

# OPTIMAL PORTFOLIO SELECTION WITH STOCHASTIC MAXIMUM DOWNSIDE RISK AND UNCERTAIN IMPLICIT TRANSACTION COSTS

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## Abstract

A multi-stage stochastic optimal portfolio policy that minimizes downside risk in the presence of uncertain implicit transaction costs is proposed. As asset returns in economic recessions and booms are characterised by extreme movements, some individual stocks show an extreme reaction while others exhibit a milder reaction. The study therefore considers a risk-averse and conservative investor who is highly concerned about the performance of his portfolio in an economic recession environment. Maximum negative deviation is taken as the downside risk and stochastic programming is applied with stochastic data given in the form of a scenario tree. A set of discrete scenarios of asset returns is considered, taking the deviation around each return scenario. Thus uncertainties of asset returns and implicit transaction costs are represented by discrete approximations of a multi-variate continuous distribution. The portfolio is rebalanced at discrete time intervals as new information on returns get realised. First-stage optimal-portfolio results show that implicit transaction costs vary from 7.1% to 16.7% of returns on investment.

## Keywords

Maximum downside risk, implicit transaction costs, discrete scenarios, uncertainty.

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## **1. INTRODUCTION**

Banks, fund-management firms, financial consulting institutions and large institutional investors are faced with the challenges of managing their funds, assets and stocks towards selecting, creating, balancing and evaluating optimal portfolios on a continual basis. Financial crises, economic imbalances, algorithmic trading and highly volatile movements of asset prices in recent times have raised high alarms about the management of financial risks. Extreme event risk is present in all areas of risk management. Whether one is concerned with market, credit, operational or insurance risk, one of the biggest challenges facing the risk manager is how to implement risk management models that allow for rare but damaging events, and permit the measurement of their consequences.

In financial markets, the stability and sustainability of future pay-offs of an investment are largely determined by extreme changes in financial conditions rather than typical movements. A decision-making process must be developed which identifies the appropriate weight each investment should have within the portfolio. The portfolio must strike what the investor believes to be an acceptable balance between risk and reward. In addition, the costs incurred in setting up a new portfolio or rebalancing an existing one must be included in any realistic portfolio selection analysis. Investment portfolios should be rebalanced to account for changing market conditions and changes in funding.

In this study, a multi-stage stochastic maximum negative deviation (SMNDTC) model with uncertain implicit transaction costs in optimal portfolio selection is proposed. Maximum negative deviation of asset returns from expected portfolio return is used as portfolio risk. The model takes into account downside risk and corresponding implicit transaction costs in trading in order to provide an investor or investment manager an option of selecting a portfolio knowing the implicit trading costs which are likely to be incurred. The study uses stochastic programming with recourse. The uncertainty about future asset returns and corresponding implicit transaction costs is captured in stages by means of scenarios. Implicit costs are taken to be random as it is in the buying or selling of assets (whose prices are stochastic) that these costs are incurred.

## **2. LITERATURE REVIEW**

It is well documented in the literature that investors generally shun positions in which they would be subjected to catastrophic losses however small the probability these losses carry. Such a “disaster avoidance motive” (Menezes et al., 1980) implies that investors care about extreme negative scenarios in investment and are averse to the risk of sharp price plunges. Hence the potential loss from extreme undesirable returns should become a significant factor in asset pricing. Asset returns in economic recessions and booms are characterised by extreme movements (Jansen & De Vries, 1991). The extreme movements of the market are not always reflected in all the individual stocks. Some individual stocks show an extreme reaction while others exhibit a milder reaction. It is in extreme cases that investors are highly concerned about the performance of their portfolios, particularly the downside movements.

The notion of tail risk or extreme downside risk has increasingly gained consideration in the asset pricing literature. In particular, contrary to the assumptions of the standard Capital Asset Pricing Model (CAPM) of Sharpe (1964) and Lintner (1965), in which portfolio risk is fully captured by the variance of the portfolio return distribution, asset returns display significant negative skewness

and excess kurtosis, both of which increase the likelihood of extreme negative returns (Richard et al., 2015). In the studies that focus directly on the likelihood of extreme returns, Ruenzi and Weigert (2013) use a copula-based approach to construct a systematic tail risk measure and show that stocks with high crash sensitivity, measured by lower tail dependence with the market, are associated with higher returns that cannot be explained by traditional risk factors, downside beta, co-skewness or co-kurtosis. These studies examine the variation in expected returns across individual stocks.

