Essays on Selling Mechanisms
with a Focus on Auctions

BY

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Statement of Originality

I hereby declare that this submission is my own work and to the best of my knowledge it contains no material previously published or written by another person. Nor does it contain any material which has been accepted for the award of any other degree or diploma at the University of Sydney or at any other educational institution, except where due acknowledgment is made in this thesis. Any contributions made to the research by others with whom I have had the benefit of working at the University of Sydney is explicitly acknowledged.
I also declare that the intellectual content of this study is the product of my own work and research, except to the extent that assistance from others in the projects conception and design is acknowledged.

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Abstract

This thesis examines auctions as a selling mechanism in various market environments. There are in total three chapters. The first two build theoretical frameworks to analyze equilibria and to investigate expected revenues in multiple period settings, while the last chapter is an empirical study on the pre-auction estimates in art auctions.

The first chapter studies sequential auctions with endogenous supply. In the setting of the model, whether more units of goods are to be auctioned is contingent on the performance of the current auction. More specifically, two sellers, each of whom owns one unit of an identical item, but have different opportunity costs, face a fixed number of bidders, each of whom has unit demand and private valuation. One unit is to be sold for certain by one of the sellers; the winner of the first auction exits and the winning bid becomes common knowledge; the second seller then may offer her item depending on the price achieved in the first auction. In other words, the sellers together endogenize the supply decision based on the information on the demand side revealed in the first auction. We consider sequential auctions in both first-price and second-price formats. We give conditions for a symmetric pure strategy equilibrium to exist for each auction format. Under the assumption of uniform distributions, we explicitly solve for the equilibrium. We show that the second-price auction format provides a higher expected payoff to both sellers than the first-price format, and that both sellers prefer that the low-cost seller conducts the first auction while the high-cost seller conducts the contingent second auction. In addition, we conclude that the expected price declines if the second auction is held due to the uncertainty of the availability of the second unit of the item. This finding provides one possible explanation for the 'declining price anomaly', a well-documented phenomenon that puzzles auction theorists.

The second chapter investigates a market in which buyers with interdependent valuation arrive over time. A seller sells a single item with a second-price auction in such a market and wishes to achieve maximum *ex ante* profit. Meanwhile, although the total number of buyers is fixed, they arrive one by one in an exogenous sequence. We attempt to answer the following two questions in this setting: What is the optimal timing of auction for the seller? How well does the auction perform compared to another simple selling mechanism, posted price sale? We first point out that with interdependent buyer valuation holding an auction that includes all buyers may not be profit maximizing, even after we have assumed that everyone is sufficiently patient and is not discounting the future transactions. We analyze both the efficiency loss and the improvement on the 'winner's curse' concern that are associated with an early auction. Using

uniform distribution examples and numerical solutions we show that under some conditions the optimal timing of auction can be earlier than the time when all buyers arrive. We conclude that the relative importance of the signals of other buyers plays a central role in characterizing the equilibrium. In the second part of the chapter, we compare auction and posted price sale. We argue that posted price sale is likely to be *ex ante* more profitable than auction when the total number of buyers is small and the signals of others are important to buyers.

In the final chapter, we empirically examine the pre-auction estimates in art auctions, which are given by art experts hired by auction houses. We first address the question of whether these pre-auction estimates are unbiased indicators of the actual hammer prices, a question raised and studied by many economists in the literature. We have collected 3,923 auction records of paintings by a well-defined group of American artists between 1987 and 2018. The sample size is large enough to allow us to be the first to adopt a nonparametric approach to test the bias in the pre-auction estimates. Since the pre-auction estimates include a low estimate and a high estimate, we use the arithmetic mean of the low and high estimates in the regression model for the test. After correcting for the sample selection bias, we find evidence of bias in the arithmetic mean as a predictor of the hammer price, although the size of the bias is small. Then we criticize the use of the arithmetic mean to test the bias of the pre-auction estimates, an approach adopted by all previous studies in the literature to our best knowledge. We build a simple model to illustrate that the distribution of the hammer price is left-skewed even if the distribution of buyer valuation is symmetric. If the pre-auction estimates are considered as a confidence interval for the hammer price, then we show that under some conditions art experts have incentives to place the low estimate closer to the mean of the hammer price. As a result, using the arithmetic mean of the low and high estimates is misleading, resulting in an upward bias. We also find regression results that support our argument. We conclude that empirically the low estimate should be given more weight compared to the high estimate when one tries to interpret the pre-auction estimates and to predict the hammer price.

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Chapter 1

Sequential Auctions with Endogenous Supply

1 Sequential Auctions with Endogenous Supply

1.1 Introduction

This chapter studies sequential auctions in a setting in which whether further auctions are to be held is contingent on the performance of the current auction. In particular, at the outset, one item is offered for sale at an auction and it is common knowledge that whether a second item will also be offered for sale at an auction depends on the auction price achieved in the first auction. This setting is meant to be a stylized description of a commonly observed feature in the real world. For example, the success of one auction in July 1994 conducted by the US Federal Communications Commission (FCC) to sell licenses for electromagnetic spectrum opened the flood gates for "spectrum auctions". Since then the FCC itself has conducted over 80 such auctions and raised over 60 billion dollars, and numerous similar spectrum auctions, emulating the FCC auction, have subsequently been conducted on six continents. In the art market, a record auction price achieved by an artist encourages owners of other works by the artist to put those on the market. In Australia, where houses are typically sold in auctions, it is common for real estate agents to distribute flyers touting a high auction price of a house and actively solicit further supply in the neighbourhood. In these examples, the supply of items to be auctioned is endogenous and contingent on the performance of similar prior auctions.

In our formulation, two distinct sellers, each of whom owns one unit of an identical item, but have different opportunity costs for the items, face a fixed number of bidders, each of whom has a unit demand and independently distributed private valuation. One unit is to be sold first by one of the sellers; the winner of the first auction exits and the winning price (but not the other bids) becomes common knowledge; the second seller then may offer her item depending on the auction price achieved in the first auction. It is common knowledge that the low-cost seller's opportunity cost for the item is zero; the cost of the high-cost seller is private information and has a known distribution with non-negative support. The order in which the sellers offer their item is endogenous. Finally, we consider sequential auctions in both the first-price and the second-price formats, each being an absolute auction (i.e., with a known reserve price of zero).

We show that a monotone pure strategy equilibrium exists in both first-price and second-price sequential auctions under some restrictions on the distributions of the valuations of bidders and the cost of the high-cost seller. Moreover, the two auction formats are *not* revenue equivalent. The information content from observing the winning bid in two auction formats is different: the first-price auction reveals only the upper bound of the highest valuation of the bidders that will remain for the second auction while the second-price auction reveals the exact highest valuation. As a result, even with identical realizations of the valuations of bidders, it is possible that the seller decides to offer the second item for sale in one auction format but not in the other format.

Under the assumption that the distributions are uniform, we give explicit constructions of the bidding function and the equilibria. We show that, when the second auction takes place, the expected price in the second auction is lower than the first. This is consistent with the observed declining price trend in sequential auctions documented in many empirical studies; the empirical finding is robust enough that it is referred to as the "declining price anomaly" or the "afternoon effect". The phenomenon is considered a puzzle: the price path in a sequential auction of identical items is predicted to be a martingale in classical sequential auctions models, with the winning price in each round the same on average with independent private valuations. In our setting, the uncertain supply in the second period leads to more aggressive bidding in the first auction and explains the declining price path. In addition, we show that ex ante both sellers prefer the order of moves such that the low-cost seller conducts the first auction and the high-cost seller decides whether to conduct the second auction. The low-cost seller takes advantage of the more aggressive bidding in the first auction when it is uncertain whether the second auction will take place. In contrast, the high-cost seller prefers to put her item for sale only when demand is sufficiently strong to cover her opportunity cost. Finally, we conclude that both sellers get a higher expected payoff with sealed second price auction format than using sealed first price auction format.

This chapter is organized as follows. Section 1.2 reviews related literature. Section 1.3 introduces the basic model. Section 1.4 studies the first-price auction format in the context, derives the symmetric pure strategy equilibrium and gives sufficient conditions for its existence. Section 1.4 also provides the explicit constructions of equilibria under the assumption of uniform distributions. Sections 1.5 mirrors the preceding section but studies the second-price auction format. Section 1.6 compares the first-price and the second-price auction formats. Section 1.7 offers concluding comments. All proofs are gathered in the Appendix.

1.2 Literature Review

Weber (1983) and Milgrom and Weber (2000)[1] pioneer the study of sequential auctions. In the case where the items are homogeneous, each buyer only demands a single unit and buyer valuations are independent and private, they argue that the Revenue Equivalence Theorem applies, so the equilibrium bidding strategy must lead to the same expected payoff as in an auction where all items are sold simultaneously. Moreover, they show that the price path in sequential auctions is a martingale. That is, the expected price of each auction is the same as the price of the previous one. The martingale property confirms the intuition of the Law of One Price. Bidders shade their bids in earlier auctions aware of the "option value" of participating in later rounds. In equilibrium, there should not exist arbitrage opportunity across auction rounds, so the expected price of each round must be the same on average. Their model has become a benchmark for studying sequential auctions.

Following the work of Webber and Milgrom, sequential auctions have been studied extensively. However, very few papers attempted to study strategic behaviours of sellers, who can observe results in earlier auction rounds and possibly use such information in later auctions. There are two important questions to be answered: to what extent are sellers able to obtain information about the remaining bidders in sequential auctions? How should sellers use such information? Katsenos (2008) studies the case where sellers can choose the reserve price before each round in two-round sequential auctions with single-unit demand. He concludes that pure strategy equilibrium only exists when the bidder with the second highest valuation can hide her information in the first round. This result implies that observing all bids in sealed price auctions or English auctions will not help sellers obtain useful information about bidders, because bidders would pool the bids in order to conceal their true valuation to protect their payoff in the next auction round. As a result, among standard auction mechanisms, only sequential Dutch auctions accommodate a pure strategy monotone equilibrium. However, Katsenos shows that although sellers are better off in the second round by choosing an optimal reserve price using information revealed in the first round, they lose more in the first round because the competition is weakened as some bidder types abstain from the first round in the equilibrium. Caillaud and Mezzetti (2004) find similar results in an example with multi-unit demand. The lesson is, in most cases, sellers are not able to gain much information in earlier rounds; when they are able, they are worse off on average if they are allowed to choose the reserve price using the information gained. Consequently, sellers have an incentive to promise not to use information

[1] This paper was written in 1982, but only got published recently.

revealed in the process of sequential auctions to set reserve prices; instead, they should commit to the pre-determined reserve prices, if they wish to set one.

Instead of setting reserve prices, strategic sellers could decide whether or not to auction the next item at all depending on information revealed in earlier auctions. However, the only works the author is aware of that allow a strategic seller to make supply decisions are by Zeithammer (2007 and 2009). In his 2007 paper, Zeithammer studies a very special discrete case where exactly two new buyers with single-unit demand arrive at each period and will only live for two periods whilst the seller lives forever. In addition, he assumes that buyer valuations can only be either high or low, two discrete values, and the seller's cost of production is in between these two values. He shows that in the equilibrium bidders may eventually stop shading their bids when the seller's payoff becomes too low to sustain the supply of items.

In his next paper, Zeithammer (2009) assumes that a single seller owns two identical items with increasing opportunity costs. The opportunity costs are publicly known by bidders. The seller observes only the winning price of the first auction and decides whether to auction the second item. Most importantly, all economic agents are assumed to discount the results of the second auction. A symmetric pure strategy equilibrium exists only with Dutch auction format if the discount factor is sufficiently low. When such an equilibrium exists, there is a threshold price in the first auction, below which the second auction is unavailable and above which the second auction is guaranteed.

The work by Zeithammer (2009) is similar to but different from this chapter. In this chapter, instead of publicly known, the opportunity costs of sellers are private information. Second, in our model discounting is not required to support the symmetric pure strategy equilibrium. In addition, a symmetric pure strategy equilibrium exists with sealed second price auctions as well as with sealed first price auctions (or equivalently Dutch auctions). Finally, there are multiple sellers in this chapter. Assuming multiple sellers with one unit of supply each instead of a single seller owning multiple items has many advantages. First, it reflects many real-world sequential auction environments. Second, by assuming multiple sellers, it is justifiable that the remaining seller can only access the publicly announced winning price in the first auction. This information disclosure assumption is important to the equilibrium and the robustness of the model. If sellers could observe every bid in the first auction, they would have strong incentives to dishonour their commitment of a zero reserve price and set an optimal reserve price that seizes all trade surplus in the second auction. The commitment issue will make buyers nervous and they may try to conceal their valuations by pooling, causing the collapse of any monotone pure strategy equilibrium. Third, with multiple sellers, it is easy to justify the heterogeneity in opportunity costs among homogeneous items. Finally, interesting questions arise with multiple sellers: who should sell in the first auction and who should sell in the second? The coordination between sellers on the order of moves reveals rich insights into endogenous supply in sequential auctions.

One interesting prediction of this chapter is that when the second auction exists, after observing the winning price in the first auction, the expected price of the second auction falls below it. Such result is consistent with observed declining price trend in sequential auctions documented in many empirical studies (Ashenfelter, 1989a; Beggs and Graddy, 1997; Ashenfelter and Genesove, 1992; Van den Berg, Van Ours, and Pradhan, 2001), which is known as "Declining Price Anomaly" or "Afternoon Effect". The phenomenon is puzzling because Weber (1983) shows that the price path in sequential auctions should be a martingale, namely, the winning prices in each round should be the same on average. He and Milgrom and Weber (2000) also point out that if buyers valuations are affiliated, prices should drift upwards instead. Many studies have tried to explain the reasons for the "Declining Price Anomaly". McAfee and Vincent (1993) show that risk aversion of buyers can make the prices decrease over time, though the condition is somehow restrictive and hard to defend: the degree of risk aversion of buyers needs to increase with their wealth. Bernhardt and Scoones (1994) assume that items in sequential auctions are stochastically equivalent and bidders do not know their own valuations of the next item until seeing it when the next auction starts. They find declining prices arise in such a situation. Rosato (2014) believes that loss aversion of buyers can rationalize the "Afternoon Effect". Chakraborty, Gupta, and Harbaugh (2006) study the case where items are heterogeneous. They conclude that in the equilibrium the order of the sale is endogenous and the seller sells the better item first in optimizing her expected payoff. It naturally results in a declining price path. Jeitschko (1999) argues that stochastic supply reduces the option value of latter auction rounds and causes declining prices. This chapter contributes by offering another reasonable explanation of the "Declining Price Anomaly", claiming it is what one should expect if sellers strategically make the supply decision using the information revealed in earlier auction rounds. Here the intuition for the declining price path lies in the uncertainty of the supply, which is very similar to Jeitschko's work; but our chapter shows that the uncertainty of the supply is endogenously created by the decision of sellers, which is related to Chakraborty et al. in the sense that it is the sellers being strategic that plays the important role.

Finally, this chapter assumes each round of the sequential auction is conducted as an absolute auction (i.e. the reserve price is zero), as in Bulow and Klemperer (1996) and Zeithammer (2009). The purpose of such an assumption is to isolate the effect of the sellers' supply decision from that of reserve prices. In addition, in many cases, a pre-committed reserve price is not credible as demonstrated by McAfee and Vincent (1997). According to them, if sellers cannot commit to never resell any unsold item and the resale is conducted quickly, auctions with reserve prices are revenue equivalent to those without them. In the context of this chapter, the sellers need to prepare the item and bear the full opportunity cost before each auction. Since the cost is already sunken when the auction starts, the sellers will have an incentive to sell and even resale their items at any positive price, satisfying the key conditions in McAfee and Vincent (1997). As a result, assuming a zero reserve price does not lose much insight into the implications of endogenous supply in sequential auctions.

1.3 The Model

Two sellers wish to sell identical items to n buyers ($n \geq 3$) via auctions. Buyers have single-unit demand. Each of them assigns a private value v to the item, and the value is independently drawn from a common distribution according to c.d.f. $F(v)$ and p.d.f. $f(v)$. The support interval of the distribution is assumed and normalized to $[0, 1]$. Each seller possesses one unit only and bears a different cost to make the item ready for sale. It is assumed that the identity of the lower cost seller is public information; and without loss of generality, her cost is normalized to zero. The cost of the other seller, denoted as c, is her private information and is drawn from a distribution with c.d.f. $G(c)$ and p.d.f. $g(c)$ on $[0, C]$. For regularity, $g(c)$ is assumed to be continuous, which rules out any distribution with mass points.[2] Sellers cannot coordinate to hold a simultaneous auction, so the items can only be sold one by one in sequential auctions. For now, it is assumed that the seller with zero cost sells the item first.[3] The winning price is disclosed to everyone at the end of the first auction, and the winner leaves. Then the other seller decides whether she will hold the second auction to sell her item. If she decides not to sell, the auction ends. Otherwise, she will have to pay her cost c and then start the second auction with the remaining $n - 1$ bidders. As discussed earlier, it is assumed that there is no reserve price in each auction, so sellers commit to selling to the winner at the winning price. In addition,

[2] The degenerated fixed value case, as a special case, is also ruled out.

[3] Later in the uniform distributions example, the chapter will show that each seller individually benefits by choosing such an order, so the order is actually endogenously determined.

all economic agents are risk neutral. All distribution functions, i.e. $F(v)$,$f(v)$, $G(c)$ and $g(c)$, are public information. Finally, it is pre-announced that either all auctions are conducted in sealed first price auctions or all auctions are conducted in sealed second price auctions. For consistency, it is assumed that sellers cannot switch auction format across auctions.

1.4 The Sealed First Price Sequential Auction

This section studies the case in which sealed first price auctions are used by sellers. If a symmetric pure strategy equilibrium exists, the solution can be derived backwardly. Sufficient conditions for its existence are given in the general case. A simple example is then proceeded with uniform distributions in which explicit solutions can be calculated. With the example, one can learn a few insights on the properties of the equilibrium.

1.4.1 The General Case

First, assume that a symmetric pure strategy equilibrium exists. It is a well-known consequence of incentive compatibility that the bidding functions of bidders must be strictly increasing.[4] At the end of the first auction, the winning price P is revealed. In the equilibrium, everyone can invert the bidding function used in the first auction β^I to work out the type of the winner. Denote the winner's type as v^*, then $v^* = {\beta^I}^{-1}(P)$.[5] Now v^* is the upper bound of the remaining bidders' values, so their distribution is updated with the following density functions: the c.d.f. is $\dfrac{F(v)}{F(v^*)}$ and the p.d.f. becomes $\dfrac{f(v)}{F(v^*)}$, both of them are on the support interval $[0, v^*]$.

If the seller holds the second auction, consider a remaining bidder with value v who might unilaterally deviate from the equilibrium strategy. She essentially faces $n-2$ competitors in a single-unit sealed first price auction. These $n-2$ competitors share the same updated value distribution as described above. Now the textbook results apply, according to Krishna (2009), when $v \leq v^*$ the equilibrium bidding function is

[4]In this paper, "increasing" refers to non-decreasing, but not strictly increasing.

[5]The existence of ${\beta^I}^{-1}(P)$ is ensured by the strict monotonicity of β^I.

$$b^I(v) = \frac{1}{\left(\frac{F(v)}{F(v^*)}\right)^{n-2}} \int_0^v td\left[\left(\frac{F(t)}{F(v^*)}\right)^{n-2}\right] = \int_0^v td\left[\left(\frac{F(t)}{F(v)}\right)^{n-2}\right] \tag{1}$$

Notice it is possible that $v > v^*$, as this bidder may have already deviated by underbidding in the first auction. The bidding strategy must cover such off-equilibrium situation. When it happens, this bidder can guarantee her winning in the second auction by bidding $b^I(v^*)$,[6] so she will not bid anything higher than $b^I(v^*)$. It can also be shown that she will not bid below $b^I(v^*)$ either. By the definition of b^I, the optimal action for a bidder of type v^* is to bid $b^I(v^*)$. This is because when she bids below that amount, in expectation, her utility loss of not winning overweighs the gain from paying less when she wins. When the bidder with $v > v^*$ bids below $b^I(v^*)$, compared to the type v^* bidder, her utility loss of not winning is even larger but the gain from paying less is the same. As a result, her expected utility loss overweighs the gain, and she should not bid below $b^I(v^*)$. In sum, when $v > v^*$, the bidder will just bid $b^I(v^*)$.

Now check the seller's decision. Denote the highest value among bidders in the second auction as Y, then the expected winning price (or the expected revenue of the seller) in the second auction is

$$\begin{aligned} R_2 &= E[b^I(Y)|v^*] \\ &= \int_0^{v^*} b^I(y)d\left[\left(\frac{F(y)}{F(v^*)}\right)^{n-1}\right] \\ &= \int_0^{v^*} \left(\int_0^y tF(t)^{n-3}f(t)dt\right)\frac{(n-1)(n-2)f(y)}{F(v^*)^{n-1}}dy. \end{aligned} \tag{2}$$

It is obvious that R_2 is a function of v^*, which is what one would expect, as the seller's expected revenue in the second auction should depend on the information revealed in the first round. The seller will offer the second auction if and only if $R_2(v^*) \geq c$. Buyers understand it, but without observing the true value of c, they can only hold a belief on the probability of the existence of the second auction as below.

Define the likelihood function for the second auction as

$$l(v^*) \equiv Pr(\text{there is the 2nd acution}|v^*)$$

[6] Actually she may lose if the highest value of other remaining bidders is v^*, but the probability of such event is 0. As a result, it will not affect her expected payoff.

then

$$\begin{aligned} l(v^*) &= Pr(c \leq R_2(v^*)) \\ &= G(R_2(v^*)) \end{aligned} \tag{3}$$

Knowing $b^I(v)$ and $l(v^*)$, the bidding function in the first auction, $\beta^I(v)$, can now be derived. There are some complications because the impacts upon a bidder's expected payoff are quite different when she deviates from $\beta^I(v)$ towards different directions. Consider a bidder with value v who only deviates in the first auction. If she bids as if her value was $z > v$ in the first auction, her value v must be below v^* if she loses in the first round. Her expected payoff is

$$\pi(z, v) = F(z)^{n-1}[v - \beta^I(z)] + Pr(A)[v - b^I(v)], \qquad z \geq v \tag{4}$$

Here $F(z)^{n-1}$ is the probability of this bidder winning the first auction, and A is the event that "the bidder loses the first auction by bidding $\beta^I(z)$, but the seller holds the second auction and this bidder wins it".

However, if the bidder underbids ($z < v$), it is possible that the bidder loses the first auction and the winning value v^* is smaller than her value v. In such a scenario, she will bid $b^I(v^*)$ in the second auction. As a result, her expected payoff is

$$\pi(z, v) = F(z)^{n-1}[v - \beta^I(z)] + Pr(B)[v - b^I(v)] + E[v - b^I(v^*)|C], \qquad z \leq v \tag{5}$$

Here event B is that "the bidder loses the first auction by bidding $\beta^I(z)$, the winning type v^* is higher than v, the seller holds the second auction and this bidder wins it", and event C is that "the bidder loses the first auction by bidding $\beta^I(z)$, the winning type v^* is lower than v, the seller holds the second auction and this bidder wins it". It is obvious that $B \cup C = A$.

In the equilibrium, every bidder should not have incentives to deviate from the bidding function. Mathematically, it is equivalent to

$$v \in \arg\max_z \ \pi(z, v), \tag{6}$$

so the first order necessary conditions for equilibrium are

$$\begin{cases} \text{the right-hand derivative} & \frac{d^+}{dz}\pi(z,v)\big|_{z=v} \le 0 \\ \text{the left-hand derivative} & \frac{d^-}{dz}\pi(z,v)\big|_{z=v} \ge 0 \end{cases} \tag{7}$$

Notice that by coincidence for both equations (4) and (5), $\frac{d\pi(z,v)}{dz}$ is the same when $z = v$.[7] This implies that

$$\left.\frac{d\pi(z,v)}{dz}\right|_{z=v} = 0$$

From the differential equation above, combined with the initial condition that the buyer with zero value must bid zero (i.e. $\beta^I(0) = 0$), the equilibrium bidding function $\beta^I(v)$ can be derived, as shown in the following proposition.

Proposition 1.1. *With sealed first price auctions, if a symmetric pure strategy exists, it is unique. In this equilibrium, the bidding function in the first auction is*

$$\beta^I(v) = \int_0^v \left[l(t)b^I(t) - l(t)t + t\right] d\left[\left(\frac{F(t)}{F(v)}\right)^{n-1}\right], \tag{8}$$

where function $l(t)$ is defined in equations (3) and (2).

The winner in the first auction leaves and the winning price P is disclosed. The winner's value v^* becomes public information, as

$$v^* = {\beta^I}^{-1}(P).$$

Then the second seller (with cost c) will hold the second auction if and only if $R_2(v^*) \ge c$, where function R_2 is defined in equation (2).

If there is the second auction, the bidding function for a remaining bidder with value v is

$$\begin{cases} b^I(v) = \int_0^v t \; d\left[\left(\frac{F(t)}{F(v)}\right)^{n-2}\right] & \textit{if } v < v^* \\ b^I(v^*) & \textit{if } v \ge v^* \end{cases} \tag{9}$$

[7] Shown in the proof of Proposition 1.1 in the appendix.

Proposition 1.1 actually covers two well-known special cases with fixed supplies. First, in a single-unit sealed first price auction $l(v^*)$ is always zero, so the only relevant bidding function is (according to equation (8))

$$\beta^I(v) = \int_0^v t \; d\left[\left(\frac{F(t)}{F(v)}\right)^{n-1}\right], \tag{10}$$

which is indeed the symmetric equilibrium bidding function in this case. Second, if there are two auctions for certain, $l(v^*) = 1$ for all v^*. Hence the first round bidding function becomes

$$\beta^I(v) = \int_0^v b^I(t) \; d\left[\left(\frac{F(t)}{F(v)}\right)^{n-1}\right]. \tag{11}$$

Again, this is exactly the well-established result in the standard two-unit sealed first price sequential auctions.

Proposition 1.1 gives the form of the unique symmetric pure strategy equilibrium if it exists. However, there could be cases where such an equilibrium cannot be established. More specifically, $\beta^I(v)$ is only derived from a necessary first order condition, and it may not satisfy the optimality/non-deviation condition $v \in \arg\max_z \; \pi(z, v)$. Intuitively, the existence of such an equilibrium must depend on the distributions of bidder values and seller cost. The following proposition provides sufficient conditions that ensure the symmetric pure strategy equilibrium exists.

Proposition 1.2. *With sealed first price auctions, the symmetric pure strategy equilibrium prescribed in Proposition (*1.1*) exists if*

1. *bidding function $\beta^I(v)$ given in equation (*8*) is increasing and*

2. *for all (v, z) such that $v < z$,*

$$F^{n-2}(z) - l(z)\left\{F(v)^{n-2}\left[1 - b^{I\prime}(v)\right] + (n-2)F(v)^{n-3}f(v)\left[v - b^I(v)\right]\right\} \geq 0 \tag{12}$$

Note that the first condition is also a necessary condition of the pure strategy equilibrium. It ensures that bidder with higher value bids higher in the auction. ${\beta^I}^{-1}$ exists under this condition, so the winner's value v^* is revealed at the disclosure of the winning price. In classic models with exogenous supply, the first condition alone is sufficient for the existence of the pure strategy equilibrium. In this model, however, the second condition is needed so that bidders' expected payoff function $\pi(z,v)$ satisfies the single crossing condition,[8] which is important to the pure strategy equilibrium. In addition, because the second condition only applies to the $v < z$ case, inequality (12) can be replaced by the following stronger but tidier version:

$$1 - l(z) + b^{I\prime}(v) l(z) - \frac{(n-2)f(v)}{F(z)} \left[v - b^I(v) \right] l(z) \geq 0 \tag{13}$$

1.4.2 The Uniform Distributions Example

Now consider a simple example in which the pure strategy equilibrium exists and the equilibrium bidding functions can be solved in analytic forms. Assume buyer values follow a uniform distribution on $[0,1]$, so $F(v) = v$ and $f(v) = 1$ on $[0,1]$. Assume the second seller's cost c also follows a uniform distribution on its support $[0,C]$, so $G(c) = \frac{c}{C}$ and $g(c) = \frac{1}{C}$ on $[0,C]$.

According to equation (1)

$$\begin{aligned} b^I(v) &= \int_0^v t \; d\left[\left(\frac{t}{v} \right)^{n-2} \right] \\ &= \frac{n-2}{n-1} v \end{aligned} \tag{14}$$

According to equation (2)

$$\begin{aligned} R_2(v^*) &= \int_0^{v^*} \frac{n-2}{n-1} y \; d\left[\left(\frac{y}{v^*} \right)^{n-1} \right] \\ &= \frac{n-2}{n} v^* \end{aligned} \tag{15}$$

According to equation (3)

$$l(v^*) = G(R_2(v^*)) = \begin{cases} \frac{n-2}{nC} v^* & \text{if } \frac{n-2}{nC} v^* \leq 1 \\ 1 & \text{if } \frac{n-2}{nC} v^* > 1 \end{cases} \tag{16}$$

Note that $v^* \leq 1$, so as long as C is sufficiently large (i.e. $C \geq \frac{n-2}{n}$), $\frac{n-2}{nC} v^* \leq 1$ always holds and $l(v^*) = \frac{n-2}{nC} v^*$ for all possible v^*. This case is relatively simple, as $l(v^*)$ is strictly increasing on its entire support. It is studied in the following part.

1.4.2.1 The Case with Sufficiently Large C

[8] This is shown in the proof of Proposition 1.2.

The Equilibrium

Now assume $C \geq \frac{n-2}{n}$, so $l(v^*) = \frac{n-2}{nC}v^*$. According to equation (8),

$$\begin{aligned} \beta^I(v) &= \int_0^v \left[\frac{n-2}{n}\frac{t}{C}\left(\frac{n-2}{n-1}t - t\right) + t\right] d\left[\left(\frac{t}{v}\right)^{n-1}\right] \\ &= \frac{n-1}{n}v - \frac{n-2}{n(n+1)C}v^2 \end{aligned} \tag{17}$$

It is easy to check that $\beta^{I''}(v) = -\frac{2(n-2)}{n(n+1)C} < 0$ and $\beta^{I'}(0) = \frac{n-1}{n} > 0$. In addition,

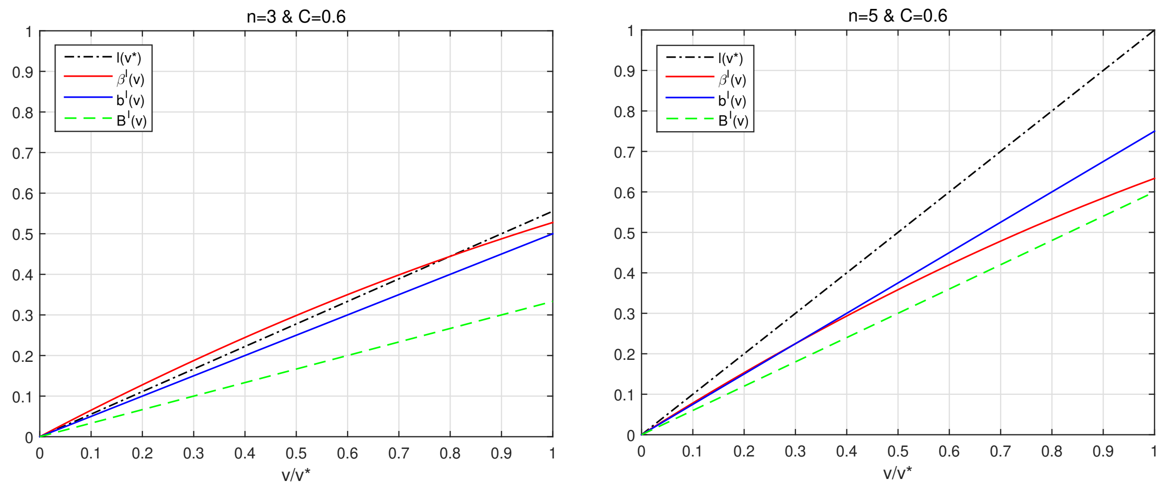
$$\begin{aligned} \beta^{I'}(1) &= \frac{n-1}{n} - \frac{2(n-2)}{n(n+1)C} \\ &\geq \frac{n-1}{n} - \frac{2(n-2)}{n(n+1)}\frac{n}{n-2} \\ &= \frac{(n-1)^2-2}{n(n+1)} \\ &> 0 \end{aligned} \tag{18}$$

The first inequality comes from $C \geq \frac{n-2}{n}$, the assumption that C is sufficiently large. The above implies that $\beta^{I'}(v) > 0$, i.e. the bidding function in the first auction is indeed increasing. Hence the first condition in Proposition 1.2 is satisfied. Now check the second condition. When $v < z$, the left-hand side of inequality (13) now becomes

$$\begin{aligned} &1 - l(z)\left[1 - \frac{n-2}{n-1} + \frac{n-2}{z}(v - \frac{n-2}{n-1}v)\right] \\ >&1 - l(z)\left[1 - \frac{n-2}{n-1} + \frac{n-2}{v}(v - \frac{n-2}{n-1}v)\right] \\ =&1 - l(z) \\ \geq&0 \end{aligned} \tag{19}$$

so the second condition in Proposition 1.2 also holds. As a result, $\beta^I(v)$ given by equation (17) is indeed the equilibrium bidding function in the first auction. One can also directly verify that $v \in \arg\max_z \pi(z, v)$ without applying Proposition (1.2). This alternative approach is provided in the appendix.

