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MINIMAL CONGESTION TREES

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

by

Shelly Jean Dawson

September 2006

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Approved by:



Joseph Chavez, Committee Chair

AUGUST 22, 2006
Date



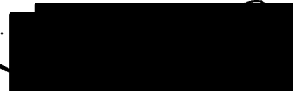
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ABSTRACT

In 2004, there was an article published by M. I. Ostrovskii titled, Minimal Congestion Trees. The intent of his paper was to consider the edge congestion problems of finite simple graphs. As a result, he developed a theorem of inequalities which estimate the minimal edge congestion for finite simple graphs. In this paper, we analyze Ostrovskii's results and then use these generic results to examine and further reduce the parameters of the inequalities for specific families of graphs, particularly complete and complete bipartite graphs. We also explore a possible minimal congestion tree for some grids while forming a conjecture for all grids.

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Chapter 1

Introduction

1.1 Background

The city of Königsberg, now called Kaliningrad, lies on both banks of the river Pregel, as well as on two islands. These two islands are connected to each other and to the river banks by seven bridges. Some people speculated that there might be some path through the city, which would cross all seven bridges only once. People tried this, but never succeeded. Leonhard Euler showed that it was impossible, and thus began Graph Theory.

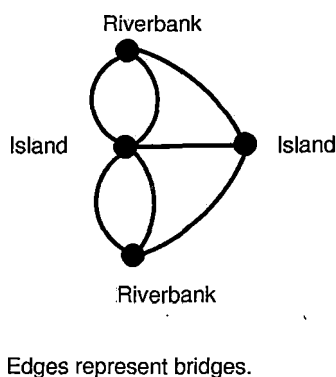


Figure 1.1: A graph representing the bridges of Königsberg.

Graph theory is one of the most widely applicable areas of mathematics. Its concepts and terminology can be used in many areas to help clarify ideas. Some examples of graphs are sets with physical links, such as electrical networks, where electrical components

are represented by vertices and the connecting wires are represented by edges. Another form of graphs are sets with logical sequencing that are used in computer flow charts where the vertices are the instructions and the edges are the flow from one instruction to the next instruction in the sequence. Graphs are also used for corporate organizational flowcharts, computer data structures, and evolutionary trees.

One aspect of graph theory is devoted to the minimization of the edge congestion over all trees of a graph. This paper will focus on M. I. Ostrovskii's [Ost04] results with finite simple graphs and minimal congestion trees. Furthermore, the paper will concentrate on the generic parameters proven by Ostrovskii for finite simple graphs and the reduction of these parameters for specific families of graphs including complete graphs, complete bipartite graphs, and grids.

1.2 Graph Theory Definitions

In this paper we will consider several types of graphs. Some graph-theoretic terminology that will be used throughout this paper follows.

Definition 1.1. *A graph $G = (V, E)$ consists of a finite set of vertices, V , and a set of edges, E , joining different pairs of distinct vertices.*

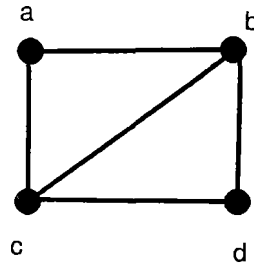


Figure 1.2: A graph G with $V = \{a, b, c, d\}$ and $E = \{(a, b), (a, c), (b, c), (b, d), (c, d)\}$.

A vertex is a point or node in a graph and an edge of a graph is a line making a connection between two vertices. A finite simple graph has no multiple edges or loops.

Definition 1.2. *A graph A is a subgraph of graph B if $E_A \subseteq E_B$ and $V_A \subseteq V_B$.*

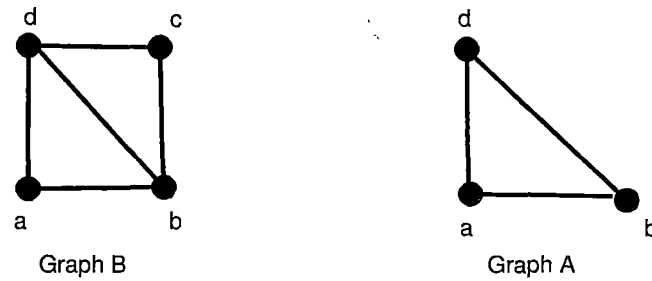


Figure 1.3: Graph A is a subgraph of Graph B.

Definition 1.3. A *connected graph* is a graph in which for any given vertex in the graph, all the other vertices are reachable from it. See Figures 1.4 and 1.5.

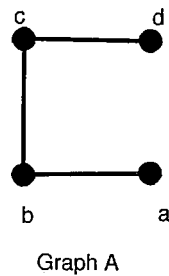


Figure 1.4: Graph A is a connected graph with four vertices, three edges, and no cycles or loops.

Definition 1.4. A *complete graph* on n vertices, denoted K_n , is a graph in which every pair of distinct vertices is adjacent. Adjacent vertices are two distinct vertices connected by an edge. Refer to Figure 1.6.

Definition 1.5. A *tree* is a connected graph in which there is only one path connecting each pair of vertices.

Definition 1.6. A *tree* of graph $G = (V, E)$ is a tree with vertex set V . A tree may contain edges that are not present in the graph G .

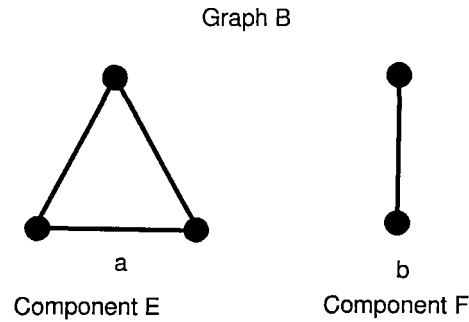


Figure 1.5: Graph B has five vertices and four edges and is not connected; it has 2 components, E and F which are two connected subgraphs; component E has one cycle.

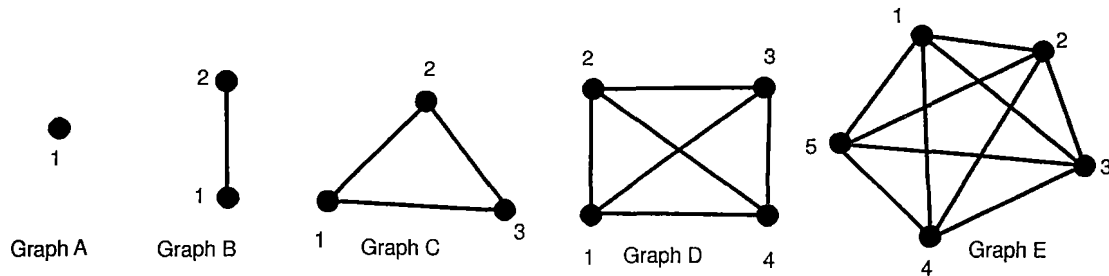


Figure 1.6: Graphs A through E are K_1 , K_2 , K_3 , K_4 , and K_5 , respectively.

Referring to Figure 1.7, notice that the edge $(1, 3)$ in the tree of Graph G is not present in Graph G. Also the edges $(3, 4)$ and $(2, 3)$ that are present in Graph G are not present in the tree of Graph G.

Definition 1.7. A *spanning tree* of a connected graph $G = (V, E)$ is a tree such that $T = (V, F)$ with $F \subseteq E$.

Refer to Figure 1.8. Notice that the spanning tree of graph G contains only edges that are present in Graph G.

Definition 1.8. For any $v \in V_G$ the *degree* of a vertex denoted d_v is the number of edges incident to a given vertex.

Referring to Figure 1.9, in graph A, vertices 1 and 3 have degree three, while vertices 2 and 4 have degree two. In graph B, vertex 1 has degree three, vertex 2 has degree four, and

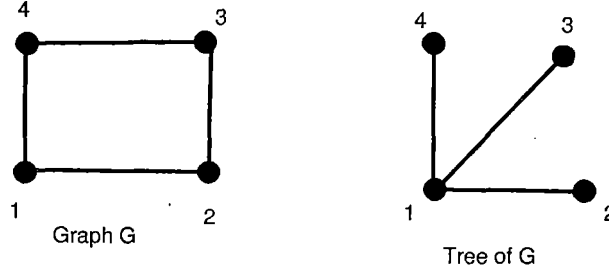


Figure 1.7: Tree of Graph G

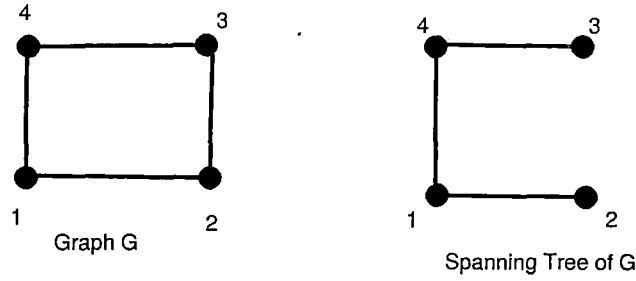


Figure 1.8: Spanning Tree of Graph G

vertices 3, 4, 5, 6, and 7 have degree 1.

1.3 Ostrovskii's Results

In this paper we will study the minimal congestion trees of finite simple graphs as presented by M.I. Ostrovskii in Minimal Congestion Trees [Ost04]. In his paper, Ostrovskii is devoted to the minimization of $ec(G : T)$. Specifically, he considers two types of edge congestion problems: minimization of the *edge congestion of G in T* over all trees with the same vertex set as G and minimization of the *edge congestion of G in T* over all spanning trees of G. His article introduces and studies the following notions:

- *tree congestion of G* is defined by $t(G) = \min \{ec(G : T) : T \text{ is a tree with } V_T = V_G\}$ where V_T is the vertex set of T and V_G is the vertex set of G ,
- *minimal congestion tree for G* is defined as any tree with $V_T = V_G$ satisfying

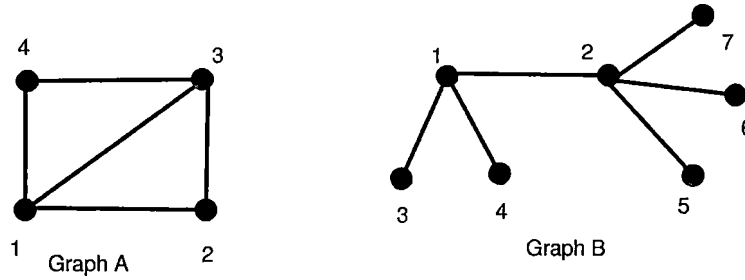


Figure 1.9: Degree of Vertex

$$ec(G : T) = t(G),$$

- *spanning tree congestion* of G , $s(G) = \min \{ec(G : T) : T \text{ is a spanning tree of } G\}$, and
- *minimal congestion spanning tree* for G is defined as a subgraph T of G such that $V_T = V_G$ and T is a tree satisfying $ec(G : T) = s(G)$.

Ostrovskii uses these notions along with the well known definition of the *cutwidth*, $cw(G) = \min \{ec(G : P) : P \text{ is a path with } V_P = V_G\}$, to develop inequalities for the congestion problems. These inequalities form a theorem which summarize estimates for $t(G)$ and $s(G)$.

Theorem 1.9. $m_G = t(G) \leq s(G) \leq |E_G| - |V_G| + 2$.

Refer to m_G in Chapter 2 for a complete definition.

1.4 My Results

In an attempt to further reduce the parameters presented by Ostrovskii, we will be

- exploring and presenting minimal congestion trees for complete graphs.
- exploring and presenting minimal congestion trees for complete bipartite graphs.
- exploring a possible minimal congestion tree for grids, if it exists.

Chapter 2

Ostrovskii's Results

Recall that Ostrovskii considers finite simple graphs and that for a graph G , by V and E , we denote the graph's vertex set and edge set, respectively. Ostrovskii used the next several definitions to lay the foundation for the definition of edge congestion which is a major component of understanding minimal congestion trees.

Definition 2.1. *A path is a sequence of consecutive edges in a graph and the length of the path is the number of edges traversed.*

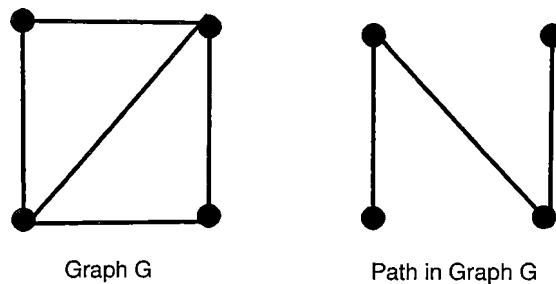


Figure 2.1: Graph G and a path in graph G of length three.

Definition 2.2. *A path joining the end vertices of g , denoted P_g , where $g \in E_G$, is called a detour for g , even in the case when $P_g = g$. See Figure 2.2.*

Definition 2.3. *Let G and H be two connected graphs with the same vertex set. An H -layout of G is a collection $P_g : g \in E_G$ of paths in H , where P_g is a path joining the*

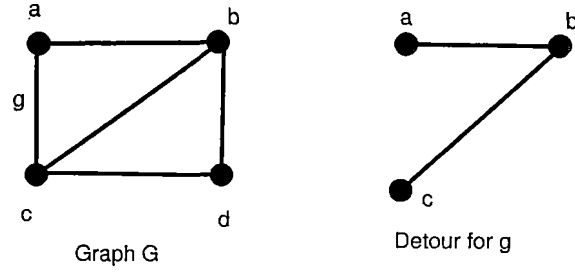


Figure 2.2: Graph G and a detour for g , P_g .

end vertices of g . Such P_g will be called *detours for g* , even in the case when $P_g = g$. See Figure 2.3.

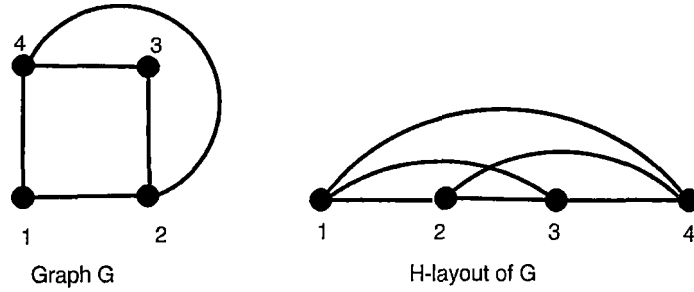


Figure 2.3: H-layout of G .

Refer to Figure 2.3. Edge $(1, 4)$ refers to the path from 1 to 4 or 1-2-3-4 in H .

Definition 2.4. m_G is the maximal number of edge disjoint paths joining u and v in Graph G among all pairs (u, v) of vertices of G .

Definition 2.5. For an edge h of H we define the congestion of L in H as the number of times h appears in L which is denoted by

$$c(h, L) = |\{P_g \in L : h \in P_g\}|.$$

Note: L is the collection of paths in an H -layout of G .