Young (1998) introduces a linear programming model which maximizes the minimum return or minimizes the maximum loss (minimax) over time periods and applies this to the stock indices of eight countries. The analysis shows that the model performs similarly to the classical mean-variance model of Markowitz (1991). Additionally, Young (1998) argues that when data is log-normally distributed or skewed, the minimax formulation might be a more appropriate method compared to the mean-variance formulation which is optimal for normally distributed data. Kamil et al. (2009) develop a single and two-stage stochastic programming model with recourse for portfolio selection in which the maximum downside deviation of asset returns from expected portfolio return is minimized. This study extends their formulation and develops a multistage stochastic maximum negative deviation (SMNDTC) model which takes into account uncertainty of implicit transaction costs and asset returns as well as recourse decisions in discrete time intervals in optimal portfolio selection.

Investment portfolios should be rebalanced to account for changing market conditions and changes in funding. The investor incurs transaction costs during initial trading and in subsequent rebalancing of the portfolio. Trading costs are either direct or indirect. Direct trading costs are observable and include brokerage commissions, market fees and taxes. Indirect costs are invisible and include bid-ask spread, market impact and opportunity costs. Some investors do not like overly high transaction costs, as these are known to erode the profits of investment. The model being proposed considers downside risk and corresponding implicit transaction costs in trading to give an investor or investment manager an option of selecting portfolios knowing the implicit trading costs which are likely to be incurred. The uncertainty about future asset returns and implicit costs is captured in stages and by means of scenarios. The main contributions of this study include:

- the development of a multi-stage stochastic maximum negative deviation model that optimizes portfolios in the presence of uncertain implicit transaction costs incurred in initial trading and in rebalancing of portfolios, and
- the development of a strategy that captures uncertainty in stock returns and in corresponding implicit trading costs in extreme downside movements of stock prices by way of scenarios.

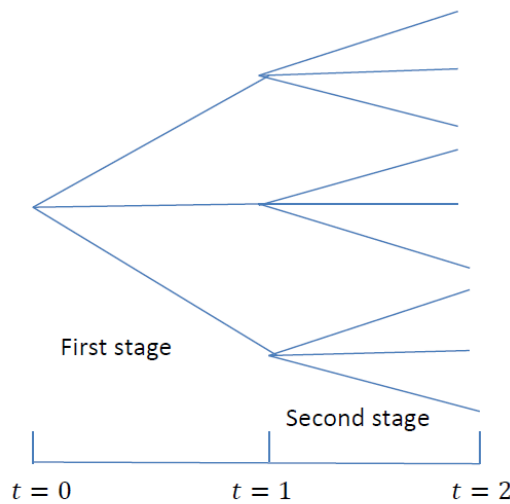
### 3. PROBLEM STATEMENT

A set of securities  $I = \{i: i = 1, 2, \dots, n\}$  is considered for an investment for the period  $[0, T]$ . The study seeks to determine a multi-period discrete-time optimal portfolio strategy subject to uncertain implicit transaction costs. The portfolio is structured in terms of asset return and downside risk measured as the maximum negative deviation of asset return from expected portfolio return. The strategy takes into account the approximate nature of a set of discrete scenarios by considering the negative deviation around each asset return scenario. Let  $R^f =$

$\{R_1, \dots, R_t\}$  be stochastic events at time periods  $t = 1, 2, \dots, \tau$ . The investment horizon  $T$  is divided into two discrete times  $T_1$  and  $T_2$  defined by  $T_1 = [0, \tau]$  and  $T_2 = (\tau, T]$ . During  $T_1$  an investor makes decisions and adjustments to his portfolio at each time-stage as new information on asset returns become available. The initial investment takes place at  $t = 0$ , with recourse decisions implemented at discrete times  $t = 1, 2, \dots, \tau$ . After  $t = \tau$ , no further decisions are implemented until investment maturity at  $t = T$ .

Buying of the initial portfolio assets and implementation of recourse decisions result in the investor incurring some transaction costs, which can erode the value of the investment. The decision process is non-anticipative, that is, a decision at a particular stage does not depend on the future realization of the random events. The recourse decision at period  $t$  is dependent on the outcome at period  $t - 1$ . Given the event history up to time  $t$ ,  $R_t$ , the uncertainty in period  $t + 1$  is characterised by finitely many possible outcomes for the observations  $R_{t+1}$ . The branching process can be represented by a scenario tree. Below is an example of a scenario tree with two-time periods and a three-three branching structure.

