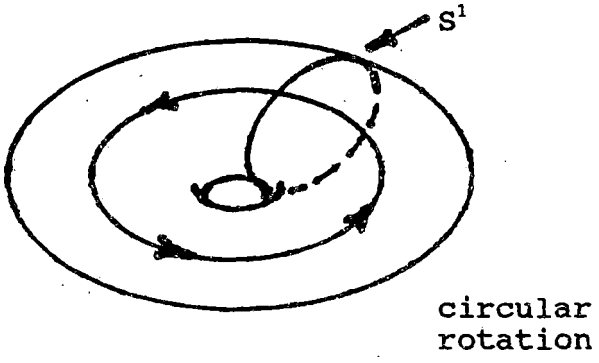
where the continuous mapping Γ is restricted to the four sides of the square [0,1]x[0,1] in the domain called a boundary. That is, $\Gamma(\gamma_i)=\gamma_i'$ and the 1-boundary, denoted by $\partial\Gamma$ is defined by the 1-cycle $\partial\Gamma=\gamma_1+\gamma_2-\gamma_3-\gamma_4$ where $\Gamma(0,0)=P_1$, $\Gamma(0,1)=P_2$, $\Gamma(1,1)=P_3$, $\Gamma(1,0)=P_4$. A 1-boundary in U is thus defined as a 1-chain in U written as a finite linear combination with integer coefficients of boundaries of such maps on rectangles.[12]

The Fundamental Group

Definition: *The fundamental group of a topological space X with base point x is defined to be the set of equivalence classes of loops at x where the equivalence relation is a homotopy.*

A loop is a path whose initial and terminal points are the same denoted by some point $x\epsilon X$. The fundamental group is denoted by $\pi_1(X,x)$.

For example, consider the fundamental group of the circle S^1, denoted by $\pi_1(S^1,x)$. Since the multiplication of arcs on the circle are <u>full</u> positive or negative revolutions (or no revolutions) then the fundamental group contains an infinite number of equivalence classes of full revolution loops and is homomorphic to the set of integers, (i.e., $\pi_1(S^1,x)\cong\mathbb{Z}$). When comparing the fundamental group of S^1 with the fundamental group of the torus, it is easy to recognize that the torus is a two-dimensional surface in $\mathbb{R}^3$, which can be viewed as a surface created by rotating a circle or $T^2=S^1xS^1$. The following depicts the torus



S^1 rotated in a circle

In either case two separate and distinct directions are defined on the torus. The circle S^1 in the figure depicts a direction we call the meridian and the circular rotation in the figure depicts a direction we call longitude. Thus, the

fundamental group of the torus, denoted by $\pi_1(T^2,x)$ also consists of an infinite number of equivalence classes of loops. The difference here is that the classes of loops and base point are defined as an ordered pair reflecting the two distinct directions. Just as $\pi_1(S^1,x)\cong\mathbb{Z}$, the fundamental group of the torus is homeomorphic to $\mathbb{Z}x\,\mathbb{Z}=\mathbb{Z}^2$, i.e. $\pi_1(T^2,x)=\pi(S^1xS^1,x)\cong\mathbb{Z}x\,\mathbb{Z}=\mathbb{Z}^2$.

One other example worth considering is the trivial case of the fundamental group. This is the fundamental group on the disk, D^2. In this case, all loops defined on any base point $x\in D^2$ are homotopic to the base point itself called the nonloop or identity loop. Therefore, $\pi_1(D^2,x)=e$.

Often the fundamental group is referred to without reference to a base point, denoted $\pi_1(X)$ when the definition of the fundamental group is applied to a path-connected topological space X, that is where any two points can be connected by a path. This is because the base point $x\in X$ does not matter since different base points result in isomorphic groups.

Let us now discuss how the fundamental group relates to the first homology group. Given a topological space X and a base point $x\in X$ there is a homomorphism from the fundamental group to the first homology group. This homomorphism takes the class of a loop, denoted by $[\gamma]$ at x to the homology

class of γ where γ is regarded as a closed 1-chain or 1-cycle.

It has been established that the first homology group is abelian. Thus the above defined homomorphism vanishes on all commutators, a b $a^{-1}b^{-1}$ of the fundamental group, implying that it also vanishes on the commutator subgroup of the fundamental group, denoted by $K=[\pi_1(X,x), \pi_1(X,x)]$. Thus K consists of all of the finite products of commutators. The quotient group $\pi_1(X,x)/K$ is often referred to as the abelianized fundamental group of X and is denoted by $\pi_1(X,x)^{ab}$. Therefore, the above defined homomorphism, h, maps the abelianized fundamental group of X into the first homology of X, i.e. $h:\pi_1(X,x)^{ab}\rightarrow H_1X$. In fact, the fundamental group determines the homology group giving the following proposition.[13]

Proposition: *If X is a path-connected space then the canonical homomorphism $h:\pi_1(X,x)^{ab}\rightarrow H_1X$ is an isomorphism.*

Intuitively, the first homology group counts the number of holes in a topological space. Since $\pi_1(X,x)/K$ and H_1X are isomorphic on a path-connected topological space X we can then say that the fundamental group, $\pi_1(X)$ also counts the number of holes in X. However, the fundamental group is able to detect and reveal far more about a topological space than do the homology groups as we shall see later in this

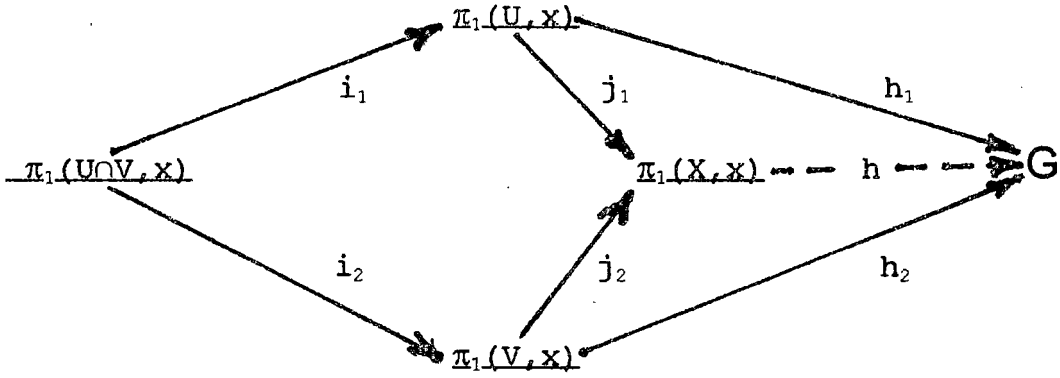
development. In particular, the fundamental group can tell us far more about knot exteriors than do the homology groups. Since the homology group is a quotient group it loses information where the fundamental group does not. Another nice feature of the fundamental group is that it is manageable for knot exterior applications.

The Seifert-Van Kampen Theorem

The following is a statement of the Seifert-Van Kampen theorem, sometimes called simply the Van Kampen Theorem.

Seifert-Van Kampen Theorem: *Let X be a space that is the union of two open subspaces U and V. Assume that the intersection of U and V is nonempty, each subspace U,V and U∩V are path connected and x is a point in the intersection. Also assume that all spaces X, U, V, and U∩V have universal coverings. If* $i_1 : \pi_1(U\cap V,x)\to\pi_1(U,X)$, $i_2:\pi_1(U\cap V,x)\to\pi_1(V,X)$, $j_1:\pi_1(U,x)\to\pi_1(X,x)$, *and* $j_2:\pi_1(V,x)\to\pi_1(X,x)$ *then for any homomorphisms* $h_1:\pi_1(U,x)\to G$ *and* $h_2:\pi_1(V,x)\to G$ *such that* $h_1\circ i_1=h_2\circ i_2$ *there is a unique homorphism* $h:\pi_1(X,x)\to G$ *such that* $h\circ j_1=h_1$ *and* $h\circ j_2=h_1$.[14]

The following diagram is an interpretation of the Van Kampen Theorem:



Clearly, the homomorphisms $h_1 \circ i_1$ and $h_2 \circ i_2$ from $\pi_1(U\cap V, x)$ *to* G are equivalent and are considered to be the same. The Van Kampen Theorem gives us a method for calculating fundamental groups as we will see. Since the Seifert-Van Kampen Theorem has been proven on simply connected topological spaces it follows that it also holds true on locally simply connected spaces since these spaces are far less restrictive. Thus the fundamental group $\pi_1(X,x)$ satisfies the definition of a universal group with the properties of the Seifert-Van Kampen Theorem.

The important hypothesis here is that all spaces are connected including $U\cap V$. The following are some corollaries and propositions as a consequence of the Van Kampen Theorem.

Corollary: *If U and V are simply connected then X is simply connected.*

Corollary: *If $U\cap V$ is simply connected, then*

$Hom(\pi_1(X,x), G) = Hom(\pi_1(U,x), G) \times Hom(\pi_1(V,x), G)$ *for any G.*

The above corollaries mean that $\pi_1(X,x)$ is the free

product of $\pi_1(U,x)$ and $\pi_1(V,x)$. Free groups are defined using the idea of free products.