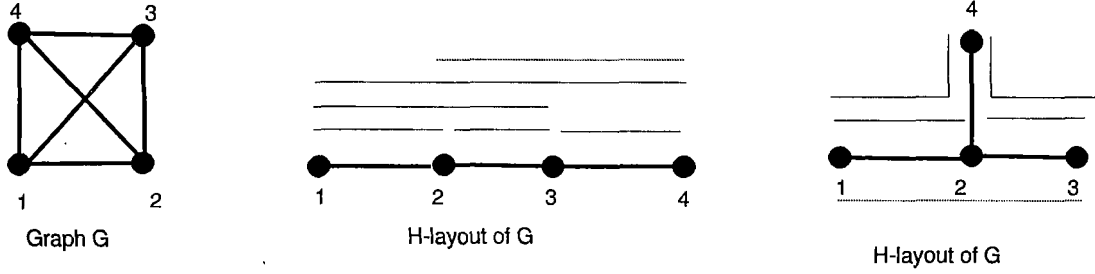


Figure 2.4: Edge Congestion of G in H where Graph G is K_4 .

Definition 2.6. *The edge congestion of G in H is defined by*

$$ec(G : H) = \max_{h \in E_H} c(h, L), \text{ where the minimum is over all } H\text{-layouts of } G.$$

Definition 2.7. *The edge congestion of G , denoted $ec(G)$, is $ec(G) = \min_H \{ec(G : H)\}$.*

For example, we can examine the complete graph, K_4 . There are two possible trees that are H – layouts of G up to isomorphism shown in Figure 2.4. To determine the edge congestion, we must look at all the paths taken to travel from one vertex to another in each of the H – layouts of G . Each of these paths is marked by a thinner line as seen in Figure 2.4. So, for each layout there is a path between vertices 1 and 2, 1 and 3, 1 and 4, 2 and 3, 2 and 4, and 3 and 4. To find the heaviest congestion in each of the layouts, one must look between any two vertices and count the number of thin lines drawn between the two vertices. Then the two vertices with the most thin lines drawn between them has the heaviest congestion. For example, in Figure 2.4 the first H – layout of G has the heaviest congestion between vertices 2 and 3. There are four thin lines drawn between vertices 2 and 3 making the congestion 4. In the second H – layout of G the congestion between vertices 1 and 2 and between vertices 2 and 3 is the heaviest and is also the same. There are three thin lines drawn between both pairs of vertices creating a congestion of 3. Since the congestion for the second H – layout of G is the minimum congestion of all H – layouts, the edge congestion of G is 3.

Using the definition of edge congestion, we can now examine the tree congestion of G , one of the main goals of Ostrovskii’s paper.

Definition 2.8. *The tree congestion of G is denoted by*

$$t(G) = \min \{ec(G : T) : T \text{ is a tree with } V_T = V_G\}$$

Any tree with $V_T = V_G$ satisfying $ec(G : T) = t(G)$ will be called a *minimal congestion tree* for G .

Let us again examine the complete graph, K_4 , in Figure 2.5. For graph G , there are only two possible non-isomorphic trees to diagram, tree A and tree B , denoted T_A and T_B , respectively.

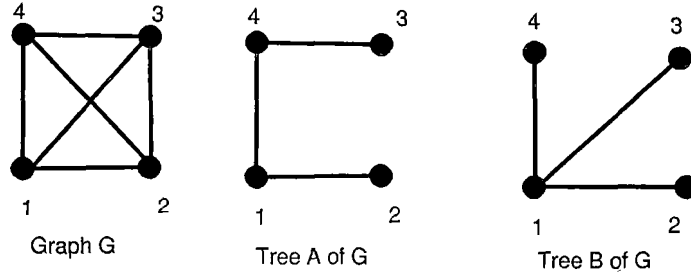


Figure 2.5: The graph of G , which is K_4 , and T_A and T_B .

The edge congestion for both trees of graph G is shown in Figure 2.6. Since the edge congestion for T_A of G is four and the edge congestion for T_B of G is three, then $t(G) = 3$ since it is the minimum edge congestion of all the trees of G . Since T_B has the minimum edge congestion for all trees of G , then it is the *minimal congestion tree* for G .

Similarly, we can define the *spanning tree congestion* of G .

Definition 2.9. The *spanning tree congestion* of G is defined by

$$s(G) = \min \{ec(G : T) : T \text{ is a spanning tree of } G\}.$$

Referring again to Figure 2.5, both T_A and T_B are spanning trees of G since they contain only edges present in graph G . Similar to trees of G , any spanning tree of G satisfying $ec(G : T) = s(G)$ will be called a *minimal congestion spanning tree* for G . So, $s(G) = 3$ and since spanning trees are a subset of all trees, then in this case $s(G) = t(G)$.

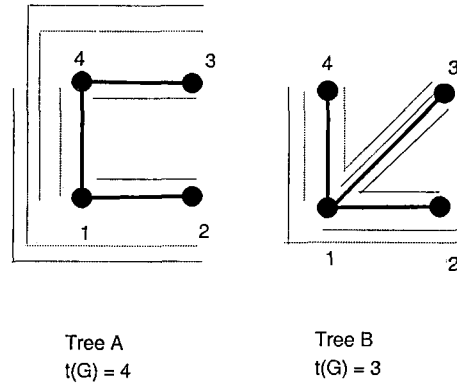


Figure 2.6: Two possible trees of K_4 , T_A and T_B .

From the definition of tree congestion we get the definition of *cutwidth* by replacing the word tree with the word path. Ostrovskii uses both cutwidth and edge congestion in his paper, but our focus will be strictly on edge congestion.

To continue, we will need a few more definitions.

Definition 2.10. *The distance between two vertices u and v in a graph G is denoted by $d(u, v)$ and is defined as the length of the shortest path between them.*

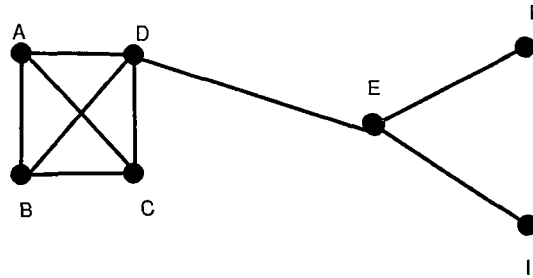


Figure 2.7: Graph G with vertices A, B, C, D, E, F , and I

In Figure 2.7, the following five paths exist between vertices A and E . Path 1 travels through vertices A, B, C, D, E which has length four; path 2 travels through vertices A, B, D, E which has length three; path 3 travels through vertices A, C, D, E which has length three; path 4 travels through vertices A, D, E which has length two; and path 5 travels through vertices A, C, B, D, E which has length four. Since path 4 has the

shortest length or path, the distance between vertices A and E is two.

Now, we can define the diameter of a graph G .

Definition 2.11. *The diameter of a graph G , such that $\text{diam}(G) = \max_{u,v} d(u,v)$, is the largest distance between any two vertices (u,v) of a graph G .*

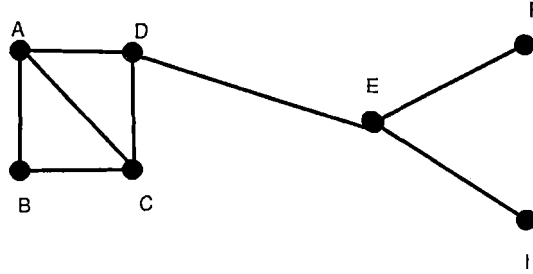


Figure 2.8: Graph G with vertices A, B, C, D, E, F , and I

All adjacent pairs of vertices will have a distance of one. For example, in Figure 2.8, the following distances represent pairs of adjacent vertices; $d(A, B) = d(A, C) = d(A, D) = d(B, C) = d(C, D) = d(D, E) = d(E, F) = d(E, I) = 1$. Distances of two, three, and four have been determined for the following pairs of vertices: $d(A, E) = d(B, D) = d(C, E) = d(D, F) = d(D, I) = d(F, I) = 2$; $d(A, F) = d(A, I) = d(B, E) = d(C, F) = d(C, I) = 3$; and $d(B, F) = d(B, I) = 4$. So, the $\text{diam}(G) = \max_{u,v} d(u,v) = 4$.

Definition 2.12. *Let u be a vertex of a tree T . If we delete all edges incident to u from T , we get a forest. Refer to Figure 2.9.*

Definition 2.13. *The weight of T at u , denoted $w(u)$, is the maximal number of vertices in a component of the forest $(T - u)$.*

In Figure 2.10 the number of vertices in components v and x is 3 and 4, respectively. Thus, $w(u) = 4$.

Definition 2.14. *A vertex v of T is called a centroid vertex if the weight of T at v is minimal, that is $w(v) \leq w(u) \forall u \in V$. Refer to Figure 2.11.*

Note that the weight of a centroid vertex is called the weight of T and is denoted by $w(T)$. Also, a graph has, at most, two centroid vertices. This occurs when T has an

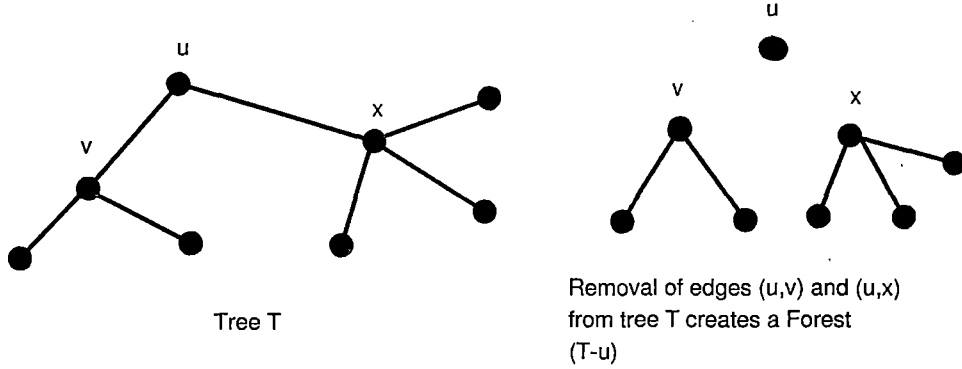


Figure 2.9: Tree T and Forest (T-u)

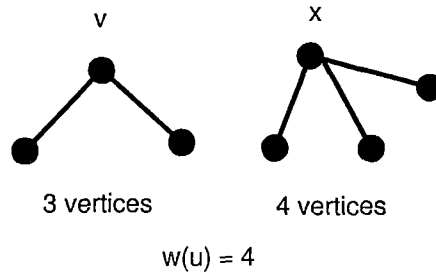


Figure 2.10: Components of Forest (T-u)

edge whose removal will split T into two components with the same number of vertices. Observe in Figure 2.12, $w(v) = 4 = w(w)$. In a case where there are two centroid vertices, $|V_T| = 2w(T)$. Trees with odd $|V_T|$ and trees with $w(T) < \frac{|V_T|}{2}$ have exactly one centroid vertex which is called the *centroid of T*. For example, removal of the edge (v, w) in Figure 2.12 splits T into two components containing the same number of vertices. Both components have four vertices. This indicates that T has two centroid vertices, v and w .

Now, for an edge $e \in E_T$, we denote the *vertex sets of the components of T* obtained after the removal of edge e as A_e and B_e . Consequently,

$$w(T) = \max \{ \min \{ |A_e|, |B_e| \} : e \in E_T \}.$$

Note that this is a second approach using the method shown in Figure 2.12 as it was used

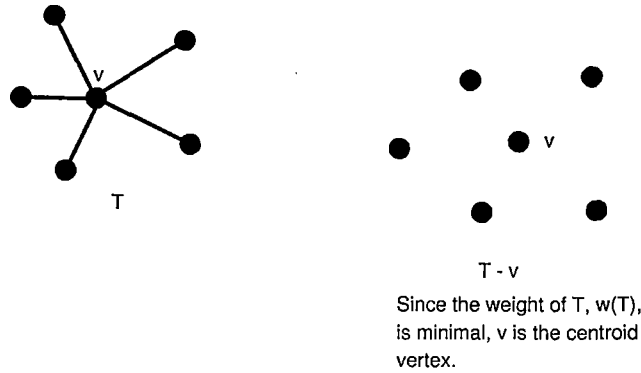


Figure 2.11: Tree T with diagram of $T - v$, where v is the centroid vertex.

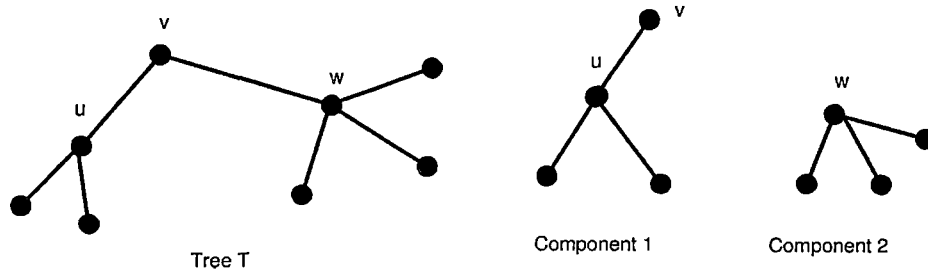
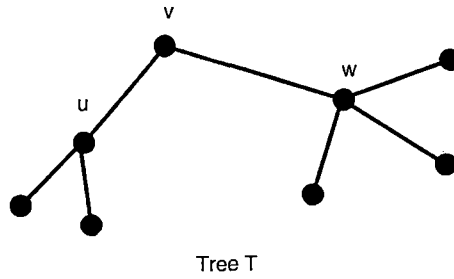
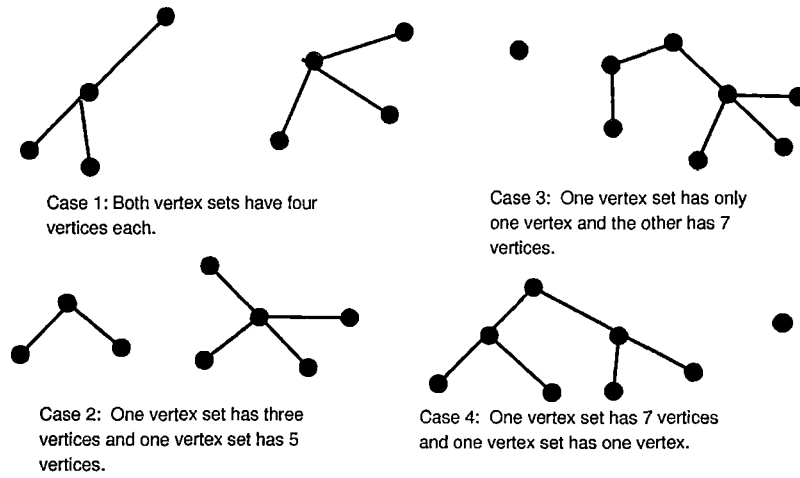


Figure 2.12: Tree T with diagram of two components obtained after removal of (v, w) .

by Ostrovskii to determine the lower bound of the inequality in his theorem. This is the weight of T when all possible combinations have been exhausted. For example, for the tree in Figure 2.13, there are four possible cases.

