



Students' Dichotomous Experiences of the Illuminating and Illusionary Nature of Pattern Recognition in Mathematics

Michael Kainose Mhlolo

To cite this article: Michael Kainose Mhlolo (2016) Students' Dichotomous Experiences of the Illuminating and Illusionary Nature of Pattern Recognition in Mathematics, African Journal of Research in Mathematics, Science and Technology Education, 20:1, 45-56, DOI: [10.1080/10288457.2016.1145446](https://doi.org/10.1080/10288457.2016.1145446)

To link to this article: <http://dx.doi.org/10.1080/10288457.2016.1145446>



Published online: 01 Mar 2016.



Submit your article to this journal [↗](#)



Article views: 4



View related articles [↗](#)



View Crossmark data [↗](#)



Students' Dichotomous Experiences of the Illuminating and Illusionary Nature of Pattern Recognition in Mathematics

Michael Kainose Mhlolo*

*Central University of Technology, Bloemfontein, South Africa. Email: mmhlolo@cut.ac.za

The concept of pattern recognition lies at the heart of numerous deliberations concerned with new mathematics curricula, because it is strongly linked to improved generalised thinking. However none of these discussions has made the deceptive nature of patterns an object of exploration and understanding. Yet there is evidence showing that pattern recognition has both positive and negative effects on learners' development of concepts. This study investigated how pattern recognition was both illuminating and illusionary for Grade 11 learners as they factorised quadratic trinomials. Psillos's four-conditions model was used to judge the reasonableness of learners' generalisations in six selected examples. The results show that pattern recognition was illuminating in the first three examples where learners made use of localised pattern recognition. In one example, pattern recognition was coincidental but not beneficial in terms of conceptual understanding. In the last two examples localised pattern recognition was at the centre of learner confusion as they failed to extend its application beyond the domain of the examples that generated the pattern. Learners' confusion with pattern recognition could be attributed to teachers' failure to meet four important conditions for good generalisations. Results from this study confirm earlier studies showing that abduced generalisations developed out of a few localised instances might be illuminating at first but might not provide the best explanation when extended beyond the localised domain. Further studies are needed that assist in developing pattern-aware teachers.

Keywords: *Mathematical patterns; trinomials; abduction; generalisation; factorisation*

Introduction

Pattern recognition and generalisation are fundamental and valuable skills in mathematics learning, with wide application in many topics, including algebra, where they are considered as the bedrock of algebraic thinking (Chua & Hoyles, 2014). According to Mulligan, English, Mitchelmore, Welsby and Crevensten (2011), mathematics learning that focuses on pattern and structure not only leads to improved generalised thinking but also can create opportunities for students to develop mathematical reasoning. The beliefs about this 'causal' relationship are so strong that mathematics is itself often defined as the 'science of pattern' and if learners are unable to recognise patterns then mathematical thinking is considered as not happening (Mason, 1996). Despite this strong view that pattern recognition creates opportunities for learners to develop mathematical reasoning, empirical evidence shows that learners can be adept at making all kinds of generalisations (Ellis, 2011), some of which are not productive in terms of being mathematically useful. In addition to students making wrong generalisations Burton (2008) points to the deceptive nature of pattern evidence itself, arguing that there are many examples where we get surprised by the unexpected turn that a pattern takes from our initial inference. We equally are surprised by the multitude of alternative and plausible patterns that we or other people might recognise from the same examples that we have worked with. These observations

point to the dichotomous nature of pattern recognition where on the one hand it can be a resource/tool for learners of mathematics, yet it is equally true that pattern recognition can be illusionary and therefore a source of confusion for learners. The dichotomous nature of pattern recognition in mathematics is rarely objectified in research on pattern generalisation. This suggests the need to further understand the circumstances under which pattern recognition can be a resource and when it is not supportive for learners doing mathematics.