It is considered that the uncertain asset returns,  $R_t$ , in period  $t$  are represented by a finite set of discrete scenarios  $\Omega = \{s: s = 1, 2, \dots, S\}$ , where the returns under a particular scenario take values  $R_s = \{R_{1,s}, R_{2,s}, \dots, R_{n,s}\}^T$  with associated probability  $p_s > 0$ , where  $\sum_{s=1}^S p_s = 1$ .



**FIGURE 1: Scenario tree**

### 3.1 Formulation of model constraints

In this problem, an investor dynamically adjusts a portfolio at successive discrete times as new information on asset returns arises. Initial portfolio selection takes place at  $t = 0$  with wealth  $W_0$  distributed among the  $n$  assets of the initial portfolio. The investor seeks to obtain an optimal strategy  $x_t = [x_{1,1,t}, x_{1,2,t}, \dots, x_{2,1,t}, x_{2,2,t}, \dots, x_{1,s,t}]^T$ ,  $t = 1, 2, \dots, \tau$ , at the end of the planning phase. It is noted that

$$\sum_{i=1}^n x_{i,t} = \sum_{s=1}^S x_{i,s,t} = 1, t = 1, 2, \dots, \tau, \tag{1}$$

where  $S$  is the total number of scenarios in period  $t$ . Let  $a_{ist} = \frac{A_{ist}}{W_t}$  and  $v_{ist} = \frac{V_{ist}}{W_t}$ , be, respectively, the buying and selling proportions of asset  $i$  of scenario  $s$  of period  $t$ , where  $A_{ist}$  is the amount of money used to buy new shares and  $V_{ist}$  is the money obtained from selling shares of asset  $i$  of scenario  $s$  of period  $t$ . It is derived that  $x_{i,s,t} = x_{i,s,t-1} + a_{i,s,t} - v_{i,s,t}$ ,  $i = 1, 2, \dots, n$ ;  $s = 1, \dots, S$ ;  $t = 1, \dots, \tau$ , and also that  $a_{i,s,t} \cdot v_{i,s,t} = 0$  since we cannot buy and sell the same asset at each time that recourse decisions are implemented.

The expected return of asset  $i$  of period  $t$  can now be stated as  $r_{it} = \sum_{s \in Q} p_s \cdot R_{ist} \cdot x_{ist}$ ,  $i = 1, \dots, n$ ;  $t = 1, \dots, \tau$ , where  $Q \subset \Omega$  is a set of scenarios of asset  $i$  of period  $t$ . Thus, the gross expected return of the portfolio of period  $t$  becomes  $r_{pt} = \sum_{s=1}^S p_s \cdot R_{ist} \cdot x_{ist}$ ,  $i = 1, \dots, n$ ;  $t = 1, \dots, \tau$ . During portfolio rebalancing, it is ensured that

$$0 \leq v_{ist} \leq x_{ist}; \quad i = 1, \dots, n; \quad t = 1, \dots, \tau \quad (2)$$

since the portfolio is self-financing and there is no additional funding to the portfolio at  $t > 0$ . Thus the volume of asset  $i$  of scenario  $s$  in period  $t$  sold for portfolio rebalancing should not exceed the volume of the asset in the portfolio.

It is observed here that  $v_{it} = \sum_{s \in Q} p_s \cdot v_{ist}$  and  $x_{it} = \sum_{s \in Q} p_s \cdot x_{ist}$ . In a self-financing portfolio being rebalanced, the amount of money gained from selling asset  $i$  of period  $t$  should be at most the amount of money used to buy asset  $j$  ( $i \neq j$ ) of the same period. This results in the constraint

$$0 \leq \sum_{i=A} a_{ist} = \sum_{j=1, j \neq i} v_{jst}, \quad i = 1, \dots, n; \quad s = 1, \dots, S \quad (3)$$

where set  $A$  contains all assets for which volumes have been bought. To avoid short-selling, the constraint

$$0 \leq x_{ist} \leq U_{ist}, \quad i = 1, \dots, n; \quad s = 1, \dots, S; \quad t = 1, \dots, \tau, \quad (4)$$