Consider the graph of a union of two circles "8" we can label U and V. These two circles share only one point we will call x, where clearly $x= U\cap V$. Let γ_1 and γ_2 be loops around the boundary of each circle U and V. It can be seen that the fundamental group of U and V are infinite cyclic, generated by classes of these loops denoted by $[\gamma_1]$ and $[\gamma_2]$. It follows that to give a homomorphism from $\pi_1(X,x)$ to any group G is the same as specifying arbitrary elements g_1 and g_2 of G called generators. In summary we can say that there exists a unique homomorphism from $\pi_1(X,x)$ to G that will map $[\gamma_1]$ to g_1 and $[\gamma_2]$ to g_2. Here an important conclusion is that <u>the fundamental group is the free group on the generators $[\gamma_1]$ and $[\gamma_2]$</u>.

If we let $[\gamma_1]$=a and $[\gamma_2]$=b be generators of the fundamental group, then every element of the fundamental group can be uniquely called a word expressed in the form $a^{m_0}b^{m_1}a^{m_2}\cdots b^{m_r}$, where m_i are integers and $0\le i\le r$. All m_i are non-zero except occasionally the first and last. The identity element is then $e=a^{\circ}b^{\circ}$ The free group of two generators can be constructed directly and algebraically as the products of words defined by juxtaposition with cancellation

of adjacent multiplicative inverses. A formal definition is provided in the next section.

The following proposition is another consequence of the Van Kampen Theorem.

Proposition: *If G is a free group on n generators, and H is a subgroup of G that has finite index d in G, then H is a free group, with dn-d+1 generators.*

The Van Kampen Theorem has given us a definition of free fundamental groups with some understanding of group generators and elements of the group called words. We will need this for our development of an algorithm for symbolic dynamical systems.

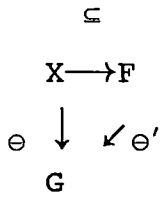
Group Presentations

In this section for group presentations we will develop techniques necessary for the development of presentations of fundamental groups of knot exteriors. This will then lead us to the calculation of symbolic dynamical systems.

From the Van Kampen Theorem we have seen some fundamental groups which are free groups. The following is a definition of a free group.

Definition: *A group F is said to be free on a subset $X \subseteq F$ if given any group G and any map, $\theta: X \to G$ there exists a unique homomorphism $\theta': F \to G$, extending θ, that is, having the property that $\theta'(x)=\theta(x)$, for all $x \in X$.*

This is the same as saying that the following diagram commutes.

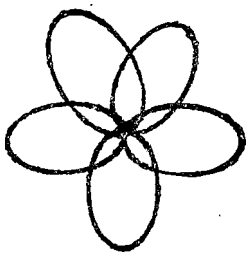


In this case X is called the basis of F and $|X|$, the cardinality of X, is called the rank of F, denoted by r(F).

Thus a group F is called free if it has a subset X where every element of F can be written uniquely, up to trivial relations, as the product of elements of X and their inverses. A trivial relation is one that is equivalent to the identity element e such as $x_1x_1^{-1}$, $x_1^{-1}x_1$, $x_2x_1x_1^{-1}x_2^{-1}$, etc. Here uniqueness simply means that if any two elements of F (called words) look different then they are different. A reduced word is one where no generators (elements of x) are adjacent to its multiplicative inverse. In other words all possible cancellations of the form $x_nx_n^{-1}=e$ have been done within the word. A reduced word $x_1x_2\cdots x_i$ is said to be cyclically reduced if $x_i\neq x_1^{-1}$.

The following describes some notation regarding free groups. A group F that is free on a subset X is denoted by F(X). A free group contains generators and only trivial relators and is denoted by $\langle x_i|\rangle$. The existence of the extended mapping Θ' implies that there are no relations in X and the uniqueness of Θ' implies that X generates F. A

comparable definition for a free abelian group is generated merely by replacing the word 'group' with the words 'abelian group' in the definition of a free group. The fundamental group of the torus is an example of a free abelian group of rank 2. Other examples of free fundamental groups are a circle of rank 1 (i.e. one trip of the path where the initial and terminal point is the same) and any bouquet of n circles all of which share only one point and are free groups of rank n. An example of a bouquet of 5 circles sharing only one point is as follows:



In this case rank n is rank 5 meaning that each circular path of the bouquet is traversed only once and in each case the common point is the initial and terminal point of each traversal. These examples of a free group are consequences of the Seifert-Van Kampen Theorem.

The following are some notably important facts for us regarding free groups. If G is any group, F is an arbitrary group and $X \subseteq F$, then we say that Hom(F,G) is the set of all homomorphisms from F to G and Map(X,G) is the set of all mappings from X into G. Now let ρ be a restriction map from the set of homomorphisms from F to G to the set of maps from

X into G defined by $\rho(\theta'(x))=\theta'(x), \forall\ x\in X$ and $\theta'\in Hom(F,G)$. Thus, ρ is surjective if an only if for all $\theta\in Map(X,G)$ there is an extension map θ' as defined in the forgoing definition of free maps and shown above. The map ρ is injective if and only if θ' exists and is unique. Therefore, to say that F is free on X is to say ρ is a bijection for any group G and vice versa. A free group may have many different bases all of which have the same number of elements. This implies that the rank of a free group is well defined. Consequently, a free group is determined up to an isomorphism by its rank.

The above comments can be summed up by the following lemma and two propositions:

Lemma: *If F is free on X, then X generates F.*

Proposition: *If F_i is free on X_i $(i=1,2)$ and $F_1\cong F_2$, then $|X_1|=|X_2|$.*

Proposition: *If F_i is free on $X_i (i=1,2)$ and $|X_1|=|X_2|$, then $F_1\cong F_2$.*

The following propositions, lemma and theorem provide us with some more useful facts about free groups.

Proposition: *Every group is isomorphic to a factor group of some free group. That is, G=Image of $\theta'\cong F(X)/Ker\theta'$.*

Lemma: *Let $a,b\in F(X)$ such that $ab=ba$. Then there is a $c\in F(X)$ such that $a=c^k$ and $b=c^h$ for some $k,h\in\mathbb{Z}$.*

In other words, this lemma says that free groups are non-commutative. Intuitively, free groups are the most non-abelian groups there are.

Proposition: *(i) In a free group F with n roots when they exist are unique, that is, if* $a,b \in F$ *satisfying* $a^n=b^n$, $n \in \mathbb{N}$, *then* $a=b$. *(ii) any element (word) w of F has only finitely many generators, that is,* $\{a \in F | a^n=w$, *for some* $n \in \mathbb{N}\}$ *is finite.*

Proposition: *Commutation is an equivalence relation on* $F \setminus \{e\}$. *That is the centralizer* $C(w):=\{w \in F |\ aw=wa$ *for any* $a \in F \setminus \{e\}\}$ *is abelian.*[15]

From here we venture into some facts about presentations of free groups. A presentation can be viewed as a convenient shorthand for specifying any particular group. If we let F(X) be a free group on X, R a subset of F, N the normal closure of R in F, and G the factor group F/N then a free presentation can be defined as follows:

Definition: *A free presentation, denoted by* $G=\langle X|R\rangle$, *is referred to simply as a presentation of G. The elements of X are called generators and those of R are called defining relators. A group G is said to be finitely presented if it has a presentation with both X and R being finite sets.*

Precisely, this means that the elements $x \in X$ generate G, the elements $r \in R$ are equal to the identity element $e \in G$, and G is the largest group with these properties. Further the defining relations $r \in R$ for G are assumed to be equal to e.

This convention allows normal algebraic manipulation. Thus a defining relation, say uv, when set equal to e, uv=e may take on the form $u=v^{-1}$, for some $u,v\in F(X)$. Groups frequently occur in the form of presentations but it is difficult to deduce properties of a group from a given presentation of it. There are a number of techniques used but none are clearly definitive in general. For our purposes we will derive presentations of knot groups using the Wirtinger process. This will be discussed later, in detail.

The following are some helpful facts about groups and their presentations. Every cyclic group is a homomorphic image of the set of integers, denoted by $\mathbb{Z}=\langle x|\rangle$, by definition. From Group Theory we know that the Kernel of a homomorphism of a cyclic group is cyclic, being either the normal closure of x^n, $n\in\mathbb{N}$ or trivial. In this notation, $\mathbb{Z}=\langle x|\rangle$, $\mathbb{Z}_n=\langle x|x^n\rangle$, $n\in\mathbb{N}$, we completely list all cyclic groups.

Proposition: *Every group has a presentation, and every finite group is finitely presented.*

The following are some important facts regarding homomorphisms.