Ellis (2011) posits that on the one hand there are potential pitfalls in the teacher's choice of examples that may hinder productive generalisation; on the other hand learners can interpret and generalise those examples in unproductive ways. In mathematics there seems to be a consensus that the use of an example is an integral part of the discipline and not just an aid for teaching and learning. With specific reference to generalisations the link is even stronger and reciprocal, given that an example is defined as any particular case of any larger class about which learners are expected to generalise and reason: concepts, representations, questions, methods, and so on (Bills & Watson, 2008). In contrast, generalisation is defined as the processes by which one derives or induces from particular cases or examples (Yilmaz, Argun & Keskin, 2009). Owing to this mutual link, generalisation is typical of example-based reasoning and the view that learners must be able to generalise assumes that this link is unproblematic. Yet current literature characterises example-based reasoning strategies as obstacles that need to be overcome (Lockwood, Ellis & Knuth, 2013). For example, Zaslavsky and Zodik (2007) warn that an example does not always fulfil its intended purpose owing to the mismatch that often occurs between the teacher's intention and students' interpretations. This suggests that there may be more to the role of examples than simply signifying an unsophisticated line of reasoning. This paper is premised on the view that the specific elements and representation of examples, and the respective focus of attention facilitated by the teacher, have a bearing on what learners notice, and consequently on their mathematical understanding. The paper raises the following specific questions:

- (1) In what way did the teacher's examples provide the learners with an opportunity to experience pattern recognition as a resource?
- (2) To what extent did learners experience pattern recognition as a source of confusion?
- (3) How could this confusion have possibly been avoided?

Conceptualisation of Mathematical Patterns

This paper acknowledges that a pattern is an ill-defined concept in mathematics (Carraher, Martinez & Schliemann, 2008), but analysis would not be meaningful without taking a clear position in terms of how patterns are being conceptualised. Mulligan et al. (2011) have described a pattern as any predictable regularity involving number, space, measure or structure, or the way in which various elements are organised and related. Most pattern recognition research has concentrated on the regularity involving number and shape yet recognising patterns in computational procedures permeates every area of mathematical thinking to such an extent that mathematics problems have been described as typically 'tame' and 'benign' (Head & Alford, 2015). Unlike the social world where there are 'wicked problems', in mathematics there are rules for classifying families of problems such that finding a solution to a new task (computational procedure) is usually a process of recognising a pattern or regularity in previously worked solutions that matches the new task. In such cases where regularity is recognised in the techniques of solving similar problems, pattern recognition is seen as a generalisation of method. This paper analyses teachers' choices of examples when factorising quadratic trinomials, hence it was considered important to look at pattern recognition from the perspective of a generalisation of method, that is, the predictable regularity in computational procedures, techniques, methods or processes for solving related quadratic trinomials.

Attributes of Useful Examples

The issue of judging the usefulness of a teacher's choice of examples in facilitating learners' pattern recognition and generalisation is central to this paper. Given this focus, there was a need to develop a tool to enable such judgement. In doing so the paper borrows from Rivera and Becker's (2007) Pattern Generalisation Scheme, which illustrates how the process of generalisation materialises from the beginning phase of noticing a regularity (abduction) in a few specific cases (examples)

to the establishment of a general form (pattern recognition) as a result of confirming it in several extensions of the pattern (induction) and then finally to the statement of a generalisation. Briefly, Rivera and Becker's (2007) Pattern Generalisation Scheme suggests that the act of generalisation starts by developing and discovering a perceived commonality (abduction), which is then verified by repeated testing (induction) leading to a generalisation. In order to decide the goodness of an abduction in relation to pattern construction and generalisation, Rivera and Becker suggest that we need to look at the characteristics of a good abductive generalisation. In doing so they borrow from Psillos (1996), who used the term 'ampliative reasoning' to describe the generalisation process and advanced the following conditions that an ampliative inference must fulfil in order to yield valid and necessary conclusions: (a) it must be non-monotonic; (b) it must deal with the cut-off point problem; (c) it must allow for vertical extrapolation; and (d) it must accommodate the eliminative dimension of ampliative reasoning. I elaborate on each of these conditions briefly.

The requirement of non-monotonicity foregrounds the necessity of stating assumptions about a pattern undergoing generalisation; it assists in confronting the biases and resolving situations of conflict between several viable claims of generalisations for the same pattern. The requirement of a cut-off point demands the justification for a global-type of generalisation (vs a local one) that holds in both specific cases and the entire class of cases. An abduced generalisation that offers the best explanation provides the cut-off point in that it can explain why the stated generalisation that depends only on a few instances (sample) actually holds for the entire class (population). The requirement for vertical extrapolation focuses on providing a generalisation that can be explained in a deeper way using perceptual knowledge or other relevant mathematical idea or concept that bears on the class. The requirement for an eliminative dimension makes it possible not only to consider several possible generalisations for a pattern, however, it also necessitates making a judgement about which one(s) will make the most sense.