is considered where  $U_{ist}$  is the maximum proportion allowed for scenario  $s$  of period  $t$  for each asset  $i$ . If  $k_{ist}$  and  $l_{ist}$  are the transaction cost rates for buying and selling, respectively, a unit volume of asset  $i$  in scenario  $s$  for portfolio rebalancing at the beginning of period  $t$ , then either  $k_{ist} \cdot a_{ist} = 0$  or  $l_{ist} \cdot v_{ist} = 0$  or both are zero. The transaction cost incurred by the investor for buying or selling asset  $i$  of scenario  $s$  in period  $t$  is given by  $k_{ist} \cdot a_{ist} + l_{ist} \cdot v_{ist}$ . Therefore, the expected transaction cost of the portfolio of period  $t$  is  $\sum_{s=1}^S p_s \{k_{ist} a_{ist} + l_{ist} v_{ist}\}$ ,  $i = 1, \dots, n$ ;  $t = 1, \dots, \tau$ . This results in the net expected portfolio return,  $N_{pt}$ , of period  $t$  as  $N_{pt} = r_{pt} - \sum_{s=1}^S p_s \{k_{ist} a_{ist} + l_{ist} v_{ist}\}$ , with the portfolio net wealth of period  $t$  given by

$$W_t = (1 + N_{pt}) \cdot W_{t-1}, \quad t = 1, 2, \dots, \tau. \quad (5)$$

Scenarios may reveal identical value for the uncertain quantities up to a certain period. Such scenarios must yield the same decisions up to that period. This results in the constraint

$$x_{ist} = x_{iht}, \quad (6)$$

for all scenarios  $s$  and  $h$  with identical past up to time  $t$ .

### 3.2 Portfolio risk

During the period  $[0; \tau]$ , the downside risk of asset  $i$  of scenario  $s$  in period  $t$  is defined as  $K_{ist} = |\min[0, R_{ist} - r_{pt}]|$ . Thus the expected downside portfolio risk at any time period  $t$  becomes  $\sum_{s=1}^S p_s K_{ist} x_{ist}$ .

This results in the expected downside portfolio risk,  $H$ , for all time periods as

$$H = \frac{1}{\tau} \sum_{t=1}^{\tau} \sum_{s=1}^S p_s K_{ist} x_{ist}. \tag{7}$$

If  $\beta_t = \sum_{s=1}^S p_s K_{ist} x_{ist}$ , the expected portfolio risk for the entire rebalancing phase becomes  $H = \frac{1}{\tau} \sum_{t=1}^{\tau} \beta_t$ .

### 3.3 The multi-stage stochastic maximum negative deviation model

The objective in this problem is to obtain an optimal portfolio that minimizes the expected portfolio risk subject to constraints describing the growth of the portfolio in all periods, a performance constraint and bounds on decision variables. Letting  $\theta$  to be the minimum desired expected portfolio mean return and  $\lambda$  to be the minimum acceptable transaction cost, the following optimization model is obtained:

$$\begin{aligned} \text{Minimize} \quad & H = \frac{1}{\tau} \sum_{t=1}^{\tau} \beta_t \tag{8} \\ \text{Subject to} \quad & N_{pt} \geq \theta \\ & W_t = (1 + N_{pt})W_{t-1}, t = 1, \dots, \tau, \\ & 0 = \beta_t - \sum_{s=1}^S p_s K_{ist} x_{ist}, i = 1, \dots, n; t = 1, \dots, \tau, \\ & 0 \leq \sum_{i \in A} a_{ist} \leq \sum_{j=1, j \neq i}^n v_{ist}, s = 1, \dots, S; t = 1, \dots, \tau, \\ & \lambda \geq \sum_{s=1}^S p_s \{k_{ist} a_{ist} + l_{ist} v_{ist}\}, i = 1, \dots, n; t = 1, \dots, \tau, \\ & 1 = \sum_{s=1}^S x_{ist}, i = 1, \dots, n; t = 1, \dots, \tau, \\ & 0 \leq v_{ist} \leq x_{ist}, i = 1, \dots, n; s = 1, \dots, S; t = 1, \dots, \tau, \\ & 0 \leq x_{ist} \leq U_{ist}, i = 1, \dots, n; s = 1, \dots, S; t = 1, \dots, \tau, \\ & x_{ist} = x_{iht}, i = 1, \dots, n; s = 1, \dots, S; t = 1, \dots, \tau. \end{aligned}$$