Lemma: *Let F,G,H be groups and $v:F\to G$, $\alpha:F\to H$ be homomorphisms such that (i) the image of v is G, denoted Im v=G and (ii) Ker v $\subseteq$ Ker α. Then there is a homomorphism $\alpha':G\to H$ such that $v\circ\alpha'=\alpha$.*

This lemma is called the induced homomorphism lemma. The following proposition by von Dyck is a result of this lemma.

Proposition: *If $G=\langle X|R\rangle$ and $H=\langle X|S\rangle$ where $R\subseteq S\subseteq F(X)$, then there is an epimorphism $\phi:G\mapsto H$ fixing every $x\in X$ such that $\ker\phi=S/R$. Conversely, every factor group of $G=\langle X|R\rangle$ has a presentation $\langle X|S\rangle$ with $R\subseteq S$.*

The following proposition gives a means by which we may test a map from the generators of a group G to another group H to determine if it extends to a homomorphism of all of G. This proposition is key to the homomorphisms we will look at for symbolic dynamical systems of knot complements.

Proposition: *If given a group presentation $G=\langle X|R\rangle$, a group H, and a mapping $\theta:X\mapsto H$, then θ extends to a homomorphism $\theta':G\mapsto H$ if and only if the result of substituting $x\theta$ for x in r yields the identity in H, $\forall\, x\in X$ and $r\in R$.*

When such a θ' exists it must be unique since X generates G. Also note that θ' is an epimorphism if and only if $\langle x\theta\rangle=H$.[16]

Another valuable tool in the development of an SDS for a knot is the use of Teitze transformations or Teitze moves. Teitze moves give us the tools needed to transform a given presentation of a group to a different presentation of the same group. The following proposition formalizes this method for us.[17]

Proposition: *Let $F\langle X|\rangle$, $G=\langle X|R\rangle$ and suppose that $w,r\in F$ with w arbitrary and $r\in R\setminus R$. If y is a symbol not in X, then both*

the inclusions $X \mapsto \langle X|R,r\rangle$ *and* $X \mapsto \langle X,y|R,y^{-1}w\rangle$ *extend to isomorphisms with domain G.*

The above proposition yields four ways to adjust a given presentation <X|R> to derive another equivalent presentation for the group. These four ways come directly from the four isomorphisms of the above proposition and are called Teitze transformations. These four moves are defined as follows where F is a free group on the subset X throughout.

Definition: *i) Adjoining a relator, denoted by R+ is defined as* $X'=X$, $R'=R \cup \{r\}$, *where* $r \in \overline{R}\backslash R$ *is the normal closure in F. ii) Removing a relator, denoted by R- is defined as* $X'=X$, $R'=R\backslash\{r\}$, *where* $r \in R \cap \overline{R\backslash\{r\}}$. *iii) Adjoining a generator, denoted by X+ is defined as* $X'=X\cup\{y\}$, $R'=R\cup\{y^{-1}w\}$, *where* $y \notin X$ *and* $w \in F$. *iv) Removing a generator, denoted by X- is defined as* $X'=X\backslash\{y\}$, $R'=R\backslash\{y^{-1}w\}$, where $y \in X$, $w \in \langle X\backslash\{y\}|\rangle$ and $y^{-1}w$ is the only member of R involving y.[18]

For example, by a finite series of Tietze moves we can show that the group G_1 presented by

$$G_1=\langle w,x,y,z|wx=y,\ xy=z,\ yz=w,\ zw=x\rangle$$

is cyclic by reducing its presentation to that of a cyclic group of order 5.

To begin with consider the first and third relators y=wx and w=yz, where by substitution gives us another relator of the group eliminating the generator y, namely w=wxz. By multiplying both sides on the left by w^{-1} we can elimi-

nate another generator, w and derive yet another relator, $w^{-1}w=w^{-1}wxz$ or $e=xz$ or $\underline{x^{-1}=z}$. By applying a comparable series of Tietze moves to the relators $z=xy$ and $x=zw$ we derive the new relator $\underline{y=w^{-1}}$. By substituting into the original presentation we can rewrite it in terms of the two generators w and x as follows: $G_1=\langle w,x|wx=w^{-1},xw^{-1}=x^{-1},\ w^{-1}x^{-1}=w,\ x^{-1}w=x\rangle$. Since $wx=w^{-1}$ can be written as $w^2=x^{-1}$ and since $w^{-1}x^{-1}=w$ can be written as $x^{-1}=w^2$ then two of the relators are the same allowing us to eliminate one in our presentation. By a similar arguement we can eliminate one of the remaining two generators since they are also equal. Thus, we have reduced our presentation to the following:

$$G_1=\langle w,x|w^2=x^{-1},\ w=x^2\rangle.$$

A close look reveals that yet another substitution can be made for w^2 since $w=x^2$ implies that $w^2=x^4$ in the second realtor. This substitution allows us to eliminate another generator, w and reduce our presentation to only one relator as follows: $G_1=\langle x|x^4=x^{-1}\rangle$ or $G_1=\langle x|x^5\rangle$ since $x^4=x^{-1}$ implyies that $x^5=e$. Thus, $G_1=\langle x|x^5\rangle$ is a presentation for an infinite cyclic group of order 5.

The following proposition describes how two presentations of a given group are related.

Proposition: *Given any two finite presentations of the same group, one can be obtained from the other by a finite sequence of Tietze transformations.*

Later we will see that Tietze moves allow us to simplify a

group presentation to one having fewer generators and relators.

In order to fully develop an algorithm for the symbolic dynamical systems for knot groups we will need presentations of commutator subgroups. The Reidemeister-Schreier theorem will allow us to accomplish this task as we shall see later.

THE FOCUS: The SDS and Directed Graph for the Trefoil

The Wirtinger Presentation

Our goal here is to present the algorithm for producing a Symbolic Dynamical System (SDS) for any non-trivial knot k. A dynamical system is defined as follows:

Definition: *A dynamical system is a pair (X,σ) consisting of a topological space X and a homeomorphism $\sigma:X\to X$. A mapping $f:(X,\sigma)\to(X',\sigma')$ of dynamical systems is a continuous function $f:X\to X'$ for which $f\circ\sigma=\sigma'\circ f$. The dynamical systems (X,σ) and (X',σ') are conjugate if there exists a mapping $g:(X',\sigma')\to(X,\sigma)$ such that $g\circ f$ and $f\circ g$ are identity functions.*[24]

The four major steps in the algorithm are developed sequentually as follows:

1) derive a presentation of a given knot group using the Wirtinger process,
2) develop a presentation for the commutator subgroup from the Wirtinger presentation using the Reidemester-Schreier Theorem with a distinguished generator.
3) develop the augmented group system and shifts of finite type from the presentation of the commutator subgroup and lastly,
4) produce the graph determined by the shifts of finite type into what is called the directed graph.

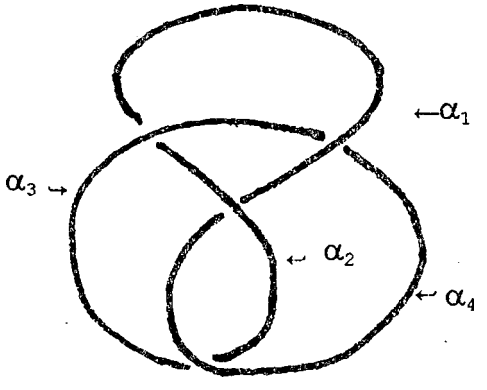
Each of the above steps will be explained in detail as the focus of this report. As each step is developed we will apply them to the simplest non-trivial knot, called the trefoil, as an ongoing example. In the end we will discuss the promise that SDS holds for discovering new knot invariants which will hopefully enhance our ability to distinguish knots.

We have not yet explained how to arrive at a presentation of the fundamental group for a knot. First we know that the fundamental group of a surface can reveal a great deal about that surface. Since a knot has no surface we must consider the complement of a knot as it would relate to an imbedding in $\mathbb{R}^3$. By creating a uniform tubular neighborhood around the knot, we then have a solid torus embedded in $\mathbb{R}^3$. By discarding this solid tube then we are left with a "tunnel" in 3-space. The 3-space without the solid tube of the knot is called the complement of the knot, denoted by $\mathbb{R}^3$-k. The hope here is to identify unique characteristics of the knot from its complement in $\mathbb{R}^3$. This can be related to the concept of identifying a person in their absence by the unique impression they left in a fingerprint.