Peirce (1966) gave three more conditions as follows: (a) a good abductive generalisation made about a pattern should be able to explain the facts, that is, there is a reliable and justifiable causal story behind why the known, including and especially the unknown, instances are the way they are; (b) the generalisation should not surprise us, that is, we expect that it will hold in the largest domain possible—we do not want to frustrate ourselves with a generalisation that seems to always fail in situations when new cases are introduced for verification; and (c) the generalisation should stand experimental verification, that is, in Psillos's (1996) terms, it is non-monotonic with a well justified cut-off point and has been vertically extrapolated. These conditions then formed the basis for judging the quality of teachers' examples towards generalisations.

Methodology

Research Design

This paper uses Qualitative Secondary Analysis of data (Irwin, 2013), which refers to the (re)using of data produced on a previous occasion to glean new social scientific and/or methodological understandings. This may involve prioritising a concept or issue that was present in the original data but was not the analytical focus at that time (Irwin, 2013). In the present analysis I prioritise learners' experiences of pattern-based thinking which was present in the original data but which was ignored previously when teachers were prioritised. Proponents of Qualitative Secondary Analysis suggest that we can come to understand re-using qualitative data not as the reuse of pre-existing data, but as a new process of recontextualising data (Savage, 2005).

Sample

Four experienced (7–10 years) high school mathematics teachers (two female and two male), in previously disadvantaged schools, were purposively sampled to take part in this study. Each of these teachers were teaching an average of 35 learners in their classes and this paper focuses on how these ±140 learners made use of pattern recognition when solving tasks.

Procedure

The author observed and video-recorded one teacher teaching five lessons on Number Patterns and the other three teachers teaching a total of 15 lessons on Functions and Algebra. In the 20 lessons there were opportunities for learners to recognise patterns and use the structures to solve related tasks.

Data Analysis

Six examples are analysed in terms of both the teacher’s choice of examples and his/her presentation as well as the learners’ interpretations. The judgements are in accordance with Psillos’s (1996) indicators to see if these generalisations meet the (a) non-monotonic, (b) cut-off point problem, (c) vertical extrapolation and (d) eliminative dimension criteria.

Validity and Reliability

A number of measures were taken to enhance the accuracy, credibility and validity of data. Firstly, participation was voluntary. During and after the lesson observations, there were frequent member checks with the participants (Lincoln & Guba, 1985). For example, during the lesson observations there was constant dialogue with participants in order to verify the researcher’s inferences. Participants were asked to read transcripts of dialogues in which they had participated in order to either agree or disagree that the summaries reflected their views, feelings and experiences. Throughout the research process the study was also subjected to peer scrutiny through conference presentations, peer discussions and research indabas.

Results

Example One

At the beginning of the lesson the teacher writes on the board $a^2 + 14a + 48$ and explains that the lesson will be on factorisation of such quadratic trinomials. He then draws these two rectangles and puts a in each of them to show the arrangement of the factors of a^2 . He then leaves the middle part of the trinomial and focuses on listing the factors of the last term (48). We pick up the discussion when the teacher points to what he calls the crux of the matter.



48	1
24	2
18	3
12	4
8	6
-48	-1
-24	-2
-18	-3
-12	-4
-8	-6

Teacher: But where is the crux of the matter? The crux of the matter is here on $14a$. If you add these two factors of 48 they must give us a positive $14a$. So we are going to choose now. Of the factors, both are factors of what, 48. So we want to put signs (pointing back into the rectangles) which will give us a positive $14a$. We want to choose factors which when you add them they give us a plus $14a$ and when we multiply them they give us a plus 48. Which are the two factors?

Downloaded by [Central University of Technology] at 07:03 21 April 2016

Class: 8 and 6

a	8	$=$	$8a$
a	6	$=$	$6a$

Teacher: The next thing we are going to do is we are going to say $a \times 8$ (showing the cross-multiplication) and we put $8a$ here (on the right-hand side of the first rectangle). Then we are going to say $6 \times a$ (again showing the cross multiplication) then we put $6a$ here. When you add $8a$ plus $6a$ what do they give us? $14a$. If you multiply 8×6 what do they give us? 48 . The fact that these two terms give us a positive 48 and the fact that this $8a$ plus $6a$ gives us $14a$, and $a \times a$ gives us a^2 , what can we say about these two numbers here? [Pointing to the binomials $(a + 8)(a + 6)$.] What can we say about these two expressions here? Are they the factors of $a^2 + 14a + 48$?