Figure 2.14 diagrams the four cases of vertex sets of the components of T . For Case 1, $\min\{|A_e|, |B_e|\} : e_1 \in E_T = 4$; for Case 2, $\min\{|A_e|, |B_e|\} : e_2 \in E_T = 3$; Case 3, $\min\{|A_e|, |B_e|\} : e_3 \in E_T = 1$; and for Case 4, $\min\{|A_e|, |B_e|\} : e_4 \in E_T = 1$. So, $w(T) = \max\{\min\{|A_e|, |B_e|\} : e \in E_T\} = 4$.

Figure 2.13: Tree T Figure 2.14: The four cases to consider in determining $w(T)$.

By using these definitions, Ostrovskii was able to develop some inequalities for the congestion problems in finite simple graphs. First, let G be a graph and U, W are two disjoint subsets of V_G . Then let $E_G(U, W)$ denote the set of all edges of graph G that have one endvertex in U and one endvertex in W . Next, suppose that T is a tree with $V_T = V_G$ such that $e \in E_T$ and let A_e and B_e be the subsets of V_T introduced above. Then e is used in $|E_G(A_e, B_e)|$ detours for edges of G . This implies some estimates for $ec(G : T)$.

Since we know that $|E_G(A_e, B_e)| \leq \Delta_G \min\{|A_e|, |B_e|\}$ where Δ_G is the degree of the largest degree vertex in Graph G . Also, $w(T) = \max\{\min\{|A_e|, |B_e|\} : e \in E_T\}$. This implies that $ec(G : T) \leq w(T) \Delta_G$.

Referring to Figure 2.15, if T is a spanning tree of G , then at least $|A_e| - 1$ of the

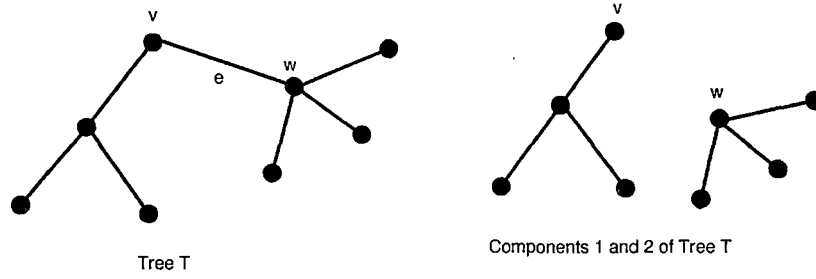


Figure 2.15: Tree T with two components, A_e and B_e , after removal of edge e .

edges of G incident to vertices of A_e have both vertices in A_e . So, $|A_e| - 1 = 3$. Similarly, $|B_e| - 1 = 3$. Therefore, in the case of T , a spanning tree of G , we can say that

$$ec(G : T) \leq w(T) \Delta_G - 2(w(T) - 1).$$

Another inequality that follows is

$$|E_G(A_e, B_e)| \leq |A_e| |B_e| \leq w(T) (|V_G| - w(T)).$$

So, $ec(G : T) \leq w(T) (|V_G| - w(T))$.

It follows from the previous inequality and $w(T) \leq \frac{|V_T|}{2}$ that

$$s(G) \leq \left\lfloor \frac{|V_T|^2}{4} \right\rfloor.$$

These inequalities draw attention to trees that have small weight. Suppose we do not assume that T is a subgraph of G , but instead we choose T to be a tree with one vertex of degree $|V_G| - 1$ and all other vertices are of degree one. Refer to Figure 2.16. Notice that vertex a of tree T has degree $3 = |V_G| - 1$ and vertices b , c , and d of Tree T all have degree one. Thus, $w(T) = 1$ for this tree and $t(G) \leq \Delta_G$.

Ostrovskii used these inequalities to obtain estimates for $t(G)$ and $s(G)$ which are contained in his theorem.

Theorem 2.15. $m_G = t(G) \leq s(G) \leq |E_G| - |V_G| + 2$.

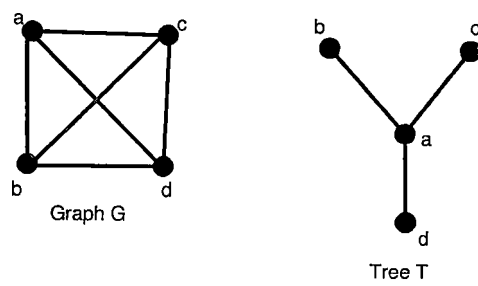


Figure 2.16: Graph G and Tree T .

Chapter 3

Complete Graphs

Definition 3.1. A complete graph on n vertices, denoted K_n , is a graph in which every pair of vertices is adjacent.

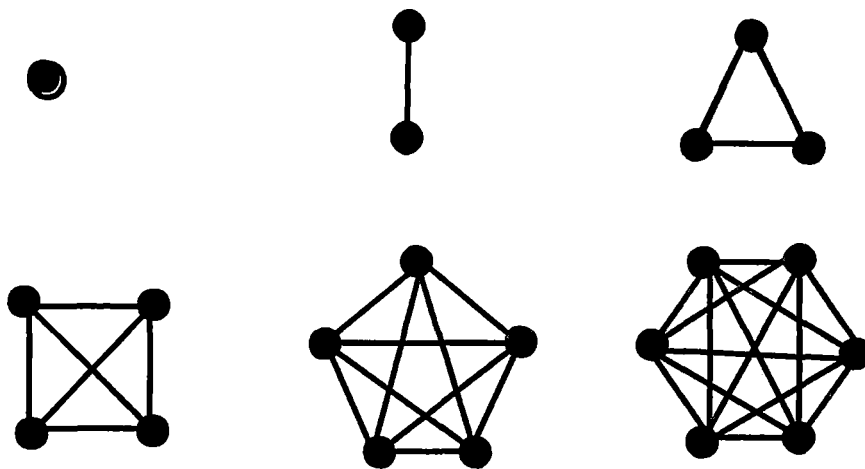


Figure 3.1: The first six complete graphs, K_1 , K_2 , K_3 , K_4 , K_5 , and K_6 .

Recall that a tree of graph G is a connected graph with the same vertex set in which there is only one path connecting each pair of vertices and a spanning tree of graph G is a tree of G that contains only edges present in graph G . Since every pair of distinct vertices is adjacent in complete graphs, then all trees of complete graphs are spanning trees. Refer to Figure 3.2, the graph of K_3 and its trees. Notice that trees A , B , and C are considered to be the same up to isomorphism since we can take the v-shaped tree and

bend it to a horizontal shape just like Tree C .

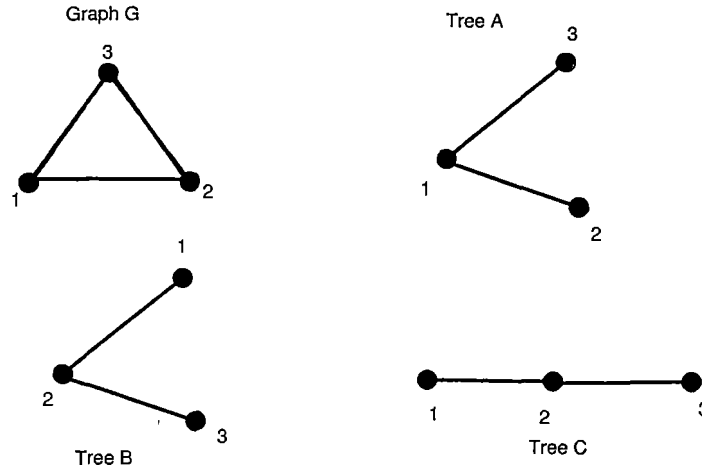


Figure 3.2: Graph G which is K_3 and some of its trees.

Consider the graph K_4 and some of its trees as seen in Figure 3.3. Tree A and Tree B are considered the same up to isomorphism since tree A can be bent into the same vertical shape as Tree B . Tree C and Tree D are also considered the same up to isomorphism. Therefore, there are only two trees to examine.

Now, let's examine complete graphs using Ostrovskii's theorem.

Recall Ostrovskii's theorem,

Theorem 3.2. $m_G = t(G) \leq s(G) \leq |E_G| - |V_G| + 2$

But, since $t(G) = s(G)$, then Ostrovskii's theorem becomes

Theorem 3.3. $m_G = t(G) = s(G) \leq |E_G| - |V_G| + 2$ for complete graphs.

Let's consider the graph K_3 as seen in Figure 3.4,

For K_3 , $m_G = 2$.

Thus, $m_G = t(G) = s(G) = 2$.

Notice that $m_G = t(G) = s(G) = 2 = n - 1$

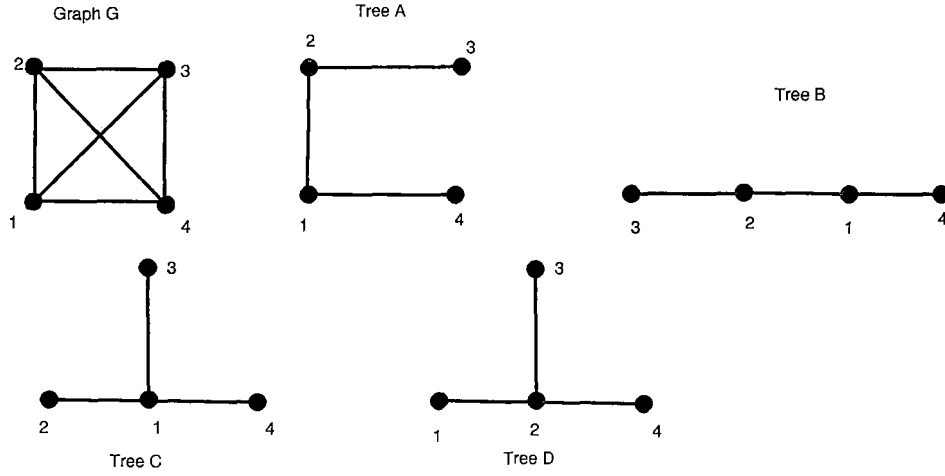


Figure 3.3: Graph G which is K_4 and some of its trees.

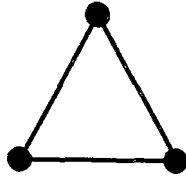


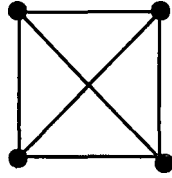
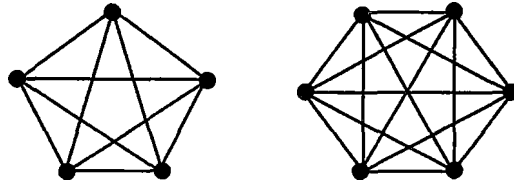
Figure 3.4: The graph K_3 .

Similarly, for graph K_4 in Figure 3.5. There are only two different trees up to isomorphism to examine. Let's consider the edge congestion of K_4 in Figure 3.7. Recall that the tree with the smallest edge congestion is called the minimal congestion tree. Tree A has an edge congestion of three and Tree B has an edge congestion of four. So, Tree A is the minimal congestion tree. Notice that the edge congestion is distributed evenly among each of the edges when the tree has this particular structure and the result is the minimal congestion tree of the graph.

For K_4 , $m_G = 3$ and $m_G = t(G) = s(G) = 3$.

Again notice that $m_G = t(G) = s(G) = 3 = n - 1$

So, for any complete graph, by Ostrovskii's theorem and the fact that $t(G) = s(G)$, it

Figure 3.5: The graph K_4 .Figure 3.6: The graphs of K_5 and K_6 , respectively.

appears that $m_G = t(G) = s(G) = n - 1$.

Now, let's consider the graph K_5 in Figure 3.6 and its trees in Figure 3.8.

Again notice that the edge congestion is distributed evenly among all the edges of Tree C . Using this type of formation of the vertices in a spanning tree of the graph results in the minimal congestion spanning tree of the graph. The center vertex of this formation is called the *centroid vertex*.

Now, we can examine K_6 in Figure 3.6 and its trees in Figure 3.9 to find similar results. Tree C has a structure similar to tree C of the graph of K_5 . Using a similar structure for tree C of the graph of K_6 also results in the minimal congestion spanning tree for the graph. Figure 3.10 demonstrates the type of structure needed to produce the minimal congestion spanning tree of the graphs of K_4 , K_5 , K_6 , K_7 , and K_8 .

For every complete graph, K_n , we can use a similar tree with a centroid vertex which will give us the minimal congestion spanning tree of the graph.

This leads to the following theorem for complete graphs.

Theorem 3.4. *For K_n , the complete graph with n vertices, the minimal congestion tree is a tree with a centroid vertex, called a parent vertex, with $n - 1$ children vertices which are each adjacent to the parent vertex, and, $t(K_n) = s(K_n) = n - 1$.*

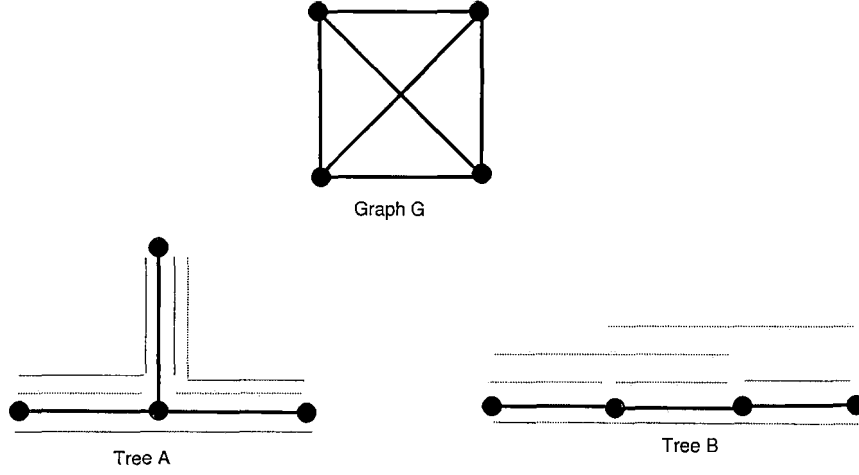


Figure 3.7: The graph of K_4 and its trees.