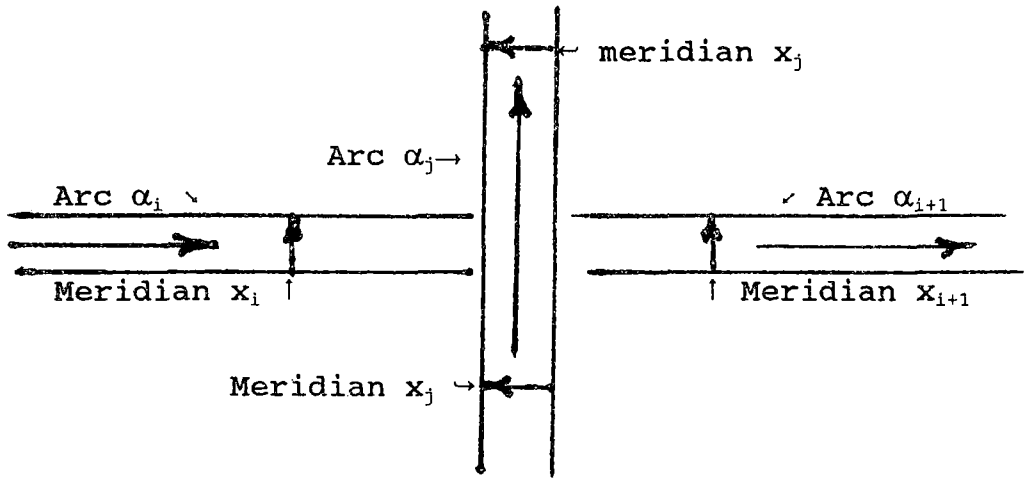
Let's now consider the fundamental group of the knot complement, denoted by $\pi_1(\mathbb{R}^3-k)$.

We mentioned earlier that Wirtinger presentations were the key for presentations of the fundamental group of knot

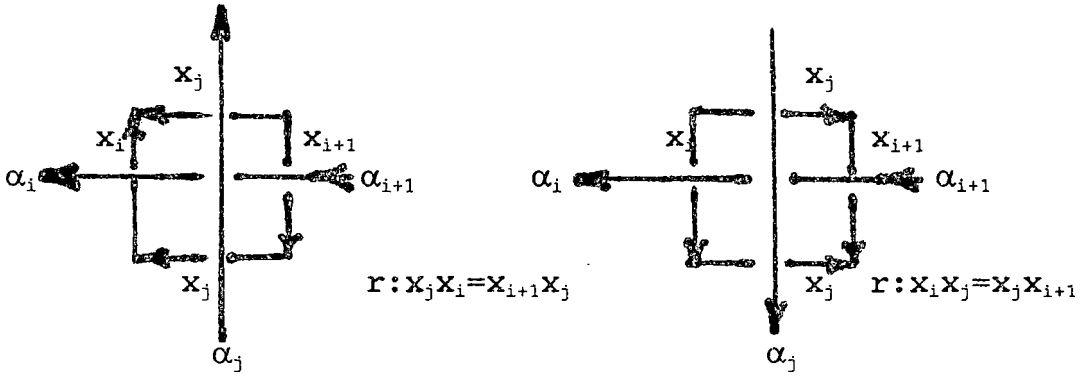
complements, so lets now look at how this works. Consider any projection of a knot onto a plane in $\mathbb{R}^3$. From this projection we identify the overpasses and underpasses at each crossing. An arc in a knot projection is that part of the projected knot that adjoins one underpass to another. Then we label each of a finite number of knot arcs by $\alpha_1,\cdots,\alpha_n$ in the projection plane. Thus, each α_i is connected to α_{i-1} and α_{i+1} (mod n) by the underpassing arcs. The union of these arcs is called a diagram of the knot *k*. For example, the arcs of a particular diagram of the figure-eight knot are shown below.



Next, we select a meridian loop x_i of the knot for each arc α_i with orientation. Imagine standing in the 'tunnel' of each arc facing with the orientation. A meridian is chosen as a clockwise loop which appears as a straight line oriented from right to left in the projection. The following is a pictorial close up of one crossing of a projection of a knot complement with orientation of arcs and related meridian.



Notice that arc α_j has an associated meridian denoted x_j. By viewing the meridian loops as oriented straight lines we realize that only two general possibilities exist as depicted below. From this we see a relation r from each that yields either $r:x_jx_i=x_{i+1}x_j$ or $r:x_ix_j=x_jx_{i+1}$ depending on the relative orientation of the crossing arcs as follows:



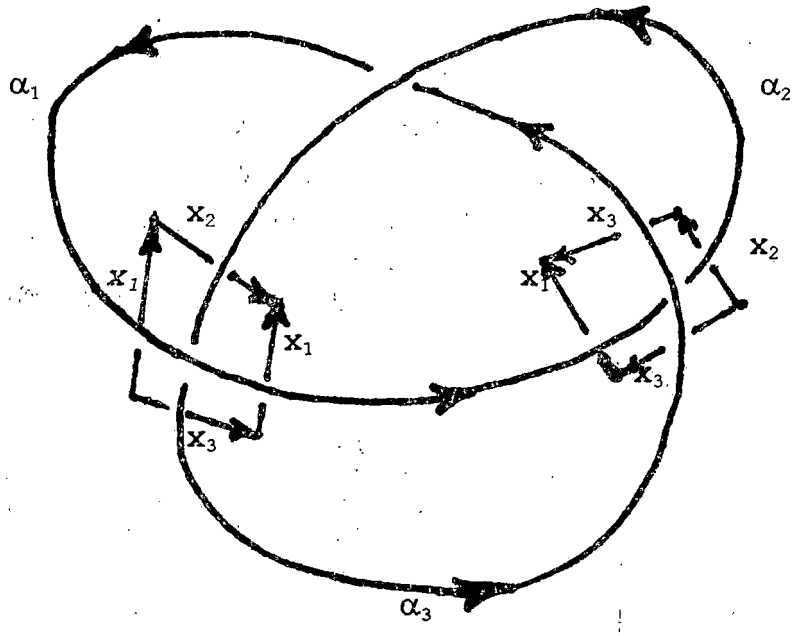
Here let r_i denote whichever of the two equations holds for any given crossing. For any knot k there are a finite number of relations say $r_1,\cdots,r_n$, where n is the number of crossings of the knot. The following theorem summarizes the above remarks, yielding the Wirtinger presentation for $\pi_1(\mathbb{R}^3-k)$. This is a consequence of the Van Kampen Theorem

for fundamental groups.[19]

Theorem: *The group $\pi_1(\mathbb{R}^3-k)$ is generated by the homotopy classes of the meridian x_i and has the following presentation $\pi_1(\mathbb{R}^3-k)=\langle x_1,\cdots,x_n|r_1,\cdots,r_n\rangle$, where any one of the r_i is a consequence of the other relations and may be omitted without effect.*

Since we will consider fundamental groups of knot complements only in $\mathbb{R}^3$, then for convience we will use the notation $\pi_1(k)$ in place of $\pi_1(\mathbb{R}^3-k)$. Where the knot is specifically known we will name it in place of the general notation of k. As mentioned earlier we will use the trefoil to demonstrate the development of each step of the SDS algorithm.

We will now develop a Wirtinger presentation of the trefoil where x_i are the generators and r_i are the relators of the presentation. We begin with a projection of the trefoil showing a chosen orientation (labeled arcs) and appropriately oriented generators (meridians). Recall from the above theorem that the relation coming from any one of the crossings may be omitted in the development of the Wirtinger presentation.



Here we write the relators from the two crossings indicated. The relators for the trefoil are $r_1: x_1x_2=x_3x_1$ and $r_2: x_3x_1=x_2x_3$. Since we now have both the generators and relators the presentation of the fundamental group of the trefoil can be written as,

$$\pi_1(\text{trefoil})=\langle x_1, x_2, x_3 \mid x_1x_2=x_3x_1=x_2x_3\rangle.$$

As mentioned earlier the trefoil is a non-trivial knot. That is equivalent to saying that π_1(trefoil) is a non-abelian group, since the only knot with abelian fundamental group is the unknot. Note that $\pi_1(\text{unknot})=\mathbb{Z}$, and is generated by a meridian of the unknot. It is worth remarking that the abelianization of any knot group is isomorphic to $\mathbb{Z}$.[20]

Tietze Moves and the Distinguished Generator

Next our goal is to find a presentation for the com-

mutator subgroup K of $\pi_1(k)$. We will use Tietze moves in a process given to us by the Reidemester-Schreier Theorem.[21] A key here is that one of the Tietze moves is designed to introduce a new generator *a*, such that $x_i=x_1a$ for some $i\in\mathbb{Z}$.

In order to find the commutator subgroup K we must first choose a generator, x called the distinguished generator and set all other generators equal to a multiple of the distinguished generator and another unknown generator. That is, we will hold that $x=x_1$, and $x_i=xa$ for some new element a and some $i\in\mathbb{Z}$. After this then we can rewrite a presentation eliminating a known generator by replacing it with its equivalent in terms of the distinguished generator.