The model (8) has a non-linear objective function and the third constraint is also non-linear. The model is transformed into a linear stochastic programming model as follows. For each scenario  $s$ , let  $M_{ist} \geq K_{ist} = |\min[0, R_{ist} - r_{pt}]|, s = 1, \dots, S$ . Then, the expected portfolio risk becomes  $G = \frac{1}{\tau} \sum_{t=1}^{\tau} \sum_{s=1}^S p_s M_{ist} x_{ist}$ , where the expected portfolio risk at any period  $t$  is given by  $\sum_{s=1}^S p_s K_{ist} x_{ist}$ . If  $Z_t = \sum_{s=1}^S p_s M_{ist} x_{ist}$ , then the expected portfolio risk for the period  $[0, \tau]$  is  $G = \frac{1}{\tau} \sum_{t=1}^{\tau} Z_t$ . The programming model (8) is transformed into the following linear stochastic model.

$$\begin{aligned}
\text{Minimize} \quad & G = \frac{1}{\tau} \sum_{t=1}^{\tau} Z_t & \text{(9)} \\
\text{Subject to} \quad & N_{pt} \geq \theta \\
& W_t = (1 + N_{pt})W_{t-1}, t = 1, \dots, \tau, \\
& 0 = Z_t - \sum_{s=1}^S p_s M_{ist} x_{ist}, i = 1, \dots, n; t = 1, \dots, \tau, \\
& 0 \leq \sum_{i \in A} a_{ist} \leq \sum_{j=1, j \neq i}^n v_{ist}, s = 1, \dots, S; t = 1, \dots, \tau, \\
& \lambda \geq \sum_{s=1}^S p_s \{k_{ist} a_{ist} + l_{ist} v_{ist}\}, i = 1, \dots, n; t = 1, \dots, \tau, \\
& 1 = \sum_{s=1}^S x_{ist}, i = 1, \dots, n; t = 1, \dots, \tau, \\
& 0 \leq v_{ist} \leq x_{ist}, i = 1, \dots, n; s = 1, \dots, S; t = 1, \dots, \tau, \\
& 0 \leq x_{ist} \leq U_{ist}, i = 1, \dots, n; s = 1, \dots, S; t = 1, \dots, \tau, \\
& x_{ist} = x_{iht}, i = 1, \dots, n; s = 1, \dots, S; t = 1, \dots, \tau.
\end{aligned}$$

The following theorem shows that models **(8)** and **(9)** yield the same optimal values.

Theorem 1

If  $x^*$  is an optimal solution to **(8)**, then  $(x^*, G^*)$  is an optimal solution to **(9)**. Conversely, if  $(x^*, G^*)$  is an optimal solution to **(9)**, then  $x^*$  is an optimal solution to **(8)**.

Proof

Without loss of generality, let  $x^* = x_{ist}^*$ . If  $x^*$  is an optimal solution to **(8)**, then  $(x^*, G^*)$  is a feasible solution to **(9)**, where  $G = \frac{1}{\tau} \sum_{t=1}^{\tau} Z_t = \frac{1}{\tau} \sum_{t=1}^{\tau} \sum_{s=1}^S p_s M_{ist} x_{ist}^* \geq \frac{1}{\tau} \sum_{t=1}^{\tau} \sum_{s=1}^S p_s K_{ist} x_{ist}^*$ . If  $(x^*, G^*)$  is not an optimal solution to **(9)**, then there exists a feasible solution  $(x, G)$  to **(9)** such that  $G < G^*$ , where

$$\begin{aligned}
G &= \frac{1}{\tau} \sum_{t=1}^{\tau} \sum_{s=1}^S p_s K_{ist} x_{ist} \\
&= \frac{1}{\tau} \sum_{t=1}^{\tau} \sum_{s=1}^S p_s \cdot |\min[0, R_{ist} - r_{pt}]| \cdot x_{ist}
\end{aligned}$$

It is observed that  $\frac{1}{\tau} \sum_{t=1}^{\tau} \sum_{s=1}^S p_s \cdot |\min[0, R_{ist} - r_{pt}]| \cdot x_{ist} = G < G^*$  and that  $G < G^* = \frac{1}{\tau} \sum_{t=1}^{\tau} \sum_{s=1}^S p_s \cdot |\min[0, R_{ist} - r_{pt}]| \cdot x_{ist}^*$ . This is a contradiction since  $x^*$  is an optimal solution of **(8)**.