Proof:

For a complete graph, the minimum edge congestion can be found in a tree with a centroid vertex where $w(t) = 1$. Let G be a complete graph with n vertices and T be a tree with the same vertex set. Let e be an edge of T and A_e and B_e be the vertex sets of the components of T obtained after the removal of e . Let $E_G(A_e, B_e)$ be the set of edges of G with one end vertex in A_e and one end vertex in B_e . Then

$$ec(G : T) = \max_e |E_G(A_e, B_e)|.$$

Case 1: Let V_T have one parent vertex and $n - 1$ children vertices.

\implies The path from the parent vertex to each child contributes one to each edge.

\implies The path from child to child has one detour through the parent vertex.

\implies The cutwidth between child and parent is $n - 1$.

Case 2: Let V_T have multiple parent vertices. Then there exists an edge, e , of T , incident to two parent vertices. Let A_e have m vertices and B_e have $n - m$ vertices. Then

$$\begin{aligned} ec(G : T) &\geq m(n - m) \\ &\geq mn - m^2 \end{aligned}$$

Now, assume that $ec(G : T) < n - 1$

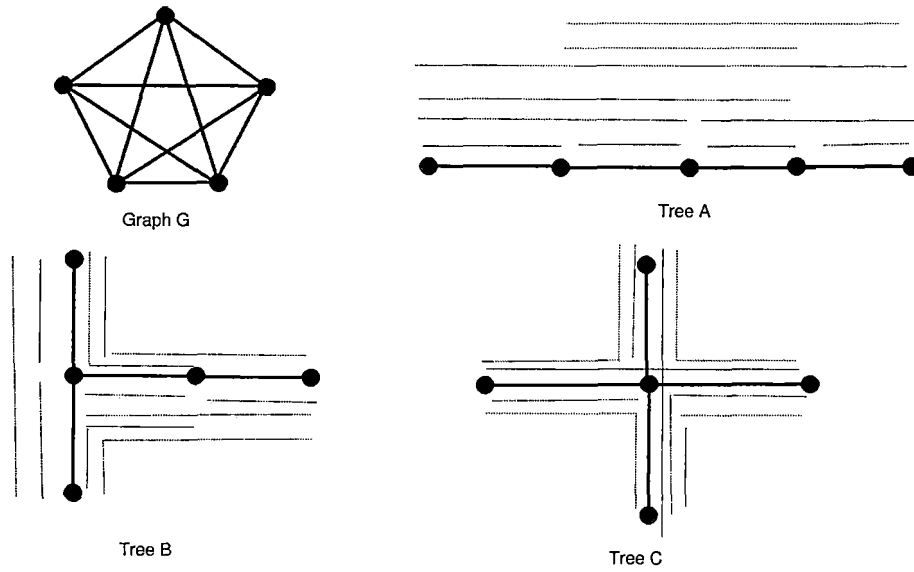


Figure 3.8: Graph G of K_5 and its trees.

$$\begin{aligned}
 \Rightarrow \quad n - 1 &> ec(G : T) \geq m(n - m) \\
 \Rightarrow \quad n - 1 &> m(n - m) \\
 \Rightarrow \quad n - 1 &> mn - m^2 \\
 \Rightarrow \quad m^2 - 1 &> mn - n \\
 \Rightarrow \quad (m + 1)(m - 1) &> n(m - 1) \\
 \Rightarrow \quad m + 1 &> n \\
 \Rightarrow \quad 1 &> n - m
 \end{aligned}$$

Since $n - m = B_e$ and B_e cannot have less than one vertex, this is a contradiction.

So, the minimum cutwidth of a complete graph is found in a tree with a centroid vertex with $w(T) = 1$.

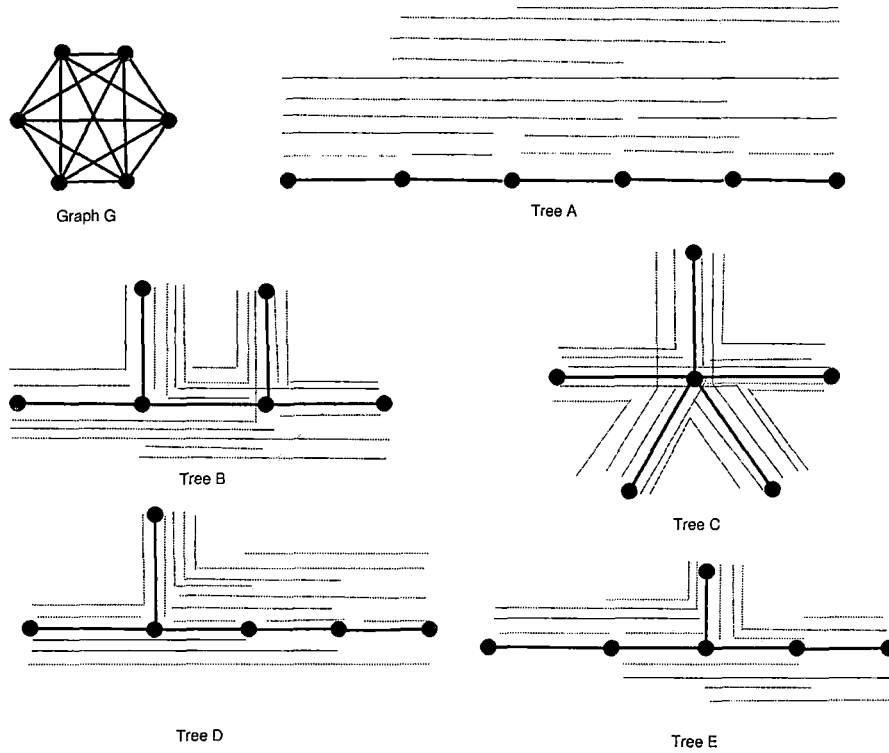


Figure 3.9: Graph G of K_6 and its trees.

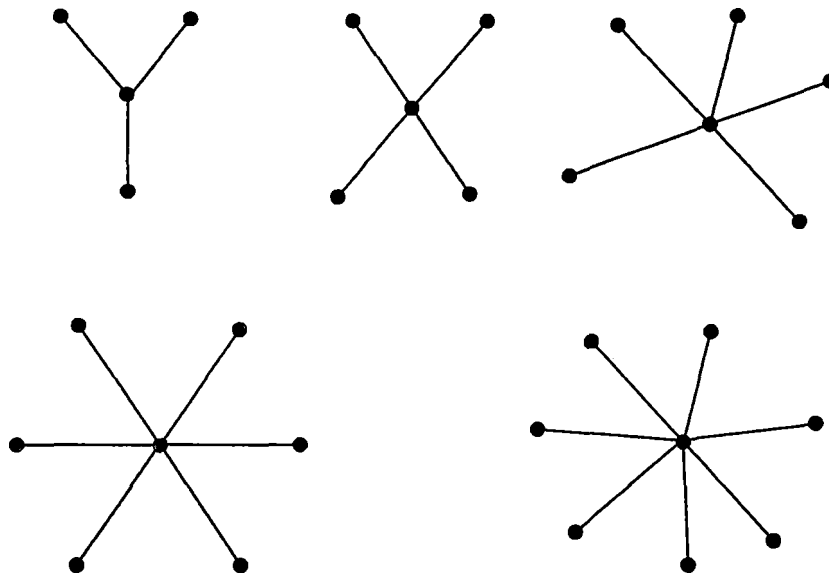


Figure 3.10: Spanning trees with centroid vertices for the complete graphs K_4 through K_8 .

Chapter 4

Complete Bipartite Graphs

Definition 4.1. A graph is bipartite if its vertices can be partitioned into two disjoint subsets U and V such that each edge connects a vertex from U to one from V . See Figure 4.1.

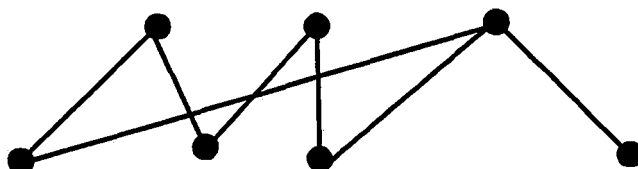


Figure 4.1: A bipartite graph.

Definition 4.2. A bipartite graph is a complete bipartite graph if every vertex in U is connected to every vertex in V . If U has n elements and V has m , then we denote the resulting complete bipartite graph by $K_{m,n}$. See Figure 4.2.

First, we will consider all the graphs $K_{m,n}$ such that m or $n = 1$. These graphs are all similar and trivial. Notice that each of the graphs and their respective trees resemble a tree with a centroid vertex as seen in complete graphs. See Figure 4.3.

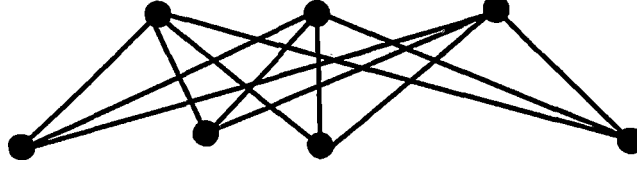


Figure 4.2: The complete bipartite graph, $K_{3,4}$.

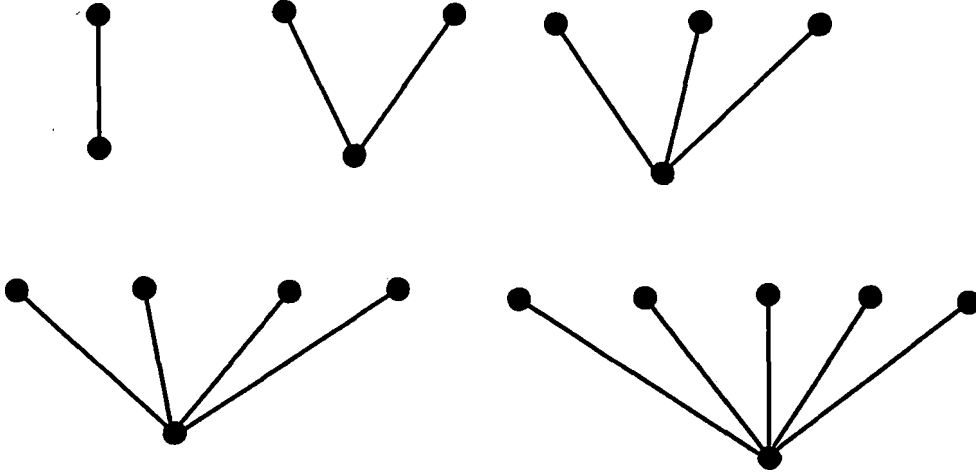


Figure 4.3: Graphs $K_{1,1}$, $K_{2,1}$, $K_{3,1}$, $K_{4,1}$, and $K_{5,1}$, respectively

Since $m_G = 1$ for all $K_{1,n}$ graphs and by Ostrovskii's theorem, then

$$\begin{aligned}
 m_G &= t(G) \leq s(G) \leq |E_G| - |V_G| + 2 \\
 \Rightarrow 1 &= t(G) \leq s(G) \leq 1 - 2 + 2 \\
 \Rightarrow 1 &= t(G) \leq s(G) \leq 1 \\
 \Rightarrow 1 &= t(G) \leq s(G) = 1
 \end{aligned}$$

These graphs are trivial since $t(G) = s(G) = 1$.

For all other complete bipartite graphs, we know the following to be true by Ostrovskii's theorem. For $K_{m,n}$, the complete bipartite graph with n and m the number of

vertices of the graph, $m_G = t(G)$.

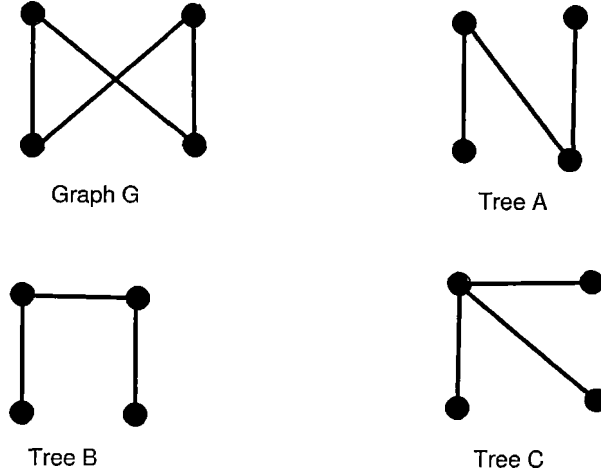


Figure 4.4: Graph G which is $K_{2,2}$ and its trees A , B , and C .

So, for $K_{2,2}$ we know that $m_G = 2$ and by Ostrovskii,

$$\begin{aligned}
 m_G &= t(G) \leq s(G) \leq |E_G| - |V_G| + 2 \\
 \Rightarrow 2 &= t(G) \leq s(G) \leq 4 - 4 + 2 \\
 \Rightarrow 2 &= t(G) \leq s(G) \leq 2 \\
 \Rightarrow 2 &= t(G) \leq s(G) = 2
 \end{aligned}$$

We can see in Figure 4.4 that the edge congestion for each of the trees is two and that $t(G) = s(G)$.

Next we can examine $K_{3,2}$ in Figure 4.5. We know that $m_G = 3$ and by Ostrovskii,

$$\begin{aligned}
 m_G &= t(G) \leq s(G) \leq |E_G| - |V_G| + 2 \\
 \Rightarrow 3 &= t(G) \leq s(G) \leq 6 - 5 + 2 \\
 \Rightarrow 3 &= t(G) \leq s(G) \leq 3 \\
 \Rightarrow 3 &= t(G) \leq s(G) = 3
 \end{aligned}$$

Again we can see that $t(G) = s(G) = 3$.

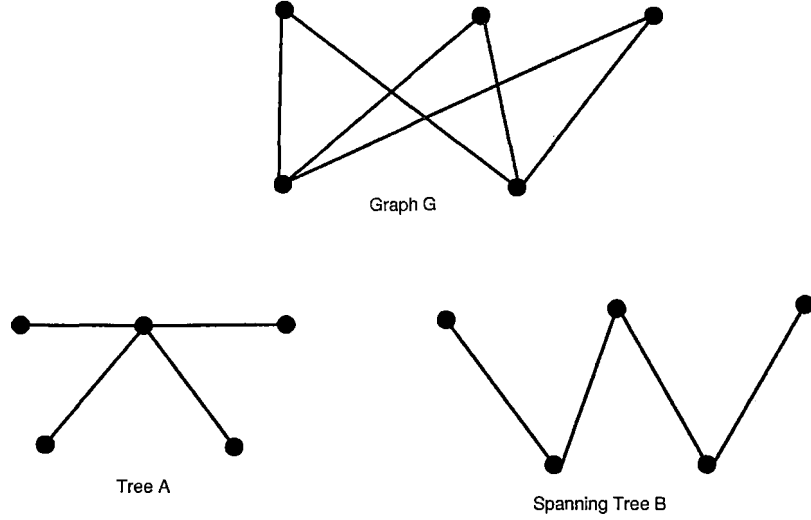


Figure 4.5: The graph $K_{3,2}$ and two of its trees.