Class: Yes.

Teacher: Can you see it? This is how you factorise the trinomials but now we are going to check. What is the next step now?

Class: [So they multiply $(a + 8)(a + 6)$ to get the initial trinomial of $a^2 + 14a + 48$.]

Example Two

Teacher: (Writes a new task on the board: $a^2 + 47a - 48$.) Who can come and try to do this one? Remember here we have a negative sign (pointing to the last term).

Learner 1: (Comes to the board and makes a table of factors of -48)

48	-1
24	-2
18	-3
12	-4
8	-6
-48	1
-24	2
-18	3
-12	4
-8	6

Class: (Following the example that they have just worked before, the class agree that the table of the required factors should be as follows:)

a	48	$=$	$48a$
a	-1	$=$	$-1a$

[So by cross multiplying $(a + 48)(a - 1)$ the class is able to show that this is the correct factorisation of $a^2 + 47a - 48$]

Teacher: Is that correct?

Class: Yes, yes, yes.

Teacher: Now I want you from that table to write as many different trinomials as possible with -48 as the third part hence can be factorised using the factors in the table.

Class: [The class is able to generate 10 different trinomials that can be factorised using the factors of -48 in the table.]

Example Three

Teacher: (Writes another task on the board: $n^2 - 16mn + 15m^2$.) Who can come and try to do it? Remember here we have a negative sign (pointing to the middle term). I want someone to come and try it. What we have done is we have only changed the sign here. (He draws the rectangles again and learners take turns to complete the table.)

$1m$	$15m$
$3m$	$5m$
$-m$	$-15m$
$-3m$	$-5m$

These are the possible factors of $15m^2$. So from this list of possible factors we are saying if we add the factors they must give us a what? A $(-16mn)$. Are we together? We are going to have an n here (putting the n in the rectangles).

n	$-m$	=	$-mn$
n	$-15m$	=	$-15mn$

Class: The factors that we want from the table are $-m$ and $-15m$.

Discussion 1: In what way did the teacher's examples provide learners with an opportunity to experience pattern recognition as a resource?

There are two main reasons why examples 1–3 are being discussed concurrently. Firstly we cannot observe and generalise a pattern in just one example because a pattern suggests regularity which implies that a phenomenon has occurred more than once. In fact the literature suggests that on average the brain will generalise a pattern after being exposed to it three times (Laughbaum, 2009). The second reason is that these are the three examples which address my first research question, that is, in what way did the teacher's examples provide the learners with an opportunity to experience pattern recognition as a resource?

We start by asking the question: what was presented by the teacher and what pattern was recognised by learners when factorising a quadratic trinomial of the form $ax^2 + bx + c$? In example one the teacher points to *the crux of the matter* being the link between two factors of c that must add up to b . He concludes that example by saying 'this is how we factorise the trinomials'. Learners check the accuracy of this procedure of factorising by multiplying the factors and are convinced that it works. So following Rivera and Becker's (2007) framework, at this stage we can say the learners have discovered a potential regularity R that suggests that when factorising a trinomial we look for factors of c that add up to b . In example 2 we can say there is induction taking place because there is now repeated testing of this regularity. When the learners check again this viable general form F and confirm that it always gives correct results (under current assumptions), then learners conclude and therefore generalise the procedure to example 3 with a slightly different task. Consistent with Rivera and Becker's (2007) framework, we can see how the procedure for factorising trinomials has moved from abduction, through induction to generalisation. In terms of answering the first research question of this paper, 'In what way did the teacher's examples provide the learners with an opportunity to experience pattern recognition as a resource?', in these three examples we see how pattern recognition indeed was a resource for learners when they were factorising quadratic trinomials. Admittedly their abductive generalisation might have been limited in that it was localised to trinomials with a leading coefficient of 1, but all the same this pattern recognition helped learners (under current assumptions) to make sense of factorising similar trinomials. The next few exercises in the textbook

which had tasks with a leading coefficient of 1 did not present any problems for learners and it can be argued that if learners were to met similar tasks in future they would be able to solve them.