Now we want to write a presentation of the commutator subgroup of our trefoil group using the Reidemester-Schreier Theorem discussed earlier. Consequently, the commutator subgroup of π_1(trefoil) will also be a nonabelian group as is needed. A consequence of the Reidemester-Schreier Theorem is that the generators of the commutator subgroup are defined by $a_i=x^{-i}ax^i$ and the defining relators are found by conjugating the relation in the last presentation by powers of x. Notice that we have already done this for the trefoil. We need only rewrite our relation in terms of a_i, $i\in\mathbb{Z}$. In reviewing $a_i=x^{-i}ax^i$ by successive values of $i\in\mathbb{Z}$ starting with i=0 our generators can be derived and by substitution our relation can be rewritten. If i=0, then

$a_0=x^{-0}ax^0=a$ implying $a_0^{-1}=a^{-1}$. If i=1, then $a_1=x^{-i}ax$ implying $a_i^{-1}=(x^{-1}ax)^{-1}=x^{-i}a^{-1}x$, etc. A quick inspection of our only relation shows that we can stop here.[30]

For example, the fundamental group of the trefoil can be presented with one meridian generator (the distinguished generator), x and some other generator *a* as $\pi_1(\text{trefoil})=\langle x,a|x^{-1}a^{-1}x^{-1}ax^2a\rangle$. By inserting $xx^{-1}=e$ in the relator between the a^{-1} and the second x^{-1} our relator is transformed into the product of conjugations of the form $a_i=x^{-i}ax^i$. Now it can be written as $x^{-1}a^{-1}xx^{-2}ax^2a$, where $x^{-1}a^{-1}x=a_1^{-1}$, $x^{-2}ax^2=a_2$ and $a=a_0$. With this we can write the relator as a presentation of the commutator subgroup *K* as $K=\langle a_i|a_{i+1}^{-1}a_{i+2}a_i, i\in\mathbb{Z}\rangle$. Later we will see how this example plays a part in the complete development of the symbolic dynamical system for the trefoil.

Augmented Group Systems and Shifts of Finite Type

To this point we have completed roughly half of our algorithm of symbolic dynamical systems with the trefoil as the example. However, before we continue some introduction to augmented group systems and shifts is necessary here. Lets begin with a definition.

Definition: *An augmented group system is a triple denoted by* (G,χ,x) *consisting of a finitely presented group G, an epimorphism* $\chi:G\to\mathbb{Z}$ *and a distinguished element* $x\in G$, *such that* $\chi(x)=1$.

We have an example of an augmented group system in our trefoil.[22] The augmented group system for the trefoil is the triple $(\pi_1(\text{trefoil}),\chi,x)$, where $\chi:\pi(\text{trefoil})\rightarrow\mathbb{Z}$ and $\chi(x)=1$.

Next lets consider the definition of a representation shift.

Definition: *A representation shift, denoted by Φ_Σ, is the set of representations $\rho:K\rightarrow\Sigma$ together with the shift map $\sigma x:\Phi_\Sigma\rightarrow\Phi_\Sigma$, defined by $\sigma_x\rho(a)=\rho(x^{-1}ax)$ for all $a\in K$, where K is the commutator subgroup of the fundamental group and Σ is the target group.*

Lets now define a shift of finite type. First let x be a distinguished generator and σ_x be a series of mappings, where $\sigma_x r(\gamma)=\rho(x^{-1}\gamma x)$ $\forall\rho\in H$ only and $x\in K$. By Van Kampen such a series of mappings would be depicted as one mapping cycle. If we first applied our set of representations Φ_Σ and then our series of mappings σ_x we would have what is called shifts of finite type. We will exemplify a shift of finite type with the trefoil when we resume the construction of its SDS.

Next we will look at permutation representations.

Definition: *A permutation representation of a group K is a homorphism $\rho:K\rightarrow S_r$, where S_r is the symmetric group operating on the set $\{1,2,\cdots,r\}$.*

We call ρ a representation of K_χ in S_r. The following proposition is well known.[23]

Proposition: *Let K be any group and let r be a positive integer. The function* $\pi:\rho\rightarrow\{g\in K|\rho(g)(1)=1\}$ *maps the set of representations* $\rho:K_\chi\rightarrow S_r$ *onto the set of subgroups* $H\leq K$ *having index* $|K:H|\leq r$. *The preimage of any subgroup of index r contains exactly (r-1)! transitive representations.*

If $\rho(K)$ operates transitively on the set $\{1,2,\cdots,r\}$ then we say the representation ρ is transitive.

Recall that a dynamical system is defined as follows:

Definition: *A dynamical system is a pair* (X,σ) *consisting of a topological space X and a homeomorphism* $\sigma:X\rightarrow X$. *A mapping* $f:(X,\sigma)\rightarrow(X',\sigma')$ *of dynamical systems is a continuous function* $f:X\rightarrow X'$ *for which* $f\circ\sigma=\sigma'\circ f$. *The dynamical systems* (X,σ) *and* (X',σ') *are conjugate if there exists a mapping* $g:(X',\sigma')\rightarrow(X,\sigma)$ *such that* $g\circ f$ *and* $f\circ g$ *are identity functions.*[24]

The following proposition follows from the last stated proposition above.

Proposition: *Let r be a positive integer. The function* $\pi:\rho\rightarrow H=\{g\in K_\chi|\rho(g)(1)=1\}$ *induces a mapping from* (Φ_r,σ_r) *onto* $(\tilde{\Phi}_r,\tilde{\sigma}_r)$.

The following corollary is a consequence of the pre-

ceding two propositions.[25]

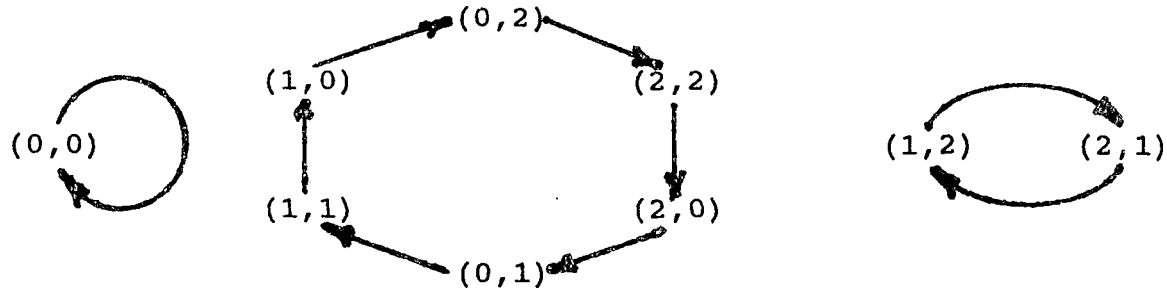
Corollary: *Let (G,χ,x) be an augmented group system and let r be a positive integer. Then the associated representation shift (Φ_r,σ_x) is finite if and only if the subgroup system $(\tilde{\Phi}_r,\tilde{\sigma}_x)$ is finite.*

The pair (Φ_Σ,σ_x) is a dynamical system where Φ_Σ is a space and σ_x is a homeomorphism which can be nicely represented by a directed graph.

The Directed Graph

The fourth part in our algorithm for the construction of a symbolic dynamical system for a given knot complement is the directed graph, denoted by $\Gamma=(V,E)$, where $V=\{\rho|\rho:K\rightarrow\Sigma\}$ is the set of vertices and $E=\{\rho\rho'|\rho'=\sigma_x\rho\}$ is the set of edges. The vertices are representations of the fundamental group in $\mathrm{Hom}(K,\Sigma)$ and are connected by an edge called the shift map, σ_x, from ρ to ρ'. It turns out that the entire directed graph is one or more series of periodic points all connected by edges. Some periodic points are of period one meaning a point is connected to itself by one edge. And it is possible that some points are non-periodic points that may be "pruned" from the directed graph.

The directed graph for the trefoil, which we will develop in detail later, is as follows:



The Trefoil and Its Dynamical System

In the algorithm for symbolic dynamical systems we now have four major steps. The first two that we discussed earlier were: 1) to write a simplified Wirtinger presentation of the fundamental group of the exterior (complement) of any given knot, k. Recall to simplify a Wirtinger presentation just solve relators for one or more generators and substitute into other relators minimizing both generators and relations, and 2) to develop a presentation for the commutator subgroup. Recall we first select a generator and designate it as a distinguished generator, equate each of the other Wirtinger generators to a multiple of the distinguished generator and some other generator. By substituting and appling Tietze moves we produce still another equivalent presentation in terms of the distinguished generator and the unknown group generator. Then by representing the group in only one relator and inserting a sequence of generator multiples equivalent to the identity in strategic positions of that relator, we can rewrite a relator as a multiple sequence of conjugate generators. The result then is the

desired presentation of the commutator subgroup.

The third step is the development of what is called the augmented group system and shifts of finite type. In essence, we develop a set of vertices and a set of edges as described above.

The fourth and final step in our algorithm to develop a symbolic dynamical system for a given knot, k is the plotting of the directed graph, Γ. This is done from the set of vertices and edges developed previously. This concludes the algorithm for a dynamical system and its directed graph which is called a symbolic dynamical system. We are now left only to conclude the construction of the symbolic dynamical system for the trefoil. This we will do in the next section.