Conversely, if  $(x^*, G^*)$  is an optimal solution of **(9)**, where

$$G = \frac{1}{\tau} \sum_{t=1}^{\tau} \sum_{s=1}^S p_s \cdot |\min[0, R_{ist} - r_{pt}]| \cdot x_{ist},$$

then  $x^*$  is an optimal solution of **(8)**. Otherwise, there exists a feasible solution  $x$  to **(8)** such that

$$\begin{aligned} G &= \frac{1}{\tau} \sum_{t=1}^{\tau} \sum_{s=1}^S p_s \cdot |\min[0, R_{ist} - r_{pt}]| \cdot x_{ist} \\ &< \frac{1}{\tau} \sum_{t=1}^{\tau} \sum_{s=1}^S p_s \cdot |\min[0, R_{ist} - r_{pt}]| \cdot x_{ist}^* \\ &= G^* \end{aligned}$$

which contradicts that  $(x^*, G^*)$  is an optimal solution to **(9)**. This completes the proof.

### 3.4 Measurement of transaction costs

Transaction costs incurred by an investor when buying or selling shares of securities at a stock market are broadly of two types, namely implicit and explicit costs. Explicit costs can easily be determined before execution of trade, as they do not rely on the trading strategy. These include market fees, clearing and settlement costs, brokerage commissions, and taxes and stamp duties. Implicit costs, on the other hand, are invisible. They depend mainly on the trading characteristics relative to the prevailing market conditions. They are strongly related to the trading strategy and provide opportunities to improve the quality of trade execution. They are of three categories, namely market impact, opportunity costs and spread. These costs can turn high-quality investments into moderately profitable investments or low-quality investments into unprofitable investments (D'Hondt & Giraud, 2008). When an investment decision is immediately executed without delay, implicit costs are largely a result of market impact or liquidity restrictions only, and defined as the deviation of the transaction price from the 'unperturbed' price that would have prevailed if the trade had not occurred. Thus, in this study, immediate execution of trade is assumed, and hence market impact accounts for the total implicit costs. Hau (2006) provides a methodology for calculating these implicit costs which this study adopts. The transaction price is taken to be the last price of the month and the spread mid-point is used as benchmark. The effective spread is then calculated as twice the distance from the mid-price measured in basis points. Thus obtaining the effective spread (implicit transaction cost) as

$$Spread^{Trade} = \frac{200 \times |P^T - P^M|}{P^M}$$

where  $P^T$  is the transaction price and  $P^M$  is the mid-point of the bid-ask spread.

## 4. DATA AND SAMPLE

The historical monthly data of securities on the Johannesburg Stock Market from January 2008 to September 2012 is considered, and the following criteria are used to select securities available for portfolio selection:

- stocks with negative mean returns for the entire period considered are excluded from the sample,
- companies which were not listed on the Johannesburg Stock Market by January 2008 and only entered afterwards are excluded, and
- securities having the highest positive mean returns for the entire period.



Empirical distributions computed from past monthly returns are taken as equi-probable scenarios. A scenario,  $R_{ist}$ , for the return of asset  $i$  of period  $t$  is calculated as  $R_{ist} = \frac{P_{i,t}}{P_{i,t-1}} - 1$ , where  $P_{i,t}$  is the historical monthly price of asset  $i$ . Five scenarios are considered for each asset return and corresponding implicit cost at each time period and the model is applied over one stage. The initial portfolio is selected from 13 assets and empirical distributions of these securities are considered. Since for each security we have 54 monthly returns for the period under study, the months are numbered from 1 to 54 and random numbers used to select asset returns and associated transaction costs to get scenarios for each asset. It is assumed that transaction costs are random since they are randomly selected together with corresponding asset returns. Thus a scenario comprises an asset return and the associated transaction cost. The transaction cost is given as a rate and scenarios are taken to be equally likely to occur. Thus, each asset's return and transaction cost scenario has a probability of occurring of  $\frac{1}{5n}$ , where  $n$  is the number of assets in the portfolio. The number of scenarios is restricted to 5, since in stochastic programming the scenario tree grows exponentially. At the end of the first stage, the investor decides on the first-stage optimal portfolio as given by the investor's chosen portfolio risk, the gross portfolio mean return or net portfolio mean return as the case may be and the portfolio transaction cost.