Next, we can consider the graph $K_{3,3}$. Since $m_G = 3$ and by Ostrovskii's theorem,

$$\begin{aligned}
 m_G &= t(G) \leq s(G) \leq |E_G| - |V_G| + 2 \\
 \Rightarrow 3 &= t(G) \leq s(G) \leq 9 - 6 + 2 \\
 \Rightarrow 3 &= t(G) \leq s(G) \leq 5 \\
 \Rightarrow 3 &= \leq s(G) \leq 5
 \end{aligned}$$

In Figure 4.6, Tree A is the minimal congestion tree and it has $ec(G : T_A) = 3$. In fact, a tree with this structure will always be the minimal congestion tree for the $K_{m,n}$ graph. Trees B , C , and D are all spanning trees of the graph with $ec(G : T_B) = 4$, $ec(G : T_C) = 5$, and $ec(G : T_D) = 5$. Tree B is the minimal congestion spanning tree for the graph. A spanning tree with the same structure as Tree B will always be the minimal congestion spanning tree for the $K_{m,n}$ graph.

Consider the graph $K_{4,2}$ in Figure 4.7. Since $m_G = 4$ and by Ostrovskii's theorem,

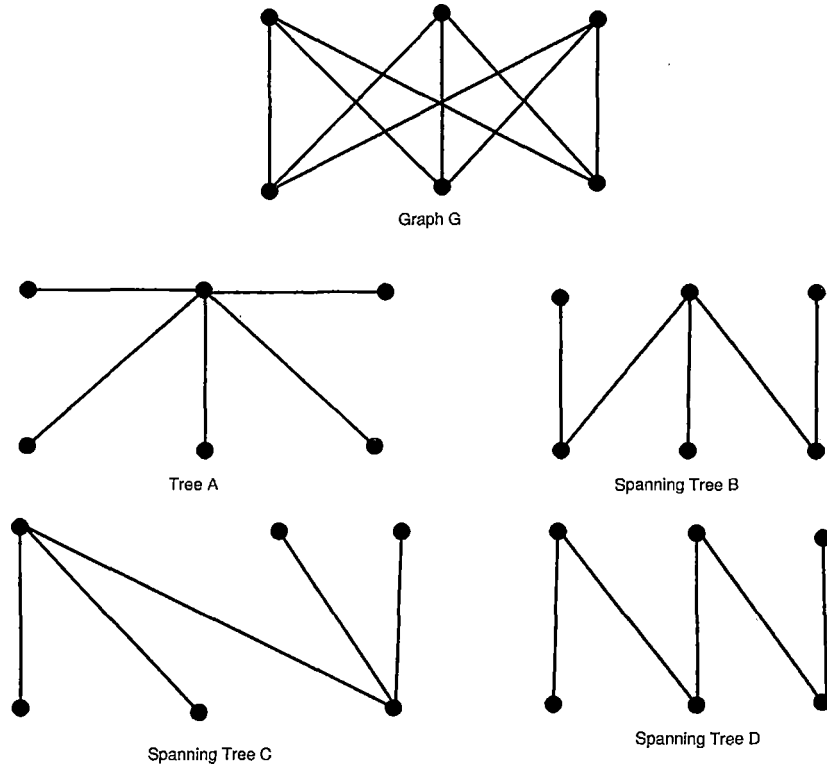


Figure 4.6: The graph $K_{3,3}$ and four of its trees.

$$\begin{aligned}
 m_G &= t(G) \leq s(G) \leq |E_G| - |V_G| + 2 \\
 \Rightarrow 4 &= t(G) \leq s(G) \leq 8 - 6 + 2 \\
 \Rightarrow 4 &= t(G) \leq s(G) \leq 4
 \end{aligned}$$

This implies that $t(G) = s(G) = 4$. Tree A is the minimal congestion tree with $ec(K_{4,2}, T_A) = 4$. Trees B , C , and D are all minimal congestion spanning trees each having edge congestion of four. Notice that Trees B and D have the same structure, that which has been pointed out previously as the structure that produces the minimal congestion spanning tree of the graph.

Consider the graph $K_{4,3}$ and its graphs as seen in Figure 4.8. Since $m_G = 4$ and by Ostrovskii's theorem,

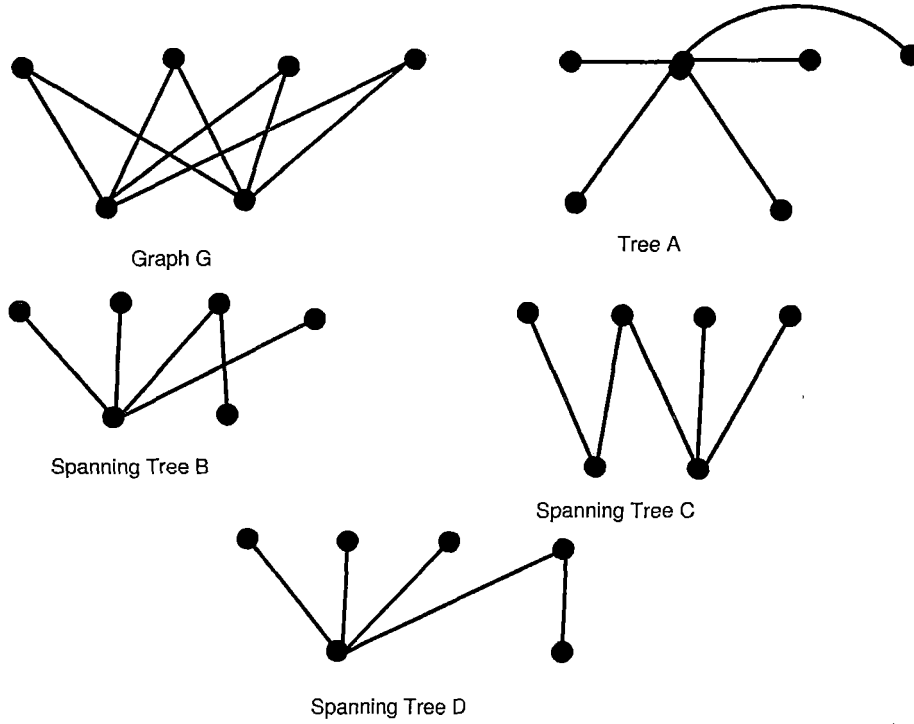


Figure 4.7: The graph $K_{4,2}$ and its trees.

$$\begin{aligned}
 m_G &= t(G) \leq s(G) \leq |E_G| - |V_G| + 2 \\
 \Rightarrow 4 &= t(G) \leq s(G) \leq 12 - 7 + 2 \\
 \Rightarrow 4 &= t(G) \leq s(G) \leq 7
 \end{aligned}$$

Since Tree A has $ec(K_{4,3}, T_A) = 4$ it is a minimal congestion tree. Note that Tree A has the desired structure for a minimal congestion tree. Tree B has $ec(K_{4,3}, T_B) = 6$, Tree C has $ec(K_{4,3}, T_C) = 7$, and Tree D has $ec(K_{4,3}, T_D) = 5$. Tree D is the minimal congestion spanning tree with an edge congestion of five. Notice that $n + m - 2 = 5$ and that Tree D is consistent in structure with the minimal congestion spanning trees examined previously.

We can find similar results for $K_{4,4}$, $K_{5,2}$, $K_{5,3}$. See Figures 4.9, 4.10, and 4.11 respectively. For $K_{4,4}$, $s(G) = 6 = n + m - 2$, for $K_{5,2}$, $s(G) = 5 = n + m - 2$, and for $K_{5,3}$, $s(G) = 6 = n + m - 2$. The following theorem summarizes the observations from the

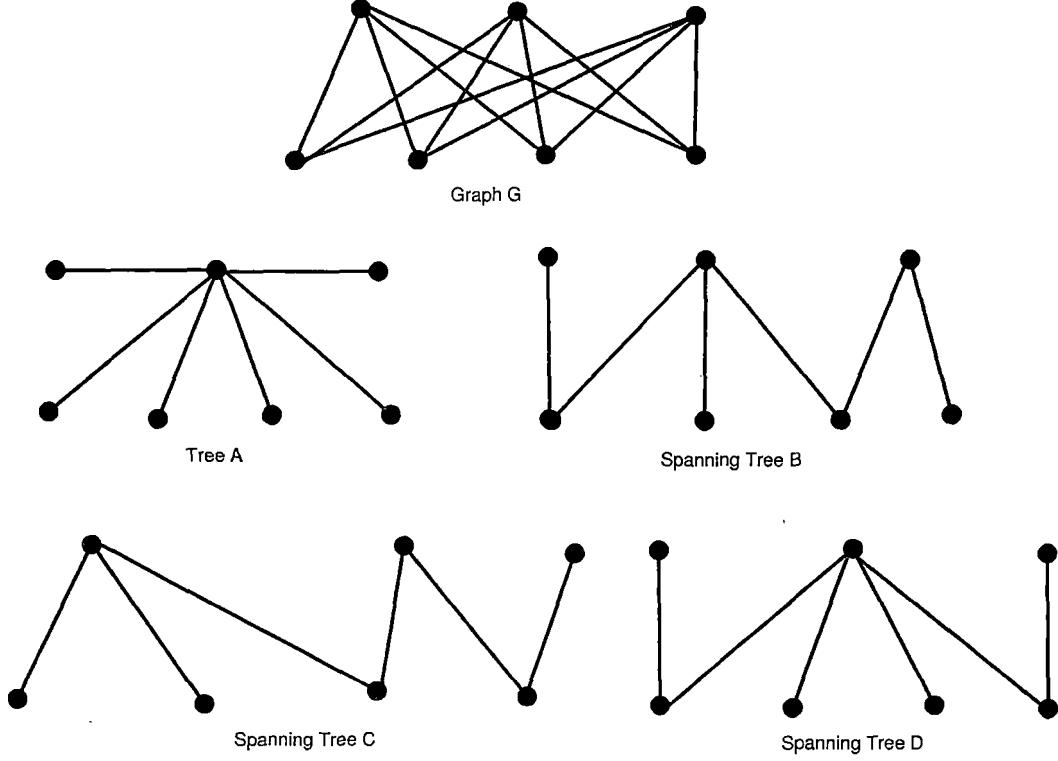


Figure 4.8: The graph $K_{4,3}$ and four of its trees.

previous examples.

Theorem 4.3. For $K_{m,n}$, the complete bipartite graph with m and n the number of vertices of the graph such that $m + n \geq 2$, $m_G = t(K_{m,n}) = \max(m, n)$.

Proof.

By Ostrovskii's theorem, we know that $m_G = t(K_{m,n})$. Now, for a complete bipartite graph, the minimum congestion tree will have $M_1, \dots, M_m \in U$ and $N_1, \dots, N_n \in V$ vertices such that M_p is a parent vertex adjacent to each of the M_i and N_j vertices, as seen in the example in Figure 4.12. Each edge incident to M_p and any M_i vertex, e_1 and e_2 in Figure 4.12, will have an edge congestion of n , one for each path between M_1 and each of the N_i vertices. The edge incident to the M_p and N_1 vertices, e_3 in Figure 4.12, will have an edge congestion of m , one for each path between the N_1 and M_i vertices. Similarly, the remaining edges, e_4, e_5, e_6 , and e_7 in Figure 4.12, will have an edge congestion of m . Thus, the $\max(m, n) = t(K_{m,n})$ by definition of tree congestion. By Ostrovskii, $m_G = t(K_{m,n})$,

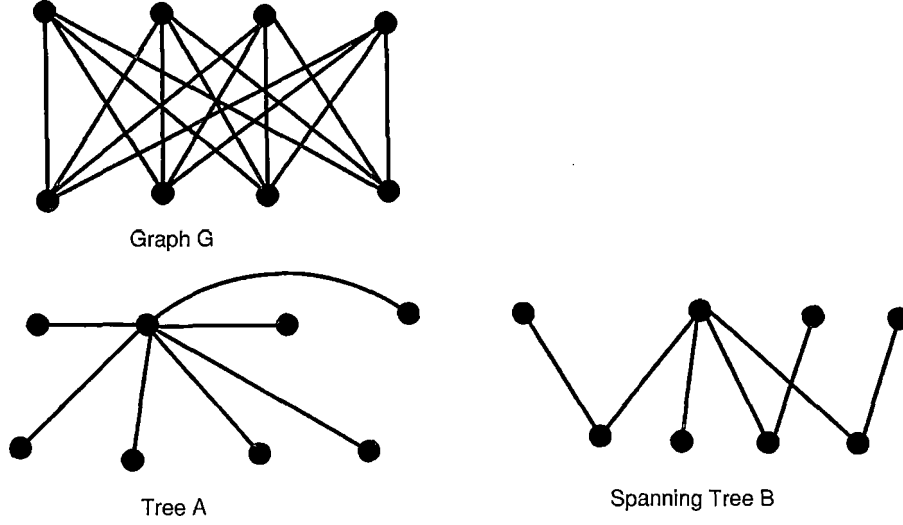


Figure 4.9: The graph $K_{4,4}$ and its minimal congestion trees.

so it follows that $m_G = t(K_{m,n}) = \max(m, n)$.

Theorem 4.4. *For $K_{m,n}$, the complete bipartite graph with m and n the number of vertices of the graph such that $m + n > 2$, $m_G = t(K_{m,n}) \leq s(K_{m,n}) = m + n - 2$.*

Proof:

First, by Ostrovskii's theorem, we know that $m_G = t(G) \leq s(G)$. Now, for a complete bipartite graph, denoted by $K_{m,n}$ such that m and n are the number of vertices of the graph and $m + n \geq 2$, the minimum edge congestion of the graph can be found in a tree with M_1, \dots, M_m and N_1, \dots, N_n vertices. The minimum edge congestion is $m + n - 2$.

Let M_1, M_2, \dots, M_m and N_1, N_2, \dots, N_n be the vertices of the complete bipartite graph $K_{m,n}$. Let T be a tree with M_p as the parent vertex of M_r vertices such that M_p is connected to each N_j vertex and no two M_i children are directly connected to the same vertex N_j . See Figure 4.13 for an example.