In fact, inequality (19) is established without using any property of G(c). In other words, the uniform distribution of the bidder values alone is sufficient to satisfy the second condition in Proposition (1.2).



(a) example 1

(b) example 2

Figure 1: Bidding functions with uniformly distributed buyer values on $[0, 1]$, when C is sufficiently large, in the sealed first price sequential auctions.

Corollary 1. With sealed first price auctions, if bidder values are uniformly distributed, regardless the distribution of seller's cost c (including its upper bound C), the symmetric pure strategy equilibrium exists if the first round bidding function $\beta^I(v)$ given in equation (8) is increasing.

Note that Corollary 1 does not put any restriction upon the distribution of seller's cost c, so it also applies to the case where C is not sufficiently large. Corollary 1 implies that when bidder values are uniformly distributed, the symmetric pure strategy equilibrium existence condition is not different between the endogenous supply model and the fixed supply model.

Figure 1 shows the equilibrium bidding functions in two examples. It is easy to check that in both cases $C = 0.6$ is sufficiently large so that the likelihood function $l(v^*)$ is always strictly increasing. In the figure, $B^I(v)$ represents the equilibrium bidding function in the first auction if there is always a second auction as in a classic sequential auction model. In both examples, $\beta^I(v)$ lies above $B^I(v)$. That is, when the second auction is contingent on the result of the first auction, bidders bid more aggressively. The intuition is that the availability of the second auction is uncertain, so the option value of the second auction is discounted and bidders value the first auction relatively more. One can also observe that $\beta^I(v)$ is concave, i.e. its slope flattens when v increases. This is because a bidder with higher value v would expect a higher winning price on average and hence a higher probability of having the second auction. The option value of the second auction is then discounted less, so the bidder would bid more similar to the case with fixed two auctions.

The Endogenous Order of Sale

Till now, this chapter has assumed that the order of sale is determined as the following: the seller with zero cost moves first and sells her item in the first auction, then the seller with cost c moves after observing the auction result in the first round. In this part, we are going to show that both sellers mutually benefit by following such an order.

If sellers follow the prescribed order of sale, denote $\pi_1^{I,0}$ as the expected payoff of the zero cost seller who moves first and $\pi_2^{I,c}$ as the expected payoff of the seller with cost c who moves second. Denote X as the highest value among all n buyers, then

$$\begin{aligned}\pi_1^{I,0} &= E\left[\beta^I(X)\right] \\ &= \int_0^1 \left(\frac{n-1}{n}x - \frac{n-2}{n(n+1)C}x^2\right) d\left(x^n\right) \\ &= \frac{n-1}{n+1} - \frac{n-2}{(n+1)(n+2)C}\end{aligned} \tag{20}$$

There is zero probability for a seller with $c \geq \frac{n-2}{n}$ to hold the second auction because the highest possible value of v^* is 1 and the highest possible $R_2(v^*)$ equals $\frac{n-2}{n}$. As a result, $\pi_2^{I,c} = 0$ for $c \geq \frac{n-2}{n}$; whilst for a seller with $c < \frac{n-2}{n}$,

$$\begin{aligned}\pi_2^{I,c} &= E\left[R_2(X) - c\mid R_2(X) \geq c\right] \\ &= \int_{\frac{nC}{n-2}}^1 \left(\frac{n-2}{n}x - c\right) d\left(x^n\right) \\ &= \frac{n-2}{n+1} - c + c^{n+1}\left(\frac{n}{n-2}\right)^n \frac{1}{n+1} \\ &> 0\end{aligned} \tag{21}$$

In sum,

$$\pi_2^{I,c} = \begin{cases} \frac{n-2}{n+1} - c + c^{n+1}\left(\frac{n}{n-2}\right)^n \frac{1}{n+1} > 0, & \text{if } c < \frac{n-2}{n} \\ 0, & \text{if } c \geq \frac{n-2}{n} \end{cases} \tag{22}$$

Now consider the case where sellers instead choose the alternative order of sale. That is, the seller with cost c moves first and the zero cost seller moves second. If the seller with cost c holds the first auction, buyers know for sure that there will be a second auction as the other seller bears zero cost and will certainly benefit from trade. As a result, $l(v^*) = 1$ for all v^* and the first round bidding function becomes

$$\begin{aligned} \beta^I(v) &= \int_0^v b^I(t)\, d\left[\left(\frac{F(t)}{F(v)}\right)^{n-1}\right] \\ &= \int_0^v \frac{n-2}{n-1} t\, d\left[\left(\frac{t}{v}\right)^{n-1}\right] \\ &= \frac{n-2}{n} v \end{aligned} \tag{23}$$

Hence the expected revenue of this seller from the first auction is

$$E\left[\frac{n-2}{n}X\right] = \int_0^1 \frac{n-2}{n} x\, d\left(x^n\right) = \frac{n-2}{n+1} \tag{24}$$

Consequently, the seller shall offer the first auction only if $c \leq \frac{n-2}{n+1}$. When this happens, the expected payoff of the zero cost seller from the second auction is

$$\begin{aligned} E\left[R_2(X)\right] &= \int_0^1 \frac{n-2}{n} x\, d\left(x^n\right) \\ &= \frac{n-2}{n+1} \end{aligned} \tag{25}$$

If $c > \frac{n-2}{n+1}$, the seller with such cost will not hold the auction even if she moves first. And everyone shall observe that there is no auction in the first round. Then the seller with zero cost holds an auction in the second round, and her expected payoff is simply that of a standard single-unit auction with n bidders. It is $\frac{n-1}{n+1}$. Without observing the true value of c, the zero cost seller believes that $c > \frac{n-2}{n+1}$ occurs with probability $1 - \frac{n-2}{(n+1)C}$, so her expected payoff when she moves secondly, denoted as $\pi_2^{I,0}$, is

$$\begin{aligned} \pi_2^{I,0} &= \frac{n-2}{n+1}\frac{n-2}{(n+1)C} + \frac{n-1}{n+1}\left[1 - \frac{n-2}{(n+1)C}\right] \\ &= \frac{n^2C - n - C + 2}{C(n+1)^2} \end{aligned} \tag{26}$$

One can check that

$$\begin{aligned} \pi_1^{I,0} - \pi_2^{I,0} &= \left[\frac{n-1}{n+1} - \frac{n-2}{(n+1)(n+2)C}\right] - \frac{n^2C - n - C + 2}{C(n+1)^2} \\ &= \frac{n-2}{(n+1)^2(n+2)C} \\ &> 0 \end{aligned} \tag{27}$$

This implies that the seller with zero cost strictly prefers moving first.

As for the seller with cost c, if she moves first, as discussed above, her expected payoff is

$$\pi_1^{I,c} = \begin{cases} \frac{n-2}{n+1} - c, & \text{if } c \leq \frac{n-2}{n+1} \\ 0, & \text{if } c > \frac{n-2}{n+1} \end{cases} \tag{28}$$

Compare $\pi_1^{I,c}$ above to $\pi_2^{I,c}$ given in equation (22), it is apparent that $\pi_2^{I,c} = \pi_1^{I,c} = 0$ when $c \geq \frac{n-2}{n}$, but $\pi_2^{I,c} > \pi_1^{I,c}$ when $c < \frac{n-2}{n}$. Hence for the seller with cost c, moving secondly weakly dominates moving first.

In conclusion, both sellers prefer such an order that the zero cost seller sells her item in the first auction and then the other seller has the option to sell her item in the second round. As a result, the order of sale needs not to be exogenously given. Instead, sellers should be able to coordinate and endogenously reach an agreement upon the order of sale.

1.4.2.2 The Case with Small C

The Equilibrium

Now consider the case where the upper bound of cost, C, is not sufficiently large. That is, $C < \frac{n-2}{n}$. In this situation, the first round bidding function is slightly more complicated, since the likelihood function has two stages as shown below.

$$l(v^*) = \begin{cases} \frac{n-2}{nC} v^*, & \text{if } v^* < \frac{n}{n-2} C \\ 1, & \text{if } v^* \geq \frac{n}{n-2} C \end{cases} \tag{29}$$

As a result, when $v < \frac{n}{n-2} C$,

$$\begin{aligned} \beta^I(v) &= \int_0^v \left[\frac{n-2}{n} \frac{t}{C} \left(\frac{n-2}{n-1} t - t \right) + t \right] d \left[\left(\frac{t}{v} \right)^{n-1} \right] \\ &= \frac{n-1}{n} v - \frac{n-2}{n(n+1)C} v^2 \end{aligned} \tag{30}$$

which is the same as equation (17). It has been shown earlier that this bidding function is increasing.

When $v \geq \frac{n}{n-2}C$, according to equation (8),

$$\begin{aligned}\beta^I(v) &= \int_0^{\frac{n}{n-2}C} \left[\frac{n-2}{n}\frac{t}{C}\left(\frac{n-2}{n-1}t - t\right) + t\right] d\left[\left(\frac{t}{v}\right)^{n-1}\right] + \int_{\frac{n}{n-2}C}^{v} \frac{n-2}{n-1} t d\left[\left(\frac{t}{v}\right)^{n-1}\right] \\ &= \frac{1}{n(n+1)}\left(\frac{n}{n-2}C\right)^n \frac{1}{v^{n-1}} + \frac{n-2}{n} v\end{aligned} \tag{31}$$

In this case, it is easy to check that $\beta^{I\prime}(v) > 0$ as long as $n \geq 3$, as

$$\begin{aligned}\beta^{I\prime}(v) &= \frac{n-2}{n} - \frac{n-1}{n(n+1)}\left(\frac{n}{n-2}C\right)^n \frac{1}{v^n} \\ &\geq \frac{n-2}{n} - \frac{n-1}{n(n+1)}\left(\frac{n}{n-2}C\right)^n \frac{1}{(\frac{n}{n-2}C)^n} \\ &= \frac{(n-1)^2 - 2}{n(n+1)} \\ &> 0\end{aligned} \tag{32}$$

In sum, when $C < \frac{n-2}{n}$, the first round bidding function is

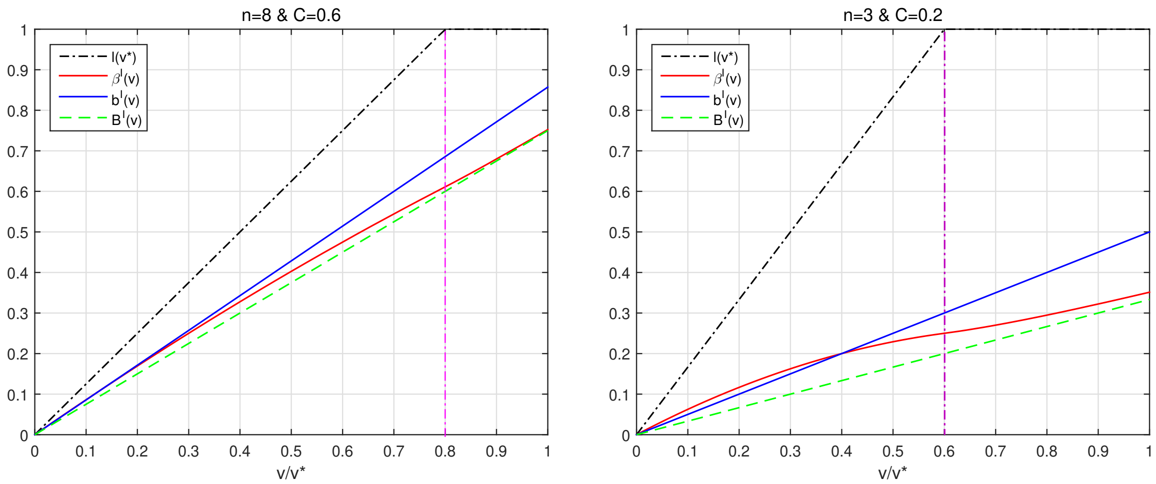
$$\beta^I(v) = \begin{cases} \frac{n-1}{n}v - \frac{n-2}{n(n+1)C}v^2, & \text{if } v < \frac{n}{n-2}C \\ \frac{1}{n(n+1)}\left(\frac{n}{n-2}C\right)^n \frac{1}{v^{n-1}} + \frac{n-2}{n}v, & \text{if } v \geq \frac{n}{n-2}C \end{cases} \tag{33}$$

and $\beta^{I\prime}(v) > 0$ for all $v \in [0, 1]$.

Apply Corollary 1, the symmetric pure strategy equilibrium exists and equation (33) is indeed the equilibrium bidding function. Also, one can directly check $\beta^I(v)$ given in equation (33) leads to an equilibrium without applying Corollary 1. This alternative approach is provided in the appendix.

Figure 2 shows the equilibrium bidding functions in two examples with small C. The likelihood function $l(v^*)$ becomes 1 when v^* is above 0.8 and 0.6 in example 1 and 2, respectively. In the figure, $B^I(v)$ represents the equilibrium bidding function in the first auction if there is always a second auction as in a classic sequential auction model. Again, in both examples, $\beta^I(v)$ lies above $B^I(v)$; and $\beta^I(v)$ gets closer to $B^I(v)$ when v increases. The same intuition applies as in the case with sufficiently large C. Figure 2b shows clearly that bidding behaviours in the first auction changes when $l(v^*)$ reaches 1. This threshold is the inflection point of $\beta^I(v)$, below which $\beta^I(v)$ is concave and beyond which $\beta^I(v)$ is convex.

The Endogenous Order of Sale



(a) example 1

(b) example 2

Figure 2: Bidding functions with each buyer's value independent uniformly distributed on $[0,1]$, when C is small, in the sealed first price sequential auctions.

This part of the paper demonstrates that when C is not sufficiently large ($C < \frac{n-2}{n}$), both sellers also prefer the following order of sale: the zero cost seller moves first and the seller of cost c moves secondly.

First, assume both sellers stick to the prescribed order. Because $c \leq C < \frac{n-2}{n}$, no matter what value c takes, it is always possible for the seller with cost c to participate in the second auction. The condition is simply $R_2(v^*) = \frac{n-2}{n-1}v^* > c$, or equivalently $v^* > \frac{n-1}{n-2}c$. Such v^* always exists, as $\frac{n-1}{n-2}c < \frac{n-1}{n} < 1$. As a result, the expected payoff of the seller with cost c if she moves secondly is

$$\begin{aligned}
\pi_2^{I,c} &= E\left[R_2(X) - c\,|R_2(X) \geq c\right] \\
&= \int_{\frac{nC}{n-2}}^{1} \left(\frac{n-2}{n}x - c\right) d\left(x^n\right) \\
&= \frac{n-2}{n+1} - c + c^{n+1}\left(\frac{n}{n-2}\right)^n \frac{1}{n+1} \\
&> 0
\end{aligned} \tag{34}$$

and the expected payoff of the zero cost seller if she moves first is

$$
\begin{aligned}
\pi_1^{I,0} &= E\left[\beta^I(X)\right] \\
&= \int_0^{\frac{n}{n-2}C} \left[\frac{n-1}{n}x - \frac{n-2}{n(n+1)C}x^2\right] d(x^n) \\
&\quad + \int_{\frac{n}{n-2}C}^{1} \left[\frac{n-2}{n}x + \frac{1}{n(n+1)}\left(\frac{n}{n-2}C\right)^n \frac{1}{x^{n-1}}\right] d(x^n) \\
&= \frac{n-2}{n+1} - \frac{n}{(n+1)(n+2)}\left(\frac{n}{n-2}C\right)^{n+1} + \frac{1}{n+1}\left(\frac{n}{n-2}C\right)^n
\end{aligned}
\tag{35}
$$

Here $\beta^I(v)$ follows equation (33).

Now study the case where sellers choose the alternative order of sale, i.e. the seller with cost c moves first and the seller with zero cost moves secondly. It the seller with cost c holds the auction, her expected revenue is $\frac{n-2}{n+1}$, as calculated in equation (24). If $c < \frac{n-2}{n+1}$, she will abstain from the auction and get zero payoff; if $c \geq \frac{n-2}{n+1}$, she will hold the first auction and obtain $\frac{n-2}{n+1} - c$ in expectation.

Now there are two scenarios.

(1) If $\mathbf{C \leq \frac{n-2}{n+1}}$

First, if $C \leq \frac{n-2}{n+1}$, then c is always smaller than $\frac{n-2}{n+1}$. The seller with cost c always holds the first auction. As a result,

$$
\pi_1^{I,c} = \frac{n-2}{n+1} - c, \qquad \text{for all } c \tag{36}
$$

and

$$
\pi_2^{I,0} = \frac{n-2}{n+1} \tag{37}
$$

which has been calculated in equation (25).

Note that[9]

$$
\begin{aligned}
\pi_1^{I,0} - \pi_2^{I,0} &= \frac{1}{n+1}\left(\frac{n}{n-2}C\right)^n - \frac{n}{(n+1)(n+2)}\left(\frac{n}{n-2}C\right)^{n+1} \\
&= \frac{1}{n+1}\left(\frac{n}{n-2}C\right)^n \left(1 - \frac{n^2}{(n-2)(n+2)}C\right) \\
&> \frac{1}{n+1}\left(\frac{n}{n-2}C\right)^n \left(1 - \frac{n}{n+2}\right) \\
&> 0
\end{aligned}
\tag{38}
$$

[9]The first inequality below is due to $C < \frac{n-2}{n}$.

In other words, the seller with zero cost strictly prefers moving first. In addition, it is obvious that $\pi_2^{I,c} > \pi_1^{I,c}$, and the gap is $c^{n+1}\left(\frac{n}{n-2}\right)^n \frac{1}{n+1}$. This implies that the seller with cost c strictly prefers moving secondly. In sum, both sellers would agree that the zero cost seller moves first and the other seller moves next, if $C \leq \frac{n-2}{n+1}$.

(2) If $\mathbf{\frac{n-2}{n+1} < C < \frac{n-2}{n}}$

In the second scenario, i $\frac{n-2}{n+1} < C < \frac{n-2}{n}$, it is possible that $c > \frac{n-2}{n+1}$ and the seller with such cost will not hold the first auction. Now the expected payoff to the seller with cost c, if she moves first, is the same as in equation (28). Without observing the true value of c, the zero cost seller believes that the first mover holds the auction at probability $\frac{n-2}{n+1}$. Hence, the expected payoff of the zero cost seller, if she moves second, should be the same as in equation (26).

Obviously $\pi_2^{I,c} > \pi_1^{I,c}$ for all c. Now compare $\pi_1^{I,0}$ and $\pi_2^{I,0}$.

$$\pi_1^{I,0} - \pi_2^{I,0} = \frac{1}{n+1}\left(\frac{n}{n-2}C\right)^n - \frac{n}{(n+1)(n+2)}\left(\frac{n}{n-2}C\right)^{n+1} - \frac{n^2C - n - C + 2}{C(n+1)^2} \tag{39}$$

It can be shown that RHS of equation (39) is strictly positive given that $\frac{n-2}{n+1} < C < \frac{n-2}{n}$ and $n \geq 3$, so the seller with zero cost strictly prefers moving first. As a result, if $\frac{n-2}{n+1} < C < \frac{n-2}{n}$, both sellers also strictly prefer the prescribed order of sale.

Summarizing the two scenarios above, it can be concluded that when C is not sufficiently large (i.e. $C < \frac{n-2}{n}$), both sellers also prefers the same order of sale. They will endogenously coordinate so that the zero cost seller moves first and the seller with higher cost moves secondly.

1.4.2.3 "Declining Price Anomaly" And Other Equilibrium Properties

Declining Price Anomaly

Many empirical studies have observed that the winning prices decline from round to round in sequential auctions. Such a phenomenon is called "Declining Price Anomaly" because the classic sequential auction theory predicts that the price path should be a martingale. In this section, with the uniform distributions example, this paper shows that a declining price path is exactly what one should expect. In the model with endogenous supply, the phenomenon is no longer an "anomaly but instead a property of the equilibrium.

First, check the case where $\mathbf{C \geq \frac{n-2}{n}}$. If there are two auction rounds, given the winning price P_1 in the first auction, one can calculate the expected price in the second auction round, $E[P_2|P_1]$. Because in the equilibrium, the first round bidding function $\beta^I(v)$ is invertible, knowing P_1 is equivalent to knowing the winner's type v^*. Hence,

$$\begin{aligned} E[P_2|P_1] &= E[P_2|v^*] \\ &= R_2(v^*) \\ &= \frac{n-2}{n}v^* \end{aligned} \tag{40}$$

According to $\beta^I(v)$ given in equation (17),

$$P_1 = \beta^I(v^*) = \frac{n-1}{n}v^* - \frac{n-2}{n(n+1)C}v^{*2} \tag{41}$$

and as long as $v^* \neq 0$,

$$\begin{aligned} P_1 - E[P_2|P_1] &= \frac{1}{n}v^* - \frac{n-2}{n(n+1)C}v^{*2} \\ &\geq \frac{1}{n}v^* - \frac{n-2}{n(n+1)}\frac{n}{n-2}v^{*2} \\ &= v^*\left(\frac{1}{n} - \frac{1}{n+1}v^*\right) \\ &> 0 \end{aligned} \tag{42}$$

That is, $E[P_2|P_1] < P1$. The equilibrium price path is not a martingale, instead, it has a declining trend.

In fact, a much stronger result can be obtained if C is large enough. If $C > \frac{(n-1)(n-2)}{n+1}$, one can check that for all v,

$$\begin{aligned} \beta^I(v) - b^I(v) &= \left[\frac{n-1}{n}v - \frac{n-2}{n(n+1)C}v^2\right] - \frac{n-2}{n-1}v \\ &> 0 \end{aligned} \tag{43}$$

In the equilibrium, the winner's value in the first auction v^* is always above that of the winner in the second round v^{**}. As a result,

$$P_1 = \beta^I(v^*) > b^I(v^*) \geq b^I(v^{**}) = P_2 \tag{44}$$

Note that equation (44) is stronger than equation (42). Equation (42) predicts that prices decline only in expectation, but it is still possible that some realizations can lead to an increasing price path. In contrast, equation (44) ensures that the winning price in the second round is always strictly lower than the price in the first auction.

Figure 1a illustrates such a case. In this example, $C = 0.6 > \frac{(n-1)(n-2)}{n+1}$, so $\beta^I(v)$ entirely lies above $b^I(v)$. In contrast, Figure 1b shows the case where $\beta^I(v)$ can be below $b^I(v)$.

Now check the case where $\mathbf{C < \frac{n-2}{n}}$. Denote the winner's value in the first auction as v^*, equation (40) still holds. $\beta^I(v)$ follows equation (33).

If $v^* \geq \frac{n}{n-2}C$, then

$$\begin{aligned} P_1 &= \frac{1}{n(n+1)}\left(\frac{n}{n-2}C\right)^n \frac{1}{v^{*n-1}} + \frac{n-2}{n}v^* \\ &> \frac{n-2}{n}v^* \\ &= E[P_2|P_1] \end{aligned} \tag{45}$$

If $v^* < \frac{n}{n-2}$, then $P_1 = \frac{n-1}{n}v^* - \frac{n-2}{n(n+1)C}v^{*2}$. Hence, inequality (42) holds. That is, $P_1 > E[P_2|P_1]$.

In sum, when $C < \frac{n-2}{n}$, the endogenous supply also predicts a declining price trend in expectation.

The intuition of the declining price trend lies within the uncertainty of the existence of the second auction. If the supply is fixed and there are two auctions for sure, the no-arbitrage opportunity principle applies so the prices across different rounds should be the same in expectation. This is because the second auction has an "option value" when buyers compete in the first round, so they shade their bids in the first round taking into account of such "option value" they must forgo if they win. As a result, each bidder bids lower in the first round, but the one with the highest value leaves the market in the second round. These two effects affect the intensity of competition and balance each other in the equilibrium. The consequence is that in expectation the winning price should be the same across different rounds. However, when sellers can choose whether or not to offer the next auction, the supply becomes uncertain. Buyers are not sure if there is going to be a second round. As a result, the "option value" of the second auction is discounted by the probability that the second seller does not hold the second auction. Consequently, in the first auction, they will shade their bids not as much as they would do in the fixed supply case. On the other hand, the winner of the first auction will still leave the market in the second auction, so the balance between the two effects is broken. Now the competition in the first round becomes more intense, so a higher price on average should be expected.

This intuition is similar to the work of Jeitschko (1999), who has also pointed out that the uncertainty of supply can explain the "Declining Price Anomaly". However, Jeitschko assumes that the uncertainty is exogenous and sellers are not strategic. This paper takes an important step further by allowing sellers to make use of the information revealed in the sequence of auctions. The supply becomes endogenous and its uncertainty is well modelled. It depends on their belief on the cost of sellers and the bid of buyers. For example, if a bidder bids high in the first auction, she would expect that the likelihood of the second auction to be high as well.

Other Equilibrium Properties

Now it is a good time to review the equilibrium property on the order of sale. It has been concluded in an earlier section that both sellers are better off with the following order: the zero cost seller moves first and then moves the seller with cost c. This result should not be surprising.

There is an advantage for the seller with a positive cost to move second, as she can get a better estimation of buyer values by observing the winning price in the first auction. With the extra information, she will be able to make a wiser choice on whether or not to sell her item to reduce the chance of loss. However, such information is valueless to the zero cost seller, because she always prefers sale.

On the other hand, the zero cost seller benefits by selling her item in the first auction. This is because bidders are not sure if there is going to be a second auction and they bid more aggressively in the first round. This is the same intuition for the declining price path. However, if the seller with cost c moves first, bidders know that the zero cost seller must hold the second auction, so there is no longer any uncertainty with the supply. As a result, the advantage of selling first vanishes, and the seller with cost c cannot benefit from it.

In sum, the first auction provides an advantage to the zero cost seller exclusively, whilst moving secondly only benefit the seller with a positive cost. Consequently, both of them will take their preferred position without any conflict.

Another interesting observation is that in the equilibrium the expected payoff of the zero cost seller, $\pi_1^{I,0}$, depends on C but not on c, while the expected payoff of the seller with cost c, $\pi_2^{I,c}$, depends on c but not on C. More specifically, it is easy to check that $\pi_1^{I,0}$ is increasing in C, and $\pi_2^{I,c}$ is decreasing in c.

Note that buyers do not know c while they bid in the first auction, it makes sense that the payoff of the seller in the first auction (with zero cost) is unaffected by c. On the other hand, buyers guide their bids with their belief up the distribution of c, so C matters. If C is higher, they would think that the likelihood of having the second auction is lower. Consequently, they will compete more aggressively in the first auction, so the expected payoff of the seller will be higher.

Once the second auction is offered, bidders compete as in a standard single-unit sealed first price auction, so neither C nor c affects their competition. However, a higher c implies less chance for the seller with such cost to make a profit and a higher chance for her to not hold the second auction at all. As a result, her expected payoff is decreasing in c.

1.4.2.4 Non-Existence of Symmetric Pure Strategy Equilibrium When Cost Is Public Information

A symmetric pure strategy equilibrium does not always exist if sellers are allowed to make supply choice based on the information revealed in earlier rounds of sequential auctions. This paper earlier has given the sufficient conditions that support the symmetric pure strategy equilibrium. In this subsection, an example is presented in which symmetric pure strategy equilibrium does not exist.

Continuing the case with uniform distributions, assume that the distribution of c degenerates to a single point. In other words, assume c is public information. Now assume that a symmetric pure strategy equilibrium exists. In this case, buyers will be able to tell the seller's decision after observing the result of the first auction. There would a threshold price P^* (and equivalently a threshold winner value x^*) such that the seller will offer the second auction if the winning price in the first auction exceeds P^*.

Mathematically, the bidding function in the second round is still $b^I(v) = \frac{n-2}{n-1}v$, and the expected revenue from the second auction is still $R_2(x^*) = \frac{n-2}{n}x^*$. As a result, the likelihood function $l(x^*)$ is not continuous, and it becomes:

$$l(x^*) = \begin{cases} 0, & \text{if } x^* < \frac{n}{n-2}c \\ 1, & \text{if } x^* \geq \frac{n}{n-2}c \end{cases} \tag{46}$$

Here $\frac{n}{n-2}c$ is the threshold winner value. If the symmetric pure strategy equilibrium exists, it is easy to check that the bidding function in the first auction must have two stages.

If $v < \frac{n}{n-2}c$, then

$$\begin{aligned} \beta^I(v) &= \int_0^v t \; d\left[\left(\frac{t}{v}\right)^{n-1}\right] \\ &= \frac{n-1}{n}v \end{aligned} \tag{47}$$

If $v \geq \frac{n}{n-2}c$, then

$$\begin{aligned} \beta^I(v) &= \int_0^{\frac{n-2}{n}c} t \; d\left[\left(\frac{t}{v}\right)^{n-1}\right] + \int_{\frac{n-2}{n}c}^{v} b(t) \; d\left[\left(\frac{t}{v}\right)^{n-1}\right] \\ &= \frac{n-2}{n}v + \frac{1}{n}\left(\frac{n}{n-2}c\right)^n \frac{1}{v^{n-1}} \end{aligned} \tag{48}$$

A necessary condition for equilibrium is that the bidding function must be increasing.[10] However, this condition is violated when v is larger than and close enough to $\frac{n}{n-2}c$, as

$$\begin{aligned} \beta^{I\prime}(\frac{n}{n-2}c) &= \frac{n-2}{n} - \frac{n-1}{n}\left(\frac{n}{n-2}c\right)^n \frac{1}{\left(\frac{n}{n-2}c\right)^n} \\ &= -\frac{1}{n} \\ &< 0 \end{aligned} \tag{49}$$

This implies that there does not exist any symmetric pure strategy equilibrium when c is publicly known.

What causes the problem with a symmetric pure strategy equilibrium in this case? A short answer is a discontinuity or a jump of the likelihood function at the threshold value. Consider a bidder with value v in the first auction. If v is below the threshold, she is not able to alter the likelihood of the second auction by marginally deviating from the equilibrium bidding function. As a result, her bid only matters if she wins; and if she wins, there certainly will not be the second round. Hence, she will simply bid as if there is just one auction at all. This is confirmed by equation (47). On the other hand, if v is just above the threshold, there will always be a second auction if this bidder deviates only marginally. As a result, she needs to consider the "option value" of the second round and shade her bid accordingly. There is a contrast around the threshold value: just below the threshold, bidders bid more aggressively as if there is only one auction, whilst above the threshold bidders shade their bids because of the availability of the second auction. Consequently, the sudden change across the threshold creates a problem, so the pure strategy equilibrium no longer exists.