Example Four

Teacher: (writes the next task $5y^2 - 13y - 28$ on the board) I want three learners to come to the board and try this example. But before that let us complete the table of factors for them. (They complete this table on the board as a class.)

1	-28
2	-14
4	-7
-1	28
-2	14
-4	7

Learner 3: (works as follows:)

$5y$	\swarrow	14	$=$	$-14y$
y	\searrow	-2	$=$	$+10y$

$$(5y - 14)(y + 2) = 5y^2 - 4y - 28$$

Learner 4: (works as follows:)

$5y$	\swarrow	-4	$=$	$-4y$
y	\searrow	-7	$=$	$+35y$

$$(5y - 4)(y + 7) = 5y^2 + 31y - 28$$

Learner 5: (works as follows:)

$5y$	\swarrow	-7	$=$	$+7y$
y	\searrow	-4	$=$	$-20y$

$$(5y + 7)(y - 4) = 5y^2 - 13y - 28$$

[So by cross-multiplication Learner 5 was able to show that $(5y + 7)(y - 4)$ is the correct factorisation of $5y^2 - 13y - 28$.]

Teacher: Who of the five learners is correct?

Class: Learner 5, yes she is correct.

Teacher: [To Learner 5.] Can you explain how you got the correct factors.

Learner 5: I just played around with the factors and then placed them in the brackets, checked by expanding the brackets and it worked.

Class: [They clap for Learner 5]. Some are mumbling saying but sir you just said that if the factors don't add to the middle term then they are wrong.

Discussion 2: In what way was pattern recognition coincidental?

Although this question was not initially raised, what emerged from example 4 deserves special attention because, instead of exposing the fallibility of the learners' generalisation, the task tended to further

confirm the regularity that had been abduced earlier. This is evident in that, when factorising the quadratic trinomial $5y^2 - 13y - 28$, the factors of -28 listed in the table had no causal connection with the middle term of $-13y$ yet it turned out coincidentally that -4 and $+7$ worked well when placed in the two brackets. So in example 4 it can be argued that Learner 5 arrives at the correct factorisation by coincidence. Diaconis and Mosteller (1989) defined a coincidence as a surprising concurrence of events, perceived as meaningfully related, with no apparent causal connection. Was this helpful for learners? The answer could be either 'yes' if we consider that the correct solution was found but in terms of conceptual understanding it might be 'no'. This is confirmed by the learners who pointed out to the teacher; 'but sir you just said that if the factors don't add to the middle term then they are wrong'. That learners were confused is further evidenced when Learner 5 is asked to explain how she got the correct factors—she says that she just played around with the figures and it worked—typical of trial and error—but we cannot encourage learners to rely on trial and error if we aim at conceptual understanding.

Following the rule that has been generalised from the first three examples we see how the other two learners 3 and 4 failed to identify factors of -28 that would add up to $-13y$, pointing to the fallibility of the generalised pattern. By clapping for Learner 5, the class therefore believed something that was wrong simply because coincidentally it worked. Burton (2008) posited that unconscious pattern recognition contains a probability of correctness, which is consciously experienced as a feeling of knowing. The closer the fit between previously learned patterns and the new incoming pattern, the greater the degree of the feeling of correctness will be. In this example we see this close fit between what the learners had generalised before and this new incoming observation, leading learners to believing this 'wrong' process to be correct. The literature, however, warns that sometimes even mathematically correct solutions to mathematically correct pattern recognition problems are not a good use of student time and sometimes they are misleading. Indeed the abduced pattern was not helpful later when the learners were faced with trinomials whose leading coefficient was something other than 1.

Example Five

Teacher: Now what I want us to do is in pairs let us try numbers 11 and 12. Can you do these ones for me in pairs? Teacher writes on the board:

(11) $d^2 - 5d + 8$

(12) $6t^2 - 19t + 15$

Class: (After some time of trial and error they complain that they cannot get the solutions.)

Teacher: Can somebody come to the board and show how you worked it.

Learner 6: (Comes to the board and creates a table with factors of 8.)