Since the paths that contain the most detours have the greatest edge congestion in a tree, then the edge congestion is the minimum cutwidth for the tree. Now, the path between M_p and each N_j contributes one to each edge between M_p and each N_j . And, the paths from the child, M_1 , to each N_j , not directly connected to M_1 , contributes $n - 1$ to the edge connecting M_p and N_1 (where N_1 is the vertex directly connected to M_1 .) The

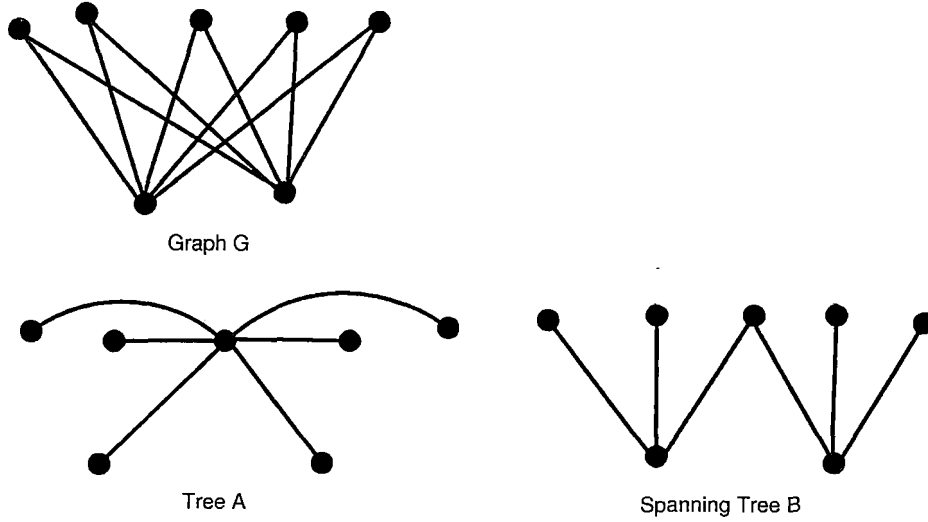


Figure 4.10: The graph $K_{5,2}$ and its minimal congestion trees.

children M_2, \dots, M_m contribute one each to the edge connecting M_p and $N_1 \Rightarrow m - 2$. Thus, the edge congestion of the path is $1 + (n - 1) + (m - 2) \Rightarrow m + n - 2$.

Since all children, M_i , detour through the parent M_p , then the path containing the detour will have the edge with the greatest congestion. And, since no two children (M_i) are directly connected to the same N_j vertex, then all the paths with detours through the parent, M_p , will have the same edge congestion.

$$\Rightarrow \text{greatest edge congestion of the tree is } m + n - 2$$

$$\Rightarrow \text{cutwidth of the tree is } m + n - 2$$

Now, assume $m + n - 2$ is not the minimum cutwidth of the tree of the graph. Then there exists a tree of the graph that has a minimum cutwidth less than $m + n - 2 \Rightarrow$ all paths on the tree have edge congestion less than $m + n - 2$.

Case 1: Assume the graph has multiple parents and the minimum cutwidth of the tree is less than $m + n - 2$. Then the edges of the tree that connect the two parents have the greatest edge congestion since every sibling must use the detour between the parents in their path. See Figure 4.14 for an example.

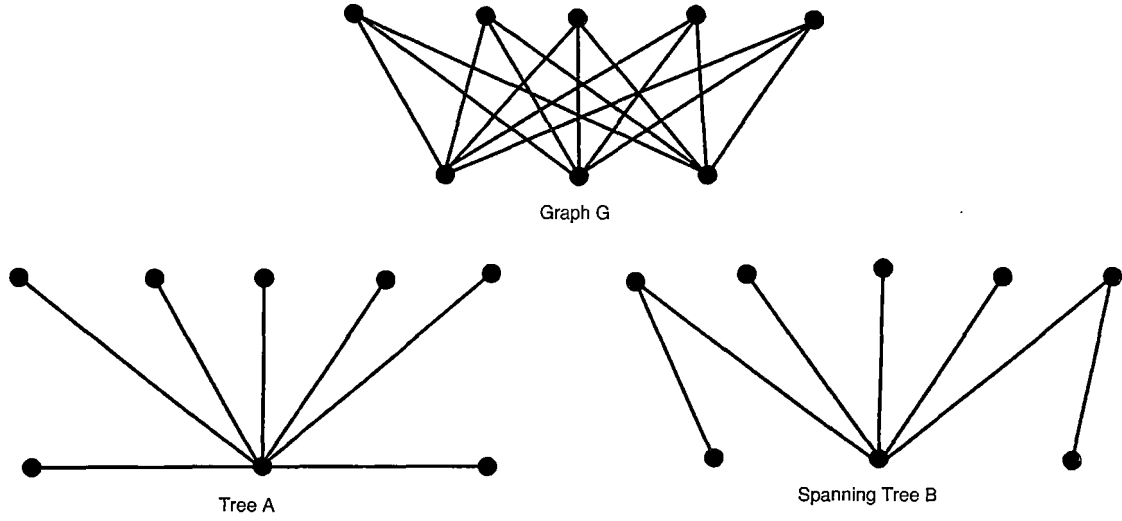


Figure 4.11: The graph $K_{5,3}$ and its minimal congestion trees.

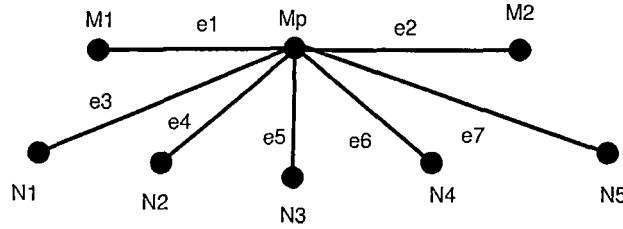


Figure 4.12: A minimal congestion tree of graph $K_{5,3}$.

Since the minimum cutwidth of the tree in Figure 4.14 is 6 and $m + n - 2 = 4 + 3 - 2 = 5$ then the minimum cutwidth of the tree is not less than $n + m - 2$ which is a contradiction.

Case 2: Assume the graph has multiple children directly connected to the same vertex N_m . See Figure 4.15 for an example. Assume the minimum cutwidth of the tree is less than $m + n - 2$.

Since the minimum cutwidth of the tree is 6 and $m + n - 2 = 3 + 4 - 2 = 5$, then the minimum cutwidth of the tree is not less than $m + n - 2$ which is a contradiction. Thus, there does not exist a path on the tree with edge congestion less than $m + n - 2$.

Therefore, $m + n - 2$ must be the minimum edge congestion of the tree of the

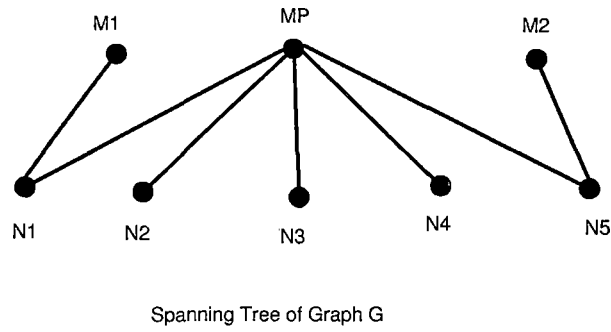


Figure 4.13: A possible spanning tree of graph G , $K_{3,5}$

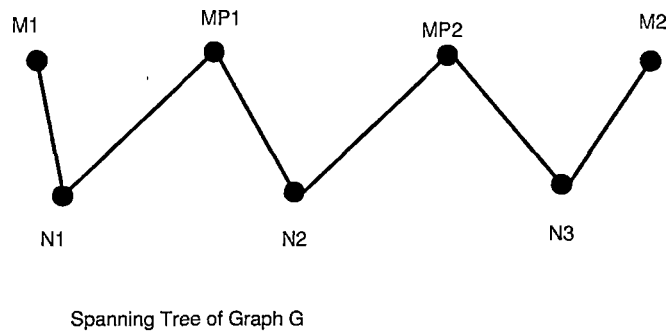
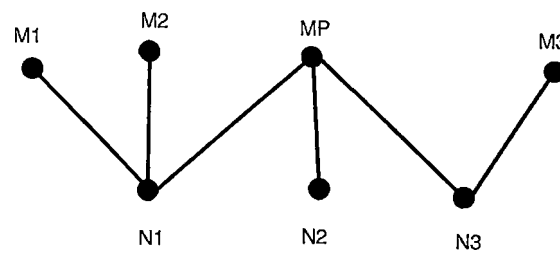


Figure 4.14: A possible spanning tree of graph G , $K_{4,3}$

graph $\Rightarrow m + n - 2$ is the minimum edge congestion of the complete bipartite graph, $K_{m,n}$ such that m and n are the number of vertices of the graph and $m + n \geq 2$.



Spanning Tree of Graph G

Figure 4.15: A possible spanning tree of graph G , $K_{4,3}$

Chapter 5

Grids

5.1 $P_2 \times P_3$ Grids

Theorem 5.1. *For G , a $P_2 \times P_3$ grid, $t(G) = s(G) = 3$.*

Proof:

Let T be a spanning tree with the same vertex set as graph G , a $P_2 \times P_3$ grid. Then T has five edges and six vertices. To form this spanning tree of G , we will remove the edges $(2, 4)$ and $(4, 6)$ which are the lower edges of the graph G . Refer to Figure 5.1.

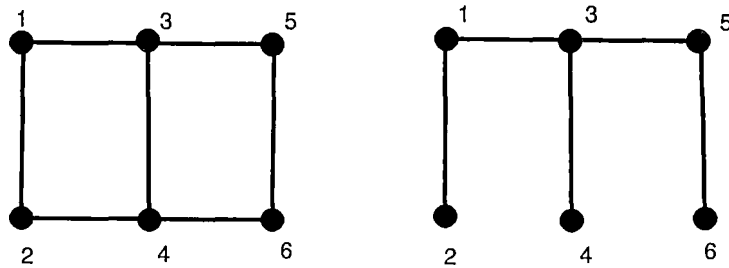


Figure 5.1: Graph G and Spanning Tree of G

Now, to determine the spanning tree congestion, each vertex must travel a path to its adjacent vertices. Specifically, the paths traveled between adjacent vertices are $1 \rightarrow 2$, $1 \rightarrow 3$, $3 \rightarrow 5$, $3 \rightarrow 4$, $5 \rightarrow 6$, $2 \rightarrow 4$, and $4 \rightarrow 6$. This implies that the greatest congestion will occur at the vertical edges where the paths between the even numbered vertices of the graph must

travel. For instance, the edge between vertices 1 and 2 has congestion two. Similarly, the edge between vertices 5 and 6 has congestion two. The edge between vertices 3 and 4 has congestion three. Refer to Figure 5.2. So, $ec(G : T) = 3$.

Recall Ostrovskii's Theorem, $m_G = t(G) \leq s(G) \leq |E_G| - |V_G| + 2$. First, we can calculate m_G , the maximal number of edge-disjoint paths joining u and v in Graph G , among all pairs (u, v) of vertices of G . Since the highest degree vertex of G is three and there are three edge-disjoint paths joining vertices 3 and 4, $m_G = 3$. Applying Ostrovskii's Theorem to a $P_2 \times P_3$ grid produces the result $m_G = t(G) = 3$. Since $m_G = t(G) = 3$ and $ec(G : T) = 3$, then $m_G = t(G) = s(G) = 3$.

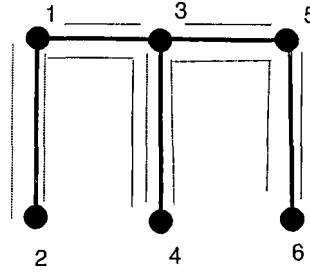


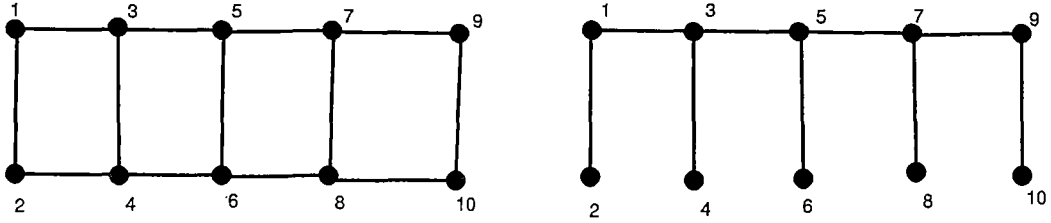
Figure 5.2: Spanning Tree of G with the paths used to calculate congestion.

It is easy to see how we can extend the case of the $P_2 \times P_3$ grid to the $P_2 \times P_4$ grid, $P_2 \times P_5$ grid, and finally to the $P_2 \times P_n$ grid. In Figures 5.3 and 5.4, the heaviest congestion is found along the interior vertical edges of the tree in exactly the same manner the congestion was calculated in a $P_2 \times P_3$ grid. This results in $ec(G : T) = 3$ for both a $P_2 \times P_4$ and a $P_2 \times P_5$ grid. Applying Ostrovskii's Theorem to both grids produces the same results as the $P_2 \times P_3$ grid, $m_G = t(G) = s(G) = 3$. These results lead to a theorem for all $P_2 \times P_n$ grids.

5.2 $P_2 \times P_n$ Grids

Theorem 5.2. *For a $P_2 \times P_n$ grid, $m_G = t(G) = s(G) = 3$.*

Proof:

Figure 5.3: $P_2 \times P_4$ Grid and Spanning TreeFigure 5.4: $P_2 \times P_5$ Grid and Spanning Tree

By Ostrovskii's Theorem we know that

$$m_G = t(G) \leq s(G) \leq |E_G| - |V_G| + 2$$

Again recall that m_G is the maximal number of edge-disjoint paths joining u and v in Graph G among all pairs (u, v) of vertices of G . For each corner vertex in Figure 5.3, specifically vertices 1, 2, 7, and 8, each vertex has two adjacent edges producing at most two edge disjoint paths between itself and any other vertex in the grid. The remaining vertices, vertices 3, 4, 5, and 6 are found between the corner vertices on the grid, each having three adjacent edges and thus at most three edge disjoint paths. Therefore, the maximal number of edge-disjoint paths joining any two vertices in G is three. This can be extended to any $P_2 \times P_n$ grid. Every $P_2 \times P_n$ grid will always have four corner vertices, each having two adjacent edges producing at most two edge disjoint paths. The remaining vertices of any $P_2 \times P_n$ grid will be found between the corner vertices in the same manner as the $P_2 \times P_4$ grid. Similar to vertices 3, 4, 5, and 6 in the $P_2 \times P_4$ grid, the remaining vertices in a $P_2 \times P_n$ grid will have three adjacent edges producing at most three edge-disjoint paths joining any two vertices in G . Now, we know that for any grid, $P_2 \times P_n$, $m_G = 3$. Since $m_G = 3$ and

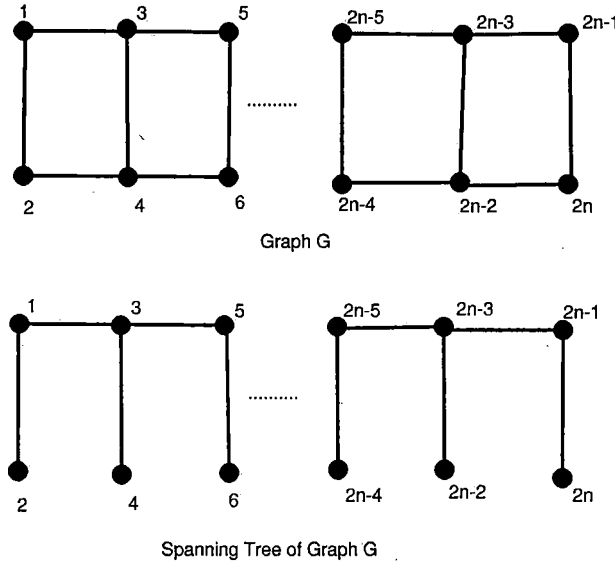
$m_G = t(G)$, then $m_G = 3 = t(G)$. Now, we will show that $t(G) = s(G)$.