[10] In this paper, "increasing" refers to non-decreasing, but not strictly increasing.

1.5 The Sealed Second Price Sequential Auction

Sealed second price auction is another popular auction format. When buyer values are independent, it is outcome equivalent to English auction (ascending auction). In this section, the paper studies the case where each auction is conducted as a sealed second price auction. Like the section with sealed first price auctions, sufficient conditions for the existence of a symmetric pure strategy equilibrium are given. If such an equilibrium exists, its general form is derived. In addition, simple examples with uniform distributions are used to gain more insights into the model.

1.5.1 The General Case

Assume that a symmetric pure strategy equilibrium exists, it can be derived by backward induction. If there is a second auction, it is a dominant strategy for every bidder to bid her true value, so the bidding function in the second round is simply $b^{II}(v) = v$ for all v.

In a symmetric pure strategy equilibrium, bidding functions of bidders must be increasing. Denote the equilibrium bidding function in the first auction as $\beta^{II}(v)$, then ${\beta^{II}}^{-1}(v)$ exists due to the monotonicity. As a result, once the winning price P_1 is announced, everyone knows the highest value among the remaining $n-1$ bidders in the second round as $v^* = {\beta^{II}}^{-1}(P_1)$. Note that it is different from the sealed first price auction case, where v^* was the upper bound of the highest value among the remaining bidders. The distribution for the rest of the $n-2$ bidder values[11] is now updated with the following density functions: the c.d.f. is $\frac{F(v)}{F(v^*)}$ and the p.d.f. becomes $\frac{f(v)}{F(v^*)}$, both of them on the support interval $[0, v^*]$.

Now check the seller's decision. Denote the second highest value among bidders in the second auction as Y_2, then the expected winning price (or the expected revenue of the seller) in the second auction is

$$\begin{aligned} R_2^{II}(v^*) &= E[b^{II}(Y_2)|v^*] \\ &= \int_0^{v^*} y \; d\left[\left[\frac{F(y)}{F(v^*)}\right]^{n-2}\right] \end{aligned} \tag{50}$$

R_2^{II} is a function of v^*, as seller's expected revenue in the second auction should depend on

[11] The winner in the first auction has left, and the highest remaining value is v^*, so there are only $n-2$ publicly unknown values.

the information revealed in the first round. The seller will offer the second auction if and only if $R_2^{II}(v^*) \geq c$. Buyers understand it, but without observing the true value of c, they can only hold a belief on the probability of the existence of the second auction as discussed below.

Define the likelihood function for the second auction as

$$l^{II}(v^*) \equiv Pr(\text{there is the 2nd acution}|v^*)$$

then

$$\begin{aligned} l^{II}(v^*) &= Pr(c \leq R_2^{II}(v^*)) \\ &= G(R_2^{II}(v^*)) \end{aligned} \tag{51}$$

Consider a bidder with value v in the first auction. Denote $\pi(z, v)$ as her expected payoff if she pretends her value to be z. The equilibrium requires that $v \in \arg\max_z \pi(z, v)$. Similar to the case with sealed first price auctions, the first round equilibrium bidding function can be derived from the following necessary first-order condition

$$\left.\frac{d\pi(z, v)}{dz}\right|_{z=v} = 0 \tag{52}$$

The equilibrium bidding function $\beta^{II}(v)$ can be solved for from the differential equation above, as shown in the following proposition.

Proposition 1.3. *With sealed second price auctions, if a symmetric pure strategy exists, it is unique. In this equilibrium, the bidding function in the first auction is*

$$\beta^{II}(v) = v + \left[l^{II'}(v)\left(1 - F(v)\right) - l^{II}(v)f(v)\right]\left[\frac{v}{f(v)} - \frac{1}{f(v)}\int_0^v t\ d\left[\left[\frac{F(t)}{F(v)}\right]^{n-2}\right]\right], \tag{53}$$

where function $l^{II}(v)$ is defined in equations (51) and (50).

The winner in the first auction leaves and the winning price P_1 is disclosed. v^*, the highest value among the $n-1$ remaining bidders, becomes public information, as

$$v^* = \beta^{II^{-1}}(P_1).$$

Then the second seller (with cost c) will hold the second auction if and only if $R_2^{II}(v^*) \geq c$, where function R_2^{II} is defined in equation (50).

If there is the second auction, the bidding function for a remaining bidder with value v is

$$b^{II}(v) = v, \qquad \text{for all } v. \tag{54}$$

Proposition 1.3 actually covers two well-known special cases with fixed supplies. First, in a single-unit sealed first price auction $l^{II}(v^*)$ is always zero and so is $l^{II'}(v^*)$. As a result, the only relevant bidding function is (according to equation (53))

$$\beta^{II}(v) = v, \tag{55}$$

which is indeed the symmetric equilibrium bidding function in this case. Second, if there are two auctions for certain, $l^{II}(v^*) = 1$ and $l^{II'}(v^*) = 0$ for all v^*. Hence the first round bidding function becomes

$$\beta^{II}(v) = \int_0^v t \; d\left[\left[\frac{F(t)}{F(v)}\right]^{n-2}\right]. \tag{56}$$

Again, this is exactly the well-established result in the standard two-unit sealed first price sequential auctions.

Proposition 1.3 gives the form of the unique symmetric pure strategy equilibrium if it exists. However, there could be cases where such an equilibrium cannot be established. More specifically, $\beta^{II}(v)$ is only derived from a necessary first order condition, and it may not satisfy the optimality/non-deviation condition $v \in \arg\max_z \; \pi(z, v)$. Intuitively, the existence of such an equilibrium must depend on the distributions of bidder values and seller cost. The following proposition provides sufficient conditions that ensure the symmetric pure strategy equilibrium exists.

Proposition 1.4. *With sealed second price auctions, the symmetric pure strategy equilibrium prescribed in Proposition (1.3) exists if*

1. *bidding function $\beta^{II}(v)$ given in equation (53) is increasing and*

2. *for all v,*

$$l^{II'}(v)\,[1 - F(v)] + f(v) - l^{II}(v)F(v) \geq 0. \tag{57}$$

Note that the first condition is also a necessary condition for a pure strategy equilibrium. It ensures that bidder with higher value bids higher in the first auction. Similar to the case with sealed first price auctions, the second condition ensures that bidders' expected payoff function $\pi(z, v)$ satisfies the single crossing condition, which is important to the existence of pure strategy equilibrium. In addition, because inequality (57) only involves the distribution of bidder values, the second condition restricts the distribution of v alone. However, the distribution of c is still important, as it enters the calculation of the first round bidding function $\beta^{II}(v)$.

1.5.2 The Uniform Distributions Example

Assume buyer values follow a uniform distribution on $[0, 1]$, so $F(v) = v$ and $f(v) = 1$ on $[0, 1]$. Assume the second seller's cost c also follows a uniform distribution on its support $[0, C]$, so $G(c) = \frac{c}{C}$ and $g(c) = \frac{1}{C}$ on $[0, C]$.

According to equation (50),

$$\begin{aligned} R_2^{II}(v^*) &== \int_0^{v^*} y \; d\left[\left(\frac{y}{v^*}\right)^{n-2}\right] \\ &= \frac{n-2}{n-1} v^*. \end{aligned} \tag{58}$$

According to equation (51),

$$l^{II}(v^*) = G(R_2^{II}(v^*)) = \begin{cases} \frac{(n-2)v^*}{(n-1))C} & \text{if } \frac{(n-2)v^*}{(n-1))C} \leq 1 \\ 1 & \text{if } \frac{(n-2)v^*}{(n-1))C} > 1 \end{cases}. \tag{59}$$

1.5.2.1 The Case with Sufficiently Large C

Note that $v^* \leq 1$, so as long as C is sufficiently large (i.e. $C \geq \frac{n-2}{n-1}$), $\frac{(n-2)v^*}{(n-1))C} \leq 1$ always holds and $l^{II}(v^*) = \frac{(n-2)v^*}{(n-1))C}$ for all possible v^*. It can be shown that in this case the symmetric pure strategy equilibrium exists.

The Equilibrium

Assuming $C \geq \frac{n-2}{n-1}$, according to equation (53),

$$\begin{aligned}\beta^{II}(v) &= v + \left[\frac{n-2}{(n-1)C}(1-v) - \frac{(n-2)v}{(n-1)C}\right]\left[v - \int_0^v t\ d\left(\left(\frac{t}{v}\right)^{n-2}\right)\right] \\ &= \left[1 + \frac{n-2}{(n-1)^2C}\right]v - \frac{2(n-2)}{(n-1)^2C}v^2.\end{aligned} \tag{60}$$

$\beta^{II}(v)$ needs to be increasing to be an equilibrium bidding function. Note that because $v \leq 1$ and $C \geq \frac{n-2}{n-1}$,

$$\begin{aligned}\beta^{II\prime}(v) &= 1 + \frac{n-2}{(n-1)^2C} - \frac{4(n-2)}{(n-1)^2C}v \\ &\geq \frac{n-4}{n-1}.\end{aligned} \tag{61}$$

As a result, $\beta^{II}(v)$ is increasing if $n \geq 4$.

What happens when $n = 3$? One can check that $\beta^{II\prime}(v) < 0$ when C is close to $\frac{n-2}{n-1}$ and v is close to 1. It implies that $\beta^{II}(v)$ given in equation (60) is not an equilibrium bidding function, and there does not exist a symmetric pure strategy equilibrium in this case. Intuitively, when the number of bidders is too small, there will be little competition in the second auction. At the same time, when C is not too large, the chance of having the second auction is relatively large. Hence, the option value of the second round is relatively high, so it will significantly affect bidding behaviours in the first round. In the first round, bidders shade their bids taking into account of the option value of the possible second auction. Importantly, bidders shade their bids by different proportions. Bidders with low values shade smaller proportions as they reckon the likelihood of having the second round is low. In contrast, bidders with high values have incentives to shade more as they believe the chance of having the second round is high. This contrast causes the problem because high-value bidders would like to shade their bid relatively more and may result in a breaking down of the monotonicity of the bidding function.

Next, it is easy to check that the second condition in Proposition 1.4 is satisfied, as the left-hand side of inequality (57) becomes

$$\begin{aligned}&l^{II\prime}(v)\,[1-v] + 1 - l^{II}(v)v \\ =&\left[1 - l^{II}(v)v\right] + l^{II\prime}(v)(1-v) \\ \geq&0.\end{aligned} \tag{62}$$

In conclusion, applying Proposition 1.4, when $n \geq 4$, $\beta^{II}(v)$ is indeed the equilibrium bidding function. One can also directly verify that $v \in \arg\max_z \ \pi(z, v)$ without applying Proposition (1.2). This alternative approach is provided in the appendix.

In fact, inequality (62) is established without using any property of G(c). In other words, the uniform distribution of bidder values alone is sufficient to satisfy the second condition in Proposition (1.4).

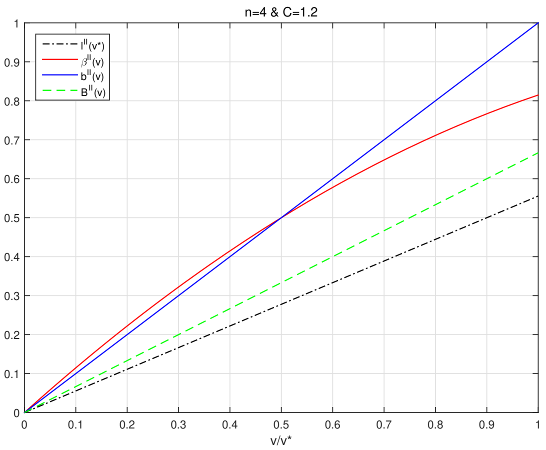
Corollary 2. With sealed second price auctions, if bidder values are uniformly distributed, regardless of the distribution of seller's cost c (including its upper bound C), the symmetric pure strategy equilibrium exists if the first round bidding function $\beta^{II}(v)$ given in equation (53) is increasing.

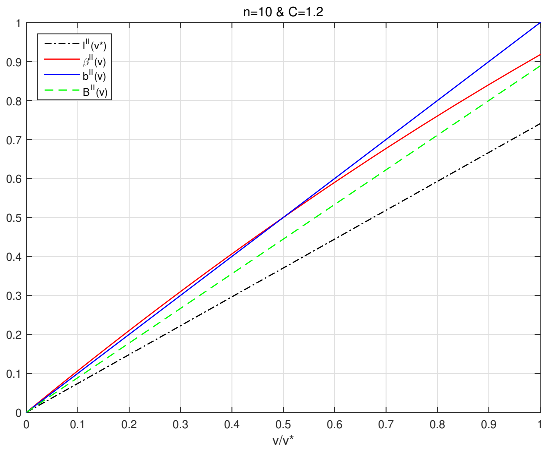
Corollary 2 implies that when bidder values are uniformly distributed, the symmetric pure strategy equilibrium existence condition is not different between the endogenous supply model and the fixed supply model.

Figure 3 shows the equilibrium bidding functions in two examples with sufficiently large C. In the figure, $B^{II}(v)$ represents the equilibrium bidding function in the first auction if there is always a second auction as in a classic sequential auction model. Again, in both examples, $\beta^{II}(v)$ lies above $B^{II}(v)$; and $\beta^{II}(v)$ gets closer to $B^{II}(v)$ when v increases. The same intuition applies as in the case with sealed first price auctions. Compared to Figure 3a, in Figure 3b bidders bid more aggressively in both rounds because there are more competitors in this case. In addition, more bidders imply on average a higher winning price in the first auction and hence a higher likelihood of the second auction. Therefore, $\beta^{II}(v)$ is closer to $B^{II}(v)$ in Figure 3b.

One important observation is that $\beta^{II}(v)$ can be higher than v for some low values of v. Why would a bidder bid higher than her valuation in the first round? First, compared to the model with a fixed supply, here a bidder has an extra incentive to imitate a higher type because it may help increase the probability of the second auction when she loses in the first round. Such an incentive is stronger when v is low and so is the probability of winning the first round. As a result, in a separating equilibrium, low type bidders need to bid aggressively to deter bidders with even lower values from imitating them. Second, it has been shown in the proof of Proposition 1.4 that such a bidder still expects a positive payoff by bidding above her valuation. The individual rational constraint is not a problem.

The Endogenous Order of Sale





(a) n = 4

(b) n = 10

Figure 3: Bidding functions with each buyer's value independent uniformly distributed on [0, 1], when C is small, in the sealed second price sequential auctions.

Now assume $n \geq 4$, so the symmetric pure strategy equilibrium exists. In this part, the paper is going to show that both sellers prefer the order of sale in which the zero cost seller moves first followed by the seller with cost c.

If sellers follow the prescribed order of sale, denote $\pi_1^{II,0}$ as the expected payoff of the zero cost seller who moves first and $\pi_2^{II,c}$ as the expected payoff of the seller with cost c who moves second. Denote X_2 as the second highest value among all n buyers, then

$$\begin{aligned}\pi_1^{II,0} &= E\left[\beta^{II}(X_2)\right] \\ &= \int_0^1 \left[\left(1+\frac{n-2}{(n-1)^2C}\right)x - \frac{2(n-2)}{(n-1)^2C}x^2\right] n(n-1)(1-x)x^{n-2}dx \\ &= \frac{n-1}{n+1} - \frac{(n-2)^2}{(n-1)(n+1)(n+2)C}.\end{aligned} \tag{63}$$

A seller with $c \geq \frac{n-2}{n-1}$ never holds a second auction because the highest possible value of v^* is 1 and the highest possible $R_2^{II}(v^*)$ equals $\frac{n-2}{n-1}$. As a result, $\pi_2^{II,c} = 0$ for $c \geq \frac{n-2}{n-1}$; whilst for a seller with $c < \frac{n-2}{n-1}$,

$$
\begin{aligned}
\pi_2^{II,c} &= E\left[R_2^{II}(X_2) - c \,|R_2^{II}(X_2) \geq c\right] \\
&= \int_{\frac{(n-1)c}{n-2}}^{1} \left(\frac{n-2}{n-1}x - c\right) n(n-1)(1-x)x^{n-2}dx \\
&= \frac{n-2}{n+1} - c + \frac{n-2}{n-1}\left(\frac{(n-1)c}{n-2}\right)^{n} - \frac{n-2}{n+1}\left(\frac{(n-1)c}{n-2}\right)^{n+1} \\
&> 0.
\end{aligned}
\tag{64}
$$

In sum,

$$
\pi_2^{II,c} = \begin{cases} \frac{n-2}{n+1} - c + \frac{n-2}{n-1}\left(\frac{(n-1)c}{n-2}\right)^{n} - \frac{n-2}{n+1}\left(\frac{(n-1)c}{n-2}\right)^{n+1} > 0, & \text{if } c < \frac{n-2}{n-1} \\ 0, & \text{if } c \geq \frac{n-2}{n-1} \end{cases}
\tag{65}
$$

Now consider the case where sellers instead choose the alternative order of sale. That is, the seller with cost c moves first and the zero cost seller moves second. If the seller with cost c holds the first auction, buyers know for sure that there will be a second auction as the other seller bears zero cost and will certainly benefit from trade. As a result, $l^{II}(v^*) = 1$ for all v^* and the first round bidding function becomes

$$
\begin{aligned}
\beta^{II}(v) &= \int_0^v t \; d\left[\left(\frac{t}{v}\right)^{n-2}\right] \\
&= \frac{n-2}{n-1}v
\end{aligned}
\tag{66}
$$

Hence the expected seller's revenue from the first auction is

$$
\begin{aligned}
E\left[\frac{n-2}{n-1}X_2\right] &= \int_0^1 \frac{n-2}{n} xn(n-1)(1-x)x^{n-2}dx \\
&= \frac{n-2}{n+1}
\end{aligned}
\tag{67}
$$

Consequently, the seller with cost c shall offer the first auction only if $c \leq \frac{n-2}{n+1}$. When this happens, the expected payoff of the zero cost seller from the second auction is

$$
\begin{aligned}
E\left[R_2^{II}(X)\right] &= \int_0^1 \frac{n-2}{n-1} xn(n-1)(1-x)x^{n-2}dx \\
&= \frac{n-2}{n+1}
\end{aligned}
\tag{68}
$$

If $c > \frac{n-2}{n+1}$, the seller with such cost will not hold the auction even if she moves first. And everyone shall observe that there is no auction in the first round. Then the seller with zero cost holds an auction in the second round, and her expected payoff is simply that of a standard single-unit auction with n bidders. It is $\frac{n-1}{n+1}$. Without observing the true value of c, the zero cost seller believes that $c > \frac{n-2}{n+1}$ occurs with probability $1-\frac{n-2}{(n+1)C}$, so her expected payoff when she moves secondly, denoted as $\pi_2^{II,0}$, is

$$\begin{aligned}\pi_2^{II,0} &= \frac{n-2}{n+1}\frac{n-2}{(n+1)C} + \frac{n-1}{n+1}\left[1-\frac{n-2}{(n+1)C}\right] \\ &= \frac{n^2C-n-C+2}{C(n+1)^2}\end{aligned} \tag{69}$$

One can check that

$$\begin{aligned}\pi_1^{II,0} - \pi_2^{II,0} &= \left[\frac{n-1}{n+1} - \frac{(n-2)^2}{(n-1)(n+1)(n+2)C}\right] - \frac{n^2C-n-C+2}{C(n+1)^2} \\ &= \frac{4(n-2)}{(n+1)(n+2)C} \\ &> 0\end{aligned} \tag{70}$$

This implies that the seller with zero cost strictly prefers moving first.

As for the seller with cost c, if she moves first, as discussed above, her expected payoff is

$$\pi_1^{II,c} = \begin{cases} \frac{n-2}{n+1} - c, & \text{if } c \le \frac{n-2}{n+1} \\ 0, & \text{if } c > \frac{n-2}{n+1} \end{cases} \tag{71}$$

Compare $\pi_1^{II,c}$ above to $\pi_2^{II,c}$ given in equation (65). When $c < \frac{n-2}{n+1}$,

$$\begin{aligned}\pi_2^{II,c} - \pi_1^{II,c} &= \frac{n-2}{n-1}\left(\frac{(n-1)c}{n-2}\right)^n - \frac{n-2}{n+1}\left(\frac{(n-1)c}{n-2}\right)^{n+1} \\ &= \left(\frac{(n-1)c}{n-2}\right)^n (n-2)\left(\frac{1}{n-1} - \frac{(n-1)c}{(n+1)(n-2)}\right) \\ &> \left(\frac{(n-1)c}{n-2}\right)^n (n-2)\left(\frac{1}{n-1} - \frac{n-1}{(n+1)(n-2)}\frac{n-2}{n+1}\right) \\ &= \left(\frac{(n-1)c}{n-2}\right)^n \frac{4n(n-2)}{(n-1)(n+1)^2} \\ &> 0\end{aligned} \tag{72}$$

As a result, $\pi_2^{II,c} = \pi_1^{II,c} = 0$ when $c \ge \frac{n-2}{n-1}$, but $\pi_2^{II,c} > \pi_1^{II,c}$ when $c < \frac{n-2}{n-1}$. Hence for the seller with cost c, moving second weakly dominates moving first.

In conclusion, both sellers prefer such an order that the zero cost seller sells her item in the first auction and then the other seller has the option to sell her item in the second round. As a result, the order of sale needs not to be exogenously given. Instead, sellers should be able to coordinate and endogenously reach an agreement upon the order of sale.

The intuition for such coordination between sellers is exactly the same as in the case with sealed price auctions. The first auction provides an advantage to the zero cost seller because bidders bid more competitively in the first auction when there is uncertainty about the existence of the second round. On the other hand, moving secondly benefits the seller with positive cost, as she can get information from the result of the first auction and maker a wiser choice on whether or not to sell her item.

1.5.2.2 The Case with Small C

Now consider the case where the upper bound of cost, C, is not sufficiently large. That is, $C < \frac{n-2}{n-1}$. If a symmetric pure strategy equilibrium exists, the likelihood function has two stages as shown below:

$$l^{II}(v^*) = \begin{cases} \frac{n-2}{(n-1)C}v^*, & \text{if } v^* < \frac{n-1}{n-2}C \\ 1, & \text{if } v^* \geq \frac{n-1}{n-2}C \end{cases} \tag{73}$$

It suggests that there is a threshold winning price (or equivalently a corresponding value v^*) in the first auction beyond which the second auction is guaranteed. When $v < \frac{n-1}{n-2}C$, according to Proposition 1.3,

$$\underline{\beta}^{II}(v) \equiv \beta^{II}(v) = \left[1 + \frac{n-2}{(n-1)^2C}\right]v - \frac{2(n-2)}{(n-1)^2C}v^2 \tag{74}$$

When $v \geq \frac{n-1}{n-2}C$, $l^{II}(v) = 1$ and $l^{II'}(v) = 0$, so

$$\begin{aligned} \bar{\beta}^{II}(v) &\equiv \beta^{II}(v) \\ &= v + [0\,(1 - F(v)) - f(v)]\left[\frac{v}{f(v)} - \frac{1}{f(v)}\int_0^v t\ d\left[\left[\frac{F(t)}{F(v)}\right]^{n-2}\right]\right] \\ &= \frac{n-2}{n-1}v. \end{aligned} \tag{75}$$

Note that $\bar{\beta}^{II}(v)$ is exactly the same as the first round bidding function in the standard case with two auctions for certain. That is, a bidder with a value higher than the threshold will simply bid as if there would certainly be a second auction.

However,

$$\begin{aligned}
&\underline{\beta}^{II}\left(\frac{n-1}{n-2}C\right)-\bar{\beta}^{II}\left(\frac{n-1}{n-2}C\right)\\
=&\frac{1}{n-1}-\frac{1}{n-2}C\\
>&\frac{1}{n-1}-\frac{1}{n-2}\frac{n-2}{n-1}\\
=&0
\end{aligned} \tag{76}$$

It implies that a bidder with a value slightly below $\frac{n-1}{n-2}C$ bids higher than another bidder with a value slightly above $\frac{n-1}{n-2}C$. The bidding function $\beta^{II}(v)$ is not increasing, so there does not exist a symmetric pure strategy equilibrium in this case.

Recall a symmetric pure strategy equilibrium exists with sealed first price auctions, even if C is small. What is the difference between sealed first price auctions and sealed second price ones that causes the different results? The key is that in the sealed second price auction the winner does not pay her own bid but the second highest bid. If a symmetric pure strategy equilibrium exists, consider a bidder with a value above the threshold that guarantees the second round. By marginally deviating from her equilibrium bid, the likelihood of the second auction does not change. If she loses, that is when the "option value" of the second round actually matters, the winning price would still be higher than the threshold and the second auction will be held. In addition, this bidder only needs to pay the second highest bid if she wins. Consequently, she does not care about what other bidders bid, and she will just bid as if she was in a standard two-units sequential auction. By contrast, if the bidder has a value below the threshold, she will be able to push up the expected likelihood of the second round by bidding slightly higher in the first auction, which is good to her if she loses. More specifically, consider this particular event in which a bidder benefits by overbidding in the first auction: (i) her bid is the second highest; (ii) her bid convinces the second seller to hold the second auction; (iii) if she had instead bid honestly, then the second seller would not hold the second auction. This is a positive probability event, hence it may be in the bidder's best interest to adjust her bidding strategy and tend to bid higher compared to the case where she cannot influence the likelihood of the second auction. It implies that a bidder with a value just below the threshold would bid higher than another bidder with a value just above the threshold. As a result, the bidding function is not increasing around the threshold value, and the equilibrium collapses.

1.5.2.3 "Declining Price Anomaly"

Just like in the case with sealed first price auctions, when sealed second price auctions are used, the declining price trend is not an anomaly but a property of the equilibrium in the model with endogenous supply.

Assume $n \geq 4$ and $C \geq \frac{n-2}{n-1}$ so that the symmetric pure strategy equilibrium exists. If there are two auction rounds, given the winning price P_1 in the first auction, one can calculate the expected price in the second auction round, $E[P_2|P_1]$. Because in the equilibrium, the first round bidding function $\beta^{II}(v)$ is invertible, knowing P_1 is equivalent to knowing the second highest value among bidders, v^*. Hence,

$$\begin{aligned} E[P_2|P_1] &= E[P_2|v^*] \\ &= R_2^{II}(v^*) \\ &= \frac{n-2}{n-1}v^* \end{aligned} \tag{77}$$

According to $\beta^{II}(v)$ given in equation (60),

$$P_1 = \left[1 + \frac{n-2}{(n-1)^2 C}\right] v^* - \frac{2(n-2)}{(n-1)^2 C} v^{*2} \tag{78}$$

so

$$P_1 - E[P_2|P_1] = \left[\frac{1}{n-1} + \frac{n-2}{(n-1)^2 C}\right] v^* - \frac{2(n-2)}{(n-1)^2 C} v^{*2} \tag{79}$$

It is easy to show that $P_1 - E[P_2|P_1] > 0$ for $v^* \in \left(0,\ \frac{1}{2} + \frac{n-1}{2(n-2)}C\right)$. But $\frac{1}{2} + \frac{n-1}{2(n-2)}C > 1$ and $v^* \leq 1$, so $P_1 > E[P_2|P_1]$ as long as v^* is non-zero. The equilibrium price path shows a declining trend in expectation. The intuition for the declining price path is exactly the same as in the case with sealed first price auctions.

1.6 Comparison Between First Price And Second Price Auctions

It is important to point out that in this paper sealed first price auctions are not revenue equivalent to sealed second price auctions. The Revenue Equivalence Theorem cannot be applied because the allocation rule is different between the auction formats, because the seller with positive cost may sell her item in one format but retain it in the other. For example, if the highest value among bidders, Y_1, is very high but the second highest value, Y_2, is very low, the seller with cost c will sell her item in the sealed first price auction case, as the winning price in the first round reveals Y_1; whilst she will not hold the second auction if sealed second price auction format is used, in which Y_2 is revealed.

Now assume $n \geq 4$ and $C \geq \frac{n-2}{n-1}$ so that symmetric pure strategy equilibrium exists with either sealed first price auctions or sealed second price auctions. We have

$$\begin{aligned}
&\pi_1^{II,0} - \pi_1^{I,0} \\
=&\left[\frac{n-1}{n+1} - \frac{(n-2)^2}{(n-1)(n+1)(n+2)C}\right] - \left[\frac{n-1}{n+1} - \frac{n-2}{(n+1)(n+2)C}\right] \\
=&\frac{n-2}{(n+1)(n+2)(n-1)C} \\
>&0.
\end{aligned} \tag{80}$$

The above inequality shows that the zero cost seller gets a strictly higher expected payoff with sealed second price auctions.

When $c \geq \frac{n-2}{n-1}$, according to equation (22) and (65), both $\pi_2^{I,c}$ and $\pi_2^{II,c}$ are zero.

When $\frac{n-2}{n} \leq c < \frac{n-2}{n-1}$, $\pi_2^{II,c} > 0 = \pi_2^{I,c}$.

When $c < \frac{n-2}{n}$,

$$\begin{aligned}
&\pi_2^{II,c} - \pi_2^{I,c} \\
=&\left[\frac{n-2}{n+1} - c + \frac{n-2}{n-1}\left(\frac{(n-1)c}{n-2}\right)^n - \frac{n-2}{n+1}\left(\frac{(n-1)c}{n-2}\right)^{n+1}\right] \\
&- \left[\frac{n-2}{n+1} - c + c^{n+1}\left(\frac{n}{n-2}\right)^n \frac{1}{n+1}\right] \\
=&\frac{c^n}{(n+1)(n-2)^n}\left[(n+1)(n-2)(n-1)^{n-1} - (n-1)^{n+1}c - n^n c\right] \\
>&\frac{c^n}{(n+1)(n-2)^n}\left[(n+1)(n-2)(n-1)^{n-1} - (n-1)^{n+1}\frac{n-2}{n} - n^n\frac{n-2}{n}\right] \\
=&\frac{c^n(n-1)^{n-1}}{(n+1)(n-2)^{n-1}}\left[3 - \frac{1}{n} - \left(\frac{n}{n-1}\right)^{n-1}\right] \\
>&0.
\end{aligned} \tag{81}$$

As shown above, in expectation the seller with cost c is strictly better off with sealed second price auctions if her cost is not too high. If her cost is very high, she shall be indifferent between two auction formats as she never enters the second round and gets zero payoff. As a result, the seller with cost c also prefers sealed second price auctions.

Why is sealed second price auctions *ex ante* superior in terms of sellers' payoff, compared to sealed first price auctions? Intuitively, it is because with sealed second price auctions the second seller can obtain better information by observing the winning price in the first auction. With sealed second price auctions, the winning price reveals the highest value among all remaining bidders; whilst in sealed first price auctions, the winning price only shows an upper bound of the highest value among remaining bidders. If the second seller could only observe the upper bound of the highest remaining value in both cases, then the Revenue Equivalence Theorem can be applied to the second round, so the expected payoff to the second seller would be the same in both auction formats. However, because the second seller gets better information with sealed second price auctions, her expected payoff in the second round must be higher. On the other hand, the expected payoff to bidders in the second round is lower with the sealed second price auctions. It implies that the option value of the second round is lower. As a result, with sealed second price auctions, bidders value the first auction relatively higher and they shall bid more aggressively. It leads to better competition and a higher price in the first round, which benefits the first seller.

In conclusion, if sellers can choose the auction format, they again are able to coordinate and both agree to adopt sealed second price auctions, which gives them higher expected payoff compared to sealed first price auctions.

1.7 Conclusion

This chapter studies a sequential auction model in which two sellers, each with one unit of the identical item, are heterogeneous in terms of their opportunity cost of selling the item. The seller with high cost has the incentive to make use of the information revealed in the first auction, so the supply of the good is contingent on the result of the earlier round. This chapter has shown that a unique symmetric pure strategy equilibrium exists under some conditions on the distributions of bidder valuations and seller costs. In such an equilibrium, both sellers prefer that the low-cost seller sells her item first followed by the high-cost seller. The high-cost seller can take advantage of observing the winning price in the first auction and then hold the second auction only if she believes the market demand is strong enough to make a profit. As a result, there is uncertainty on the availability of the second auction, so bidders bid more aggressively in the first auction as the "option value" of the second auction is discounted by its uncertainty. The low-cost seller benefits from the enhanced competition in the first auction.