#### 4.1 Model application and results

The study considers an investor who has R10 000 to spend on the initial portfolio. The optimal portfolios describing the first-stage efficient frontier are shown in TABLE 1. The phrase 'D. Lim' stands for 'diversification limit'.

**TABLE 1: Stage 1 optimal portfolios: risk, cost and expected return unconstrained**

<i>D. Lim</i>	<i>Gross mean</i>	<i>Net mean</i>	<i>Risk</i>	<i>Cost</i>	<i>% Cost</i>	<i>Net wealth</i>
0.100	0.031	0.027	0.034	44.82	12.9	10265.78
0.125	0.034	0.029	0.031	55.15	14.7	10286.85
0.150	0.036	0.030	0.029	63.49	16.7	10297.01
0.175	0.038	0.032	0.027	60.04	15.8	10317.61
0.200	0.039	0.036	0.025	36.48	7.7	10357.12
0.225	0.040	0.036	0.025	39.47	10.0	10359.09
0.250	0.040	0.036	0.024	42.45	10.0	10361.05
0.275	0.041	0.035	0.024	44.28	12.1	10362.78
0.300	0.041	0.036	0.024	46.10	12.1	10364.50
0.325	0.041	0.037	0.023	47.93	9.8	10366.23
0.350	0.042	0.037	0.023	44.71	11.9	10372.89
0.375	0.042	0.038	0.023	38.98	9.5	10382.03
0.400	0.042	0.039	0.022	33.24	7.1	10391.16

Source: Authors' analysis

As portfolios become less diversified, there is an increase in both the gross portfolio mean return and the net portfolio mean return. The risk declines and the net wealth increases with increasing diversification limit. However, the total implicit transaction cost rises from R44.82 to R63.49 as the diversification limit increases from 0.1 to 0.15 respectively. Thereafter, the transaction cost declines in a fluctuating pattern to a lowest value of R33.24 obtained when the diversification limit is 0.4. Efficient frontiers of net mean portfolio returns and gross mean portfolio returns reveal the impact of neglecting implicit transaction costs in portfolio selection. The total implicit transaction costs incurred to achieve each optimal portfolio vary from 7.1% to 16.7% of returns on investment.

Analysis of portfolio composition of assets in optimal portfolios is carried out and the information is shown in TABLE 2. It is evident from the table results that the SMNDTC model allocates the maximum weight possible to each of the selected assets except when an even distribution is impossible. The model reflects consistency in selection of assets as portfolios become less diversified.

**TABLE 2: Assets percentage composition: Stage 1 optimal portfolios**

<i>D. Lim</i>	<i>AVI</i>	<i>ASR</i>	<i>APN</i>	<i>CSB</i>	<i>CLS</i>	<i>CML</i>	<i>PNC</i>	<i>CPI</i>	<i>IPL</i>
0.100	10	10	10	10	10	10	10	10	10
0.125	12.5	12.5		12.5	12.5	12.5	12.5	12.5	12.5
0.150	15	15		15		10	15	15	15
0.175	17.5	17.5		12.5			17.5	17.5	17.5
0.200	20	20					20	20	20
0.225	22.5	22.5					22.5	10	22.5
0.250	25	25					25		25
0.275	27.5	27.5					27.5		17.5
0.300	30	30					30		10
0.325	32.5	32.5					32.5		2.5
0.350	35	35					30		
0.375	37.5	37.5					25		
0.400	40	40					20		

Source: Authors' analysis

#### 4.1.1 Sensitivity analysis

Sensitivity analysis of the SMNDTC model is done by calculating the sensitivity index (SI) for each parameter. The methodology by Hoffman and Gardner (1983) and Bauer and Hamby (1991) is employed. Output percentage difference is calculated by varying one input parameter at a time, from its minimum value (zero in this case) to its maximum value (parameter value in optimal portfolio). Sensitivity indices are obtained as follows:

$$\text{Sensitivity Index (SI)} = \frac{D_{\max} - D_{\min}}{D_{\max}}$$

where  $D_{\min}$  and  $D_{\max}$  are the minimum and maximum output values respectively. We consider maximum output values of 0.1, 0.15, 0.2 up to 0.3, which are the imposed diversification limits. The model being proposed is stochastic and works by replacing one parameter by a 'less profitable' one when we assign a weight of zero to the asset of the optimal portfolio. This happens because of the condition imposed by the model that the sum of assets' weights be unity.