Figure 5.3 contains a spanning tree of the grid $P_2 \times P_4$ which is representative of a $P_2 \times P_n$ grid. Three edges must be removed from the grid in Figure 5.3 to create the tree since the grid has three cycles. By removing the lower edges of the cycles, specifically the edges between the even numbered vertices, to find the edge congestion, each vertex will contribute one to each adjacent edge. Now, each vertex on the lower edge must use a detour to get to the vertex that is adjacent to it's right. For example, vertex 2 must detour through vertices 1 and 3 to get to vertex 4; vertex 4 must detour through vertex 3 and vertex 5 to get to vertex 6; and vertex 6 must detour through vertex 5 and vertex 7 to get to vertex 8. So, each vertex that uses a detour will contribute one to the vertical edge adjacent to it and one to the vertical edge just to the right of it. For example, vertex 2 will contribute one to the vertical edge between vertices 1 and 2 and one to the vertical edge between vertices 3 and 4; vertex 4 will contribute one to the edge between vertices 3 and 4 and one to the edge between vertices 5 and 6; and vertex 6 will contribute one to the edge between vertices 5 and 6 and one to the edge between vertices 7 and 8. Using this method, the outermost vertical edges will have a congestion of two, the interior vertical edges will have a congestion of three, and the top edges will have a congestion of two. Thus, the edge congestion of this $P_2 \times P_4$ spanning tree is three.

This method can easily be applied to any $P_2 \times P_n$ grid resulting in an edge congestion of three as seen in Figure 5.5. Thus, $m_G = t(G) = s(G) = 3$ for any $P_2 \times P_n$ grid.

5.3 $P_3 \times P_n$ Grids

We can now extend the $P_2 \times P_3$ grid and $P_2 \times P_n$ grid cases to the $P_3 \times P_n$ grid case. Let us first consider the $P_3 \times P_4$ grid in Figure 5.6. Each of the corner vertices which are numbered 1, 3, 10, and 12, have two adjacent edges which produce at most two edge disjoint paths. The vertices adjacent to the exterior edges, vertices 2, 4, 6, 7, 9, and 11, each have three adjacent edges which produce at most three edge disjoint paths. The remaining vertices, 5 and 8, have four adjacent edges each. However, there at most three edge disjoint paths since any path from vertex 5 to vertex 8 must travel through one of three edges, (4, 7), (5, 8), or (6, 9). Therefore, $m_G = 3$. Similarly, for any $P_3 \times P_n$ grid $m_G = 3$.

Figure 5.5: $P_2 \times P_n$ Grid and Spanning Tree

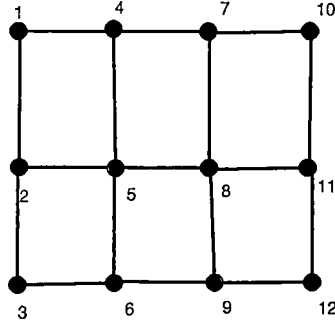
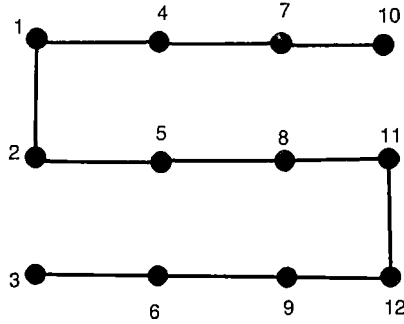
By Ostrovskii's Theorem, $m_G = t(G) = 3$. Consider several spanning trees of a $P_3 \times P_4$ grid and their congestion as. Tree A in Figure 5.7 has $ec(G : T_A) = 5$, Tree B in Figure 5.8 has $ec(G : T_B) = 4$, and Tree C in Figure 5.9 has $ec(G : T_C) = 3$. Now, since $m_G = t(G) = 3$ and $ec(G : T_C) = 3$, then $m_G = t(G) = s(G) = 3$. But, we can take any $P_3 \times P_n$ grid and use the same formation as Tree C of Graph G . This leads to the following theorem.

Theorem 5.3. *For a grid $P_3 \times P_n$, $m_G = t(G) = s(G) = 3$.*

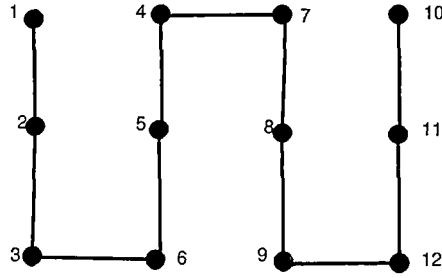
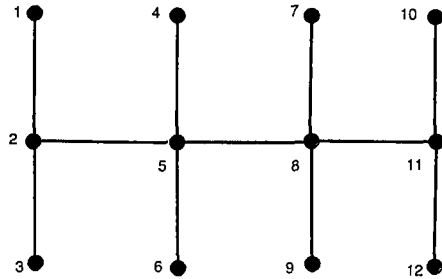
Proof:

We know that for any grid, $P_3 \times P_n$, $m_G = 3$. Thus, we can use Ostrovskii's Theorem in the same manner it was used for a $P_3 \times P_4$ grid to show that for any $P_3 \times P_n$ grid, $m_G = 3 = t(G)$. Then we will show that $t(G) = s(G)$. Consider $P_3 \times P_n$ grid in Figure 5.10 and the spanning tree in Figure 5.11 for the same grid.

Figure 5.11 is a spanning tree of the grid in Figure 5.10 which is representative

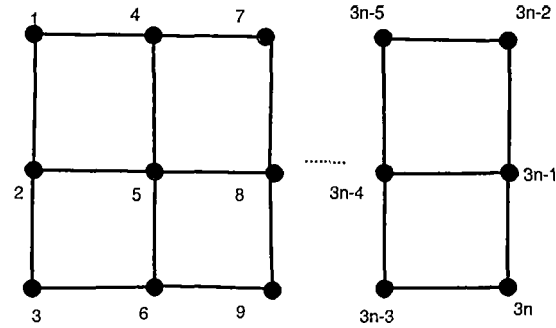
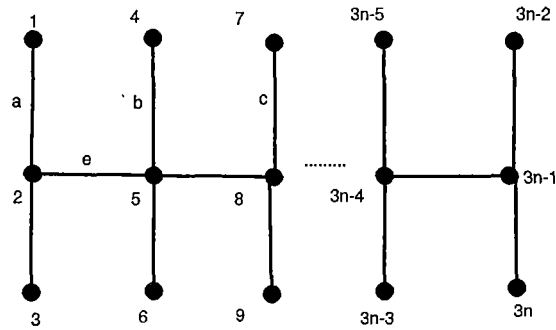
Figure 5.6: $P_3 \times P_4$ GridFigure 5.7: Tree A of a $P_3 \times P_4$ Grid

of a $P_3 \times P_n$ grid. Six edges must be removed from the grid in Figure 5.10 to create the spanning tree in Figure 5.11 since the grid has six cycles. By removing the upper and lower edges of the grid, we generate one spanning tree of the grid. In this spanning tree, each vertex will contribute one to the edge congestion of every incident edge. All of the vertices from the upper and lower edges must use a detour to get to the vertex to its right since those are the edges that were removed. For example along the upper edge of the tree, the path from vertex 1 to vertex 4 must detour through edges a , e_1 , and b . Similarly, the path from vertex 4 to vertex 7 must detour through edges b , e_2 , and c , and so on. The paths from the lower edges must take detours similar to the detours taken by the upper edges described above. So, each vertex that uses a detour will contribute one to the incident vertical edge, one to the e edge used in the detour, and one to the vertical edge just to its right. By this method, the heaviest congestion will occur at the e edges (since the upper and lower edges

Figure 5.8: Tree B of a $P_3 \times P_4$ GridFigure 5.9: Tree C of a $P_3 \times P_4$ Grid

each take detours that contribute to the e edge, for a total of 2 at each e edge) and at the interior vertical edges (since the path from vertex 1 to vertex 4 and the path from vertex 4 to vertex 7 will each contribute one to edge b , for a total of 2 at each of the interior vertical edges.) For each of the e edges, we get $1 + 2 = 3$ and for the interior vertical edges, we get $1 + 2 = 3$. This generates an edge congestion of the spanning tree of 3. (Note: The outside vertical edges of the spanning tree are trivial since the edge congestion will never exceed two since only one detour uses this edge.)

Consider the Figures 5.10 and 5.11 representing the $P_3 \times P_n$ grid. Extending the results of the $P_3 \times P_4$ grid to a $P_3 \times P_n$ grid did not affect or change the edge congestion since it just increases the number of interior vertical edges and e edges. The detours taken through these edges are the same as the detours taken through the edges of the subgraph, the $P_3 \times P_4$ grid. Therefore the edge congestion is still three.

Figure 5.10: Graph G : $P_3 \times P_n$ GridFigure 5.11: Tree of G

Although it seems as if we should be able to extend this method to any $P_m \times P_n$ grid, I have found that computing m_G for larger grids results in $m_G = 4$ at most. With this in mind, we can consider some larger grids using trees similar to the ones seen earlier. Let's first examine Figure 5.12, the $P_4 \times P_5$ grid.

By Ostrovskii's theorem, since $m_G = 4$, then $m_G = t(G) = 4$. Recall that $|E_G|$ and $|V_G|$ are the number of edges and vertices of Graph G , respectively.

$$\text{Then, } |E_G| - |V_G| + 2 = 23 - 20 + 2 = 5.$$

$$\text{Since } m_G = t(G) \leq s(G) \leq |E_G| - |V_G| + 2 \Rightarrow 4 \leq s(G) \leq 5.$$

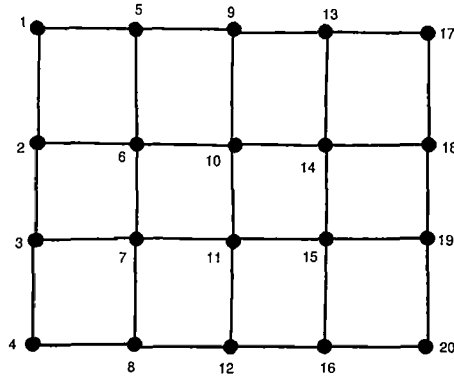
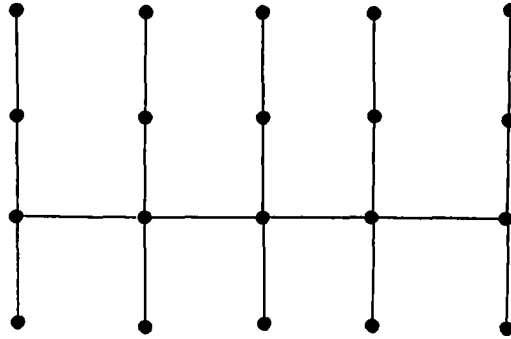


Figure 5.12: Graph G

So, $s(G)$ can be either 4 or 5. Constructing a spanning tree with a structure similar to Spanning Tree C of a $P_3 \times P_4$ grid as seen in Figure 5.9 results in an edge congestion of five. Next, we can examine Figure 5.14, a $P_4 \times P_6$ grid.

Figure 5.13: Tree A with $ec(G : T_A) = 5$.

By Ostrovskii's Theorem, $m_H = t(H) = 4$ and $|E_G| - |V_G| + 2 = 38 - 24 + 2 = 16$.

This gives the parameter $4 \leq s(H) \leq 16$.

Again, constructing a spanning tree similar to Tree C in Figure 5.9 results in a spanning tree with an edge congestion of five. Refer to Figure 5.15.

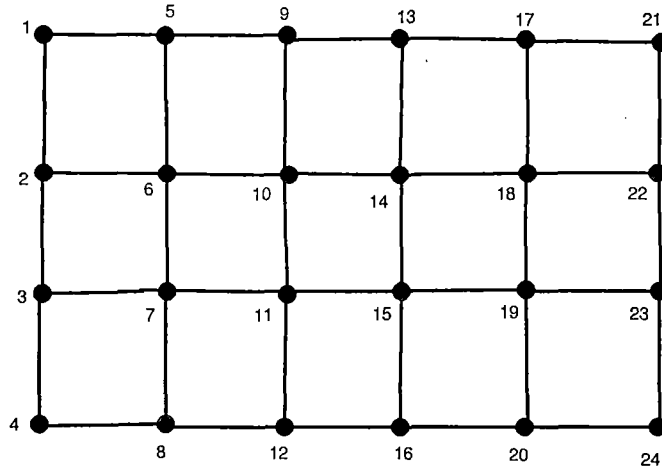


Figure 5.14: $P_4 \times P_6$ Grid, denoted Graph H.

Constructing similar spanning trees to Tree C in Figure 5.9 for a $P_4 \times P_7$ grid, $P_5 \times P_6$ grid, and $P_5 \times P_7$ grid resulted in an edge congestion of five for each of the spanning trees constructed. Refer to Figures 5.16 through 5.21.

Following the same process, spanning trees for a $P_6 \times P_7$ grid and a $P_7 \times P_8$ grid were constructed similar to the Spanning Tree C in Figure 5.9. But, instead of an edge congestion of five, the spanning trees of each grid resulted in an edge congestion of seven. Two more than the anticipated congestion of five. Refer to Figure 5.22 through Figure 5.25.