1	8
-1	-8
2	4
-2	-4

(Learner 6 checks from the list of factors but does not appear to get what he is looking for, that is, factors of 8 that will give the middle term of -5 .)

Class: (Meanwhile the class is following this working and comparing with what they wrote in their books in pairs.)

Sir there are no factors of 8 that can add up to -5 . This method is now confusing us.

Teacher: Okay let's see how you worked number 12 in your books. Can someone come to the board and show us what you did.

Example Six

Learner 7: (Comes to the board and creates a table with factors of 15.)

3	5
-3	-5
1	15
-1	-15

(Learner 7 checks from the list of factors but does not appear to get what he is looking for, that is, factors of 15 that will give the middle term of -19 .)

Class: Sir so where are we going wrong now? (Bell rings to end the lesson.)

Teacher: Let's leave it like this for today we will continue where we left tomorrow. In that case I'm not going to give you homework until we clarify this matter.

Discussion 3: To what extent did learners experience pattern recognition as a source of confusion?

The second research question is addressed in these two examples (5 and 6) which I argue were at the centre of learners' confusion. While the generalised procedure of finding two factors of c that would add up to b had worked nicely so far, it was now clear that this procedure was not always true for all trinomials of the form $ax^2 + bx + c$. Despite the fact that the learners followed the abduced procedure conscientiously, when attempting to factorise $d^2 - 5d + 8$ and $6t^2 - 19t + 15$, it did not help them to get factors of the last term that would add up to the middle term. When they checked in the table of factors for a pair that might coincidentally fit in the brackets as the previous example, luck was not on their side either. Hence I argue that pattern recognition was now at the centre of their confusion about the pattern of factorising trinomials.

There are a number of possible reasons why learners were getting confused with these two examples. Firstly the generalisation students made about the process of factorising quadratic trinomials was based on too limited examples and this might have led to learners forming a concept image or figural concept which was different from (a gap) a concept definition (Vinner, 1991). Concept images can be founded on too limited an exploration of the examples encountered so that the features of the examples that are not part of the concept are retained in the concept image (Watson & Mason, 2005). In this case students retained a concept image suggesting that the last term in every quadratic trinomial will always have factors that add up to the coefficient of the middle term. This, however, is not true for all such trinomials. Therefore the few examples that were used by the teacher had noise in them, which is described by Skemp (1969) as the possession of conspicuous attributes that are not essential to the concept. However, because the examples worked well in all cases of factorable trinomials with a leading coefficient of 1, they fit into the category described by Rissland-Michener (1978) as start-up examples, which help motivate basic definitions but are not sufficient for concept formation. Another reason why learners experienced confusion, is that the teacher did not provide some counter-examples. Skemp (1969) wrote about the learning of mathematical concepts through abstraction from examples and advised that non-examples should be included. Counter- or non-examples are useful in drawing attention to the distinction between essential and non-essential attributes of the concept and hence in refining its boundaries. Both examples 5 and 6 could have been used as counter-examples instead of being used to test students' understanding. Example 5 is not factorable and example 6 requires a different schema to factorise it and so their use would have enabled the learners to refine the boundaries for their generalisations.

Discussion 4: How could this confusion have possibly been avoided?

In order to address my third research question, I go back to the criterion for assessing a good generalisation. The first requirement is that an abduced regularity must be non-monotonic, that is, it must state the assumptions about a pattern undergoing generalisation. In order to understand why it was important for an assumption to be stated and why the procedure worked in the first examples that the class worked with, let us look at the method of factorising trinomials as an example. Samson, Cheriya-parambil and du Toit (2011) provide the alternative method of splitting the middle term in its more simplified but comprehensive form. Their method follows an observation in mathematics that when we multiply two binomials, for example, $(a + 3)(a + 4)$, we end up with two centre terms $(+3a)$ and $(+4a)$ that need to be combined. So when we factorise (which is like reversing the process of expansion), it makes sense to find these centre terms to help us see the numbers that factor correctly. In general, given a quadratic trinomial of the form $ax^2 + bx + c$ (which can be factorised), the middle coefficient b of our trinomial must be the sum or difference of two factors of the value ac . So to factorise this trinomial we try to find two terms whose sum is b and whose product is ac the master product. This will lead to the quadratic trinomial being factorised pairwise. Following this argument let us take the example of $6t^2 - 19t + 15$ that presented problems for learners, and split the middle term $-19t$ into $-9t$ and $-10t$ whose product is $90t^2$. Re-write the quadratic trinomial as $6t^2 - 9t - 10t + 15$ and the pair-wise factorisation will be $3t(2t - 3) - 5(2t - 3)$. We then take out the greatest common factor $(2t - 3)$, which is common in both parentheses, implying that $(3t - 5)(2t - 3)$ are the factors that we are looking for.