As a side note recall that the Wirtinger presentation allowed us to disregard one crossing in our development of the presentation. We ignored the upper crossing and used the two lower crossings. If we had used the upper and lower right crossing our above relator would appear as $a_{i+1}\, a_{i+2}^{-1}\, a_i^{-1}$ which is equivalent to $a_{i+1}^{-1}\, a_{i+2}\, a_i$ since both are assumed to be equivalent to 1. Alternately, if we had used the upper and lower left crossings the relator would look like the inverse of the relator we developed. That is, $a_{i+1}^{-1}a_{i+2}a_i=a_i^{-1}a_{i+2}^{-1}a_{i+1}=1$ and both agree with the general presentation since inverses of generators are assumed. Intuitively, these facts seem appropriate once we recognize that

the standard projection of the trefoil is symmetric about any of three lines through the center and any one of the three crossings, preserving orientation. Since we can physically replace any crossing with any other by a reflection and since the commutator subgroup invariantly describes the underlying knot, then it seems totally appropriate that the three relators discussed above are equivalent.

The Trefoil and Its Directed Graph

We will now conclude the construction of the SDS for the trefoil by the development of its directed graph. Recall that our shift of finite type Φ_Σ involves a finite group Σ. For our work here we will consider only the case where $\Sigma=\mathbb{Z}_3$. Recall we have $a_{i+1}^{-1}\, a_{i+2}\, a_i=1$ from our general presentation of the commutator subgroup which leads us to $a_{i+2}=a_{i+1}a_i^{-1}$ and $a_2=a_1a_0^{-1}$, when $i=0$. Given that ρ is a homomorphism and since $a_2=a_1a_0^{-1}\in K$ is mapped by ρ to $\mathbb{Z}_3$ then

$$\rho(a_2)=\rho(a_1a_0^{-1})=\rho(a_1)+\rho(a_0^{-1})\in\mathbb{Z}_3.$$

This equation is then equivalent to $\rho(a_2)=\rho(a_1)-\rho(a_0)$ since $\rho(a_0^{-1})\in\mathbb{Z}_3$ is equivalent to the additive inverse of $\rho(a_0)\in\mathbb{Z}_3$. Moreover, as a consequence of the relation $a_{i+1}^{-1}\, a_{i+2}\, a_i$, the representation ρ is determined by its values on a_0 and a_1 through induction. Thus the representations ρ are in one-to-one correspondence with $\mathbb{Z}_3x\mathbb{Z}_3$. In other words, the vertex set for our directed graph Γ is

$V=\{(\rho\{a_0),\rho(a_1))\}\cong\mathbb{Z}_3x\mathbb{Z}_3$ and the edge set comes from the map $\rho'=\sigma_x\rho$. More specifically, $\rho\rho'$ is an edge if and only if $\rho'=\sigma_x\rho$. Now since $\sigma_x\rho(a_i)=\rho(x^{-1}a_ix)=\rho(a_{i+1})$ then the edge set becomes $E=\{\rho\rho'|\rho'(a_0)=\rho(a_1),\ \rho(a_2)=\rho'(a_1)\}$. Thus, if ρ is the representation given by (1,0), where $\rho(a_0)=1$ and $\rho(a_0)=0$ then the shift $\sigma_x\rho$ is from (1,0) to the vertex $(\sigma_x\rho(a_0),\ \sigma_x\rho(a_1))$ $=(\rho(x^{-1}a_0x),\rho(x^{-1}a_1x))=(\rho(a_1),\rho(a_2))$. Since $\rho(a_2)=\rho(a_1)-\rho(a_0)$ we have $\sigma_x=(1,0)=(0,2)$ as an example. And since $\rho(a_1)-\rho(a_0)=\rho(a_2)$ in our case, then the edge set can be expressed as $E=\{\rho\rho'|\rho'(a_0)=\rho(a_1),\ \rho'(a_1)=\rho(a_1)-\rho(a_0)\}$.

Representation shifts $\rho'=\sigma_x\rho$ are calculated in the following manner.

Since our target group is $\mathbb{Z}_3$ it follows that we will have nine elements in our set of vertices we designate as $\rho_i, i=0,\cdots,8$ which are all completely determined by values on a_0 and a_1. Each ρ_i represents an ordered pair such that $\rho_i=(\rho_i(a_0),\rho_i(a_1))$, where each $\rho_i(a_0)$ and $\rho_i(a_1)$ is an element of $\mathbb{Z}_3$. From this we can list the set of nine vertices that are determined by our system as $V=\{(0,0),(1,0),(2,0),(0,1),(1,1),(2,1),(0,2),(1,2),(2,2)\}$, where $\rho_0=(0,0),\rho_1=(1,0)$, etc. In other words the ρ_i is the i plus first representation in the set of vertices. Finding

the oriented edges of our edge set for the directed graph we must select a vertex ρ_i and find another in our set of vertices ρ_i', such that $\rho_i'=\sigma_x\rho_i$. Recall that there is an edge $\rho_i\rho_i'$, if and only if $\rho_i'=\sigma_x\rho_i$, giving $\rho_i'(a_k)=\sigma_x\rho_i(a_k)=\rho_i(x^{-1}a_kx)=\rho_i(a_{k+1})$. Let $\rho_0=(0,0)$ be our starting point where $\rho_0(a_0)=\rho_0(a_1)=0\ \in\mathbb{Z}_3$. Note that $\sigma_x\rho_0=\rho_0$. This shows that there is a loop edge connecting ρ_0 to itself and so ρ_0 is a vertex of period one.

Our next step is to consider a new vertex say $\rho_1=(1,0)$, where $\rho_1(a_0)=1$, $\rho_1(a_1)=0$ and $0,1\ \in\mathbb{Z}_3$. Our calculations are as follows:

$$\sigma_x\rho_1=(\rho_1(a_1),\ \rho_1(a_1)-\rho_1(a_0))=(0,2)=\rho_6.$$

$$\sigma_x\rho_6=(\rho_6(a_1),\ \rho_6(a_1)-\rho_6(a_0))=(2,2)=\rho_8.$$

$$\sigma_x\rho_8=(\rho_8(a_1),\ \rho_8(a_1)-\rho_8(a_0))=(2,0)=\rho_2.$$

$$\sigma_x\rho_2=(\rho_2(a_1),\ \rho_2(a_1)-\rho_2(a_0))=(0,1)=\rho_3.$$

$$\sigma_x\rho_3=(\rho_3(a_1),\ \rho_3(a_1)-\rho_3(a_0))=(1,1)=\rho_4.$$

$$\sigma_x\rho_4=(\rho_4(a_1),\ \rho_4(a_1)-\rho_4(a_0))=(1,0)=\rho_1.$$

Notice that a cycle of period 6 has been completed since our calculations have brought us back to our beginning vertex namely, ρ_1. Note that there are only two vertices whose edges have not yet been determined, they are $\rho_5=(2,1)$ and $\rho_7=(1,2)$. This time we select ρ_5 letting $\rho_5(a_0)=2$ and

$\rho_5(a_1)=1$. From this we have

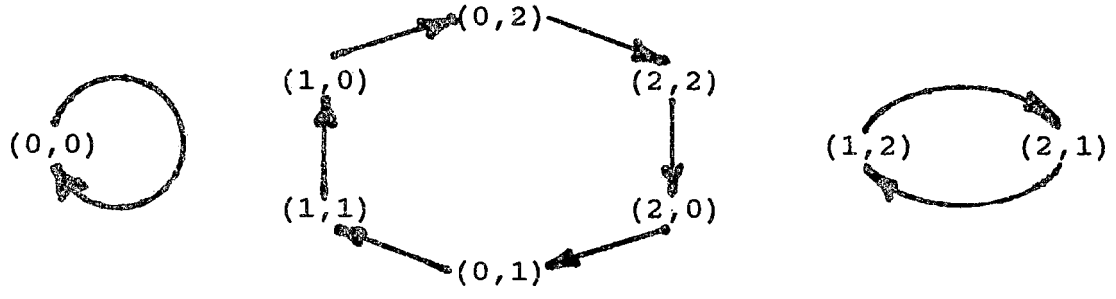
$$\sigma_x\rho_5=(\rho_5(a_1),\ \rho_5(a_1)-\rho_5(a_0))=(1,2)=\rho_7.$$

$$\sigma_x\rho_7=(\rho_7(a_1),\ \rho_7(a_1)-\rho_7(a_0))=(2,1)=\rho_5.$$

This completes a third and the last cycle since the edges have been determined on all of the vertices.

Our directed graph for the trefoil resulted in three parts, each regarded as a unique bi-infinite path with different periods. The first is period 1, the second is period 6, and the third is period 2. We can determine the affect of σ_x directly from a directed graph.

The following is the complete directed graph Γ for the trefoil.[31]



From the directed graph we can see that the representation shift $\Phi_{Z/3}$ for the trefoil contains 1 fixed point, 2 points of period 2 and 6 points of period 6.