Similar results were calculated for other grids leading to the following pattern.

$$P_4 \times P_5 \text{ grid, } s(P_4 \times P_5) = 5$$

$$P_4 \times P_6 \text{ grid, } s(P_4 \times P_6) = 5$$

$$P_4 \times P_7 \text{ grid, } s(P_4 \times P_7) = 5$$

$$P_5 \times P_6 \text{ grid, } s(P_5 \times P_6) = 5$$

$$P_5 \times P_7 \text{ grid, } s(P_5 \times P_7) = 5$$

$$P_6 \times P_7 \text{ grid, } s(P_6 \times P_7) = 7$$

$$P_7 \times P_8 \text{ grid, } s(P_7 \times P_8) = 7$$

$$P_8 \times P_9 \text{ grid, } s(P_8 \times P_9) = 9$$

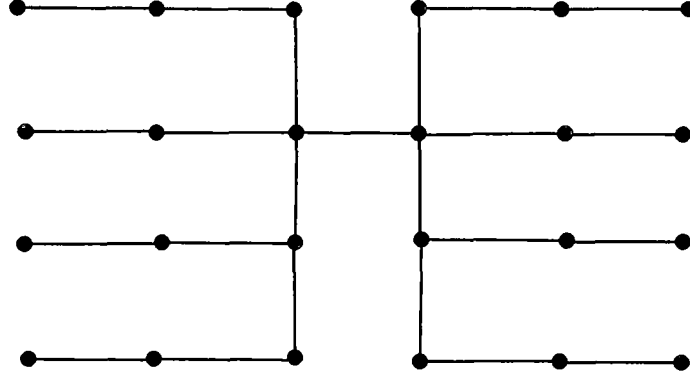


Figure 5.15: Spanning Tree C with $ec(H : T_C) = 5$.

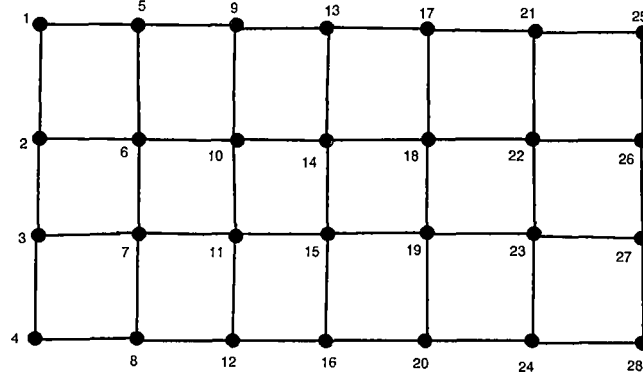


Figure 5.16: $P_4 \times P_7$ Grid, denoted Graph I.

$P_9 \times P_{10}$ grid, $s(P_9 \times P_{10}) = 9$

$P_{10} \times P_{11}$ grid, $s(P_{10} \times P_{11}) = 11$

$P_{11} \times P_{12}$ grid, $s(P_{11} \times P_{12}) = 11$

Comparing the results from these grids and the similarly constructed spanning trees leads to the following conjecture:

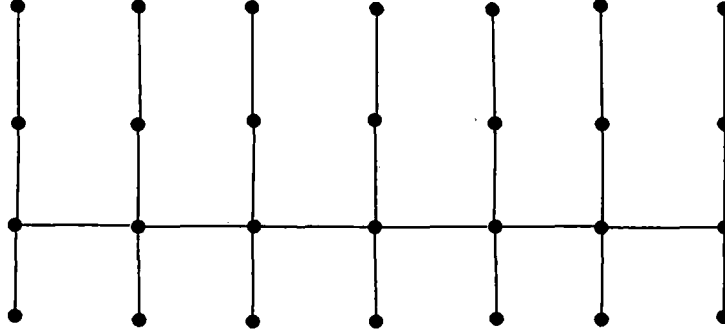


Figure 5.17: Spanning Tree C with $ec(I : T_C) = 5$.

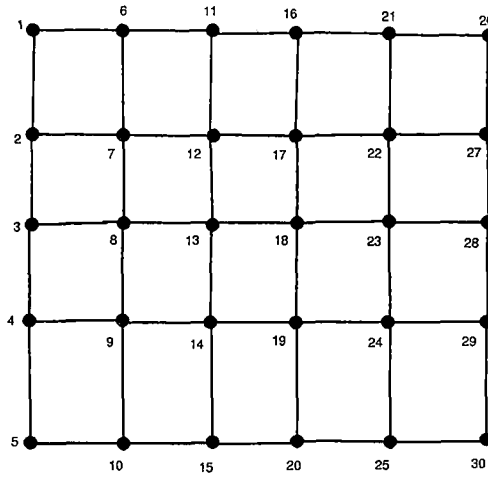


Figure 5.18: $P_5 \times P_6$ Grid, denoted Graph J.

Conjecture: For $P_m \times P_n$ grids with $m \times n$ vertices and $m \leq n - 1$

$$s(P_m \times P_n) = \begin{cases} m + 1 & m \text{ even;} \\ m & m \text{ odd.} \end{cases}$$

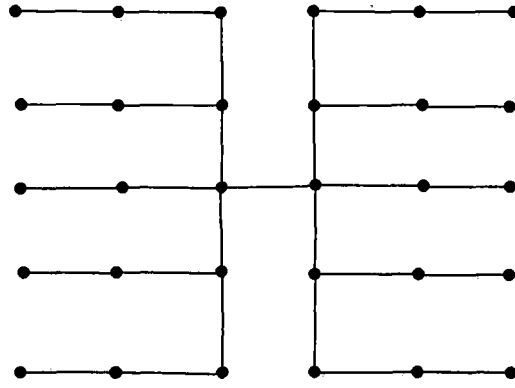


Figure 5.19: Spanning Tree C with $ec(J : T_C) = 5$.

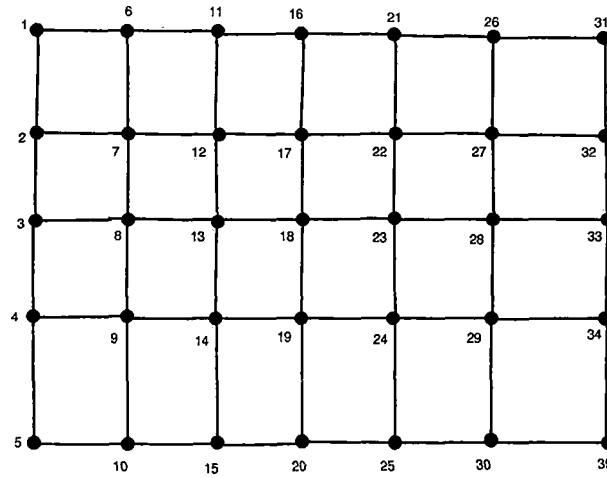


Figure 5.20: $P_5 \times P_7$ Grid, denoted Graph K .

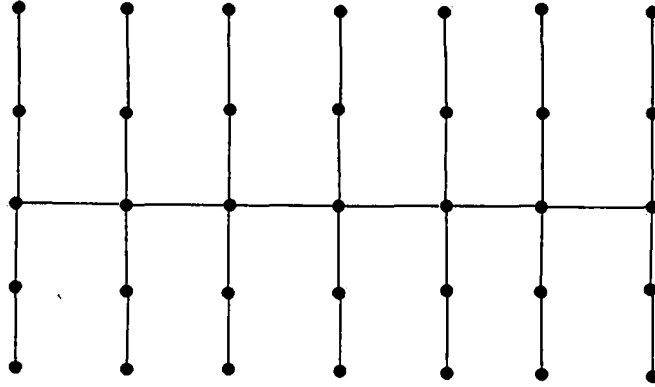


Figure 5.21: Spanning Tree C with $ec(K : T_C) = 5$.

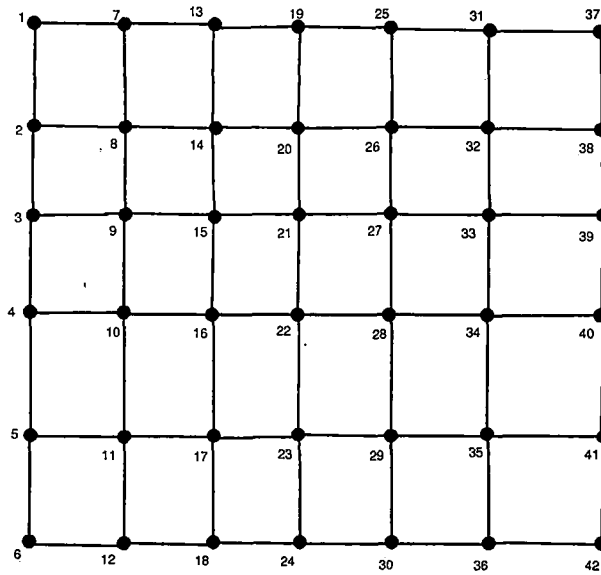


Figure 5.22: $P_6 \times P_7$ Grid, denoted Graph L .

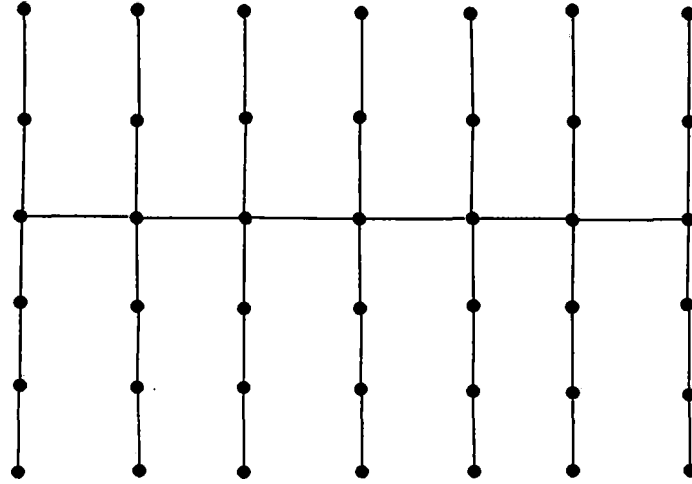


Figure 5.23: Spanning Tree C with $ec(L : T_C) = 7$.

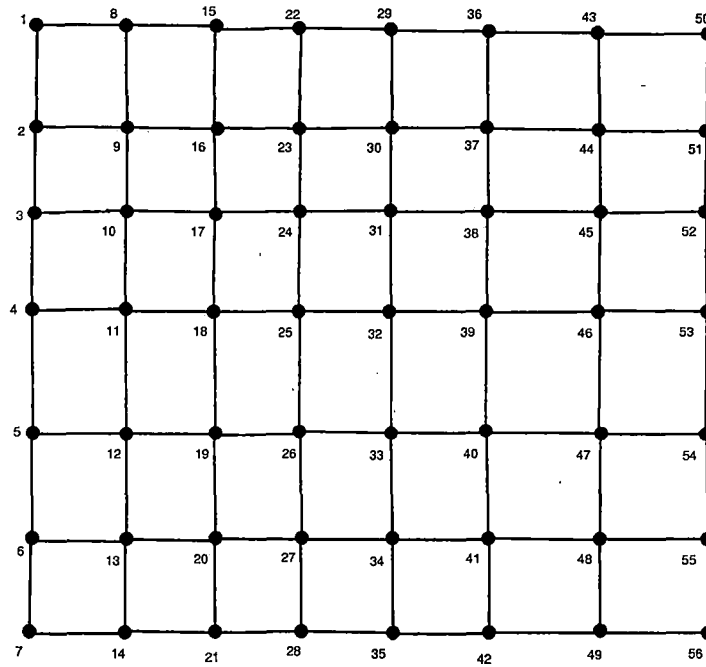


Figure 5.24: $P_7 \times P_8$ Grid, denoted Graph O .

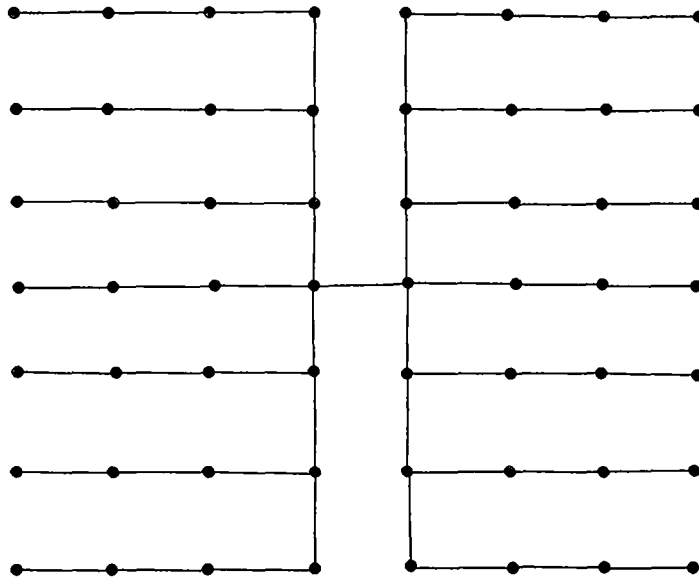


Figure 5.25: Spanning Tree C with $ec(O : T_C) = 7$.

Chapter 6

Conclusion

In conclusion, we have studied Ostrovskii's paper, Minimal Congestion Trees. The main purpose of this paper was to consider several edge congestion problems: minimization of $ec(G : T)$ over all trees with the same vertex set as graph G and minimization of $ec(G : T)$ over all spanning trees of graph G which resulted in Ostrovskii's Theorem. Ostrovskii's Theorem provides inequalities that summarize estimates for the edge congestion problems, $t(G)$ and $s(G)$.

We were able to use Ostrovskii's Theorem to find specific values for tree congestion and spanning tree congestion of two families of graphs, complete and complete bipartite graphs. For a complete graph with n vertices, we found the specific structure of the minimal congestion tree. It is a tree with a centroid vertex, called a parent vertex, and $n - 1$ children vertices which are each adjacent to the parent vertex, and, $t(K_n) = s(K_n) = n - 1$. For a complete bipartite graph we were able to find the structure of the minimal congestion tree and the minimal congestion spanning tree. We also found that $m_G = t(K_{m,n}) = \max(m, n)$ and $m_G = t(G) \leq s(G) = m + n - 2$ for m and n the number of vertices of $K_{m,n}$.

In addition, we were able to find specific values for tree congestion and spanning tree congestion for two families of grids, $P_2 \times P_n$ grids and $P_3 \times P_n$ grids. For any $P_2 \times P_n$, $t(G) = s(G) = 3$ and for any $P_3 \times P_n$, $t(G) = s(G) = 3$. We also formed a conjecture for all grids, for $P_m \times P_n$ grids with $m \times n$ vertices and $m \leq n - 1$

$$s(P_m \times P_n) = \begin{cases} m + 1 & m \text{ even;} \\ m & m \text{ odd.} \end{cases}$$

The next edge congestion problem one might consider is proving the conjecture

above; i.e., finding the tree congestion and spanning tree congestion for all grids. One possibility to consider in solving this edge congestion problem may be to expand upon the $P_2 \times P_n$ and $P_3 \times P_n$ grid solutions.

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