Such bidding behaviours of buyers in the equilibrium also implies a declining price trend, given that two auctions are held. This provides an explanation of the "Declining Price Anomaly", which has been documented in many empirical studies on sequential auctions. This chapter has shown that the declining price trend is merely a property of the equilibrium path when the supply in sequential auctions is endogenous. It is a predicted result of the strategic interactions between bidders and sellers.

However, a symmetric pure strategy equilibrium does not always exist in the model. For example, if the cost of the high-cost seller is public information, a symmetric pure strategy equilibrium does not exist. In addition, when sealed second price auctions are used, there is no symmetric pure strategy equilibrium if the upper bound of the cost of the high-cost seller is too low.

The sealed first price sequential auctions and the sealed second price sequential auctions are not revenue equivalent when the supply is contingent on the performance of the first auction. This is because the allocation rule is different between the two mechanisms. Note that the winning price in the first auction reveals the upper bound of the highest value among remaining bidders, while in the second price auction it shows the highest remaining value itself. As a result, it is possible that the second seller holds the second auction in one format but abstains from the second round in another format. This chapter has shown that if a symmetric pure strategy equilibrium exists in both mechanisms, then both sellers obtain higher expected payoffs by choosing the sealed second price auction format. Intuitively, in sealed second price auctions, the second seller obtains better information on the values of remaining bidders, so she is able to make a better entry decision. In addition, bidders bid more competitively in the first auction as the "option value" of the second auction shrinks due to the fact that the second seller extracts more from the second round. Hence, the greater competition in the first auction benefits the first seller. The implication is that one should expect the first price auctions to be less common in practice than the second price auctions in the context where the supply is endogenous. Only when the upper bound of seller's cost is low and a symmetric pure strategy equilibrium does not exist with second price auctions, the first price auction format may be chosen at the consideration of ease of implementation.

For simplicity, this chapter assumes that there are only two sellers. Further study will attempt to extend the model by assuming there are m sellers ($m > 2$). Sellers' opportunity costs of selling their items are different and privately known. One may further assume that the opportunity cost of seller i is drawn from a distinct distribution with c.d.f. $G_i(c)$. In addition, all these distribution functions are assumed to be ranked in terms of either first order stochastic dominance, i.e., for any $c \geq 0$,

$$G_1(v) \leq G_2(v) \leq \cdots \leq G_m(v),$$

or in other stochastic orders (e.g. hazard rate dominance or likelihood ratio dominance), so that the model studied in this chapter becomes a special case.

Chapter 2

Interdependent Value Auctions with Arriving Buyers

2 Interdependent Value Auctions with Arriving Buyers

2.1 Introduction

In this chapter, we study the environment in which a seller of one indivisible item faces buyers with interdependent values who do not arrive at the market at the same time. It is commonly observed that in reality buyers may enter a market at various time points as they take different lengths of time to conduct research or to settle financial arrangements. In addition, if the pre-auction exhibition period for the item is extended, the information may be able to reach a broader audience and hence attract more buyers to the market. One prominent example is the auctions on online platforms such as eBay. Items listed on these platforms get viewed by different potential buyers all the time.

In classic auction theory models, all potential buyers are assumed to be present in the auction. There are two implicit arguments supporting such a simplification. First, if the buyer values are private, the equilibrium bidding behaviour of a bidder is only affected by the other participating bidders, regardless of the number of buyers who are not present in the auction. As a result, it is not necessary to consider the other potential buyers who may have been excluded from the auction. The second implicit argument is that the seller in fact has a strong incentive to wait and include all potential buyers in the auction as this is the way to maximize the seller's expected payoff, as long as waiting is not costly and as long as the buyer values are private.

However, if waiting is costly or the buyer values are interdependent or other complications on the seller's side is added, these two implicit arguments are no longer valid. Therefore, there have been many studies in auction theory literature on arriving buyers. For example, Gallien and Gupta (2007) study online auctions with randomly arriving buyers. They show with numerical experiments that if the buyers are time sensitive the seller may significantly increase her expected payoff by adding a buyout option on top of the auction. Said (2011) assumes that new buyers and new differentiated goods may randomly arrive at the market and finds that bidders shade their bids according to the anticipated future dynamics of the market. In a more recent work of his, Said (2012) shows that a sequence of ascending auctions with price clocks maximizes the revenue if buyers arrive randomly to compete for an uncertain number of perishable goods.

All these studies share a common assumption: the buyer values are private. In this chapter, we assume that the buyer values are interdependent but keep the other settings as simple as possible. It is assumed in this chapter that the seller only has one indivisible item to sell and the buyers arrive one by one in an exogenous sequence with the total number of potential buyers known. Also, we assume that there is zero cost for the seller to wait and everyone does not discount the future. With these assumptions, we can isolate the effect of interdependent buyer values when buyers are arriving at various time points. We attempt to answer two questions in this chapter.

First, it is questionable whether the seller should wait until all potential buyers have arrived before holding the auction. Intuitively, holding the auction with all potential buyers increases the competition and ensures the efficiency of the selling mechanism. However, with interdependent buyer values, a bidder bids conservatively because of the 'winner's curse' concern. Holding an early auction with a smaller number of buyers helps mitigate the 'winner's curse' effect because winning the auction only implies that other bidders who have arrived early have inferior signals but does not affect the distribution of the signals of those who are excluded from the auction. As a result, bidders in an early auction will bid more aggressively as they hold higher expected values of the item after taking into account the 'winner's curse' problem. It is then ambiguous whether an early auction is overall *ex ante* more profitable, hence the first main question we attempt to answer is: **what is the optimal timing of auction that maximizes the expected payoff to the seller?** We start with a simplified case in which the seller is not allowed to set a reserve price. In this case, we show that it is possible that an early auction with a subset of all buyers is optimal in maximizing the seller's expected payoff. The intuition here aligns with the study by Campbell and Levin (2006). They conclude that when buyers have interdependent values, allowing a buyer whose value is not the highest to win the item is likely to improve the seller's revenue because it increases all buyers willingness to pay. We also study the case in which the seller is allowed to choose both the timing of the auction and the reserve price. The dynamics become more complicated as a positive reserve price allows the possibility of retaining the item in the first auction and re-auctioning it later to the remaining buyers. More importantly, when it comes to the second auction, the reserve price of the first auction becomes a manipulative tool that can shape the perception of the remaining buyers on the signals of those who failed in the first auction. We use numerical examples to illustrate that the seller is even more likely to prefer holding an early auction if she is allowed to set a reserve price. In addition, the relative importance of the signals of others in a buyer's valuation plays a central role in characterizing the optimal reserve price and the optimal timing of the auction.

Second, it is observed that in Australian real estate market and many other markets while auctions are a popularly adopted selling mechanism, there are still a few sellers who choose to use the posted price sale to sell their items. The second question we attempt to answer is: **how well does auction perform compared to the posted price sale when the buyers arrive over time with interdependent values? Which of the two selling mechanism provides a higher expected revenue to the seller?** There are some studies in the literature comparing auction and posted price sale, but none of them has looked at the scenario with interdependent buyer values. For example, Wang (1993) argues that with private buyer values the posted price sale may be more profitable only if conducting an auction is costly. He also shows that auction is likely to generate higher revenues if the private buyer values are more dispersed. Vakrat and Seidmann (1999) find empirical evidence that the winning price of an object in an online auction is on average 25% lower than the catalog price of the identical object sold at the same websites. They use a simple model with private buyer values to explain the phenomenon, arguing that the reason for such a price gap is the cost of participating in an auction. Khezr and Sengupta (2013) investigate the situation in which buyers' valuation only depends on their own signals and the seller's private information. They conclude that auction with a disclosed reserve price always outperforms the posted price sale. In a more recent work, Khezr (2014) keeps the same assumptions on the buyer values and acknowledges that buyers may arrive randomly over time. He shows that the posted price sale may be a more profitable option when the seller is expecting a small number of arrivals.

In this chapter, we compare the performance of sealed-bid second price auction (with a reserve price) and posted price sale when buyers with interdependent values arrive at the market over time. Both selling mechanisms have their own advantages and drawbacks, which make the comparison a project of interest. Intuitively, the auction suffers from the 'winner's curse' concern so buyers in the auction bid more conservatively. On the other hand, an optimal reserve price in the auction is a more efficient device to maximize the revenue as all buyers in the auction are symmetric *ex ante*, while the posted price is less effective because the same price applies to all buyers who observe different information due to their various arriving time. We show with examples that both the total number of buyers and the relative importance of other buyers' signals together determine which selling mechanism is more *ex ante* profitable. In general, auction tends to be better if the total number of buyers is large and if other buyers' signals are unimportant.

The rest of this chapter is organized as the following. The basic model and the market environment is introduced and defined in Section 2.2. In Sections 2.3 and 2.4, we study the optimal timing of auction. Section 2.3 covers the absolute auction case (i.e. without a reserve price), but then in Section 2.4 the reserve price is allowed. In Section 2.5, we compare auction and posted price sale in terms of the expected payoff to the seller. Section 2.6 concludes the chapter and discusses the limitations. Most of the proofs are in the Appendix.

2.2 The Basic Model

A seller wishes to conduct an auction to sell an indivisible item. There are N potential buyers who will arrive at the market one by one each time in an exogenous sequence. The buyers hold interdependent values on this item. Each of the buyers receives a private signal $t_i \in [0, T]$, which is independently drawn from the same distribution identified by C.D.F. $F(t)$ and P.D.F. $f(t)$. Therefore, the vector that contains everyone's signal $\mathbf{t} = (t_1, t_2, \ldots, t_N)$ follows a joint distribution; we denote its C.D.F. as $H(\mathbf{t})$ and its P.D.F. as $h(\mathbf{t})$. Buyer i values the item according to the valuation function $v_i(\mathbf{t}) : [0, T]^N \to \mathbb{R}$. The valuation function is assumed to be symmetric across buyers: $v_i(\mathbf{t}) = u(t_i, \mathbf{t}_{-i})$. In addition, the valuation function is increasing in t_i: $\frac{d}{dt_i}u(t_i, \mathbf{t}_{-i}) > 0$ and weekly increasing in other arguments: $\frac{d}{dt_j}u(t_i, \mathbf{t}_{-i}) \geq 0, j \neq i$. $u(\mathbf{t})$ is also assumed to be second order continuously differentiable, C^2. All buyers are assumed to be risk-neutral. All economic agents are assumed to be sufficiently patient so they do not discount future payments.

The seller's value of keeping the item is normalized to 0, so her objective is to maximize the expected revenue from the auction. For simplicity, we assume the auction is conducted in a sealed-bid second price format. The seller cannot change the auction format, but she can choose the timing of the auction. Namely, the seller can choose n ($\leq N$) such that the auction is conducted immediately after the n^{th} potential buyer has arrived but before the $(n+1)^{th}$ one does.

2.3 Model 1 - the Absolute Auction Case

We first check the simplest case in which the seller does not set a reserve price, i.e. the seller conducts an absolute auction. Therefore, the only choice variable the seller has is the timing of the auction, or equivalently the number of bidders included in the auction, n ($2 \leq n \leq N$)[12].

[12] n needs to be at least 2 to make the second price auction to work.

Given n, it is easy to show that the bidding function of bidder i with signal t_i, taking into account of the full spectrum of all signals $\mathbf{t}$ $(t_1, t_2, \ldots, t_N)$ conditional on t_i is the highest among the first n signals $(t_1, t_2, \ldots, t_n)$, is

$$\begin{aligned} b(t_i, n) &= E[u(t_i, \mathbf{t}_{-i}) \mid t_j \le t_i, j \ne i, j \le n] \\ &= \int_S u(t_i, \mathbf{t}_{-i}) h(\mathbf{t}_{-i} \mid t_j \le t_i, j \ne i, j \le n) d\mathbf{t_{-i}} \end{aligned} \tag{82}$$

where $S = \{\mathbf{t}_{-i} \mid t_j \le t_i, j \ne i, j \le n\}$. It is clear that the bidder discounts other bidders' signals up to her own signal level to avoid the winner's curse problem. However, she does not discount the signals of the buyers who are excluded from the auction because of their late arrival. This is because however high their signals are, these buyers do not affect bidder i's chance of winning the auction at all. The following proposition shows that the bidders bid more aggressively when the auction is conducted earlier with fewer potential buyers.

Proposition 2.1. *The bidding function given in Equation* 82 *is decreasing in* n*. That is,*

$$b(t_i, n_1) \le b(t_i, n_2), \qquad \forall\ 2 \le n_2 < n_1 \le N$$

Proof. See Appendix. □

Intuitively, when making a bid, the bidder takes into account the fact that winning the auction implies that all other bidders' signals are below her own signal. Therefore, the bidder should discount the signals of other participants in the auction to avoid the so-called 'winner's curse'. When the auction is held earlier with fewer participants, winning the auction sends a better message on other buyers' signals. It is possible that the bidder could win the auction not because other's signals are not as high as hers but because she is lucky as the buyers with high signals have not arrived. As a result, the bidder should submit a higher bid when n is smaller.

The expected payoff[13] to the seller is

$$\begin{aligned} U(n) &= \int_0^T b(t, n) d\left(H_n^{(2)}(t)\right) \\ &= \int_0^T b(t, n) \cdot h_n^{(2)}(t) dt \end{aligned} \tag{83}$$

where $H_n^{(2)}(t)$ and $h_n^{(2)}(t)$ are the C.D.F. and P.D.F.of the second highest signal among all t_i, $1 \le i \le n$, respectively. The seller wishes to maximize the expected payoff by choosing the optimal timing (or equivalently, the optimal size of bidders n) of the auction:

$$\max_{n \in \mathbb{N}} U(n) \tag{84}$$

[13]In the model the payoff is the same as the revenue as the seller's value of the item is 0.

Equation 83 shows the trade-off when the seller holds the auction at an earlier time. Proposition 2.1 concludes that $b(t,n)$ is decreasing in n, so the seller can benefit from an early auction since the bidders bid more aggressively. However, when the seller only sells to a subset of all buyers, inefficiency occurs as it is possible that the item is allocated to a buyer who does not value it the most. The efficiency loss indirectly reduces the seller's payoff as there would be less value to be extracted if the winner does not value the item the highest. Such a loss can be seen from $h_n^{(2)}(t)$ in Equation 83, since it is well-known that $h_n^{(2)}(t)$ is more concentrated on its left tail as n decreases. It reflects the fact that less competition in the auction due to a smaller number of bidders is likely to result in a lower winning price. In sum, whether the seller can improve her expected payoff by conducting the auction before all N potential buyers arrive depends on which of these two opposite effects dominates the other.

We now use a simple example to illustrate that the seller's optimal choice on n can vary in different situations. Assume each buyer's signal t is independently and uniformly distributed on $[0,1]$. That is, $f(t) = 1$ and $F(t) = t$ for all $t \in [0,1]$. In addition, assume a simple linear valuation function:

$$u(t_i, \mathbf{t}_{-i}) = t_i + \gamma \sum_{j \neq i} t_j, \quad \gamma > 0 \tag{85}$$

According to Equation 82, the bidding function for bidder i becomes

$$\begin{aligned} b(t_i, n) =& t_i + \gamma \left((n-1) \int_0^{t_i} x \cdot \frac{1}{t_i} dx + (N-n) \int_0^1 x dx \right) \\ =& \left(1 + \frac{\gamma}{2}(n-1)\right) t_i + \frac{\gamma}{2}(N-n) \end{aligned} \tag{86}$$

Following Equation 83 the expected payoff to the seller can be calculated as below

$$\begin{aligned} U(n) &= \int_0^1 \left(\left(1 + \frac{\gamma}{2}(n-1)\right) t_i + \frac{\gamma}{2}(N-n) \right) n(n-1)t^{n-2}(1-t)dt \\ &= \left(1 + \frac{\gamma}{2}(n-1)\right) \frac{n-1}{n+1} + \frac{\gamma}{2}(N-n) \end{aligned} \tag{87}$$

and

$$\frac{d}{dn} U(n) = \frac{2(1-\gamma)}{(1+n)^2} \tag{88}$$

so we have the following conclusion:

Proposition 2.2. *When the valuation function takes a simple linear form as defined in Equation 85 and each buyer's signal is independently and uniformly distributed on $[0,1]$, then the seller's optimal choice of the timing of the auction is the following:*

$$\begin{cases} \textit{if } \gamma < 1, & \textit{the seller waits for all buyers;} \\ \textit{if } \gamma = 1, & \textit{the seller is indifferent with the size of arrivers;} \\ \textit{if } \gamma > 1, & \textit{the seller prefers conducting the auction as early as possible (with the first 2 buyers).} \end{cases}$$

Proof. If $\gamma < 1$, Equation 88 shows that $\frac{dU(n)}{dn} > 0$. Therefore, it is optimal for the seller to set n as large as possible. The seller shall wait for all N buyers to arrive before conducting the auction. Notice that $\frac{dU(n)}{dn} = 0$ if $\gamma = 1$ and that $\frac{dU(n)}{dn} < 0$ if $\gamma > 1$, it is trivial to see other conclusions in Proposition 2.2. □

γ measures the weights a buyer allocates to the signals of others. It measures the relative importance of other buyers' opinions on the item. According to Proposition 2.2, if each buyer's signal is equally important when the seller holds the auction at an earlier time (i.e. reduces n) the effect of more aggressive bids from bidders exactly offsets the effect of efficiency loss. As a result, the seller would be indifferent with the timing of the auction. However, if the buyers are more confident with their own signals and believe they are more important than those received by others in determining the value of the item, the effect of efficiency would dominate, so the seller should wait for all buyers to arrive before holding the auction. On the other hand, if it is believed that the signals of others are more important, the seller would benefit from an earlier auction with fewer participants because the effect of more aggressive bidding would dominate.

In their well-cited work, Bulow and Klemperer (1996) prove that one extra buyer is so valuable that it is worth more than optimal mechanism design with fewer buyers. As a result, the seller should try her best to increase the number of buyers in the market. They also show that this result also holds with the interdependent value model, the only requirement is that all buyers are 'serious': the lowest possible buyer value is not less than the value by the seller. In our model, this requirement is satisfied, so the conclusion by Bulow and Klemperer (1996) is valid. At first glance, it seems puzzling because Proposition 2.2 argues that the seller should limit the auction to fewer buyers when $\gamma > 1$. However, there is actually no contradiction. This is because what Bulow and Klemperer (1996) emphasize is the number of all potential buyers existing in the market, whose signals and competing actions are taken into account by a bidder in the auction. In our model, their conclusion should be interpreted as the following: the seller should try to increase the total number of potential buyers, N; if N can be increased, then simply auctioning to all $N + 1$ buyers is more profitable than in the situation with only N buyers in total regardless how optimal the timing of the auction is.

In our model, A bidder in the auction is aware of the existence of those $N - n$ buyers who are excluded from the auction. She also takes into account their signals when calculating her bid. The main point of our model is to discusses what timing (or equivalently, n) optimizes the expected seller payoff, given a **fixed** N. The discussion is within the category of selling mechanism design itself.

In fact, in their study of the optimality of auction with interdependent buyer values, Campbell and Levin (2006) make a similar argument. They conclude that the seller's revenue can be improved by allowing a buyer whose signal is not the highest to win the item. This is because such a mechanism, in general, will increase all buyers willingness to pay, possibly enough to offset the loss to the seller of not always selling to the buyer with the greatest willingness to pay.

2.4 Model 2 - When the Seller Can Set a Reserve

In the previous section, it is assumed that the seller does not set a public reserve price in the auction. However, in practice, the reserve price is ubiquitous as it is a useful tool to improve the seller's expected payoff. In this section, we allow the seller to set a reserve price R in the initial auction as well as the timing of the auction (or equivalently, the number of the bidders, n). This change adds complication to the selling mechanism because it creates the possibility that the item fails to be sold in the initial auction. As a result, we add a few new assumptions to model the situation after the item passes in. For simplicity, we assume those who attended the initial auction leave the market afterward, searching items available in other markets. Additionally, we assume that if the item fails to be sold in the initial auction, the seller simply waits for all the remaining buyers to arrive and sells it again in an absolute auction (i.e. with 0 reserve price) to the newly arrived $N - n$ bidders (note that the first n buyers have exited the market). We also assume that these $N - n$ buyers only observe the reserve price R but not the actual bids in the first auction.

Theoretically speaking, the seller does not necessarily need to wait till the end to hold the second auction. She could again choose an optimal timing of the second auction after the initial one fails. However, allowing the seller to choose the timing of the second auction complicates the calculation without adding new insights into the dynamics of the model. The key dynamic we want to study is how the seller would choose the timing of the initial auction given that the item can be sold again if the initial auction fails. It is also reasonable to assume that the reserve price in the second auction is 0. This is because there will not be any more buyers to come after the second auction. Since it is assumed that all bidders who have attended the auction will leave the market, the seller cannot commit to a positive reserve price at the end of the second auction as it is always better to sell the item at any positive price than retaining it. In practice, the winning price of an auction is usually published if it exceeds the reserve price and the item gets sold. However, if the item fails to be sold, the highest bid in most cases is not revealed or documented. As a result, the bidders in the second auction are assumed to observe only the previous reserve price in our model.

Given n, bidders in the first auction still follow the bidding function $b(t_i, n)$ given in Equation 82. Because the valuation function $u(t_i, \mathbf{t}_{-i})$ is assumed to be increasing in t_i, it is easy to check that $b(t_i, n)$ is increasing in t_i: $\frac{d}{dt_i} b(t_i, n) > 0$. Therefore, for the seller, setting reserve R is equivalent to setting the type of buyer that will just bid R. Denote the signal received by such a buyer as $t^R \in [0, T]$, then $b(t^R; n) = R$.

For a bidder in the second auction, she observes the reserve price R in the previous auction and understands that the highest bid was below R. In other words, she knows that the signals of the previous n buyers are all below t^R. As a result, she will update the distribution of these signals $(t_1, t_2, \ldots, t_n)$ accordingly and also discount the signals of her remaining competitors up to her own signal to avoid the 'winner's curse'. Now it is easy to show that a bidder i in the second auction will submit her bid according to the following bidding function.

$$\begin{aligned}\beta(t_i, N-n, t^R) =& E[u(t_i, \mathbf{t_{-i}}) \mid t_j \leq t^R, j \leq n; t_k \leq t_i, k \neq i, n < j \leq N] \\ =& \int_Q u(t_i, \mathbf{t_{-i}}) h(\mathbf{t_{-i}} \mid t_j \leq t^R, j \leq n; t_k \leq t_i, k \neq i, n < j \leq N) d\mathbf{t_{-i}}\end{aligned} \tag{89}$$

where $Q = \left\{\mathbf{t_{-i}} \mid t_j \leq t^R, j \leq n; t_k \leq t_i, k \neq i, n < j \leq N\right\}$. It is trivial to see that $\beta(t_i, N - nt^R)$ is increasing in t^R as the valuation function $u(\mathbf{t})$ is increasing in $\mathbf{t}$.

Given n and t^R, the expected payoff to the seller is

$$\begin{aligned}U_R(t^R, n) = & H_n^{(1)}(t^R) \int_0^T \beta(t, N-n, t^R) h_{N-n}^{(2)}(t) dt \\ & + \left(H_n^{(2)}(t^R) - H_n^{(1)}(t^R)\right) \cdot b(t^R, n) \\ & + \int_{t^R}^T b(t, n) h_n^{(2)}(t) t\end{aligned} \tag{90}$$

where $H_n^{(1)}(t)$ is the C.D.F. of the highest signal from the first n buyers; $H_n^{(2)}(t)$ and $h_n^{(2)}(t)$ are the C.D.F. and P.D.F.of the second highest signal from the first n buyers, respectively; $h_{N-n}^{(2)}(t)$ is the P.D.F. of the second highest signal from the $N-n$ buyers excluded from the first auction. In addition, define

$$h_{N-n}^{(2)}(t) = 0, \qquad \text{if } N - n < 2. \tag{91}$$

The first term in Equation 90 calculates the expected revenue from the second auction if the first auction fails to sell the item. However, if n is large ($n > N - 2$) so that there are not enough new buyers to conduct the second auction with, the seller gets 0 revenue in the second stage. Equation 91 makes sure such a situation is handled in our results. The second term in Equation 90 represents the case in which the highest bid in the first auction exceeds the reserve price but the second highest does not so the item is sold at the reserve price. The last term in Equation 90 deals with the case in which the item is sold in the first auction above the reserve price.

The seller wishes to maximize the expected payoff by choosing the optimal timing (or equivalently, the optimal size of bidders n) of the auction and the optimal reserve price R:

$$\max_{(t^R,n)} U_R(t^R, n) \qquad s.t.\ n \in \mathbb{N}, t^R \in [0, T]. \tag{92}$$

There are several effects involving the reserve price R (or equivalently, t^R) that affect the seller's expected payoff. First, a higher reserve price will benefit the seller when the item is sold at the reserve price in the first auction, as shown in the second term in Equation 90. Second, a higher reserve price reduces the chance of the item being sold in the first auction. However, whether this effect will increase or decrease the seller's payoff is ambiguous, given that there is a second auction with the remaining buyers if the item passes in in the first auction. For example, if n is small, then a higher reserve price is likely to increase the probability of the item being sold to the seller with the highest signal, improving the efficiency of the selling-mechanism and increasing the expected payoff to the seller. On the other hand, if n is large, then a higher reserve price tends to reduce efficiency. These first two effects also exist with independent private value (IPV) auction models. The next effect, however, is unique to the interdependent value assumption in our model. A higher reserve price creates a better perception of the signals of the first n buyers when the first auction fails to sell the item. The $N - n$ new buyers in the second auction will think the reason previous n buyers failed to reach the reserve price may not be their low signals but instead a high reserve price set by the seller. Since these signals are value relevant to the new buyers, a higher reserve price helps increase the value of the item estimated by new buyers, who in turn will bid more aggressively in the second auction.

Next, we take the same example as in the previous section to illustrate dynamics the seller faces when she is allowed to set a reserve price. We assume that each buyer's signal t is independently and uniformly distributed on $[0, 1]$ and that the valuation function follows Equation 85. Define $m = N - n$, the bidding function in the first auction is the same as in Equation 86. It can be re-written as

$$b(t_i, n) = \left(1 + \frac{\gamma}{2}(n-1)\right) t_i + \frac{\gamma}{2} m \tag{93}$$

When $m \geq 2$ ($n \leq N-2$), according to Equation 89 the bidding function in the second auction becomes

$$\begin{aligned}\beta(t_i, m, t^R) =& t_i + \gamma \left((m-1) \int_0^{t_i} x \cdot \frac{1}{t_i} dx + n \int_0^{t^R} x \cdot \frac{1}{t^R} dx \right) \\ =& \left(1 + \frac{m-1}{2}\gamma\right) t_i + \frac{n\gamma}{2} t^R\end{aligned} \tag{94}$$

The expected payoff to the seller can be calculated from Equation 90 as below

$$\begin{aligned}U_R(t^R, n) = (t^R)^n &\int_0^1 \left(\left(1 + \frac{m-1}{2}\gamma\right) s + \frac{n\gamma}{2} t^R \right) m(m-1)s^{m-2}(1-s)ds \\ &+ n(t^R)^{n-1}(1-t^R) \left(\left(1 + \frac{\gamma}{2}(n-1)\right) t^R + \frac{\gamma}{2} m \right) \\ &+ \int_{t^R}^1 \left(\left(1 + \frac{\gamma}{2}(n-1)\right) s + \frac{\gamma}{2} m \right) n(n-1)s^{n-2}(1-s)ds\end{aligned} \tag{95}$$

When $m < 2$ ($n > N-2$), the expected payoff to the seller is

$$\begin{aligned}U_R(t^R, n) =& n(t^R)^{n-1}(1-t^R) \left(\left(1 + \frac{\gamma}{2}(n-1)\right) t^R + \frac{\gamma}{2} m \right) \\ &+ \int_{t^R}^1 \left(\left(1 + \frac{\gamma}{2}(n-1)\right) s + \frac{\gamma}{2} m \right) n(n-1)s^{n-2}(1-s)ds \\ =& \left(1 + \frac{n-1}{2}\gamma\right) \left(\frac{n-1}{n+1} + (t^R)^n - \frac{2n}{n+1}(t^R)^{n+1} \right)\end{aligned} \tag{96}$$

The following proposition gives the optimal reserve price for a given n.

Proposition 2.3. *For a given $n \in [2, N]$, the optimal reserve type is*

$$t^{R^*}(n) = \begin{cases} 1 - \frac{(N-n-3)\gamma+4}{(n-3)(N-n+1)\gamma+4(N-n+1)}, & \text{if } n \leq N-2; \\ \frac{1}{2}, & \text{if } n > N-2. \end{cases} \tag{97}$$

Proof. See Appendix. □

Figure 4 shows the optimal reserve curves when γ takes different values with $n = 10$. Please note that the curves are plotted as if n is a continuous variable for illustration purpose, although n can only take integer values. When n is larger than 8, the seller cannot organize a second auction if the item is not sold in the initial auction. In this case, the optimal reserve type is $\frac{1}{2}$ for all values of γ, which is simply a horizontal line in the figure. It is clear that regardless of the value of γ, the optimal reserve type is higher than this $\frac{1}{2}$ level when $n \leq 8$. This is because the seller is happy to set a higher reserve when there is a second chance to sell the item if the item fails to be sold due to the reserve price. In addition, when the second auction takes place, a higher reserve price can increase the estimated value of the item among the remaining bidders, as we discussed earlier.

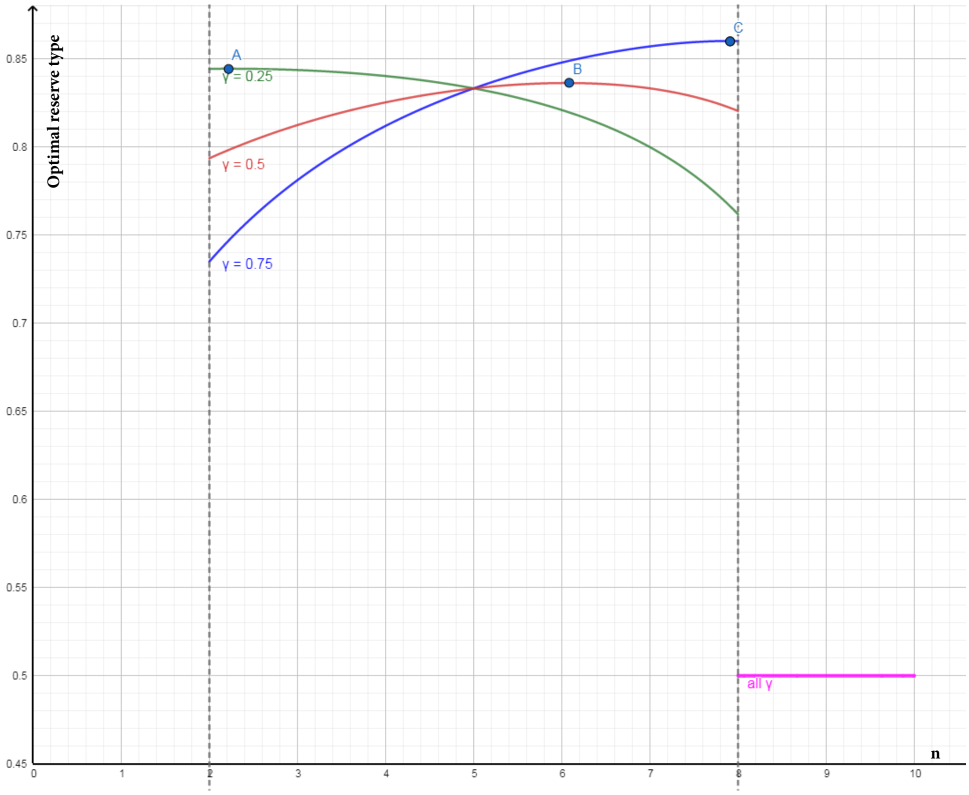


Figure 4: Optimal reserve type curves with with each buyer's signal independent uniformly distributed on $[0, 1]$, when $N = 10$, across three γ values, in the sealed-bid second price auction.