Recall that we denote the augmented group system by $\mathcal{G}=(G,\chi,x)$. Representation shifts operating on $\mathcal{G}$ with a target group Σ are denoted by $\Phi_\Sigma(\mathcal{G})$. A dynamical system is

a pair $(\Phi_\Sigma(\mathbf{g}),\sigma_x)$ defined to be a topological space plus a homeomorphism σ_x that maps a representation shift to itself, $\sigma_x:\Phi_\Sigma \mapsto \Phi_\Sigma$, where Φ_Σ is a set of ordered pair each in Σ called vertices. Finally, a special kind of dynamical system is one that can be completely described by a finite graph Γ. The bi-infinite paths in Γ corresponds to the representations ρ. This special kind of dynamical system is called a symbolic dynamical system and is generally denoted in the abreviated form by $\Phi_\Sigma(\mathbf{g}_k)$, where k represents the exterior of a given knot.

CONCLUSION

Here it seems appropriate to summarize the foregoing and close with some information on recent research that has gone beyond this application of symbolic dynamical systems. First, the algorithm for developing symbolic dynamical systems will be summarized in a nontechnical narrative in my own words. The steps in a practical application are then reviewed as we display the development of a symbolic dynamical system in the section entitled Another Knot? - Why Not? Lastly, we talk briefly about a different kind of knot invariant called entropy that takes the structure of a symbolic dynamical system into higher dimensional knot theory.

The Algorithm in Summary

A known knot is selected then imbedded in $\mathbb{R}^3$. This is followed by selecting a fixed neighborhood of the knot that envelops the entire knot. We now extract a solid tube containing the knot with a neighborhood and consider the complement of the knot. Such an extraction leaves a 'tunnel shaped' void of the knot in $\mathbb{R}^3$. Thus, the fundamental group is defined in the exterior of the knot and can be presented in terms of meridian loops.

Here we turn to the Wirtinger process for a presentation from a projection of the knot. The practice here is to orient the knot projection, select meridian loops viewed as

straight lines on the projection orient them from right to left while facing with the orientation of the knot. These meridian loops are to be labeled in a manner corresponding to the arcs of the knot projection determined from underpass to underpass. A presentation of the fundamental group of the knot is written using meridians as generators while the relations of the presentation are taken from the relationships among the meridians at all but one of the crossings in the projection.

Next with some algebra we solve for and substitute generators among relators. Then by using Tietze moves we combine relators, equate them to the identity and eleminate as many generators and relators as is possible. In so doing we have simplified the presentation to the minimum.

The Reidemester-Schreier Theorem enables us to write a presentation for the commutator subgroup K of our fundamental group. The generators of K are denoted by a_i where $i=1,2,\cdots$. Each a_i is equal to the conjugation $x^{-i}ax^i$ from the prior presentation. We write the presentation of the commutator subgroup by rewriting the conjugations of previous relations in terms of the new generators.

Now we have a presentation of the commutator subgroup, and let Φ_Σ denote the set of all representators of K into Σ. Next we define a mapping based on the distinguished generator x and denoted by σ_x. Recall that the pair (Φ_Σ, σ_x) is called a dynamic system, where Σ is a selected target

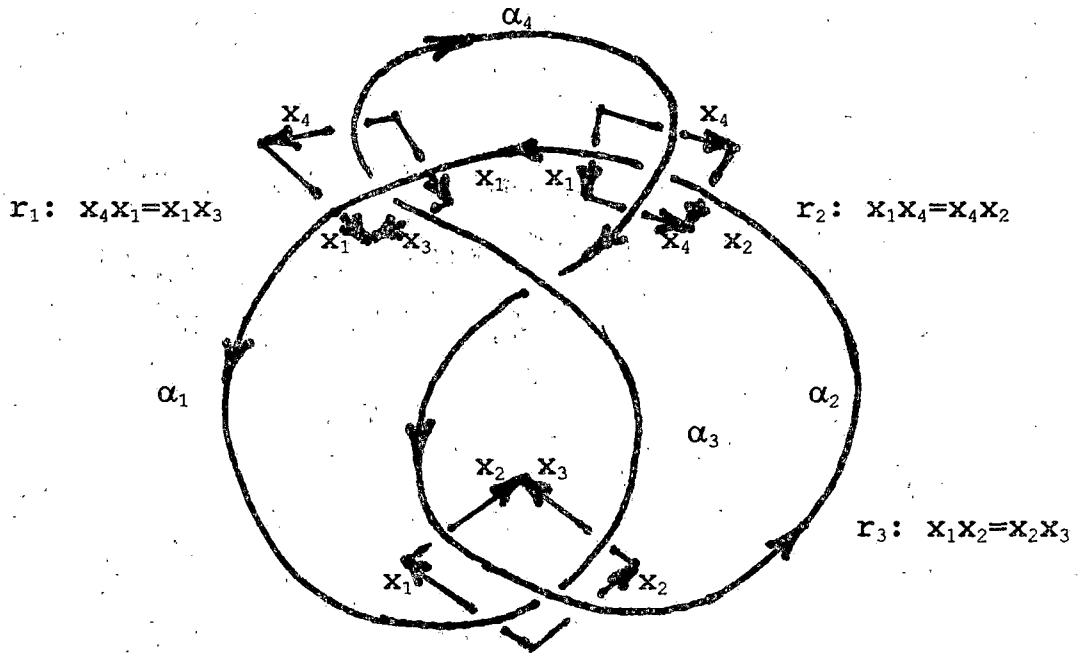
group. For our Trefoil, example the target group selected was the cyclic group $\mathbb{Z}_3$. This then allowed us to define a representation shift ρ such that $\sigma_x\rho(a)=\rho(x^{-1}ax)$. With the representation shift defined all that is left to do is to consider each vertex ρ and calculate the map $\sigma_x\rho$ to some terminal point ρ'. This we call an edge. Once we have collected complete sets of all possible vertices and edges it is then a simple matter to graphically represent our dynamical system with what we call a directed graph and denote by Γ. The directed graph is drawn by plotting all possible vertices and connecting them in accordance with the direction determined by the calculations of $\sigma_x\rho$ from our set of edges. The directed graph will depict one or more bi-infinite paths consisting of several loops each with a discernible period. The directed graph of a dynamical system is a symbolic depiction of that system we call a symbolic dynamical system. The foregoing is a summary of the approach to an active development of a symbolic dynamical system for any known knot.

Another Knot? - Why Not?

Lets take a summary look at the development of the symbolic dynamical system for another non-trivial knot, called the figure-eight knot. the figure-eight knot is classified as a 4_1 knot, the target group Σ we will use is

$\mathbb{Z}_3$ and we can denote the symbolic dynamical system by $\Phi_{Z/3}(g)$.

From the standard projection of the figure-eight knot we develop a presentation for the fundamental group of the knot exterior using the Wirtinger process as follows:



A Wirtinger presentation of the figure-eight knot is

$$\pi_1(4_1)=\langle x_1,x_2,x_3,x_4 | x_4x_1=x_1x_3,\ x_1x_4=x_4x_2,\ x_1x_2=x_2x_3\rangle$$

From r_1 and r_2 we have $x_3=x_1^{-1}x_4x_1$, $x_2=x_4^{-1}x_1x_4$, $x_3^{-1}=x_1^{-1}x_4^{-1}x_1$ and $x_2^{-1}=x_4^{-1}x_1^{-1}x_4$. Rewriting r_3 we have $x_3^{-1}x_2^{-1}x_1x_2=e$, then by substituting we can write our only relator as follows:

$$x_1^{-1}x_4^{-1}x_1x_4^{-1}x_1^{-1}x_4\ x_1\ x_4^{-1}x_1x_4=e$$

Notice we now have only two generators and one relator, such that $\pi_1(4_1)=\langle x_1,x_4 | x_1^{-1}x_4^{-1}x_1\ x_4^{-1}x_1^{-1}x_4x_1x_4^{-1}x_1x_4\rangle$. Next select x_4 as our distinguished generator such that $x_4=ax$, where a is

another group generator and $x=x_1$.

Note that $x_4^{-1}=x^{-1}a^{-1}$. Again by substitution we have,

$$\pi_1(4_1)=\langle x,a|x^{-1}x^{-1}a^{-1}xx^{-1}a^{-1}x^{-1}axxx^{-1}a^{-1}xax\rangle \text{ or}$$

$$\pi_1(4_1)=\langle x,a|x^{-2}a^{-1}x^2x^{-2}a^{-1}x^2x^{-3}ax^3x^{-2}a^{-1}x^2x^{-1}ax\rangle$$

by strategic insertions of $x^ix^{-i}, i\in\mathbb{Z}$ such that our relator is presented as the product of conjugates. By the Reidemester-Schreier Theorem we can now write a presentation for the commutator subgroup, such that the generators are defined by $a_i=^{-i}ax^i$, $i\in\mathbb{Z}$. Thus, $a_0=a$, $a_1=x^{-1}ax$, $a_2=x^{-2}ax^2$, $a_3=x^{-3}ax^3$ and $a_0^{-1}=a^{-1}$, $a_1^{-1}=x^{-1}a^{-1}x$, $a_2^{-1}=x^{-2}a^{-1}x^2$, $a_3^{-1}=x^{-3}a^{-1}x^3$. Rewriting our relator with the above substitutions we now have $a_2^{-1}\cdot a_2^{-1}\cdot a_3\cdot a_2^{-1}\cdot a_1$, which will be the relator for the presentation of the commutator subgroup, $K=\langle a_1,a_2^{-1},a_3|a_2^{-1}a_2^{-1}a_3a_2^{-1}a_1\rangle$. In general, we have $K=\langle a_i|a_{i+1}^{-1}a_{i+1}^{-1}a_{i+2}a_{i+1}^{-1}a_i\rangle$. By setting our relator equal to the identity and solving for a_{i+2} we have $a_{i+1}^{-2}a_{i+2}a_{i+1}^{-1}a_i=1$ gives $a_{i+2}=a_{i+1}^2a_i^{-1}a_{i+1}$.