This results in a relative sensitivity value. Hence,  $D_{\min}$  value of the model output is a relative value. Thus, applying the above method yields a relative sensitivity analysis of the model, the results of which are shown in TABLE 3.

**TABLE 3: SMNDTC model sensitivity analysis**

<i>D. Lim</i>	<i>Parameter</i>	<i>Proxy</i>	<i>Max value</i>	<i>Cost SI</i>	<i>Risk SI</i>	<i>Wealth SI</i>	
0.10	X1		0.10	0.0357	0.3235	0.0027	
	X2			-0.0076	0.0000	0.0026	
	X3			-0.0036	0.0588	0.0004	
	X4			0.4931	0.0882	-0.0011	
	X5	X7(10)		0.0134	0.0294	0.0008	
	X6			0.0442	0.1176	0.0007	
	X8			0.2700	-0.0294	0.0008	
	X11			0.0192	0.2353	0.0018	
	X12			0.0451	0.1765	0.0018	
	X13			0.0013	0.2353	0.0001	
	0.15	X1		0.15	0.0128	0.3793	0.0019
		X2			-0.0331	0.0000	0.0018
		X4			0.4971	0.1379	-0.0038
X6		X5(10)		0.0217	0.1724	-0.0011	
X8				0.2608	0.0000	-0.0009	
X11				-0.0047	0.2759	0.0006	
X12				0.0227	0.2414	0.0006	
0.20	X1		0.20	-1.1239	0.3600	0.0075	
	X2			-1.2303	-0.0400	0.0074	
	X8	X4(20)		-0.5482	-0.0400	0.0039	
	X11			-1.1645	0.2800	0.0058	
	X12			-1.1009	0.2400	0.0058	
0.25	X1		0.25	0.0436	0.4583	0.0021	
	X2	X11(25)		-0.0707	0.0000	0.0020	
	X8			0.6620	0.0000	-0.0023	
	X12			0.0683	0.2917	0.0001	

<i>D. Lim</i>	<i>Parameter</i>	<i>Proxy</i>	<i>Max value</i>	<i>Cost SI</i>	<i>Risk SI</i>	<i>Wealth SI</i>
0.30	X1		0.30	0.4583	0.4583	0.0025
	X2	X11(10)		0.0417	0.0417	0.0024
	X8			0.0417	0.0417	-0.0028
	X12			0.3333	0.3333	0.0000

Source: Authors' analysis

It is observed that, for each diversification limit considered, the same asset is being chosen as the proxy. However, these proxies vary from one diversification limit to the other. The results show small percentages of model variability of optimal wealth due to changes in input parameter value. The wealth sensitivity index ranges from -0.38% to 0.75%. This is a good indication that the model output values are not significantly influenced by specific input values. Some wealth sensitivity indices are negative, implying that the respective proxies result in better wealth. However, the cost sensitivity indices vary from -1.2303 to 0.6620. The negative indices show that the proxies result in higher implicit transaction costs compared to assets in optimal portfolios. All such proxies cause a decline in portfolio wealth, since all corresponding wealth S.I. values are positive.

## 5. CONCLUSION

In this study, a multi-stage stochastic maximum downside risk model that incorporates uncertainty of asset returns and implicit transaction costs is proposed. The model best applies to periods of economic recessions which are characterised by extreme movements in asset prices. In such times, investors are highly concerned about the performance of their portfolios, particularly the downside movements. The contribution of this study includes:

- the development of a multi-stage stochastic maximum negative deviation model that optimizes portfolios in the presence of uncertain implicit transaction costs incurred in initial trading and in subsequent rebalancing of portfolios, and
- the development of a strategy that captures uncertainty in stock returns and in corresponding implicit trading costs in extreme downside movements of stock prices by way of scenarios.

The methodology allows investors and investment managers to decide on optimal portfolios realizing the associated implicit transaction costs. It is a linear programming model and hence it is feasible for large-scale portfolio selection, as it reduces considerably the time needed to reach a solution. It is, however, left for further research to obtain a model that captures both implicit and explicit transaction costs in uncertain market environments.

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