The maximum point of each curve is labelled in the figure. When $\gamma = 0.25$ we observe that the optimal reserve type decreases as n increases. In contrast, the optimal reserve type increases with n when $\gamma = 0.75$. Recall that setting a higher reserve price brings at least two opposite effects to the seller's expected payoff. First, it affects the efficiency of the selling mechanism, creating a probability that the seller does not accept the highest bid in the first auction and has to sell to a lower bid in the second auction. Specifically, the possible payoff loss caused by the inefficiency due to the reserve price becomes worse when n increases, as there will be fewer remaining buyers in the second auction if the first auction fails to sell the item. As a result, when n increases, the seller may want to reduce the reserve price to mitigate the issue. Second, a higher reserve price improves the belief of the $N - n$ new buyers in the second auction about the signals of the first n buyers who failed in the first auction. Therefore, it increases the estimated value of the item among bidders in the second auction since the value depends on the signals of the first n buyers as well. As a result, when n increases, this effect of belief improvement becomes stronger, so the seller may wish to set a higher reserve price. Also, coefficient γ determines the overall strength of this belief improvement effect, as γ measures how important the signals of others are in the valuation function. When γ is sufficiently large, the belief improvement effect can be strong enough to dominate the efficiency concern so that the seller will raise the reserve when n increases. In our example, $\gamma = 0.75$ illustrates such a case. On the other hand, when $\gamma = 0.25$, the green curve in Figure 4 shows that the effect of belief improvement is not strong enough to offset the effect of inefficiency problem: the optimal reserve type keeps decreasing as n rises.

Another choice variable the seller has is the timing of the auction or the number of bidders included in the first auction n. Bring Equation 97 into $U_R(t^R, n)$, we obtain the seller's maximum expected payoff given n:

$$U^*(n) \equiv U_R\left(t^{R^*}(n), n\right) \tag{98}$$

The optimal n^* can be calculated by maximizing $U^*(n)$ subject to $n \in [2, N]$ and $n \in \mathbb{N}$. Then the optimal reserve type is $t^{R^*}(n^*)$ from Equation 97. We actually obtain n^* and t^{R^*} in a two-stage process, but the following proposition states that this method is equivalent to maximizing the expected payoff with respect to (t^R, n) simultaneously.

Proposition 2.4. *Denote* $S \equiv \underset{2\leq n\leq N,\ n\in\mathbb{N}}{\operatorname{argmax}} U^*(n)$, *where* $U^*(n)$ *is defined in Equation* 98. *Define* $t^{R^*}(n)$ *as in Equation* 97, *then* $\{\left(t^{R^*}(n^*), n^*\right) \mid n^* \in S\}$ *is the solution set to*

$$\max_{(t^R,n)} U_R(t^R, n) \qquad s.t.\ n \in \mathbb{N}, t^R \in [0, T].$$

Proof. See Appendix. □

When $n > N - 2$ (i.e. $m < 2$), $t^{R*} = \frac{1}{2}$ according to Proposition 2.3. From Equation 96, we obtain

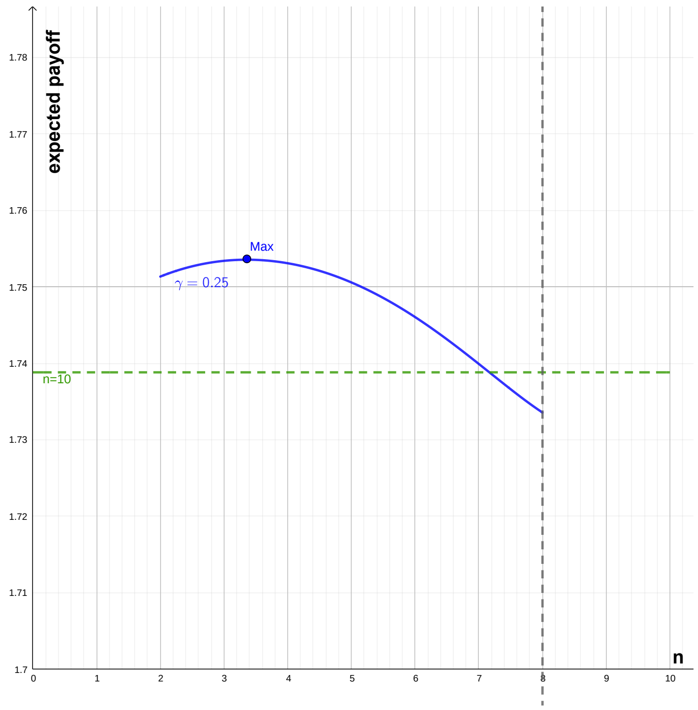
$$U^*(n) = \frac{1 + \frac{n-1}{2}\gamma}{n+1}\left(n - 1 + \left(\frac{1}{2}\right)^n\right) \tag{99}$$

One can verify that $U^*(n)$ given in Equation 100 is increasing in n. As a result, the maximum expected payoff to the seller when $n > N - 2$ is at $n = N$ and its maximum value is

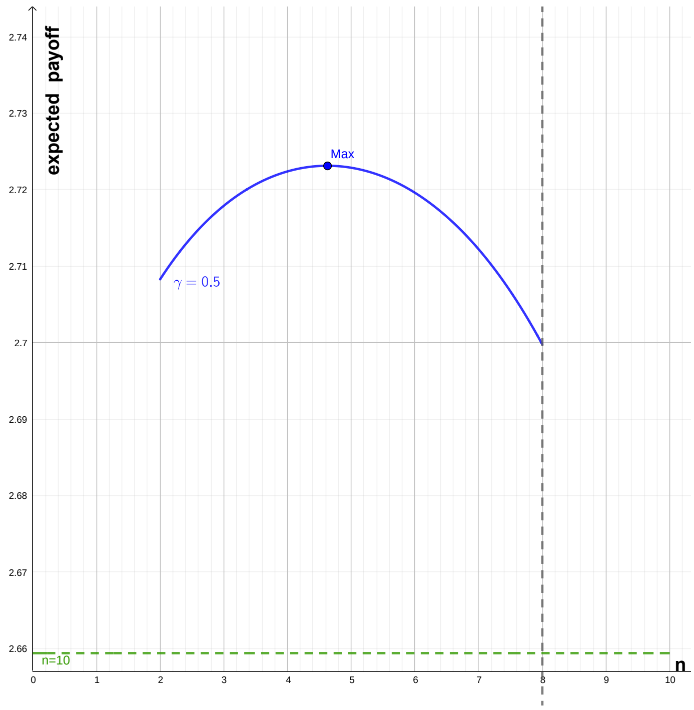
$$\begin{aligned}\max_{n>N-2} U^*(n) &= U^*(N) \\ &= \frac{1 + \frac{N-1}{2}\gamma}{N+1}\left(N - 1 + \left(\frac{1}{2}\right)^N\right)\end{aligned} \tag{100}$$

When $n \leq N-2$, we cannot get an explicit analytical solution, but the numerical solution is calculated. Figure 5 shows the curve of $U^*(n)$, the maximum expected payoff to the seller as a function of n , across four different values of γ when $N = 10$. The green horizontal dashed line in the figure represents the seller's expected payoff when $n = N$, which is her optimal choice if $n > N - 2$. In all four sub-figures, it is clear that the maximum point of the $U^*(n)$ curve when $n \leq N-2$ (the blue curve) is higher than $U^*(N)$, so the seller's optimal choice of n never exceeds $N - 2$ to ensure there are enough remaining bidders to conduct a second auction with in case the item is not sold in the first auction.

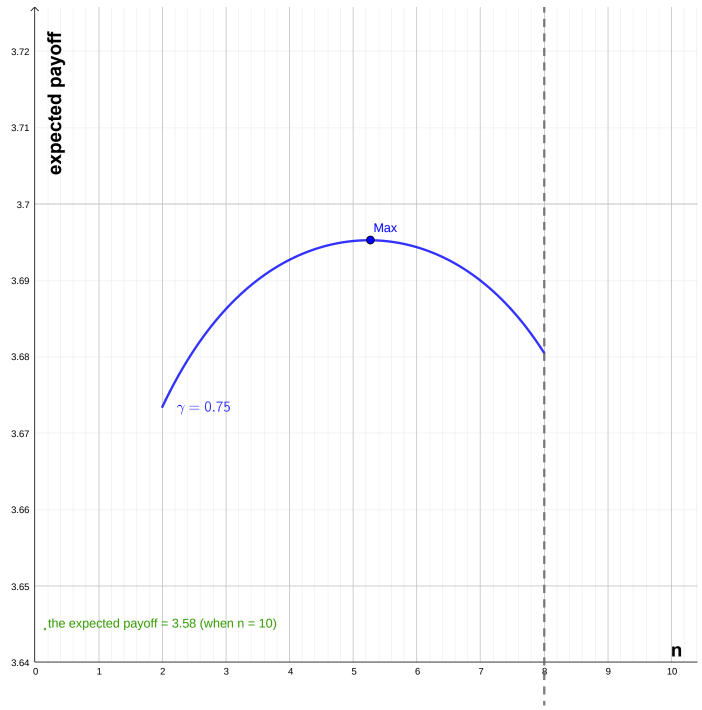
Another observation is that the curve of $U^*(n)$ is not monotonic while $2 \leq n \leq N - 2$. Unless γ is very close to 1 (for example, in Subfigure d $\gamma = 0.98$), the optimal n is strictly less than $N - 2$. For example, $n^* = 3$ when $\gamma = 0.25$, $n^* = 5$ when γ takes a value between 0.5 and 0.75. It seems that the seller somehow balances the number of bidders in two auctions with her optimal choice of n. The intuition here is not straightforward, as the impact of n (or the timing of the first auction) on seller's expected payoff is more complicated in Model 2, because there could be a second auction. Holding an earlier auction (or choosing a smaller n) affects the bidding behaviours of buyers in both auctions: bidders in the first auction will bid more aggressively because the 'winner's curse' problem is less severe with a smaller n, as discussed in Model 1. Meanwhile, buyers in the second auction are less influenced by the previous reserve price because the number of buyers who failed to reach that reserve is smaller. As a result, a smaller n may cause buyers in the second auction to bid less aggressively if the reserve is high, but more aggressively if the reserve price is low. In addition, changing n will also affect the efficiency of the selling mechanism. However, the direction of the efficiency change is ambiguous



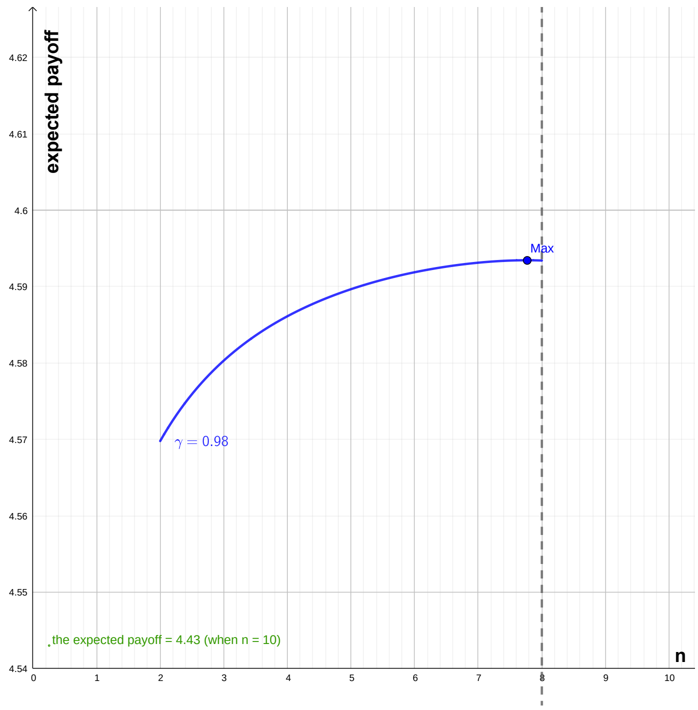
(a) $\gamma = 0.25$



(b) $\gamma = 0.5$



(c) $\gamma = 0.75$



(d) $\gamma = 0.98$

Figure 5: Maximum expected payoff to the seller, $U^*(n)$, with each buyer's signal independent uniformly distributed on $[0, 1]$, when $N = 10$, in the sealed-bid second price auction.

when n is reduced, which is a key difference from Model 1. For example, if n is already small, reducing n improves efficiency. On the other hand, if n is large, how the efficiency changes when n decreases depends on the reserve price level.

Recall Proposition 2.2 in Model 1 predicts that the seller should not conduct an early auction ($n < N$) if $\gamma < 1$. In contrast, the results of Model 2, in which the seller is allowed to set a reserve price, show that even when $\gamma < 1$ the seller can benefit from an early auction (i.e. $n^* < N$). This is because in the new model the seller has a second chance to sell the item so auctioning the item to only a subset of buyers is not as inefficient as in Model 1. Such a result makes our study more applicable to the practice as in most real-life examples buyers conduct extensive research before the auction and put more weight on their own signals (i.e. $\gamma < 1$).

2.5 Model 3 - Posted Price vs Auction

In this section, we compare the posted price sale against auction in our environment with interdependent buyer values. Both selling mechanisms are described below.

We define the auction mechanism as the following. For simplicity, in this section, we assume that the seller simply waits until all N buyers arrive and hold a sealed-bid second price auction with a reserve price. We do not allow the seller to choose the timing of the auction here because the choice of n complicates the calculation but does not give much insight into the key difference between the auction and the posted price sale. In addition, holding one auction to all potential buyers with a reserve is a popular selling mechanism in both practice and theory.

For the posted price sale, we assume the seller chooses a public price P for the item at the beginning and commits to it. As assumed in the basic model, N buyers arrive one by one in an exogenous sequence. Upon arrival, if the item is still available, the ith buyer observes two pieces of information: First, she observes the number of buyers who arrived before her, or equivalently her own position in the sequence, i. Second, she sees the posted price P. Then she decides whether to purchase the item at price P immediately. If she decides to purchase the item, the transaction takes place and the market closes. Otherwise, buyer i leaves the market and the seller waits for the next buyer to come. We understand there might be a commitment issue when buyer N refuses the offer because the seller may wish to sell the item at any positive price as it is her last chance to make any revenue. However, a similar commitment issue also exists with the reserve price in the auction. We believe that it is fair to assume that the seller can fully commit to the reserve price and the posted price in the two selling mechanisms, respectively.

Assume the signal of buyers is independent and identically distributed, if the seller chooses to **auction the item**, her optimal expected payoff, U_A^*, is a well-known result.

$$U_A^* = \max_{t^R} U_A(t^R) \tag{101}$$

where $U_A(t^R)$ is the expected payoff to the seller given the reserve type t^R

$$U_A(t^R) = \left[F_N^{(2)}(t^R) - F_N^{(1)}(t^R)\right] B(t^R) + \int_{t^R}^{T} B(t) dF_N^{(2)}(t) \tag{102}$$

and $B(t_i)$ is the bidding function by a bidder with signal t_i

$$B(t_i) = E\left[u(t_i, \mathbf{t}_{-i}) \mid t_j \le t_i, j \ne i\right] \tag{103}$$

In the **posted price sale**, the expected value of the item to buyer 1, who arrives first, is

$$v_1(t_1) = E\left(u(t_1, \mathbf{t}_{-i})\right) \tag{104}$$

Buyer 1 purchases the item if and only if $v_1(t_1) \ge P$, so if she does not buy the item others can infer that $v_1(t_1) < P$. It is easy to check that $v_1(t_1)$ is increasing in t_1, so $v_1^{-1}(P)$ exists on $[v_1(0), v_1(T)]$. Define $\bar{t}_1 \equiv v_1^{-1}(P)$, when those who arrive later see the item is still available, they can infer that $t_1 < \bar{t}_1$.

For buyer i ($i \ge 2$), when she arrives, if the item is still available, her expected value of the item is

$$v_i(t_i) = E\left[u(t_i, \mathbf{t}_{-i}) \mid t_j < \bar{t}_j, \forall j < i\right] \tag{105}$$

where $\bar{t}_j = v_j^{-1}(P)$. It is easy to check that $v_i(t_i)$ is increasing, so its inverse function exists on $[v_i(0), v_i(T)]$. Define $\bar{t}_i \equiv v_i^{-1}(P)$, when those who arrive later see the item is still available, they can infer that $t_1 < \bar{t}_1$.

For a given price P, the expected payoff to the seller is

$$\begin{aligned} U_S(P) &= \left[1 - \Pr(t_i < \bar{t}_i, \forall i)\right] P \\ &= \left[1 - H(\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_N)\right] P \\ &= \left[1 - \prod_{i=1}^{n} F(\bar{t}_i)\right] P \end{aligned} \tag{106}$$

and the maximum expected payoff is

$$U_S^* = \max_{P} U_S(P) \tag{107}$$

We use the same example from the previous two sections to illustrate how the seller's expected payoff is affected by the two different selling mechanisms. Again, the signal of buyers is independently and uniformly distributed on [0.1]. The valuation function follows Equation 85.

For the auction, the optimal reserve type $t^{R^*} = \frac{1}{2}$ according to Proposition 2.3 and the maximum expected payoff to the seller is also calculated in Equation 100.

$$U_A^* = \frac{1 + \frac{N-1}{2}\gamma}{N+1}\left(N - 1 + \left(\frac{1}{2}\right)^N\right) \tag{108}$$

For the posted price sale,

$$\begin{aligned} v_1(t_1) =& E\left[t_1 + \gamma \sum_{j=2}^{N} t_j\right] \\ =& t_1 + \gamma(N-1)\int_0^1 t dt \\ =& t_1 + \frac{N-1}{2}\gamma \end{aligned} \tag{109}$$

and for $P \in \left[\frac{\gamma(N-1)}{2}, \frac{\gamma(N-1)}{2} + 1\right]$,

$$\bar{t}_1 = P - \frac{\gamma(N-1)}{2}. \tag{110}$$

When $i > 1$, we have

$$\begin{aligned} v_i(t_i) =& E\left[t_i + \gamma \sum_{j \neq i} t_j \;\middle|\; t_j < \bar{t}_j, \forall j < i\right] \\ =& t_i + \gamma(N-i)E(t) + \gamma \sum_{j=1}^{i-1} E(t_j \mid t_j < \bar{t}_j) \\ =& t_i + \frac{\gamma(N-i)}{2} + \gamma \sum_{j=1}^{i-1} \int_0^{\bar{t}_j} t \cdot \frac{1}{\bar{t}_j} dt \\ =& t_i + \frac{\gamma(N-i)}{2} + \frac{\gamma}{2} \sum_{j=1}^{i-1} \bar{t}_j \end{aligned} \tag{111}$$

It is easy to check that for $P \in \left[\frac{\gamma(N-1)}{2}, \frac{\gamma(N-1)}{2} + 1\right]$,

$$\bar{t}_i = P - \frac{\gamma(N-i)}{2} - \frac{\gamma}{2} \sum_{j=1}^{i-1} \bar{t}_j. \tag{112}$$

The following proposition states that the optimal P must fall into the interval $\left[\frac{\gamma(N-1)}{2}, \frac{\gamma(N-1)}{2} + 1\right]$, so we can use Equations 110 and 112 to help calculate the optimal P and corresponding highest expected seller payoff.

Proposition 2.5. *When the signal of buyers is independently and uniformly distributed on* $[0, 1]$ *and the valuation function follows Equation* 85*,* P^* *that maximizes the expected seller payoff in a posted price sale,* $U_S(P)$*, must fall in* $\left[\frac{\gamma(N-1)}{2}, \frac{\gamma(N-1)}{2}+1\right]$.

Proof. See Appendix. □

Now we only consider the situation with $P \in \left[\frac{\gamma(N-1)}{2}, \frac{\gamma(N-1)}{2}+1\right]$. We have

$$\begin{aligned} \bar{t}_{i+1} - \bar{t}_i &= \frac{\gamma}{2} - \frac{\gamma}{2}\bar{t}_i \\ \Rightarrow \bar{t}_{i+1} &= \frac{\gamma}{2} + \left(1 - \frac{\gamma}{2}\right)\bar{t}_i \end{aligned} \tag{113}$$

for all $1 \le i \le N-1$, then

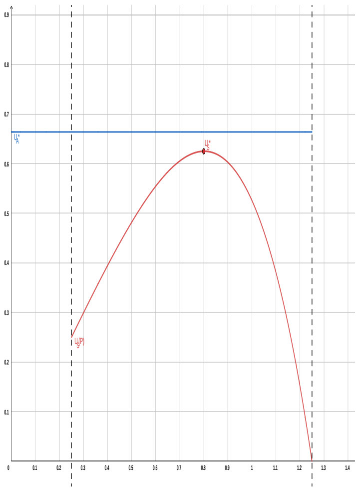
$$\begin{aligned} \bar{t}_i =& \frac{\gamma}{2} + \left(1 - \frac{\gamma}{2}\right)\left(\frac{\gamma}{2} + \left(1 - \frac{\gamma}{2}\right)\bar{t}_{i-2}\right) \\ =& \frac{\gamma}{2}\left[1 + \left(1 - \frac{\gamma}{2}\right) + \cdots + \left(1 - \frac{\gamma}{2}\right)^{i-2}\right] + \left(1 - \frac{\gamma}{1}\right)^{i-1}\bar{t}_1 \\ =& \frac{\gamma}{2} \cdot \frac{1 - (1-\gamma/2)^{i-1}}{1 - (1-\gamma/2)} + \left(1 - \frac{\gamma}{1}\right)^{i-1}\bar{t}_1 \\ =& 1 - \left(1 - \frac{\gamma}{2}\right)^{i-1}\left[1 - P + \frac{\gamma}{2}(N-1)\right] \end{aligned} \tag{114}$$

for all $1 \le i \le N$. The expected payoff to the seller is

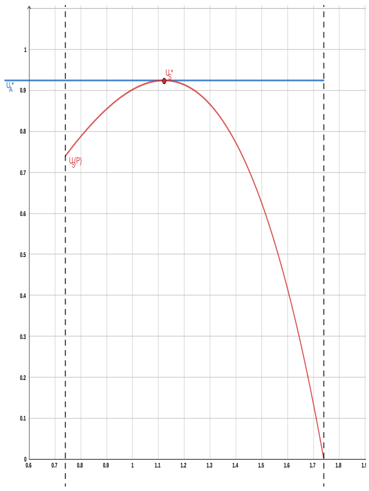
$$U_S(P) = \left[1 - \prod_{i=1}^{n} \bar{t}_i\right] P \tag{115}$$

The explicit analytical solution to $\max_P U_S(P)$ is hard to obtain so we show its numerical solutions with comparison to the optimal auction payoff in Figure 6. In Figure 6, the horizontal blue line represents the optimal expected payoff to the seller with the auction mechanism, U_A^*. The red curve shows the expected payoff to the seller in the posted price sale with a price P. The two vertical dashed lines identify the interval $\left[\frac{\gamma(N-1)}{2}, \frac{\gamma(N-1)}{2}+1\right]$. We only consider this interval according to Proposition 2.5.

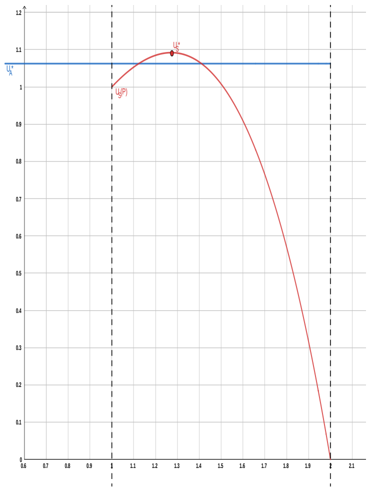
Before checking the results in detail from Figure 6, it is useful to intuitively understand the advantages and drawbacks of the two selling mechanisms. With the interdependent buyer values, a bidder in the auction must take into account the 'winner's curse'. She needs to discount all other bidders' signals up to her own signal when making her bid. On the other hand, a buyer in the posted price sale is not subject to the 'winner's curse' problem because she does not need to discount the signals of those who would arrive after her. Being able to purchase the item only implies that buyers who arrived earlier failed to outbid the posted price P. In



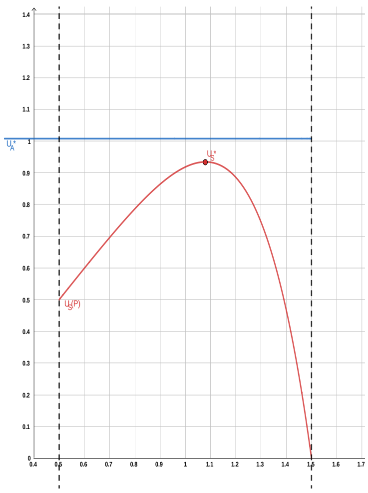
(a) $N = 3,\ \gamma = 0.25$



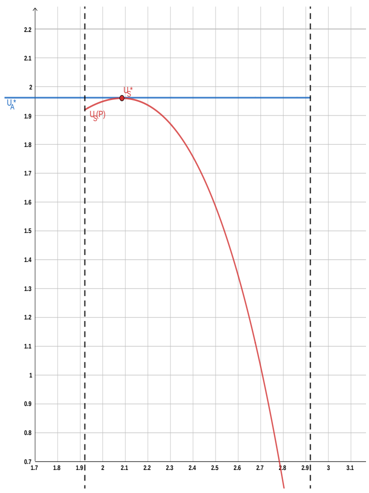
(b) $N = 3,\ \gamma = 0.74$



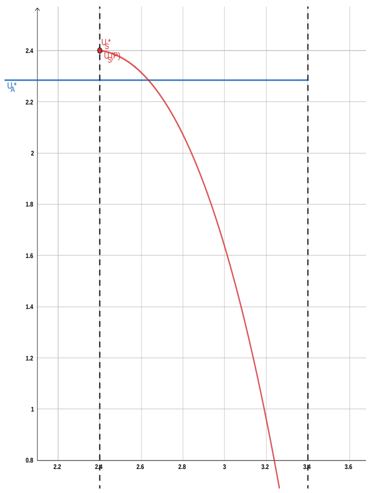
(c) $N = 3,\ \gamma = 1$



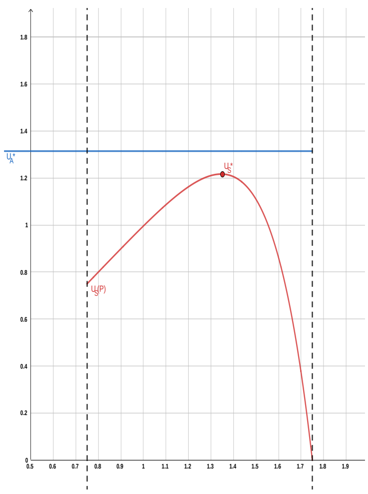
(d) $N = 5,\ \gamma = 0.25$



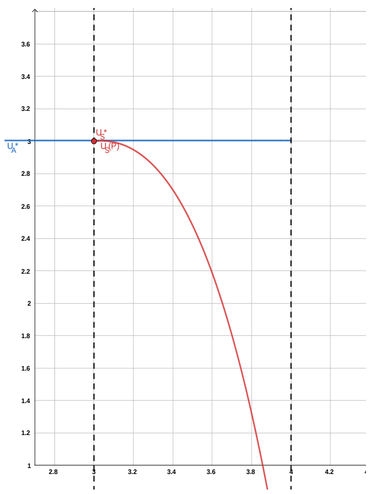
(e) $N = 5,\ \gamma = 0.96$



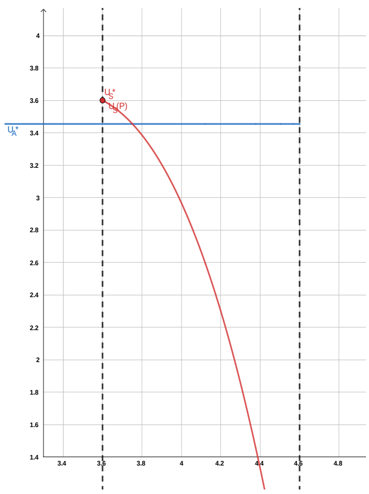
(f) $N = 5,\ \gamma = 1.2$



(g) $N = 7,\ \gamma = 0.25$



(h) $N = 7,\ \gamma = 1$



(i) $N = 7,\ \gamma = 1.2$

Figure 6: Comparison between the expected seller payoff using the posted price sale, $U_S(P)$, and the expected seller payoff using the sealed-bid second price auction, U_A^*; with each buyer's signal independent uniformly distributed on $[0, 1]$.

comparison, buyers in auction bids less aggressively than buyers in the posted price sale. Note that γ measures the relative importance of other buyers' signals to a buyer when she evaluates the item. As a result, when γ is big, the 'winner's curse' effect is strong so the posted price sale tends to be more profitable than the auction. In addition, when the total number of buyers, N, is smaller, it is likely that the highest signal among all buyers is lower. Consequently, the 'winner's curse' effect is stronger and the posted price sale also tends to be more profitable than the auction.

Another key difference between the auction and the posted price sale is that buyers in the auction are in symmetric positions with identical information available to each of them. In contrast, buyers in the posted price sale observe different information depending on their positions of arrival. For example, a buyer who arrives late observes the purchase decision of those who have arrived before her. Additionally, the item may be sold in earlier rounds even before the arrival of a particular buyer. As a result, the reserve price R or the reserve type t^R applies in the same way to all the buyers in the auction, so the optimal reserve level t^{R^*} can efficiently maximize the expected payoff. In comparison, to effectively maximize the expected payoff in the posted price sale, each buyer should be given a unique offer because they observe different information. However, the seller is only allowed to commit to one posted price P to all buyers in the model, so the optimal P^* does not maximize the expected payoff as efficiently as t^{R^*} does in the auction. When the total number of the buyers, N, is large, such a difference is amplified so that the auction tends to be more profitable than the posted price sale.

Figure 6 confirms our intuitions above. First, in Subfigures (a), (d), and (g), we observe that the function curve of $U_S(P)$ lies entirely below U_A^* when $\gamma = 0.25$, regardless the number of total buyers, N. That is, when γ is small the adverse effect of the 'winner's curse' is weak, so the auction tends to be superior to the posted price sale in generating payoff to the seller. Furthermore, given a fixed N, when γ increases the curve of $U_S(P)$ locates relatively higher to U_A^*. Subfigures (c), (f) and (i) illustrate that the posted price sale can be more profitable as long as γ is sufficiently high. Second, Subfigures (b), (e) and (h) show the cases in which both selling mechanisms generate the same profit and the seller is indifferent between them. When $N = 3$, γ needs to be 0.74; when $N = 5$, γ needs to be 0.96; when $N = 7$, γ needs to be 1. Apparently, when N increases, the posted price sale becomes more disadvantaged because a single price is applied to more buyers with different information. As a result, in order to completely offset such a disadvantage and to make the two selling mechanisms equally profitable, γ needs to be larger to make the auction suffer from a stronger effect of the 'winner's curse' consideration.

Finally, Subfigure (h) plots a special case: although the auction and the posted price sale lead to the same expected payoff, the posted price sale actually gives a deterministic payoff. In this situation, the optimal price is set to be the item value to the first buyer assuming her signal is of the lowest type. As a result, the item is surely sold to the first buyer at that price.

One phenomenon our model can help explain is that in the real estate market many sellers choose the posted price sale mechanism to sell their houses after their houses earlier passed in at auctions. According to our model, if γ is small and/or if the number of buyers remaining is still large, the auction is likely to be a more profitable option. In the real estate market, it is reasonable to assume that γ is not too small as there are many common aspects of a house that every buyer values. After a failed auction, it is also reasonable to assume that the number of buyers who are still interested in the property is small. As a result, the seller is likely to conclude that the posted price sale is a better option in such a situation.