It should be clear from this rule that, as long as the leading co-efficient (a) is equal to (1) , then the master product ac will always be equal to c . Hence in the examples that the learners initially worked with (which all had a leading coefficient of 1), the factorisation always worked even though the master product ac was never mentioned and the assumption of the leading coefficient of 1 was not stated. In fact another assumption that should have been stated is that the trinomial should be able to be factorised because not all quadratic trinomials of the form $ax^2 + bx + c$ can be factorised. For example $d^2 - 5d + 8$ cannot be factorised. This failure to state the assumptions did not assist the learners in confronting other examples. This confirms Psillos's (1996) observation that an abductive generalisation of a pattern that offers the best explanation (in this case when the leading coefficient is 1) can still be shown false if additional or different assumptions (leading coefficient not equal to 1 , or not factorable) are made that would necessitate developing a different generalisation.

Following Psillos's second requirement, an abductive generalisation that offers the best explanation should provide the cut-off point in that it should explain why the stated generalisation that depends only on a few instances (sample) actually holds for the entire class (population). This requirement demands that we think about a generic example which makes explicit the reasons for the truth of an assertion and provides justification for a global-type generalisation that holds in both specific cases and the entire class of cases. Similarly Peirce (1966) cautioned that we do not want to frustrate ourselves with a generalisation that seems to always fail in situations when new cases are introduced for verification. Hence there should be a reliable and justifiable causal story behind why the known instances are the way they are. In all the examples 1–6 learners were not provided with such a reliable justification as to why things worked the way they did. The focus was more on procedural understanding rather than conceptual understanding.

The third requirement in Psillos's model is that of vertical extrapolation which focuses our attention on whether or not an abductive generalisation draws on the deeper structure of the available and unavailable cues. In this case learners were not provided with the deeper mathematical structures and ideas but instead a superficial generalisation that could only be generalised locally for trinomials with a leading coefficient of 1 and not further. Comparing with the way the method of splitting the middle term is described earlier, it can be concluded that the way the teacher guided the learners into generalising the procedure for factorising trinomials clearly lacked such deep perceptual knowledge.

The final requirement for eliminative dimension suggests that we need to consider several possible generalisations for a pattern before making judgement about which one will make the most sense. Consistent with this requirement, an abductive generalisation of a pattern that offers the best explanation should be chosen from several plausible ones and judged most tenable on the basis that it provides a maximal understanding of the pattern beyond what is superficially evident. In this case there

was no consideration of other plausible ones (such as counter-examples with no factors or those with coefficients other than 1), hence the generalisation could not provide maximal understanding of the pattern beyond those trinomials with a leading coefficient of 1.

Conclusion

This paper investigated ways in which patterns were both illuminating and illusionary for learners factoring trinomials. Psillos's (1996) four conditions were applied in judging the quality of examples chosen by the teacher and the explanations thereof. Examples 1–3 selected from classroom interactions show that learners made use of a localised pattern recognition when factorising trinomials with a leading coefficient of 1. In example 4 the learners find the correct solution coincidentally. However in examples 5 and 6 it is evident that the localised pattern recognition is now at the centre of their confusion as they try to extend its application beyond the domain of the examples from which it was generated. This confirms the argument that patterning can be leading and misleading at the same time. Thus, the findings of this case study indicate that there is a need to provide increased support to teachers and learners in an effort to enhance their knowledge of patterning. Teachers who are aware of these four conditions in relation to the formation of a generalisation about a pattern should be capable of exercising judgement about which examples and counter-examples will offer the best explanation; it will also enable them to separate 'good' from 'bad' potential explanations. Further studies are needed that might assist in creating 'pattern aware' learners who can make informed decisions about patterning activities and recognise worthwhile pattern learning experiences. Additionally such studies would also inform the development of curriculum resource material on patterning as well as support teacher professional development about patterning.