From here we can produce the directed graph. We choose $\Sigma = \mathbb{Z}_3$ and define our vertex and edge sets to be $V=\{\rho|\rho:\mapsto\mathbb{Z}_3\}$ and $E=\{\rho\rho'|\rho'(a_0)=\rho(a_1),\ \rho'(a_1)=3\rho(a_1)-\rho(a_0)\}$.

Note that since ρ is a homomorphism then

$$\rho(a_{i+2}) = \rho(a_{i+1}^{2}a_i^{-1}a_{i+1})$$

$$\rho(a_{i+2}) = \rho(a_{i+1}^{2})+\rho(a_i^{-1})+\rho(a_{i+1})$$

$$\rho(a_{i+2}) = 2\rho(a_{i+1})-\rho(a_i)+\rho(a_{i+1})$$

$$\rho(a_{i+2}) = 3\rho(a_{i+1})-\rho(a_i)$$

$$\rho(a_{i+2}) = -\rho(a_i)\in\mathbb{Z}_3$$

Since $3\rho(a_{i+1})-\rho(a_i)$ is in $\mathbb{Z}_3$ then we have that $3\rho(a_{i+2})$ will always equal zero. Thus, $\rho(a_{i+2})=-\rho(a_i)$.

Given that $\rho\in Hom(K,\mathbb{Z}_3)$ it then follows that the set of vertices has nine elements, ρ_i, $i=1,\cdots,9$ all of which are completely determined by values on a_0 and a_1. That is, each ρ_i represents an ordered pair such that $\rho_i=(\rho_i(a_0),\rho_i(a_1))$, where $\rho_i(a_0),\rho_i(a_1)\in\mathbb{Z}_3$. Again, we can now list our set of nine vertices as follows:

$V_0=\{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0),(2,1),(2,2)\}$

where $\rho_1=(0,0),\rho_2=(0,1)$ and so on.

In order to find the oriented edges of our directed graph, Γ we must select a ρ_i and find a ρ_i'. But first recall that there is an edge $\rho_i\rho_i'$, where $a_k=x^{-k}a_0x^k$ if and only if $\rho_i'=\sigma_x\rho_i$. Thus, we have $\rho_i'(a_k)=\sigma_x\rho_i(a_k)=$ $\rho_i(x^{-1}a_kx)=\rho_k(a_{k+1})$. Given that $\rho_1=(0,0)$ then let ρ_1 be our

starting point where $\rho_1(a_0)=\rho_1(a_1)=0\in\mathbb{Z}_3$. Thus, $\sigma_x\rho_1=(\rho_1(a_1),-\rho_1(a_0))=(0,0)$. We have determined that ρ_1 is a vertex of period one since it maps to itself, $\sigma_x:\rho_1\rightarrow\rho_1$. Lets now consider $\rho_2=(0,1)$, where $\rho_2(a_0)=0$ and $\rho_2(a_1)=1\in\mathbb{Z}_3$. It follows that

$$\sigma_x\rho_2=(\rho_2(a_1),\ -\rho_2(a_0))=(1,0)=\rho_4,$$

$$\sigma_x\rho_4=(\rho_4(a_1),\ -\rho_4(a_0))=(0,2)=\rho_3,$$

$$\sigma_x\rho_3=(\rho_3(a_1),\ -\rho_3(a_0))=(2,0)=\rho_7,$$

$$\sigma_x\rho_7=(\rho_7(a_1),\ -\rho_7(a_0))=(0,1)=\rho_2,$$

Notice that we have completed a cycle of period four on ρ_2. We still have four vertices left whose edges are yet to be determined.

Lets consider the first vertex with an undetermined edge, $\rho_5=(1,1)$, where $\rho_5(a_0)=\rho_5(a_1)=1\in\mathbb{Z}_3$. It follows that

$$\sigma_x\rho_5=(\rho_5(a_1),\ -\rho_5(a_0))=(1,2)=\rho_6,$$

$$\sigma_x\rho_6=(\rho_6(a_1),\ -\rho_6(a_0))=(2,2)=\rho_9,$$

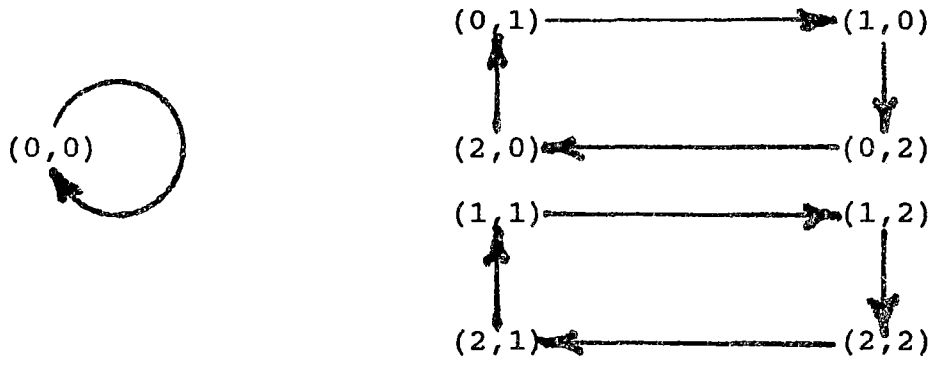
$$\sigma_x\rho_9=(\rho_9(a_1),\ -\rho_9(a_0))=(2,1)=\rho_8,$$

$$\sigma_x\rho_8=(\rho_8(a_1),\ -\rho_8(a_0))=(1,1)=\rho_5,$$

Again we have clearly completed another cycle but this time we have completed the determination of all of the oriented edges for the nine vertices. This information can now be graphed as a symbolic representation of the fundamental

group of the figure-eight knot.

Thus, the directed graph for the figure-eight knot is,



Directed graph Γ - Figure-eight knot (4_1)

Recall that the symbolic dynamical system, $\Phi_{Z/3}(g)$ for the trefoil had one shift of period 6, another of period 2 and one fixed point. Compare with $\Phi_{Z/3}(g)$ for the figure-eight having two shifts of period 4 and one fixed point. Since symbolic dynamical systems are knot invariants then the difference in the two systems of these knots implies they are distinct.

The next section is provided to show that the value of symbolic dynamical systems has not stopped here but has also revealed knot invariants in higher dimensional knots.

Entropy: A Knot Invariant

The use of symbolic dynamical systems has been carried generally to higher dimensional knots. As it happens, entropy is a knot invariant for higher dimensional knots, denoted by n-knots, as determined from dynamical systems.

By generalizing the definition of an embedded 1-dimen-

sional knot we can say that an n-knot is a smoothly embedded n-sphere, also denoted by k, in $\mathbb{R}^{n+2}$ space. Similarly we can define the exterior of the knot. That is, let N(k) be a neighborhood of k that is diffeomorphic to $\mathbb{R}^n \times D^2$, then the closure of X(k) of $\mathbb{R}^{n+1}$-N(k) is called the exterior of the knot.[34]

The fundamental group and augmented group systems are likewise supported by higher dimensional knot definitions. Thus representative shifts can be calculated and directed graphs produced. It has been determined that the shifts described by a directed graph has an entropy equal to log λ, where λ is the Perron eigenvalue of the adjacent matrix of the directed graph. Not only do conjugate shifts have the same entropy but so do finitely equivalent shifts.[33]

There is a corollary that gives the promise of the discovery of further invariant group systems that apply to knot theory. That corollary is:

Corollary: *Let (G,χ,x) be an augmented group system and let r be a positive integer. The entropy, denoted by $h(\Phi_r)$ of the associated representation shift (Φ_r, σ_χ) is an invariant of the group system (G,χ), i.e. the entropy depends only on the isomorphism class of the group system.*

This corollary is a very powerful tool in that it can be used to define a sequence of entropy invariants for higher

dimensional knots.

This very brief and informal discussion on entropy was included to demonstrate the extent of influence symbolic dynamical systems are currently having in Knot Theory. It was also intended to point out that this fledgling field of study holds a tremendous potential for discovery and further expansion.

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