2.6 Conclusion

In this chapter, we have studied auctions in a single item market with arriving buyers who have interdependent values for this item. We attempt to answer the following two questions in such an environment: What is the optimal timing of auction? Compared to posted price sale, does auction perform better in terms of the expected payoff to the seller?

In answering the first question, we first study a simplified case without any reserve price and show that a seller may prefer an early auction with fewer bidders because the 'winner's curse' problem is mitigated. The relative importance of other buyers' signals in determining the buyer valuation plays a central role in characterizing the equilibrium. With our uniform distribution linear valuation example, it is concluded that the seller prefers an early auction if the weight coefficient for other buyers' signals, γ, is larger than 1. The seller can also be indifferent about the timing of auction if γ is equal to 1. Once the seller is allowed to set a reserve price, the dynamics become more complicated. This is because the seller has a second chance to sell the item if it is passed-in at the first auction. Setting an optimal reserve price and choosing the optimal timing of auction both require the seller to balance the efficiency loss and the 'winner's curse' concern. Although we cannot obtain an explicit analytical solution, numerical examples are presented to illustrate that under certain conditions an early auction with an optimal reserve price does maximize the expected payoff to the seller. It is also evident that the seller may prefer an early auction even if γ is smaller than 1, after she is allowed to set a reserve price. We also point out that our findings are not contradictory to the well-known result from Bulow and Klemperer (1996), which emphasizes the superior benefit from one extra buyer. An early auction indeed excludes some buyers from bidding, but it does not remove them entirely from the market. In other words, while the subset of buyers are bidding in the auction, they are still taking into account the existence of those who are willing to buy the item but have not arrived yet. In essence, an early auction is merely a variant of selling mechanism but not an effort to reduce the size of the market demand.

In answering the second question, the optimal expected revenue from the auction is readily available, while the posted price sale needs more attention. The key dynamics with the posted price sale is that buyers who arrive at different time points observe different information. If a buyer sees the item is still available after her arrival, she can infer that all buyers who have come before did not accept the price offer so she can obtain inferences on their signals. This results in an information asymmetry: the later a buyer arrives the more information she will get. Assuming the seller commits to the posted price without changing it, such a price applies to all buyers who have asymmetric information so the seller cannot set an optimal price that efficiently maximizes the revenue. On the other hand, the posted price sale avoids the 'winner's curse' problem so the buyers are more willing to pay compared to those in the auction. Again, numerical examples are presented as we cannot get an explicit analytical solution for the optimal expected revenue from the posted price sale. Our examples illustrate that the posted price sale tends to be more profitable when the total number of buyers is small and when γ is big.

To our best knowledge, this chapter provides the first study in the literature on the optimal timing of auction and on the comparison between auction and posted price sale in an environment with arriving buyers who have interdependent values. We give theoretical analysis and numerical examples to show that under certain conditions it is optimal to hold an early auction without waiting for all potential buyers to arrive. Our findings may be applied to guide sellers and auction houses to decide the timing of auctions in practice. Our model also provides a rationale for exclusive auction/sale events observed in some real-world markets. In addition, the comparison between auction and posted price sale offers an explanation of why some sellers choose to sell their items (e.g. houses) by posting a price instead of holding an auction.

We understand there are some limitations to our study in this chapter. First, it is assumed that buyers arrive one by one in an exogenous sequence for calculation ease. Although the main results are unlikely to be affected, it is ideal to assume that the buyers enter the market at a stochastic arrival rate. The key limitation is the exogeneity of the sequence. It will be interesting to allow buyers to make a decision on the timing of their own arrival. This will add extra strategic interactions among buyers themselves and also between buyers and the seller. Further research needs to be carried out in this direction. Second, when we compare auction and posted price sale, we simply assume that the seller holds the auction after every buyer arrives. We do this in order to capture the main difference between auction and posted price sale as well as to simplify calculations. However, we have shown that such an auction mechanism may not be optimal because an early auction may lead to a higher expected revenue. As a result, our comparison essentially is one between posted price sale and the 'standard' auction used in practice. We can try adding the early auction option into the comparison in future studies. Finally, due to the complexity of the models, we cannot obtain explicit analytical solutions, so we rely on numerical examples to visualize the results. In future studies, it is worth trying to prove some general characteristics of the solutions that do not rely on any particular distribution or valuation function.

Chapter 3

Art Auction Pre-Sale Estimates Interpretation: An Empirical Study

3 Art Auction Pre-Sale Estimates Interpretation: An Empirical Study

3.1 Introduction

Most fine works of art are sold in auctions, in which their monetary values are established by the highest bids from the buyers. Before auctions, auction houses send their specialists to evaluate the item and then publish a low estimate and a high estimate of its market value. A question that draws much attention of researchers is whether these pre-sale estimates are unbiased indicators of the item to be sold in art auctions.

Milgrom and Weber (1982) argue that committing to truthfully reveal all private information before an auction is the best strategy for the seller because of the linkage principle. Ashenfelter (1989b) also agrees that pre-sale estimates should be unbiased in predicting prices in auctions given that the market is efficient and competitive. However, the efficiency assumptions can be violated in practice, resulting in biased pre-sale estimates. Specifically, auction houses that act as the agent for sellers may hold different interests from the sellers' when organizing the auctions, maximizing their own expected profit. For example, the auction houses may have an incentive to lower the appraised values to reduce the insurance costs in case the items were damaged or stolen. In addition, an indicator of performance called 'value realized rate' is published after major auctions, which measures the total sale revenue realized in the auctions as a proportion to the pre-sale estimates. Auction houses may wish to enhance their public image by strategically lowering the pre-sale estimates to achieve a higher value realized rate. Ashenfelter and Graddy (2003) documented some anecdotal evidence on such agency problems. Ekelund, Jackson, and Tollison (2013) point out that 'conservative forecasting' can be one explanation for the downward bias in pre-sale estimates. Furthermore, Mei and Moses (2002) show that auction houses may use the pre-sale estimates to manipulate beliefs of buyers, assuming that the buyers are credulous. Another factor that can possibly lead to the bias of the pre-sale estimates is the restriction set by auction houses on the reserve price. All major auction houses stipulate that the reserve price must not exceed the pre-sale low estimate of the item. Ashenfelter, Graddy, and Stevens (2002) argue that an upwardly biased low estimate may occur if the seller wishes to set a higher reserve price and negotiates with her auction house to raise the low estimate.

Mixed results are found in numerous empirical studies that try to test the bias of the pre-sale estimates in art auctions. Abowd and Ashenfelter (1988) find that the pre-sale estimates are unbiased and that they perform much better than hedonic price functions in predicting prices. Ashenfelter (1989b) shows that the auction houses are truthful in reporting pre-sale estimates. Czujack and Martins (2004) conclude that the auction houses have given unbiased predictions for the works that have been sold, using the data on 675 Picasso paintings in auctions conducted between 1975 and 1994. McAndrew, Smith, and Thompson (2012) analyzed a set of French Impressionist paintings listed in auctions from 1985 to 2001 and find no evidence of bias in the pre-sale estimates published by the auction houses. On the other hand, Beggs and Graddy (1997) show that there are systematic under-predictions in a group of impressionist and modern artworks while over-predictions occur in another group of works in contemporary art. Bauwens and Ginsburgh (2000) also find evidence of a significant bias in the pre-sale estimates, but they also conclude that the bias is small in size, studying 1,621 English silver teapots auctioned in London between 1976 and 1990. Mei and Moses (2002) use a combined large data set of paintings to show an upward bias in the pre-sale estimates for expensive paintings.Ekelund, Jackson, and Tollison (2013) conclude that the pre-sale estimates are biased downward, based on the sale records on paintings by a group of eight American artists.

In this chapter, we have constructed a relatively large data set and re-visit the question that how one should interpret the pre-sale estimates in art auctions. We follow Ekelund, Jackson, and Tollison (2013) and believe that the eight artists included in their samples are a well defined and acknowledged group of American artists who banded together and enjoyed similar fames. Paintings by them constitute good samples with limited heterogeneity, so we decide to base our study on them. We collected information on their paintings listed in auctions between 1987 and 2018 and constructed a data set of 3923 observations with detailed records, while Ekelund, Jackson, and Tollison (2013) only obtained 557 workable observations. We consider this rich and unique data set with a sufficiently large sample size as one of our contributions.

There are two main issues with the previous empirical studies in the literature. First, all of them take parametric approaches, in which particular specifications of the 'true regression model' are assumed to enable hypothesis testing. Consequently, the test results inevitably depend on the regression equations. The potential misspecification problem could be the reason why different studies disagree on the bias of the pre-sale estimates in art auctions. To address this problem, we adopt a nonparametric method to re-visit the unbiased hypothesis test on the pre-sale estimates, following other standard assumptions in the literature. One challenge we face is the sample selection issue in auction sales because the hammer price is not recorded if one item fails to meet the reserve price and 'passes in'. We follow Das, Newey, and Vella (2003) and use a two-step approach, which can be seen as an extension of the standard sample selection model proposed by Heckman (1979), to correct for the sample selection bias when running the nonparametric regression. In addition, parametric methods are also carried out in comparison using the same data set in this chapter. We conclude that evidence of the bias is not strong. In addition, among the few popular specifications we tested, the nonparametric model seems to agree that a simple linear model is a good one. Although parametric regressions show that the bias exists, the size of the bias is small.

The second issue is that previous studies do not fully use the information embedded in the pre-sale estimates. As stated before, the pre-sale estimate for an item is a price interval, consist of a low estimate and a high estimate. However, when testing the bias of the pre-sale estimates, only one variable, which solely represents the pre-sale estimates by itself, is included in the regression models. Thus, a formal process of transforming the interval estimate into a point estimate is necessary. Surprisingly, none of the previous studies in the literature pays enough attention to this step; most of them just intuitively use the arithmetic mean of the low and the high estimates as the point estimate for the test. The underlying assumption of the arithmetic mean approach is that the low and the high estimates are symmetric about the mean of the price distribution estimated by the art experts. However, this assumption is problematic as the art experts are likely to have other priorities when making estimates.

The pre-sale estimate for an item is essentially a confidence interval of its market price if we assume that the art experts must ensure the actual hammer price falls between the low and the high estimates with at least a certain probability level(i.e. the confidence level). It is reasonable to assume that the art experts' main objective is to narrow the span of such a confidence interval. We show that the distribution of the hammer price is left-skewed in an English auction with symmetrically distributed independent private values. As a result, the low and the high estimates provided by the art experts will not be symmetric about the mean of the price. More specifically, we predict that the arithmetic mean of the low and the high estimates is larger than the mean of the price if the confidence level used by the art experts stays below a certain threshold.[14] Therefore, we conclude that it is important to include both the low and the high estimates as two variables in the regression model if one wishes to fully interpret the pre-sale estimates. Based on the same data set, we find empirical evidence to suggest that a relatively higher weight should be assigned to the low estimate to predict the hammer price.

[14]The threshold is given as c^* in Proposition 3.2.

In Section 2, the data is described. In Section 3, we apply both nonparametric and parametric methods to test whether the arithmetic mean of the low and the high estimates is biased. In Section 4 we use a simple theoretical model to illustrate that a symmetric distribution of buyer values leads to an asymmetric distribution of the auction price, which could be a source of the bias detected in Section 3. We also predict that the arithmetic mean of the low and the high estimates overestimates the mean of the auction price if the confidence level is below a threshold. In Section 5 we include both the low and the high estimates in the regression models and show that empirically the low estimate should be given a higher weight when interpreting the pre-sale estimates in art auctions. Section 6 concludes this chapter and discusses the limitations.

3.2 Data

All paintings in our data set are created by eight American artists. They are Robert Henri (1865 - 1929), Arthur B. Davies (1862 - 1928), William Glackens (1870 - 1938), Ernest Lawson (1873 - 1939), George Luks (1867 - 1933), Maurice Prendergast (1858 - 1924), Everett Shinn (1876 - 1953), and John Sloan (1871 - 1951). This group of artists are called 'The Eight' by art collectors. They are also identified as the 'Ashcan School' after they banded together for a group exhibition at the Macbeth Galleries in New York. We believe that observations about the auction records of their paintings will make a good sample because the authors share similar fames and skills, which will greatly reduce heterogeneities within the data.

Ekelund, Jackson, and Tollison (2013) obtained a data set on the paintings by 'The Eight', but they only achieved 557 workable observations with complete information.[15] We decided to construct our own data set and coded a program that can automatically search through auction data websites[16] and fetch targeted information. We successfully compiled a much larger data set that contains complete auction records on 3923 paintings by 'The Eight' auctioned between 1987 and 2018.[17] The sample distribution of the paintings by author is summarized in Table 1. The Sotheby's and the Christie's are the two major auction houses in our sample. They conducted auctions for 1,029 and 1,018 pieces of paintings from our sample, respectively.

Each observation provides the following detailed information: the title of the painting,

[15] There are in total 2500 observations in their data set, many of them with missing information. I tried to request the access to their data but unfortunately did not get any response

[16] Most of the data are fetched from www.askart.com.

[17] I give special thanks to my colleague Julian Kuan, who coded the web crawler program and helped a lot in cleaning the data.

Table 1: Observations by author

	Artist								
	Henri	Davies	Glackens	Lawson	Luks	Prendergast	Shinn	Sloan	Total
Observation	589	565	465	486	441	356	643	378	3923
Percentage	15%	14.4%	11.9%	12.4%	11.2%	9.1%	16.4%	9.6%	100%

Table 2: Partition of observations

	Passed-in	HP < Low	Low ≤ HP ≤ High	High < HP	Total
Observation	1031	1221	931	740	3923
Percentage	26.3%	31.1%	23.7%	18.9%	100%

*Low - the low estimate; High - the high estimate; HP - the hammer price

author, size (in $inch^2$), media (for example, canvas, paper, board and so on), colour style (multi-coloured or monochrome), whether the piece was signed by the artist, auction house, date of auction, low and high estimates of the price, whether the piece was sold or passed in, and the hammer price[18] if the painting was sold. The pre-sale estimates and the hammer prices are all converted to their dollar values in the year 2018, using the Consumer Price Index published by the U.S. Federal Reserve Bank.

One complication with the data is that the highest bid (the auction price[19]) was not recorded if the price did not meet the secret reserve and the item was passed in. Table 2 shows the partition of all observations by the level of the highest bids relative to the secret reserve and the pre-sale estimates. It shows that 26.3% of paintings failed to be sold in auctions, hence the highest bids for them are not observable in our data.

3.3 Testing for the bias of the pre-sale estimates

In this section, we test whether the pre-sale estimates are unbiased predictions of realized auction prices. Denote Y_i as the hammer price of painting i, L_i and H_i as the low estimate and the high estimate, respectively, for painting i. We first follow the standard approach in the

[18]Some observations only record the actual price paid by the winner in the auction. It differs from the hammer price by a pre-determined proportional charge called 'buyer premium'. The 'buyer premium' is published by auction houses, so the hammer price can be recovered from the actual price paid by the purchaser.

[19]All the auctions were conducted in the open English auction format, so the highest bid determined the market value of the painting.

literature to use the arithmetic mean of L_i and H_i to represent the pre-sale estimates in the regression models: $M_i \equiv \frac{1}{2}(L_i + H_i)$. Unbiasedness requires that the expectation of the hammer price is equal to the pre-sale estimate:

$$E(Y_i \mid M_i) = M_i \tag{116}$$

The hammer price Y_i can only be observed if the painting is sold in the auction. As a result, a formal sample selection model can be written down as follows:

$$d_i = 1(Z_i'\gamma + \varepsilon_i > 0), \qquad \text{selection process} \tag{117}$$

$$Y_i \cdot d_i = Y_i^* = (F(M_i) + u_i)d_i, \qquad \text{price equation} \tag{118}$$

Equation 117 formulates the selection process: dummy variable d_i is equal to one if painting i is sold and zero if it passes in. Z_i is a variable vector that includes all relevant variables that may influence the sale result in the auction. All variables in Z_i come from the information set Ω that is available to the art experts before the auction. In our data set, they include the following regressors: painting size in square inches, signature dummy variable,[20] four medium dummy variables,[21] colour dummy variable,[22] seven author dummies,[23] two auction house dummies,[24] recession dummy,[25] seven time dummies,[26] the low estimate and the high estimate. ε_i is a random disturbance with zero conditional mean $E(\varepsilon_i \mid \Omega) = 0$ and a finite variance.

Equation 118 describes the relationship between the hammer price and the pre-sale estimate for painting i, where F is a function of M_i. u_i is the error term in the price equation, also satisfying the zero conditional mean assumption. Y_i^* is the observed hammer price variable; it takes the value of zero for paintings that failed to be sold in the auction. As long as u_i and ε_i are correlated, directly running a regression with observed Y_i^* causes a sample selection bias. The standard practice to correct this sample selection bias is to either use a two-step

[20] It is equal to one if the painting was signed by the artist and zero otherwise.

[21] They are board, paper, canvas and panel, with other hard materials as the base group.

[22] It is equal to one if the painting shows multiple colours and zero otherwise.

[23] Arthur B. Davies is the base artist.

[24] They are Sotheby's and Christie's, with all other smaller auction houses as the base group.

[25] It takes the value of one if the painting was auctioned in a recession and value of zero otherwise. The recession periods are identified by the National Bureau of Economic Research (NBER). There were three recessions in total in the time horizon of our data set.

[26] The time horizon of our data set is divided into eight four-year periods. The base period is from 2014 to 2018.

Heckman (1979) estimator using the inverse Mills ratio (IMR) or take a Maximum Likelihood (ML) approach.

Assume a simple linear model for the price equation: $F(M_i) = \beta_0 + \beta_1 M_i$, then we can conclude

$$E(Y_i \mid M_i, d_i = 1, Z_i) = \beta_0 + \beta_1 M_i + \rho\sigma \text{IMR}_i \tag{119}$$

where ρ denotes the correlation between u_i and ε_i, σ denotes the standard error of u_i. The inverse Mills ratio is defined as $\text{IMR}_i = \frac{\varphi(t_i)}{\Phi(t_i)}$, with φ and Φ being the standard normal density and distribution functions, where t_i is the predicted value from the probit equation 117. Alternatively, one can obtain ML estimators by maximizing the following likelihood function:

$$\begin{aligned} L = &\prod_{d=0} \Phi(-Z_i'\gamma) \\ &\times \prod_{d=1} \left[\frac{1}{\sigma}\varphi\left(\frac{Y_i - \beta_0 - \beta_1 M_i}{\sigma}\right)\right] \Phi\left[\left(Z_i'\gamma + \rho\left(\frac{Y_i - \beta_0 - \beta_1 M_i}{\sigma}\right)\right)(1-\rho^2)^{-\frac{1}{2}}\right] \end{aligned} \tag{120}$$

In general, the ML estimators are more efficient than the two-step Heckman estimators. As a result, we only present ML estimation results in this section.

With the simple linear specification, the unbiasedness hypothesis we need to test is $H_0 : \beta_0 = 0, \beta_1 = 1$. Another commonly used specification of the price equation in the literature is in log form:

$$\ln Y_i \cdot d_i = \ln Y_i^* = (\beta_0 + \beta_1 \ln M_i + u_i) d_i \tag{121}$$

Assuming the distribution of u_i is exponential normal, the unbiasedness hypothesis we need to test is $H_0 : \beta_0 = 0, \beta_1 = 1$.

Table 3 summarizes the regression results of the selection model with the level form price equation and the log form price equation in comparison. Robust standard errors are used in the regressions because heteroskedasticity is detected in Breusch-Pagan test. Both specifications show significant correlation ρ between the error terms in the selection equation and the price equation. This confirms that selection bias will occur if the regression is run directly on the observed hammer price, so the two-stage selection model is absolutely necessary to correct for such a bias. With the selection equation, we find that the low and the high pre-sale estimates do not affect the probability of sale. Also, the colour style of the paintings and the medium type do not affect the probability of sale, expect that paintings on canvas tend to have a higher passed-in rate. Paintings by George Luks, Maurice Prendergast and Everett Shinn seem to be

Table 3: MLE results for the two parametric models

	Level model				Log model			
	Coeff.	R.S.E.	95% C.I.		Coeff.	R.S.E.	95% C.I.	
Price equation								
constant	-3087.695	6424.547	-15679.58	9504.186	-0.276***	0.063	-0.401	-0.152
M	1.142***	0.119	0.909	1.375				
$\ln(M)$					1.032***	0.006	1.021	1.043
Selection equation								
recession	-0.178**	0.079			-0.200***	0.076		
Sotheby's	0.258***	0.064			0.294***	0.060		
Christie's	0.164***	0.062			0.169***	0.058		
signature	0.062	0.062			0.140**	0.059		
size	-0.0001*	0.000			-0.000**	0.000		
Henri	0.172**	0.086			0.176**	0.083		
Glackens	-0.73	0.089			-0.069	0.084		
Lawson	-0.141	0.088			-0.198**	0.083		
Luks	-0.258***	0.086			-0.247***	0.083		
Prendergast	-0.376***	0.105			-0.391***	0.097		
Shinn	-0.221***	0.082			-0.222***	0.081		
Sloan	0.029	0.095			-0.012	0.092		
board	-0.047	0.096			-0.037	0.092		
paper	-0.106	0.114			-0.057	0.110		
canvas	-0.223**	0.113			-0.200*	0.108		
panel	-0.112	0.146			-0.109	0.136		
coloured	-0.006	0.067			-0.023	0.066		
low estimate	-0.000	0.000			-0.000	0.000		
high estimate	0.000	0.000			-0.000	0.000		
year90	0.111	0.098			0.195**	0.092		
year94	-0.115	0.099			-0.074	0.092		
year98	0.163	0.096			0.233***	0.090		
year02	0.053	0.093			0.161*	0.088		
year06	0.244**	0.086			0.361***	0.084		
year10	0.001	0.086			0.076	0.083		
year14	-0.104	0.083			-0.099	0.078		
constant	0.766***	0.144			0.609***	0.139		
Equation interdependence								
ρ	-.0329**	.015			-0.569***	0.059		

* p < 0.10;   ** p < 0.05;   *** p < 0.01

harder to sell, while Robert Henri's works enjoy a better chance to be sold in auctions.[27] As one would expect, paintings auctioned during recession periods have a slimmer chance to meet the reserve price. Not surprisingly, the two big auction houses, the Sotheby's and the Christie's, performed better compared to the other smaller ones. There is some evidence that paintings with the signature of the artists are more likely to be sold, but the two different price equation specifications disagree on its significance.

Apparently, the log model rejects the hypothesis that the pre-sale estimates are unbiased in predicting the hammer price at 5% significance level. Table 3 shows that β_0 is significantly negative and the 95% confidence interval for β_1 is above 1. After transforming the log equation to the level form, the regression results for the price equation can be written as

$$E(Y_i \mid M_i) = 0.76 M_i^{1.03} \tag{122}$$

The multiplier coefficient (0.76) shows an upward bias, while the power coefficient (1.03) indicates a downward bias. Ekelund, Jackson, and Tollison (2013) also find the two types of biases, but they conclude both types lead to an underestimation bias. With our result, the overall direction of the bias depends on which effect dominates the other. For example, when the value of the painting is relatively low (i.e. M_i is small), the multiplier effect is dominating so the bias is an overestimation. When the value of the painting is high, we expect an underestimation bias as the power coefficient shall have a larger effect.

With the level model, the evidence of the bias is weaker and not so obvious. If we independently test $H_0 : \beta_0 = 0$ and $H_0 : \beta_1 = 1$, we cannot reject these two null hypotheses at 5% significance level. However, when we test the joint hypothesis, the unbiasedness of the pre-sale estimates are rejected at a significant level as low as 2%. In addition, the size of the bias is actually very small, which is demonstrated in Table 4. If the hypothesis is that the pre-sale estimate undervalues the paintings by 2%, then such a hypothesis can only be rejected at 10% significance level. Furthermore, we cannot reject the hypothesis that the pre-sale estimate is downwardly biased by only 3%. In conclusion, the level model finds that there exists a downward bias in the pre-sale estimates, but the size of the bias is within 3%, a relatively small level.

The level model and the log model are two popular specifications widely used in the literature. The results we find are similar to the findings in the previous studies. Compared to

[27]Rober Henri was the leader and teacher of this artist group. It is not surprising to see that his paintings are sold with a higher probability in auctions.

Table 4: Joint hypothesis test results on different bias levels

Hypothesis		χ^2	p-value
	$\beta_1 = 1$	7.92	0.019
$\beta_0 = 0$ &	$\beta_1 = 1.02$	4.79	0.091
	$\beta_1 = 1.03$	3.52	0.172

Ekelund, Jackson, and Tollison (2013), who studied the same group of artists, our results differ by concluding that the size of the bias is smaller and that the bias could be an overestimation. Because our data set has a much larger sample size, we believe our findings are persuasive. However, there is one major issue with such a parametric approach. The regression results and the test results heavily rely on the specification of the 'true model'. The different findings we obtained with the level model and the log model, even though the differences are not alarmingly big, illustrate how the specification of the regression equation can alter the final conclusion. As a result, we attempt a nonparametric approach in the rest of the section.

The nonparametric analysis does not assume a pre-determined form of the regression equation. Instead, the function form is derived from the data. This will avoid the potential misspecification problem but will also require a much larger sample size. Our data contains nearly 4,000 observations, allowing us to conduct the first nonparametric regression on this topic in the literature.[28]

Another challenge with the nonparametric approach is that neither standard Heckman estimators nor the ML methods are applicable to correct for the selection bias. Das, Newey, and Vella (2003) proposed a two-stage method for the nonparametric analysis, which can be seen as an extension of the Heckman (1979) sample selection model. The structure of the model is identical to equations 117 and 118, but nonparametric regressions are conducted to assess them. Das, Newey, and Vella (2003) prove that

$$E(Y_i \mid M_i, d_i = 1, Z_i) = F(M_i) + \lambda(p) \tag{123}$$

where $\lambda(p)$ is a function of the propensity score p from the selection equation 117. We follow this method and run a nonparametric estimation with local-linear Epanechnikov kernel regressions. The main results are presented in Table 5.

[28] To our best knowledge there is not a study that addresses the bias of the pre-sale estimates in auctions using a nonparametric model.

Table 5: Nonparametric estimation results

	Estimate	S.E.	95% C.I.	
Price equation				
M_i	1.139 ***	0.098	0.976	1.379
p	-54791.68 ***	15859.26		
Selection equation				
M_i	4.37e-07	2.23e-06		

Price equation estimates are averages of derivatives; * p < 0.10; ** p < 0.05; *** p < 0.01

One important assumption that ensures the validity of this nonparametric selection model is that M_i does not influence the selection result (i.e. whether or not the painting will be sold in the auction) given all other variables from the information set Ω. Table 5 shows that M_i is insignificant in the selection equation (Equation 117). This confirms that our nonparametric model is indeed valid. Also, $\lambda(p)$ is very significant in the price equation (Equation 118). It implies that such a sample selection model is necessary, otherwise, a selection bias will occur. Finally, the average of the derivative of the hammer price with respect to M_i is estimated to be 1.139, and its 95% confidence interval covers the point 1. As a result, on average we do not reject the hypothesis that M_i is unbiased in its marginal effect. However, we are not able to test whether there exists a constant bias that applies to M_i in all levels. This is because this nonparametric approach for the selection model can only identify function $F(M_i)$ up to an additive constant term, as shown by Das, Newey, and Vella (2003).

Figure 7 helps us visualize how the price equation looks like on the dimension of M_i. The curve of the nonparametric margins shows the accumulated estimated marginal effect of M_i on the hammer price. In fact, the figure is supposed to display the 95% confidence intervals for each circled points on that curve, but the 95% confidence intervals are too small relative to the scope of the vertical axis to be seen. Figure 7 also shows that the curve of the margins stays fairly close to the 45-degree straight line, reflecting the fact that the marginal effect is not significantly different from 1. In addition, although the curve of the margins is slightly curved, it seems that a level linear model can still be a good approximation. On the other hand, the fitted curve of the log model (from the parametric regressions) differs from the curve of the nonparametric margins with big discrepancies.

In conclusion, the nonparametric results seem to agree more with the parametric level model.

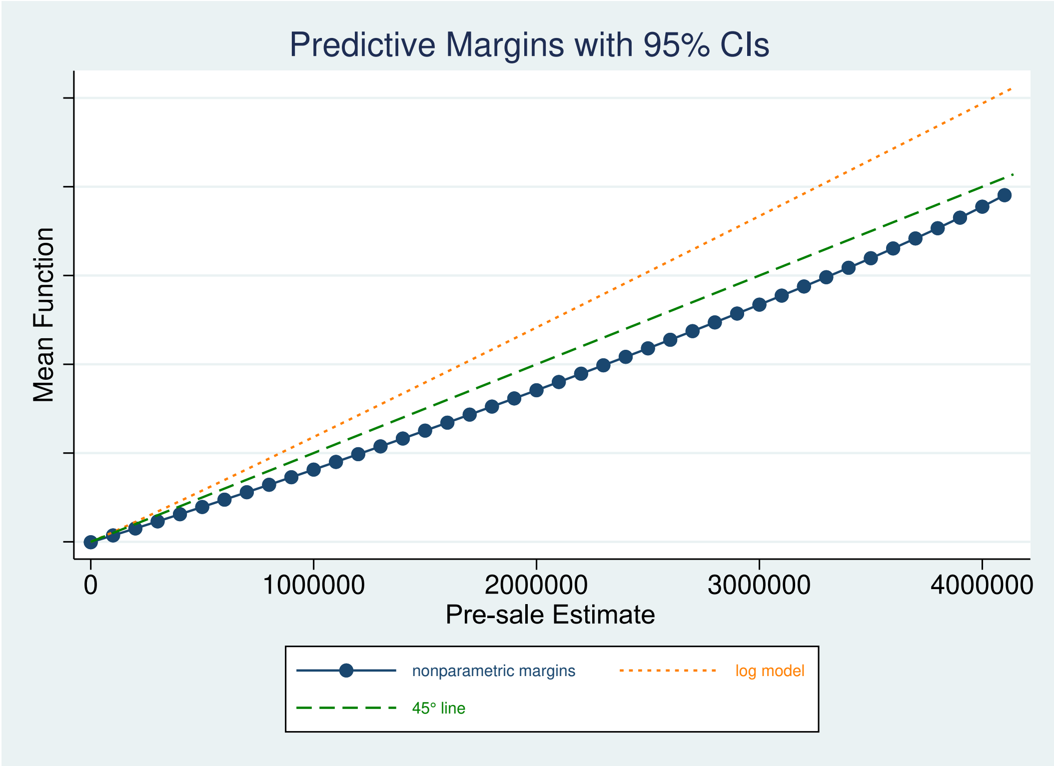


Figure 7: Estimated margins from the nonparametric model

The evidence of bias is weak, as we cannot reject the hypothesis that the pre-sale estimate has a one-to-one marginal effect on the hammer price. The parametric level model has a similar result as the null hypothesis that $\beta_1 = 1$ cannot be rejected independently. Unfortunately, we are not able to test whether the pre-sale estimate is biased by a constant difference using the nonparametric method due to its limitation. However, we notice that the curve of the nonparametric margins on the pre-sale estimate is nearly linear, so it makes sense to believe the parametric level model provides a good approximation. In addition, if the 'true model' is close enough to linear, the parametric level model is more efficient compared to the nonparametric method. Based on our data set, we believe the findings from the parametric level model is more persuasive and conclude that there is some evidence of bias in the pre-sale estimates, but the size of the bias is small. The nonparametric method is a good complementary tool to check the non-linearity of the 'true model' and to re-examine the bias in the marginal effect. Although it is less efficient because part of the data is used to identify the function form, it can be extremely useful with other data set in which the 'true model' exhibits significant non-linear features.