Disclosure Statement

The author declares that he has no financial or personal relationships which might have inappropriately influenced the writing of this article.

References

- Bills, L., & Watson, A. (2008). Editorial introduction. *Educational Studies in Mathematics*, 69, 77–79.
- Burton, R.A. (2008). *On being certain: Believing you are right even when you're not*. New York: St Martin's Press.
- Carraher, D.W., Martinez, M.V., & Schliemann, A.D. (2008). Early algebra and mathematical generalization. *ZDM Mathematics Education*, 40, 3–22.
- Chua, B.L., & Hoyles, C. (2014). Generalisations of linear figural patterns in secondary school mathematics. *The Mathematics Educator*, 15, 1–30.
- Diaconis, P., & Mosteller, F. (1989). Methods for studying coincidences. *Journal of the American Statistical Association*, 84, 853–861.
- Ellis, A.B. (2011). Generalizing-promotions actions: How classroom collaborations can support students' mathematical generalizations. *Journal for Research in Mathematics Education*, 42, 308–345.
- Head, B.W., & Alford, J. (2015). Wicked problems: Implications for public policy and management. *Administration & Society*, 47, 711–739.
- Irwin, S. (2013). Qualitative secondary data analysis: Ethics, epistemology and context. *Progress in Development Studies*, 13, 295–306.
- Lauthbaum, E.D. (2009). Generalizing patterns in algebra for long-term memory and understanding. *Communicator*, 34(20), 30–34.
- Lincoln, Y.S., & Guba, E.G. (1985). *naturalistic inquiry*. Newbury Park, CA: Sage.
- Lockwood, E., Ellis, A., & Knuth, E. (2013). Mathematicians' example-related activity when proving conjectures. In S. Brown, G. Karakok, K.H. Roh & M. Oehrtman (Eds.), *Proceedings for the sixteenth special interest group on research on undergraduate mathematics education*. Denver, CO: Northern Colorado University.
- Mason, J. (1996). Expressing generality and roots of algebra. In N. Bednarz, C. Kieran & L. Lee (Eds.), *Approaches to algebra: Perspectives for research and teaching* (pp. 65–86). Dordrecht: Kluwer.

- Mulligan, J., English, L., Mitchelmore, M. Welsby, S., & Crevensten, N. (2011). *An evaluation of the pattern and structure of a mathematics awareness program in the early school years*. Retrieved from <http://eprints.qut.edu.au/47315/>
- Peirce, C.S. (1966). The non-euclidean geometry. In A.W. Burks (Ed.), *Collected papers of Charles Sanders Peirce* (p. 72), Vol. 8. Cambridge, MA: Harvard University Press.
- Psillos, S. (1996). *Ampliative reasoning: Induction or abduction?* Contributing paper to the ECAI '96 workshop on abductive and inductive reasoning, P. Flach & A. Kakas (Eds.), Budapest.
- Rissland-Michener, E. (1978). Understanding understanding mathematics. *Cognitive Science*, 2, 361–383.
- Rivera, F., & Becker, J. (2007). Abduction-induction (generalisation) processes of elementary majors on figural patterns in algebra. *The Journal of Mathematical Behaviour*, 26(2), 140–155.
- Samson, D., Cheriyaarambil, K., & du Toit, S. (2011). Factorisation—Variations on a theme. *Learning and Teaching Mathematics*, 11, 18–21.
- Savage, M. (2005). Revisiting classic qualitative studies. *Forum: Qualitative Social Research*, 6, article 31.
- Skemp, R. (1969). *The psychology of learning mathematics*. Harmondsworth: Penguin.
- Vinner, S. (1991). The role of definitions in teaching and learning. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 65–81). Dordrecht: Kluwer Academic.
- Watson, A., & Mason, J. (2005). *Mathematics as a constructive activity: Learners generating examples*. Mahwah, NJ: Erlbaum.
- Yilmaz, R., Argun, Z., & Keskin, M. (2009). What is the role of vizualisation in generalisation processes: The case of pre-service secondary mathematics teachers. *Humanity and Social Sciences Journal*, 4, 130–137.
- Zaslavsky, O., & Zodik, I. (2007). Mathematic teachers' choices of examples that potentially support or impede learning. *Research in Mathematics Education*, 9, 143–145.