3.4 A theory model on the interpretation of the pre-sale estimates

Both the parametric and nonparametric approaches in the previous section to test the bias of the pre-sale estimates in auctions face a critical problem: only the arithmetic mean of the low and the high estimates (M_i) are used in the price equation (Equation 118) and the unbiasedness hypothesis only involves M_i. Such an approach fails to use the full information conveyed by the pre-sale estimates and will lead to misleading interpretations.

The low (L_i) and the high (H_i) estimates together constitute an interval estimate, M_i is a valid indicator for the bias test only under the assumption that L_i and H_i are symmetric about the actual mean of the distribution of the hammer price believed by the art experts. However, such an assumption is likely to be violated as we illustrate with the following simple model.

We assume that one item is to be sold in an open English auction.[29] There are $n(> 3)$ bidders to participate in the auction, each of them holds a private value, v_i, for the item.[30] We assume the art experts are competent and objective. It is also assumed in this model that after evaluating the item and considering all the market information available, the art experts correctly identify that the values of the bidders are independently and identically distributed according to a distribution function $F(v)$ and a density function $f(v)$. We also assume that the low and the high pre-sale estimates made by the art experts constitute a confidence interval. In other words, the art experts need to ensure that the realized highest bid in the auction lies between the the low and the high estimate with a probability instructed by their auction houses. Finally, we assume that the goal of the art experts is to minimize the span of such a confidence interval under the constraint of the confidence level required by their auction houses.

We believe this model reasonably describes the task of the art experts and captures the main issues on how the pre-sale estimates are made in a simplified framework. The auction houses apparently wish their published pre-sale estimates to be able to predict the auction prices with a decent probability to establish or enhance their reputation in the market. At the same time, if the art experts can make the span between the low and the high estimates as small as possible, it will further demonstrate their expertise and competency, which can potentially attract more clients and profits in the future. On the hand other, the art experts have little incentive to make the low and the high estimate to be symmetric about the expected mean of the hammer

[29] Open English auctions are the most popular auction format, if not the only one, in selling works of fine art.

[30] We believe that the independent private value (IPV) model is the simplest one that describes the art auctions well.

price.

In our analysis, we do not care if the item can be sold or not, so we do not consider the reserve price. Therefore, for simplicity, we call the highest bid the 'hammer price', denoted as P, even though the hammer does not actually fall if the item passes in. We first start with the situation in which the number of bidders n is known.

The cdf of the hammer price P is

$$\begin{aligned} G_n(p) &= \Pr(P \leq p) \\ &= F_n^{(2)}(p) = F^n(p) + n\,(1 - F(p))\,F^{n-1}(p) \end{aligned} \tag{124}$$

where $F_n^{(2)}$ is the CDF of the second highest order statistic among all n bidder values. The density function of P is then the PDF of the second highest order statistic among all v:

$$\begin{aligned} g_n(p) &= f_n^{(2)}(p) \\ &= n(n-1)\,(1 - F(p))\,F^{n-2}(p)f(p) \end{aligned} \tag{125}$$

We have the following proposition on the distribution of the hammer price.

Proposition 3.1. *Assume the density function $f(v)$ is symmetric on $\mathbb{R}$. Then the distribution defined by the density function $g_n(p)$ given by Equation 125 is left-skewed.*

Proposition 3.1 shows that even if the distribution of the bidder value is symmetric we will end up with a left-skewed distribution for the hammer price. Such an asymmetry will influence the art experts in making the low and the high pre-sale estimates as summarized in Proposition 3.2, which is the main result of our theoretical model.

Proposition 3.2. *Assume the density function $f(v)$ is symmetric and log-concave on $\mathbb{R}$. Also, $f(v)$ is assumed to be continuous where $f(v) > 0$. Then, for the distribution function $G_n(x)$ given by Equation 124, there exists c^*, l^* and h^* on $\mathbb{R}$ such that*

1. $(l^*, h^*) \in \underset{(L,H)}{\operatorname{argmin}} \mid H - L \mid$ *s.t.* $\int_L^H d\,(G_n(x)) = c^*$

2. $\frac{l^*+h^*}{2} = \int_{\mathbb{R}} x d\,(G_n(x))$

For any $0 < c < c^*$, if $(\hat{l}, \hat{h}) \in \underset{(L,H)}{\operatorname{argmin}} \mid H - L \mid$ s.t. $\int_L^H d\,(G_n(x)) = c$, then

$$\frac{\hat{l} + \hat{h}}{2} > \frac{l^* + h^*}{2} = \int_{\mathbb{R}} x d\,(G_n(x)) \tag{126}$$

Proposition 3.2 states that there exists a threshold confidence level c^* such that if the art experts minimize the span between the low and the high estimates then the arithmetic mean of them is precisely the unbiased predictor for the hammer price. Furthermore, as long as the confidence level required by the auction houses is below the threshold level c^*, if the art experts minimize the span between the low and the high estimates, the arithmetic mean of them is a biased estimator of the hammer price. The bias is an upward bias.

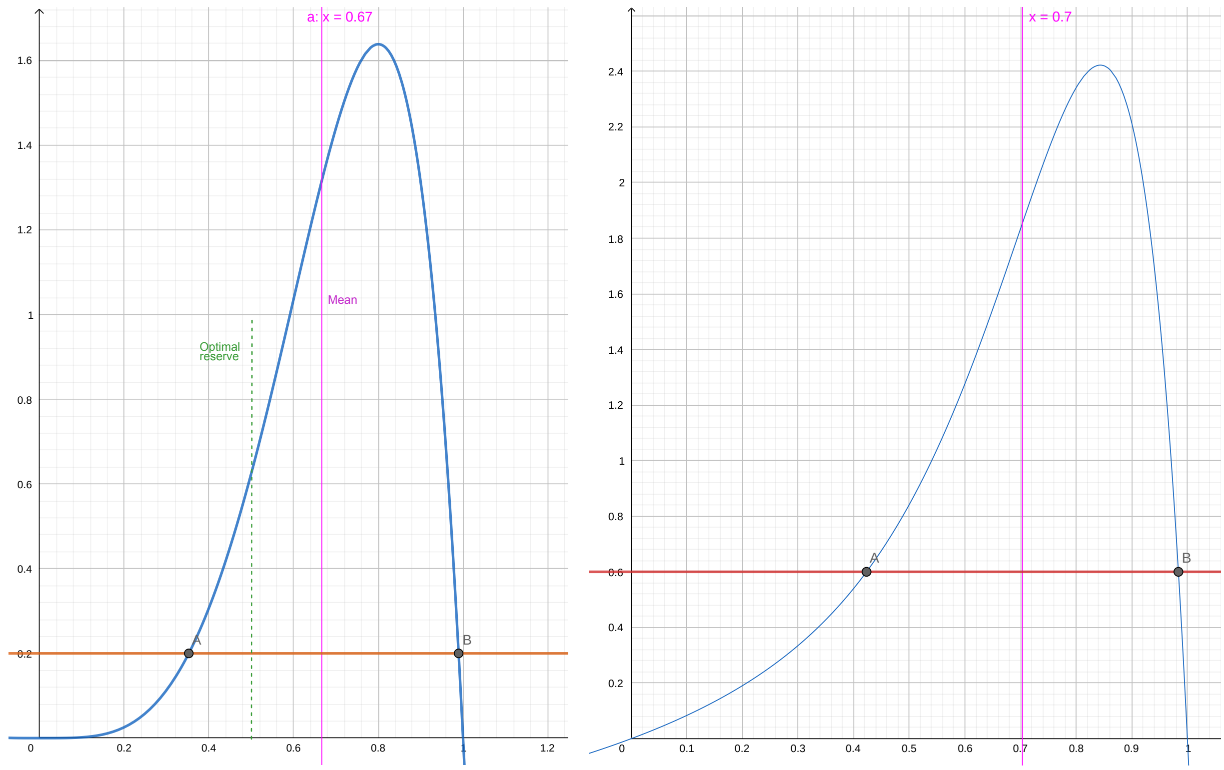
The requirement of $f(v)$ being log-concave is not too restrictive. A non-negative function $f : \mathbb{R}^n \to \mathbb{R}_+$ is log-concave if its domain is a convex set, and if it satisfies the inequality

$$f\left(\theta x + (1-\theta)\, y\right) \geq f(x)^{\theta} f(y)^{1-\theta}, \qquad \text{for all } x, y \in Dom(f) \text{ and } 0 < \theta < 1. \tag{127}$$

In fact, many commonly used distributions, such as uniform distribution, normal and multivariate normal distributions, exponential distribution and logistic distribution, are log-concave. The purpose of this requirement is to ensure that the distribution of the hammer price is strong unimodal.

The main point of Proposition 3.2 is that using the arithmetic mean of the low and the high estimates as the predictor of the hammer price is naturally biased due to the skewness of the underlying distribution, if the confidence level is below some threshold. In other words, the bias detected in the previous section and in other empirical studies in the literature does not necessarily imply that the art experts or the auction houses are incompetent or not objective, as it might merely be a result of using a carelessly chosen point estimator to represent an interval estimator. One important condition for such a bias is a low confidence level relative to some threshold. We can observe some evidence from Table 2 supporting such a phenomenon. As shown in Table 2, only 23.7% of the 3,923 observations ended up with a hammer price that was between the low and the high pre-sale estimates. It implies that the confidence level the art experts have in mind while making the low and the high estimates is roughly 23.7%. Although we are not able to gauge the threshold level c^*, we feel comfortable to believe that a confidence level as low as 23.7% is very likely below c^*. Also, while the highest bid in 57.4% of the observations is below the low estimate, only 18.9% of the observations are sold at a hammer price higher than the high estimate. Such an asymmetry supports the prediction in Proposition 3.1 that the distribution of the hammer price is left-skewed.

So far we have assumed that the number of bidders, n, is known by the experts. Now we relax this assumption by assuming that n is distributed on $[3, N]$ with probability function $\Pr(n) = k(n)$ such that $\sum_{n=3}^{N} k(n) = 1$. Art experts do not observe the realization of n before



(a) $n = 5$ (b) n is evenly distributed on {3,4,5,6,7,8,9,10}

Figure 8: The density function curve for the hammer price with each buyer's value independent uniformly distributed on $[0, 1]$, in the open English auction.

the auction. As a result, the CDF of the hammer price P is

$$\begin{aligned} G(p) &= \Pr(P \leq p) \\ &= \sum_{n=3}^{N} k(n) \cdot \Pr(P \leq p \mid n) = \sum_{n=3}^{N} k(n) F_n^{(2)}(p) \\ &= \sum_{n=3}^{N} k(n) G_n(p) \end{aligned} \tag{128}$$

$G(p)$ is simply a linear combination of $G_n(p)$. With the linearity, the proof of Proposition 3.2 remains valid with respect to $G(p)$. As a result, Proposition 3.2 also applies to the case with random n.

Figure 8 illustrates the main points of Proposition 3.2 with two simple numerical examples. In both subfigures, $f(v)$ is assumed to be a uniform distribution density function on $[0, 1]$. Subfigure (a) shows the density function for the hammer price when $n = 5$, while Subfigure (b) illustrates the case in which n is evenly distributed on $[3, 10]$. In both cases, it is clear that the distribution of the hammer price is left-skewed and unimodal. In addition, points A and B present the location of l^* and h^*, respectively. Points A and B are symmetric about the mean of the hammer price, which is represented by a vertical line. The size of the area surrounded by the curve of the hammer price density function between point A and point B is the confidence level. It is also clear in the figures that when the confidence level is reduced, the distance between A and B also decreases and the midpoint of A and B shifts to the right-hand side of the mean of the hammer price. In Subfigure (a), the optimal reserve price is also marked with a dashed vertical line. Recall most auction houses stipulate that the reserve price must be lower than the low estimate. It is reasonable to assume that auction houses will not exclude the optimal reserve so they must set a confidence level so that the low estimate is higher than the optimal reserve. In the case illustrated by Subfigure (a), this consideration leads to a relatively low confidence level and an upward bias if the mean of the low and the high estimates is used to predict the hammer price.

3.5 Empirical evidence of the theory

The previous section proves that using the arithmetic mean alone in the regression models cannot fully utilize the information conveyed in the pre-sale estimates. Consequently, it will result in bias and leads to problems in interpreting the pre-sale estimates. In this section, we replace the arithmetic mean in the regression models by the low estimate (L_i) and the high estimate (H_i). We show that the low estimate should be given a higher weight when one attempts to predict the hammer price.

The selection model now becomes

$$d_i = 1(Z_i'\gamma + \varepsilon_i > 0), \qquad \text{selection process} \tag{129}$$

$$Y \cdot d_i = Y_i^* = (F(L_i, H_i) + u_i)\, d_i, \qquad \text{price equation} \tag{130}$$

For the parametric level model, it is assumed that

$$F(L_i, H_i) = \beta_0 + \beta_1 L_i + \beta_2 H_i; \tag{131}$$

for the parametric log model, we assume

$$\ln(Y_i^*) = (\beta_0 + \beta_1 \ln(L_i) + \beta_2 \ln(H_i) + v_i)\, d_i; \tag{132}$$

for the nonparametric model, function $F(L_i, H_i)$ remains unspecified until it is identified by the data.

Table 6: Estimation results of models including L and H

	Level model(FGLS)				Log model(FGLS)				Nonparametric model	
	Coeff.	R.S.E.	95% C.I.		Coeff.	R.S.E.	95% C.I.		Estimate	B.S.E.
Price equation										
L	0.872***	0.157	0.564	1.179	0.696***	0.110	0.480	0.913	0.797**	0.390
H	0.287***	0.104	0.083	0.492	0.337***	0.111	0.118	0.554	0.304	0.265
constant	94.477	141.756			0.015	0.076			N/A	
Selection										
IMR	94.477	141.756			-0.790***	0.092				
p									-44852.13***	14203.72

Nonparametric estimates are averages of derivatives; * p < 0.10; ** p < 0.05; *** p < 0.01

We encounter a heteroskedasticity issue, which is detected by the BreuschPagan test. Heteroskedasticity arises in our model because the variance of the error term in the price equation (Equation 130) is expected to be high when the value of the painting is high and/or the market information for the painting is not sufficient to provide an accurate estimate. To tackle the heteroskedasticity issue, apart from adopting the robust standard error, we use Feasible Generalized Least Squares (FGLS) to model the variance function of the error term u_i:

$$Var(u_i \mid \Omega) = \sigma^2 V(L_i, H_i) \tag{133}$$

L_i and H_i enter the variance function not only because they together indicate the price level of the painting but also because $H_i - L_i$ reflects how uncertain the art experts are about the market value of the painting.

Table 6 summarizes the estimation results of different models with both L_i and H_i included in the price equation. First, both the parametric log model and the nonparametric model conclude that the two-stage selection model is crucial to correct for the selection bias, as the inverse Mills ratio (IMR) in the log model and the propensity score p in the nonparametric model are significant, implying a correlation between error terms u_i and v_i. However, IMR is insignificant in the level model, so the sample selection bias may not be a problem if we think the level model is the 'true specification'.

The key results we focus in this section is the weights we should assign to the low and the high estimates in predicting the hammer price. Both the parametric models find significant weights attached to L_i and H_i. All three models show that a higher point estimate is given to the coefficient of the low estimate. This confirms the prediction of our theoretical model in the previous section. That is, the arithmetic mean of the low and the high estimates tend to overestimate the hammer price; to construct an unbiased estimator, higher weight should be assigned to the low estimate. In particular, according to the level model, one dollar increase in the low estimate is associated with more than 87 cents increase in hammer price *ceteris paribus*. On the other hand, only a 29-cent increase in hammer price is expected if the high estimate is increased by one dollar *ceteris paribus*. The figures of the marginal effects are 70 cents versus 34 cents in the log model and 80 cents versus 30 cents in the nonparametric model. The marginal effect of the low estimate is estimated to be about twice as much as that of the high estimate across all three regression models we studied.

Formally, we can test the null hypothesis $H_0 : L = H$ against the alternative $H_a : L > H$ for all three models. We reject the null hypothesis at 1% significance level with the level model. We reject the null hypothesis at 5% significance level in the log model and in the nonparametric model.

In sum, our data provide strong evidence that a higher weight should be given to the low pre-sale estimate when one attempts to predict the hammer price. The findings support the predictions in Proposition 3.2 in our theoretical model. We conclude that the arithmetic mean of the low and the high estimate is not a proper predictor of the hammer price in the art auctions, because it tends to overestimate the hammer price. Therefore, the bias detected in Section 3 and other empirical studies in the literature does not necessarily imply that the art experts are incompetent or not objective.

3.6 Conclusion

In this chapter, we study how the pre-sale estimates in art auctions should be interpreted. The data set we rely on is constructed by fetching auction records from auction data websites with an automatic searching program. The data set contains 3,923 observations on paintings by a well-defined group of American artists which were listed in auctions over the past 32 years.

The first part of this chapter re-visits the question of whether the pre-sale estimates are biased, which is heavily debated in the literature. We contribute to this topic in the following two ways. First, we believe our data set is superior to many other data sets used in previous studies. The sample size is sufficiently large to support better estimations using various regression models. In addition, the artists included in our data belong to a unique school and they show a unified understanding of art creation in their paintings during the same short time period. Therefore, our data suffer less of a heterogeneity problem faced by many studies in the literature. Second, we are the first to adopt a nonparametric approach to analyze the bias in the pre-sale estimates in art auctions; while all previous studies take parametric models by assuming a particular specification of the 'true model'. The parametric approach has a potential mis-specification problem which may lead to misleading results as the estimation heavily relies on the assumption of the structure of the equations. It could be a source of the disagreement about the existence of bias among different studies. We use the nonparametric model proposed by Das, Newey, and Vella (2003) to address the sample section issue in the data set and compare the results to the findings from two popular parametric models. We conclude that there is evidence of bias if the arithmetic mean of the low and the high pre-sale estimates is used to predict the hammer price. There can be two types of biases, one is upward and the other is downward. The overall direction of the bias depends on the which dominates the other. However, the overall size of the bias is small. In addition, the nonparametric model does not show much nonlinearity in the price equation, so the parametric level model can be a good approximation if the sample size is not big enough to support a nonparametric regression.

The second part of this chapter questions the use of a carelessly selected point estimator to represent the pre-sale estimates in the bias test. The pre-sale estimates published by auction houses include a low and a high estimate. In most of the previous studies, including the first part of this chapter, the arithmetic mean of the low and the high estimates are used in the regression models to test for its bias in predicting the hammer price. However, such a point estimator cannot fully represent the information conveyed by the low and the high estimates which should be considered as an interval estimator. In particular, the arithmetic mean is unbiased only if the low and the high estimates are symmetric about the mean of the hammer price, a condition that is unlikely to meet in practice.

We use a simple theoretical model to show that the arithmetic mean of the low and the high estimates is expected to overestimate the hammer price in practice. We assume that the art experts' objective is to minimize the span between the low and the high estimates while ensuring the realized hammer price to fall into the pre-sale estimate range with some probability level required by their auction houses. We show that the distribution of the hammer price in an English auction with independent private values (IPV) is left-skewed and strong unimodal if the distribution of the bidder value is symmetric and log-concave. Essentially the skewness of the hammer price distribution leads to the asymmetry in the low and the high estimates about the mean of the hammer price because the art experts seek to include the most dense part of the distribution in the estimate range. The theoretical model predicts that the arithmetic mean of the low and the high estimates will overestimate the hammer price if the confidence level adopted by the art experts is below some threshold. Few studies in the literature use the geometric mean instead, in principle the geometric mean of the low and the high estimates faces the same problem as with the arithmetic mean. We do not explicitly demonstrate the bias associated with the geometric mean because it is not popular in the literature and also it is difficult to conclude a clear direction of the bias with it.

Guided by the conclusion of the theoretical model, we replace the arithmetic mean in the empirical models by two separate variables: the low estimate and the high estimate. The results from both parametric and nonparametric models suggest that a higher weight should be assigned to the low estimate in order to correctly predict the hammer price. This finding confirms the prediction of our theoretical model. We conclude that the use of the arithmetic mean of the low and the high estimates will result in an upward bias in predicting the hammer price even if the art experts are competent and if the auction houses are truthful. The bias identified in the previous studies does not necessarily imply that the art experts or the auction houses provide misleading pre-sale estimates. On the contrary, it could be because we have been interpreting the pre-sale estimates in a misleading way.

One main limitation of the empirical models in this chapter is that our nonparametric method for the two-stage selection model cannot completely identify the price equation (Equation 118). An additive constant is left out for identification, so we are not able to conduct any test on the constant term in the price equation. In other words, we are not able to test whether the pre-sale estimates are biased by a fixed margin on average, using the nonparametric model. Importantly, our empirical findings rely on the data set, which only contains paintings by 'The Eight', a group of American artists. The results may not be applicable to other groups of artwork. For example, we find that there is little non-linearity in the price equation using the nonparametric model. This observation suggests that a parametric level model can be a good approximation. However, in other art markets, the non-linearity of the 'true model' can be severe, so the parametric level model may give misleading results. Finally, both our theoretical model and the empirical models are the most simplified version. One possible extension for further study is to take into account the fact that many similar paintings (or works of art in general) are gathered together and auctioned sequentially in one auction event. The number of similar paintings in the same auction event and the order of auction may be important explanatory variables that influence the realized hammer price. A more detailed data set and a more comprehensive theoretical model need to be constructed to address such a consideration.

4 Appendix

Proof of Proposition 1.1

Proof. Consider a bidder with value v and follow the notations used in equation (4). Denote Y_1 and Y_2 as the highest and the second highest value among the other $n-1$ bidders, respectively. Then the p.d.f. for the joint distribution of Y_1 and Y_2 is $f_{1,2}(y_1, y_2) = (n-1)(n-2)f(y_1)f(y_2)F^{n-3}(y_2)$. The p.d.f. for Y_1 is $f_1(y_1) = (n-1)F^{n-2}(y_1)f(y_1)$.

$$\begin{aligned} Pr(A) &= Pr(Y_2 < v \le z < Y_1)E[l^I(Y_1)|Y_2 < v \le z < Y_1] \\ &= \int_z^1 \int_0^v l^I(y_1)(n-1)(n-2)f(y_1)f(y_2)F^{n-3}(y_2)dy_2dy_1 \\ &= (n-1)F^{n-2}(v)\int_z^1 l^I(y_1)f(y_1)dy_1 \end{aligned} \tag{134}$$

According to equation (4), when $z \ge v$,

$$\pi(z,v) = F(z)^{n-1}[v - \beta^I(z)] + (n-1)F^{n-2}(v)\int_z^1 l^I(y_1)f(y_1)dy_1 \left[v - b^I(v)\right] \tag{135}$$

In this case, define $\bar{\pi}(z,v) \equiv \pi(z,v)$.

When $z \le v$, following the notations in equation (5), one can check that

$$\begin{aligned} Pr(B) &= \int_v^1 \int_0^v l^I(y_1)(n-1)(n-2)f(y_1)f(y_2)F^{n-3}(y_2)dy_2dy_1 \\ &= (n-1)F^{n-2}(v)\int_v^1 l^I(y_1)f(y_1)dy_1 \end{aligned} \tag{136}$$

and that

$$E[v - b^I(v^*)|C] = \int_z^v \left[v - b^I(y_1)\right] l^I(y_1)(n-1)F^{n-2}(y_1)f(y_1)dy_1 \tag{137}$$

According to equation (5),

$$\begin{aligned} \pi(z,v) =& F(z)^{n-1}[v - \beta^I(z)] + \left[v - b^I(v)\right](n-1)F^{n-2}(v)\int_v^1 l^I(y_1)f(y_1)dy_1 \\ &+ \int_z^v \left[v - b^I(y_1)\right] l^I(y_1)(n-1)F^{n-2}(y_1)f(y_1)dy_1 \end{aligned} \tag{138}$$

In this case, define $\underline{\pi}(z,v) \equiv \pi(z,v)$.

The bidder will truthfully bid according to her type only if

$$\left.\frac{d}{dz}\bar{\pi}(z,v)\right|_{z=v} \le 0$$

and

$$\left.\frac{d}{dz}\underline{\pi}(z,v)\right|_{z=v} \ge 0$$

.

It is easy to check that

$$
\begin{aligned}
&\frac{d}{dz}\bar{\pi}(z,v)\Big|_{z=v}\\
=&(n-1)F^{n-2}(v)f(v)\left[v-\beta^I(v)\right]-F^{n-1}(v)\beta^{I\prime}(v)-(n-1)F^{n-2}(v)f(v)l^I(v)\left[v-b^I(v)\right]\\
=&\frac{d}{dz}\underline{\pi}(z,v)\Big|_{z=v}
\end{aligned}
\tag{139}
$$

As a result, the equilibrium requires

$$
(n-1)F^{n-2}(v)f(v)\left[v-\beta^I(v)\right]-F^{n-1}(v)\beta^{I\prime}(v)-(n-1)F^{n-2}(v)f(v)l^I(v)\left[v-b^I(v)\right]=0 \tag{140}
$$

Rearrange the above differential equation, it becomes

$$
\begin{aligned}
&\left[F^{n-1}(v)\beta^I(v)\right]'=(n-1)F^{n-2}(v)f(v)\left[l^I(v)b^I(v)-l^I(v)v+v\right]\\
\Rightarrow&F^{n-1}(v)\beta^I(v)=F^{n-1}(0)\beta^I(0)+\int_0^v\left[l^I(t)b^I(t)-l^I(t)t+t\right](n-1)F^{n-2}(t)f(t)dt
\end{aligned}
\tag{141}
$$

It is obvious that in the equilibrium a bidder with $v=0$ bids 0, so the unique solution to the differential equation is

$$
\beta^I(v)=\int_0^v\left[l(t)b^I(t)-l(t)t+t\right]d\left[\left(\frac{F(t)}{F(v)}\right)^{n-1}\right]. \tag{142}
$$

If a symmetric pure strategy equilibrium exists, the above is the unique bidding fuction in the first auction. The rest of the equilibrium has been analyzed preceding Proposition 1.1.

□

Proof of Proposition 1.2

Proof. If $\beta^I(v)$ is increasing, ${\beta^I}^{-1}$ exists, so $\pi(z,v)$ is equivalent to $\Pi(\beta,v)$, where $\beta\equiv\beta^I(z)$ and $\Pi(\beta,v)=\pi({\beta^I}^{-1}(\beta),v)$. In addition, $\frac{dz}{d\beta}>0$.

When $z\le v$,

$$
\begin{aligned}
&\frac{\partial}{\partial\beta}\Pi(\beta,v)\\
=&\frac{\partial}{\partial z}\underline{\pi}(z,v)\frac{dz}{d\beta}\\
=&\left((n-1)F^{n-2}(z)f(z)\left[v-\beta^I(z)\right]-F^{n-1}(z)\beta^{I\prime}(z)-(n-1)F^{n-2}(z)f(z)l^I(z)\left[v-b^I(z)\right]\right)\frac{dz}{d\beta}
\end{aligned}
\tag{143}
$$

so

$$
\begin{aligned}
&\frac{\partial^2}{\partial v \partial \beta}\Pi(\beta, v) \\
=&\frac{\partial^2}{\partial v \partial z}\underline{\pi}(z, v)\frac{dz}{d\beta} \\
=&\left((n-1)F^{n-2}(z)f(z) - (n-1)F^{n-2}(z)f(z)l^I(z)\right)\frac{dz}{d\beta} \\
=&\left((n-1)F^{n-2}(z)f(z)\left[1 - l^I(z)\right]\right)\frac{dz}{d\beta} \\
\geq& 0
\end{aligned}
\tag{144}
$$

Hence, $\Pi(\beta, v)$ satisfies the single crossing condition when $z \leq v$.

When $z \geq v$,

$$
\begin{aligned}
&\frac{\partial}{\partial \beta}\Pi(\beta, v) \\
=&\frac{\partial}{\partial z}\bar{\pi}(z, v)\frac{dz}{d\beta} \\
=&\Big((n-1)F^{n-2}(z)f(z)\left[v - \beta^I(z)\right] - F^{n-1}(z)\beta^{I\prime}(z) - (n-1)F^{n-2}(v)f(z)l^I(z)\left[v - b^I(v)\right]\Big)\frac{dz}{d\beta}
\end{aligned}
\tag{145}
$$

so

$$
\begin{aligned}
&\frac{\partial^2}{\partial v \partial \beta}\Pi(\beta, v) \\
=&\frac{\partial^2}{\partial v \partial z}\bar{\pi}(z, v)\frac{dz}{d\beta} \\
=&\Big((n-1)F^{n-2}(z)f(z) - (n-1)F^{n-2}(v)f(z)l^I(z)\left[1 - b^{I\prime}(v)\right] \\
&- (n-1)(n-2)F^{n-3}(v)f(v)f(z)l^I(z)\left[v - b^I(v)\right]\Big)\frac{dz}{d\beta} \\
=&\Big\{F^{n-2}(z) - l(z)\Big(F(v)^{n-2}\left[1 - b^{I\prime}(v)\right] + (n-2)F(v)^{n-3}f(v)\left[v - b^I(v)\right]\Big)\Big\}(n-1)f(z)\frac{dz}{d\beta}
\end{aligned}
\tag{146}
$$

If

$$
F^{n-2}(z) - l(z)\Big(F(v)^{n-2}\left[1 - b^{I\prime}(v)\right] + (n-2)F(v)^{n-3}f(v)\left[v - b^I(v)\right]\Big) \geq 0,
$$

then $\frac{\partial^2}{\partial v \partial \beta}\Pi(\beta, v) \geq 0$ and $\Pi(\beta, v)$ satisfies the single crossing condition when $z \geq v$.

In sum, when inequality (12) holds for all (z, v) such that $z \geq v$, $\Pi(\beta, v)$ satisfies the single crossing condition, so theo the Single Crossing Condition Sufficiency Theorem[31] can be applied.

[31] There are several versions on single crossing condition theorems. Theorem 4.2 in Milgrom (2004) is applied here in this paper.

Since the bidding function $\beta^I(v)$ is the solution to the first order condition of $\max_\beta \Pi(\beta, v)$, as long as it is increasing, it must be the case that $\beta^I(v) \in \arg\max_\beta \ \Pi(\beta, v)$.

The above has shown that incentive compatibility constraint for the equilibrium is satisfied. It is trivial to check that individual rational constraint also holds.

□

Proof of Proposition 1.3

Proof. Follow the notations in the section preceding Propostion 1.3. Denote Y_1 and Y_2 as the highest and the second highest value among the other $n-1$ bidders, respectively.

$$\pi(z, v) = Pr(Y_1 < z)E[v - \beta^{II}(Y_1)|Y_1 < z] + Pr(A)E(v - Y_2|A) \tag{147}$$

Here A is the event that "the bidder loses the first auction by bidding $\beta^{II}(z)$, but the seller holds the second auction and this bidder wins it".

If $z \leq v$, then $A = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5$. A_1 is the event that "$z < Y_2 < v < Y_1$ and the seller holds the second auction"; A_2 is the event that "$Y_2 < z < v < Y_1$ and the seller holds the second auction"; A_3 is the event that "$z < Y_2 < Y_1 < v$ and the seller holds the second auction"; A_4 is the event that "$Y_2 < z < Y_1 < v$ and the seller holds the second auction". $A_1,\ A_2,\ A_3,\ A_4$ are mutually exclusive, and obviously A_5 is of probability 0.

Now it is easy to write

$$\begin{aligned}
&Pr(A)E(v - Y_2|A) \\
=& \int_z^v \int_v^1 l^{II}(y_2)(v - y_2)(n-1)(n-2)f(y_1)f(y_2)F^{n-3}(y_2)dy_1dy_2 \\
&+ \int_0^z \int_v^1 l^{II}(z)(v - y_2)(n-1)(n-2)f(y_1)f(y_2)F^{n-3}(y_2)dy_1dy_2 \\
&+ \int_z^v \int_{y_2}^v l^{II}(y_2)(v - y_2)(n-1)(n-2)f(y_1)f(y_2)F^{n-3}(y_2)dy_1dy_2 \\
&+ \int_0^z \int_z^v l^{II}(z)(v - y_2)(n-1)(n-2)f(y_1)f(y_2)F^{n-3}(y_2)dy_1dy_2 \\
=& \int_0^z l^{II}(z)(v - y_2)\left[1 - F(z)\right](n-1)(n-2)f(y_2)F^{n-3}(y_2)dy_2 \\
&+ \int_z^v l^{II}(y_2)(v - y_2)\left[1 - F(y_2)\right](n-1)(n-2)f(y_2)F^{n-3}(y_2)dy_2
\end{aligned} \tag{148}$$

Then,

$$\begin{aligned}\pi(z,v) = &\int_0^z \left[v - \beta^{II}(y_1)\right](n-1)F^{n-2}(y_1)f(y_1)dy_1 \\ &+ \int_0^z l^{II}(z)(v-y_2)\left[1-F(z)\right](n-1)(n-2)f(y_2)F^{n-3}(y_2)dy_2 \\ &+ \int_z^v l^{II}(y_2)(v-y_2)\left[1-F(y_2)\right](n-1)(n-2)f(y_2)F^{n-3}(y_2)dy_2\end{aligned} \tag{149}$$

In this case, denote $\underline{\pi}(z,v) \equiv \pi(z,v)$.

If $z \geq v$, event A is "$Y_2 < v \leq z < Y_1$ and the seller holds the second auction", so

$$\begin{aligned}&Pr(A)E(v - Y_2|A) \\ =&\int_0^v \int_z^1 l^{II}(v-y_2)(n-1)(n-2)f(y_1)f(y_2)F^{n-3}(y_2)dy_1dy_2 \\ =&\int_0^v l^{II}(z)(v-y_2)\left[1-F(z)\right](n-1)(n-2)f(y_2)F^{n-3}(y_2)dy_2\end{aligned} \tag{150}$$

Then,

$$\begin{aligned}\pi(z,v) = &\int_0^z \left[v - \beta^{II}(y_1)\right](n-1)F^{n-2}(y_1)f(y_1)dy_1 \\ &+ \int_0^v l^{II}(z)(v-y_2)\left[1-F(z)\right](n-1)(n-2)f(y_2)F^{n-3}(y_2)dy_2\end{aligned} \tag{151}$$

In this case, denote $\bar{\pi}(z,v) \equiv \pi(z,v)$.

The bidder will truthfully bid according to her type only if

$$\left.\frac{d}{dz}\bar{\pi}(z,v)\right|_{z=v} \leq 0$$

and

$$\left.\frac{d}{dz}\underline{\pi}(z,v)\right|_{z=v} \geq 0$$

In addition, one can check that

$$\begin{aligned}&\left.\frac{d}{dz}\bar{\pi}(z,v)\right|_{z=v} \\ =&\left.\frac{d}{dz}\underline{\pi}(z,v)\right|_{z=v} \\ =&(n-1)F^{n-2}(v)f(v)v - \beta^{II}(v)(n-1)F^{n-2}(v)f(v) \\ &+ \left(l^{II\prime}(v)\left[1-F(v)\right] - l^{II}(v)f(v)\right)\int_0^v (v-y_2)(n-1)(n-2)f(y_2)F^{n-3}(y_2)dy_2\end{aligned} \tag{152}$$

so the equilibrium requires

$$\begin{aligned}&(n-1)F^{n-2}(v)f(v)v - \beta^{II}(v)(n-1)F^{n-2}(v)f(v) \\ &+ \left(l^{II\prime}(v)\left[1-F(v)\right] - l^{II}(v)f(v)\right)\int_0^v (v-y_2)(n-1)(n-2)f(y_2)F^{n-3}(y_2)dy_2 \\ =&0\end{aligned} \tag{153}$$

Rearranging the equation above gives the unique solution for $\beta^{II}(v)$:

$$
\begin{aligned}
&\beta^{II}(v)\\
=&v+\left(l^{II\prime}(v)\left(1-F(v)\right)-l^{II}(v)f(v)\right)\left(\frac{v}{f(v)}-\frac{1}{f(v)}\int_0^v y_2 d\left(\left(\frac{F(y_2)}{F(v)}\right)^{n-2}\right)\right)
\end{aligned}
\tag{154}
$$

Therefore, if a symmetric pure strategy equilibrium exists, the above is the unique bidding function in the first auction. The rest of the equilibrium has been analyzed preceding Proposition 1.3.

□

Proof of Proposition 1.4

Proof. If $\beta^{II}(v)$ is increasing, ${\beta^{II}}^{-1}$ exists, so $\pi(z,v)$ is equivalent to $\Pi(\beta,v)$, where $\beta \equiv \beta^{II}(z)$ and $\Pi(\beta,v)=\pi({\beta^{II}}^{-1}(\beta),v)$. In addition, $\frac{dz}{d\beta}>0$.

When $z \geq v$,

$$
\begin{aligned}
&\frac{\partial^2}{\partial v \partial \beta}\Pi(\beta,v)\\
=&\frac{\partial^2}{\partial v \partial z}\bar{\pi}(z,v)\frac{dz}{d\beta}\\
=&\frac{dz}{d\beta}\Big\{(n-1)F^{n-2}(z)f(z)\\
&+\left(l^{II\prime}(z)\left(1-F(z)\right)-l^{II}(z)f(z)\right)\int_0^v (n-1)(n-2)F^{n-3}(y_2)f(y_2)dy_2\Big\}\\
=&\frac{dz}{d\beta}(n-1)F^{n-2}(z)\left\{f(z)\left[1-l^{II}(z)\right]+l^{II\prime}(z)\left[1-F(z)\right]\right\}\\
\geq&0,
\end{aligned}
\tag{155}
$$

as $l^{II}(z)\leq 1$, $F(z)\leq 1$ and $l^{II\prime}(z)\geq 0$. Therefore, $\Pi(\beta,v)$ satisfies the single crossing condition when $z\geq v$.

When $z \leq v$,

$$\begin{aligned}
&\frac{\partial^2}{\partial v \partial \beta}\Pi(\beta, v) \\
=&\frac{\partial^2}{\partial v \partial z}\pi(z, v)\frac{dz}{d\beta} \\
=&\frac{dz}{d\beta}\Big\{(n-1)F^{n-2}(z)f(z) - l^{II}\left[1-F(z)\right](n-1)(n-2)F^{n-3}(z)f(z) \\
&+ \Big(l^{II\prime}(z)\left[1-F(z)\right] - l^{II}(z)F(z)\Big)\int_0^z (n-1)(n-2)F^{n-3}(y_2)f(y_2)dy_2 \\
&+ l^{II}(z)\left[1-F(z)\right](n-1)(n-2)F^{n-3}(z)f(z)\Big\} \\
=&\frac{dz}{d\beta}(n-1)F^{n-2}(z)\Big\{l^{II\prime}(z)\left[1-F(z)\right] + f(z) - l^{II}(z)F(z)\Big\}
\end{aligned} \tag{156}$$

As long as for all z,

$$l^{II\prime}(z)\left[1-F(z)\right] + f(z) - l^{II}(z)F(z) \geq 0,$$

then $\frac{\partial^2}{\partial v \partial \beta}\Pi(\beta, v) \geq 0$ and $\Pi(\beta, v)$ satisfies the single crossing condition.

In conclusion, when inequality (57) holds, $\Pi(\beta, v)$ satisfies the single crossing condition, so the Single Crossing Condition Sufficiency Theorem[32] can be applied. Note that $\beta^{II}(v)$ given in equation (53) is the solution to the first order condition of $\max_{\beta}\Pi(\beta, v)$. According to theorem, as long as $\beta^{II}(v)$ is increasing, it must be the case that $\beta^{II}(v) \in \arg\max_{\beta}\Pi(\beta, v)$.

The above has shown that incentive compatibility constraint for the equilibrium is satisfied. It is necessary to check that individual rational constraint also holds. With bidding function $\beta^{II}(v)$ given in equation (53), the expected payoff to a bidder with value v is

$$\begin{aligned}
\pi(v, v) =& vF^{n-1}(v) - \int_0^v \beta^{II}(y_2)(n-1)F^{n-2}(y_2)f(y_2)dy_2 \\
&+ \int_0^v l^{II}(v)(v-y_2)[1-F(v)](n-1)(n-2)F^{n-3}(y_2)f(y_2)dy_2
\end{aligned} \tag{157}$$

and

$$\begin{aligned}
\frac{d}{dv}\pi(v, v) =& F^{n-1}(v) + (n-1)F^{n-2}(v)f(v)v - \beta^{II}(v)(n-1)F^{n-2}(v)f(v) \\
&+ l^{II}(v)(v-v)[1-F(v)](n-1)(n-2)F^{n-3}(v)f(v) \\
&+ \int_0^v l^{II\prime}(v)(v-y_2)[1-F(v)](n-1)(n-2)F^{n-3}(y_2)f(y_2)dy_2 \\
&+ \int_0^v l^{II}(v)[1-F(v)](n-1)(n-2)F^{n-3}(y_2)f(y_2)dy_2 \\
&- \int_0^v l^{II}(v)(v-y_2)f(v)(n-1)(n-2)F^{n-3}(y_2)f(y_2)dy_2
\end{aligned} \tag{158}$$

[32] There are several versions on single crossing condition theorems. Theorem 4.2 in Milgrom (2004) is applied here in this paper.

but one can check that

$$\begin{aligned}
&\beta^{II}(v)(n-1)F^{n-2}(v)f(v)\\
=&(n-1)F^{n-2}(v)f(v)v\\
&+\int_0^v l^{II\prime}(v)(v-y_2)[1-F(v)](n-1)(n-2)F^{n-3}(y_2)f(y_2)dy_2\\
&-\int_0^v l^{II}(v)(v-y_2)f(v)(n-1)(n-2)F^{n-3}(y_2)f(y_2)dy_2
\end{aligned} \tag{159}$$

so

$$\begin{aligned}
\frac{d}{dv}\pi(v,v) =&F^{n-1}(v)+\int_0^v l^{II}(v)[1-F(v)](n-1)(n-2)F^{n-3}(y_2)f(y_2)dy_2\\
\geq&0
\end{aligned} \tag{160}$$

Since $\pi(0,0)=0$, $\pi(v,v)\geq 0$ for any $v\in[0,1]$. The individual rational constraint is satisfied.

□

Equilibrium Check in the $C\geq\frac{n-2}{n}$ Case With Sealed First Price Auctions

With sealed first price auctions, when $C\geq\frac{n-2}{n}$, the first round bidding function $\beta^I(v)$ is given in equation (17). Therefore, when $z\geq v$,

$$\begin{aligned}
&\frac{d}{dz}\pi(z,v)\\
=&\frac{d}{dz}\bar{\pi}(z,v)\\
=&\frac{n-2}{nC}z^n-(n-1)z^{n-1}+(n-1)vz^{n-2}-\frac{n-2}{nC}v^{n-1}z\\
\leq&\frac{n-2}{nC}z^n-(n-1)z^{n-1}+(n-1)zz^{n-2}-\frac{n-2}{nC}z^{n-1}z\\
=&0
\end{aligned} \tag{161}$$

When $z\leq v$,

$$\begin{aligned}
&\frac{d}{dz}\pi(z,v)\\
=&\frac{d}{dz}\underline{\pi}(z,v)\\
=&\frac{(n-1)(n-2)}{nC}z^n-(n-1)\left(1+\frac{n-2}{nC}v\right)z^{n-1}+(n-1)vz^{n-2}
\end{aligned} \tag{162}$$

so there are only three roots for $\frac{d}{dz}\underline{\pi}(z,v)=0$. They are $z=0$, $z=v$, and $z=\frac{nC}{n-2}\geq 1$.

In addition,

$$
\begin{aligned}
&\frac{d^2}{dz^2}\underline{\pi}(z,v)\bigg|_{z=v}\\
=&\frac{(n-1)(n-2)}{C}z^{n-1}-(n-1)^2\left(1+\frac{n-2}{nC}v\right)z^{n-2}+(n-1)(n-2)vz^{n-3}\bigg|_{z=v}\\
=&\frac{(n-1)(n-2)}{nC}v^{n-1}-(n-1)v^{n-2}\\
\geq&0
\end{aligned}
\tag{163}
$$

Therefore, $\frac{d}{dz}\pi(z,v) \geq 0$ when $z \leq v$. In sum, it is optimal for every bidder to truthfully bid according to their own value, so $\beta^I(v)$ given in equation (17) is indeed an equilibrium bidding function.

Equilibrium Check in the $C \leq \frac{n-2}{n}$ Case With Sealed First Price Auctions

With sealed first price auctions, when $C \leq \frac{n-2}{n}$, the first round bidding function $\beta^I(v)$ is given in equation (33). When $v \leq \frac{n}{n-2}C$, $\beta^I(v)$ is the same as in the $C \geq \frac{n-2}{n}$ case. In a similar way, $\beta^I(v)$ can be checked to be the equilibrium bidding function in the first round.

Now consider the case where $v > \frac{n}{n-2}C$. If $z > v$,

$$
\begin{aligned}
&\frac{d}{dz}\pi(z,v)\\
=&\frac{d}{dz}\bar{\pi}(z,v)\\
=&(n-1)z^{n-2}v-z^{n-1}\left(\frac{n-2}{n}-\frac{n-1}{n(n+1)}\left(\frac{n}{n-2}C\right)^n\frac{1}{z^n}\right)\\
&-(n-1)v^{n-2}\left(v-\frac{n-2}{n-1}v\right)-(n-1)z^{n-2}\left(\frac{1}{n(n+1)}\left(\frac{n}{n-2}C\right)^n\frac{1}{z^{n-1}}+\frac{n-2}{n}z\right)\\
=&(n-1)z^{n-2}v-\frac{n-2}{n}z^{n-1}-v^{n-1}-\frac{(n-1)(n-2)}{n}z^{n-1}\\
<&(n-1)z^{n-2}z-\frac{n-2}{n}z^{n-1}-z^{n-1}-\frac{(n-1)(n-2)}{n}z^{n-1}\\
=&0
\end{aligned}
\tag{164}
$$

If $z \leq v$ (and $C \leq \frac{n-2}{n}$), then

$$\begin{aligned}
&\frac{d}{dz}\pi(z,v) \\
=&\frac{d}{dz}\underline{\pi}(z,v) \\
=&(n-1)z^{n-2}v-(n-1)z^{n-2}\left(\frac{1}{n(n+1)}\left(\frac{n}{n-2}C\right)^{n}\frac{1}{z^{n-1}}+\frac{n-2}{n}z\right) \\
&-z^{n-1}\left(\frac{n-2}{n}-\frac{n-1}{n(n+1)}\left(\frac{n}{n-2}C\right)^{n}\frac{1}{z^{n}}\right)-\left(v-\frac{n-2}{n-1}z\right)(n-1)z^{n-2} \\
=&0
\end{aligned} \tag{165}$$

In conclusion, it is optimal for every bidder to truthfully bid according to their own value, so $\beta^{I}(v)$ given in equation (33) is indeed an equilibrium bidding function.

Equilibrium Check in the $C \geq \frac{n-2}{n-1}$ Case With Sealed Second Price Auctions

With sealed second price auctions, when $C \geq \frac{n-2}{n-1}$, the first round bidding function $\beta^{II}(v)$ is given in equation (53). If $z \leq v$, then

$$\begin{aligned}
&\frac{d}{dz}\pi(z,v) \\
=&\frac{d}{dz}\underline{\pi}(z,v) \\
=&(n-1)z^{n-2}v-(n-1)z^{n-2}\left[\left(1+\frac{n-2}{(n-1)^2C}\right)z-\frac{2(n-2)}{(n-1)^2C}z^2\right] \\
&+\left[\frac{n-2}{(n-1)C}(1-z)-\frac{n-2}{(n-1)C}z\right]\left[(n-1)z^{n-2}v-(n-2)z^{n-1}\right] \\
=&\left[1+\frac{n-2}{(n-1)C}(1-2z)\right](n-1)z^{n-2}v+\frac{2(n-2)}{C}z^{n}-\left(n-1+\frac{n-2}{C}\right)z^{n-1} \\
\geq&\left[1+\frac{n-2}{(n-1)C}(1-2z)\right](n-1)z^{n-2}z+\frac{2(n-2)}{C}z^{n}-\left(n-1+\frac{n-2}{C}\right)z^{n-1} \\
=&0
\end{aligned} \tag{166}$$

If $z > v$, then

$$
\begin{aligned}
&\frac{d}{dz}\pi(z,v)\\
=&\frac{d}{dz}\bar{\pi}(z,v)\\
=&(n-1)z^{n-2}v-(n-1)z^{n-2}\left[\left(1+\frac{n-2}{(n-1)^2C}\right)z-\frac{2(n-2)}{(n-1)^2C}z^2\right]\\
&+\left[\frac{n-2}{(n-1)C}(1-z)-\frac{n-2}{(n-1)C}z\right]v^{n-1}\\
<&(n-1)z^{n-2}z-(n-1)z^{n-2}\left[\left(1+\frac{n-2}{(n-1)^2C}\right)z-\frac{2(n-2)}{(n-1)^2C}z^2\right]\\
&+\left[\frac{n-2}{(n-1)C}(1-z)-\frac{n-2}{(n-1)C}z\right]z^{n-1}\\
=&0
\end{aligned}
\tag{167}
$$

In conclusion, it is optimal for every bidder to truthfully bid according to their own value, so $\beta^{II}(v)$ given in equation (53) is indeed an equilibrium bidding function.

Proof of Proposition 2.1

Proof. According to Equation 82, notice $n_1 > n_2$, we have

$$
\begin{aligned}
&b(t_i,n_2)-b(t_i,n_1)\\
=&\int_{S_2} u(t_i,\mathbf{t}_{-i})h(\mathbf{t}_{-i}\mid t_j\le t_i, j\ne i, j\le n_2)d\mathbf{t_{-i}}\\
&-\int_{S_1} u(t_i,\mathbf{t}_{-i})h(\mathbf{t}_{-i}\mid t_j\le t_i, j\ne i, j\le n_1)d\mathbf{t_{-i}}\\
=&\left(\int_{S_1} u(t_i,\mathbf{t}_{-i})h(\mathbf{t}_{-i}\mid t_j\le t_i, j\ne i, j\le n_2)d\mathbf{t_{-i}}+\int_{S_3} u(t_i,\mathbf{t}_{-i})h(\mathbf{t}_{-i}\mid t_j\le t_i, j\ne i, j\le n_2)d\mathbf{t_{-i}}\right)\\
&-\left(\int_{S_1} u(t_i,\mathbf{t}_{-i})\kappa(\mathbf{t_{-i}})d\mathbf{t_{-i}}+\int_{S_1} u(t_i,\mathbf{t}_{-i})h(\mathbf{t}_{-i}\mid t_j\le t_i, j\ne i, j\le n_2)d\mathbf{t_{-i}}\right)\\
=&\int_{S_1}\left((u(t_i,\mathbf{t}_{-i})-u(t_i,\mathbf{t}_{-i})\right)h(\mathbf{t}_{-i}\mid t_j\le t_i, j\ne i, j\le n_2)d\mathbf{t_{-i}}\\
&+\left(\int_{S_3} u(t_i,\mathbf{t}_{-i})h(\mathbf{t}_{-i}\mid t_j\le t_i, j\ne i, j\le n_2)d\mathbf{t_{-i}}-\int_{S_1} u(t_i,\mathbf{t}_{-i})\kappa(\mathbf{t_{-i}})d\mathbf{t_{-i}}\right)\\
=&\int_{S_3} u(t_i,\mathbf{t}_{-i})h(\mathbf{t}_{-i}\mid t_j\le t_i, j\ne i, j\le n_2)d\mathbf{t_{-i}}-\int_{S_1} u(t_i,\mathbf{t}_{-i})\kappa(\mathbf{t_{-i}})d\mathbf{t_{-i}}
\end{aligned}
\tag{168}
$$

where $\kappa(\mathbf{t_{-i}})=h(\mathbf{t}_{-i}\mid t_j\le t_i, j\ne i, j\le n_1)-h(\mathbf{t}_{-i}\mid t_j\le t_i, j\ne i, j\le n_2)$, $S_1=\{\mathbf{t}_{-i}\mid t_j\le t_i, j\ne i, j\le n_1\}$, $S_2=\{\mathbf{t}_{-i}\mid t_j\le t_i, j\ne i, j\le n_2\}$, so $S_1\subseteq S_2$ and

$S_3 \equiv S_2 \setminus S_1 = \{\mathbf{t}_{-i} \mid t_j \leq t_i, j \neq i, j \leq n_2; t_j > t_i, j \neq i, n_2 < j \leq n_1\}$. We have assumed that $u(t_i, \mathbf{t}_{-i})$ is weakly increasing in $\mathbf{t}_{-i}$, so

$$\begin{aligned}
&b(t_i, n_2) - b(t_i, n_1) \\
=&\int_{S_3} u(t_i, \mathbf{t}_{-i}) h(\mathbf{t}_{-i} \mid t_j \leq t_i, j \neq i, j \leq n_2) d\mathbf{t_{-i}} - \int_{S_1} u(t_i, \mathbf{t}_{-i}) \kappa(\mathbf{t_{-i}}) d\mathbf{t_{-i}} \\
\geq& \int_{S_3} u(\underbrace{t_i, \ldots, t_i}_{n_1}, t_{n_1+1}, \ldots, t_N) h(\mathbf{t}_{-i} \mid t_j \leq t_i, j \neq i, j \leq n_1) d\mathbf{t_{-i}} \\
&- \int_{S_1} u(\underbrace{t_i, \ldots, t_i}_{n_1}, t_{n_1+1}, \ldots, t_N) \kappa(\mathbf{t_{-i}}) d\mathbf{t_{-i}} \\
=&0
\end{aligned} \tag{169}$$

The last equation can be shown to hold because each t_j in $\mathbf{t}$ is independent and identically distributed. □

Proof of Proposition 2.3

Proof. When $n > N - 2$, from Equation 96 it is easy to confirmt that

$$\frac{d}{dt^R} U_R(t^R, n) = n \left(1 + \frac{n-1}{2}\gamma\right) (t^R)^{n-2}(1 - 2t^R). \tag{170}$$

Therefore, $\frac{d}{dt^R} U_R(t^R, n) > 0$ when $t^R < \frac{1}{2}$; and $\frac{d}{dt^R} U_R(t^R, n) < 0$ when $t^R > \frac{1}{2}$. It follows that fix n the maximum of $U_R(t^R, n)$ takes place at $t^R = \frac{1}{2}$.

When $n \leq N - 2$, from Equation 95 we can obtain

$$\begin{aligned}
\frac{d}{dt^R} U_R(t^R, n) =&n \left[1 + \frac{n-1}{2}\gamma - \frac{m}{2}\gamma + \frac{m-1}{m+1}\left(1 + \frac{m-1}{2}\gamma\right)\right] (t^R)^{n-1} \\
&+ (n+1) \left(\frac{n}{2}\gamma - \frac{2n}{n+1}(1 + \frac{n-1}{2}\gamma)\right) (t^R)^n
\end{aligned} \tag{171}$$

where $m = N - n$. It is easy to see that $t^R = 0$ and $t^R = 1 - \frac{(N-n-3)\gamma+4}{(n-3)(N-n+1)\gamma+4(N-n+1)}$ are the only two turning points. By checking the curvature of $U_R(t^R, n)$, notice that

$$\begin{aligned}
\frac{d^2}{dt^{R2}} U_R(t^R, n) =&n(n-1) \left[1 + \frac{n-1}{2}\gamma - \frac{m}{2}\gamma + \frac{m-1}{m+1}\left(1 + \frac{m-1}{2}\gamma\right)\right] (t^R)^{n-2} \\
&+ (n+1)n \left[\frac{n}{2}\gamma - \frac{2n}{n+1}\left(1 + \frac{n-1}{2}\gamma\right)\right] (t^R)^{n-1},
\end{aligned} \tag{172}$$

we can confirm that $t^R = 1 - \frac{(N-n-3)\gamma+4}{(n-3)(N-n+1)\gamma+4(N-n+1)}$ is indeed the maximum point when $n \leq N - 2$. □

Proof of Proposition 2.4

Proof. Denote the solution set to the maximization problem 92 as O. Take $(\hat{t}, \hat{n}) \in O$ but not in $\{(t^{R^*}(n^*), n^*) \mid n^* \in S\}$. If $\hat{n} \in S$, it must be $\hat{t} \neq t^{R^*}(\hat{n})$. However, in the Proof of Proposition 2.3 it has been shown that $t^{R^*}(\hat{n})$ is the only maximum point of $U^R(t^R, \hat{n})$. Therefore, $U^R(\hat{t}, \hat{n}) < U^R(t^{R^*}(\hat{n}), \hat{n})$, contradiction to the assumption that $(\hat{t}, \hat{n})$ maximizes $U_R(t^R, n)$. If $\hat{n} \notin S$, take $n^* \in S$, then we have $U^R(\hat{t}, \hat{n}) \leq U^R(t^{R^*}(\hat{n}), \hat{n}) < U^R(t^{R^*}(n^*), n^*)$. This is contradictory to $(\hat{t}, \hat{n})$ maximizing $U_R(t^R, n)$. In sum, we conclude that $O \subseteq \{(n^*, t^{R^*}(n^*)) \mid n^* \in S\}$.

Assume there exists $n^* \in S$ but $(t^{R^*}(n^*), n^*) \notin O$. If there exists $(\hat{t}, n^*) \in O$, then $\hat{t} \neq t^{R^*}(n^*)$. We have $U^R\left(t^{R^*}(n^*), n^*\right) > U^R(\hat{t}, n^*)$, as it has been shown that $t^{R^*}(\hat{n})$ is the only maximum point of $U^R(t^R, \hat{n})$ in the Proof of Proposition 2.3. It is contradictory to the fact that $(\hat{t}, n^*) \in O$. If there does not exist $(\hat{t}, n^*) \in O$, then for any $(\hat{t}, \hat{n}) \in O$, $\hat{n} \neq n^*$. It follows that $U^R(t^{R^*}(n^*), n^*) \geq U^R(t^{R^*}(\hat{n}), \hat{n}) \geq U^R(\hat{t}, \hat{n})$, so $(t^{R^*}(n^*), n^*)$ must also be in O, which is a contradiction to our assumption. In sum, we conclude that $O \supseteq \{(n^*, t^{R^*}(n^*)) \mid n^* \in S\}$.

In conclusion, $O = \{(n^*, t^{R^*}(n^*)) \mid n^* \in S\}$. □

.

Proof of Proposition 2.5

Proof. First, note that if $P > \frac{\gamma(N-1)}{2} + 1$, $\bar{t}_i = 1$ for all i. In addition, $P > v_i(t_i)$ for all i. As a result, the seller fails to sell the item and gets 0 payoff if she sets the price $P > \frac{\gamma(N-1)}{2} + 1$. On the other hand, if $P \leq \frac{\gamma(N-1)}{2}$, then $v_1(t_1) \geq P$ for all t_1. As a result, the item is sold to the first buyer for sure and the payoff is simply P. In this situation, setting $P < \frac{\gamma(N-1)}{2}$ is suboptimal. In conclustion, the optimal P that maximizes the the expected seller payoff must fall into the following interval $\left[\frac{\gamma(N-1)}{2}, \frac{\gamma(N-1)}{2} + 1\right]$. □

.

Proof of Proposition 3.1

Proof. It can be shown that $g_n(p)$ given in Equation 125 is a special case of function $g_F(x;a,b)$ defined by Jones (2004) as

$$g_F(x;a,b) = \frac{1}{B(a,b)} f(x) F^{a-1}(x) \left(1 - F(x)\right)^{b-1} \tag{173}$$

where $B(a,b)$ is the beta function, when $a = n-1$ and $b = 2$.

According to Jones (2004), if the distribution defined by the CDF $F(x)$ is symmetric in $\mathbb{R}$, the sign of $a-b$ determines the skewness of the density function $g_F(x;a,b)$. In the case of $g_n(p)$, $a - b = n - 3 > 0$. As result, $g_n(p)$ is left-skewed. □

Proof of Proposition 3.2

Proof. First, as shown in the proof of Proposition 3.1, $g_n(p)$ given in Equation 125 is a special case of function $g_F(x;a,b)$. As a result, the work by **?** shows that (in their Theorem 7.2) $g_n(p)$ is strong unimodal and continuous because $f(v)$ is log-concave. Therefore, for any $z \in (0, \max g_n(p))$, there exist two and only two different solutions for p to the equation $g_n(p) = z$. Denote the two solutions as l and h with $l < h$. Note that l and h are functions of z.

Define s as $s \equiv \int_l^h g_n(x)dx$, it is easy to verify that (l,h) is a solution to the maximization problem

$$\begin{aligned} &\max_{(L,H)} \mid H - L \mid \\ s.t. \quad &\int_L^H d\left(G_n(x)\right) = s \end{aligned} \tag{174}$$

Define $m \equiv \frac{1}{2}(l+h)$, it is apparent that m is a continuous function of z. Denote p^* as the peak of $g_n(p)$ as it is unimodal, then $\lim_{z \to g_n(p^*)} m(z) = p^*$. In addition, because $f(v)$ is symmetric on $\mathbb{R}$, $\lim_{z \to 0} m(z) = \int_{\mathbb{R}} x f(x) dx$.

According to Proposition 3.1, $g_n(p)$ is left-skewed, so $\int_{\mathbb{R}} x g_n(x)dx < p^*$. The definition of $g_n(p)$ implies that $\int_{\mathbb{R}} x f(x)dx < \int_{\mathbb{R}} x g_n(x)dx$, so we have $\int_{\mathbb{R}} x f(x)dx < \int_{\mathbb{R}} x g_n(x)dx < p^*$. According to the intermediate value theorem, there exists z^* such that $0 < z^* < g_n(p^*)$ and $m(z^*) = \int_{\mathbb{R}} x g_n(x)dx$.

Let $l^* = l(z^*)$, $h^* = h(z^*)$ and $c^* = \int_{l*}^{h^*} g_n(x)dx$, then

1. $(l^*, h^*) \in \underset{(L,H)}{\operatorname{argmin}} \mid H - L \mid$ s.t. $\int_L^H d\left(G_n(x)\right) = c^*$
2. $\frac{l^*+h^*}{2} = \int_{\mathbb{R}} x d\left(G_n(x)\right)$

Note s is also a continous function of z. It is easy to show that $s(z)$ is decreasing in z and that $\lim\limits_{z \to g_n(p^*)} = 0$ due to the unimodality of $g_n(p)$. By the intermediate value theorem, for any c that $0 < c < c^*$, there exists $\hat{z}$ such that $s(\hat{z}) = c$. By construction, $(\hat{l}, \hat{h}) \in \underset{(L,H)}{\operatorname{argmin}} \mid H - L \mid$ s.t. $\int_L^H d\left(G_n(x)\right) = c$.

Since $g_n(p)$ is strong unimodal and left-skewed, the work by **?** and Jones (2004) show that $l^* < \hat{l} < \hat{h} < h^*$ and that for any z that $z^* \leq z \leq \hat{z}$, $\frac{d}{dl} g_n(l(z)) + \frac{d}{dh} g_n(h(z)) < 0$. Then, one can show that $l'(z) + h'(z) > 0$. As a result,

$$\begin{aligned}
\frac{1}{2}(\hat{l} + \hat{h}) &= \frac{1}{2}\left(\left(l^* + \int_{z^*}^{\hat{z}} l'(x)dx\right) + \left(h^* + \int_{z^*}^{\hat{z}} h'(x)dx\right)\right) \\
&= \frac{1}{2}\left(l^* + h^* + \int_{z^*}^{\hat{z}} \left(l'(x) + h'(x)\right) dx\right) \\
&> \frac{1}{2}(l^* + h^*) \\
&= \int_{\mathbb{R}} x d\left(G_n(x)\right)
\end{aligned} \tag{175}$$

□

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