BAYES FACTORS FOR GROUPED DATA

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Abstract

In this paper we apply Bayes factors to grouped data. Group testing is where units are pooled together and tested as a group rather than individually. The Bayes factor is the ratio of the posterior probabilities of the null and the alternative hypotheses divided by the ratio of the prior probabilities for the null and the alternative hypotheses. A beta prior will be used, also known as a conjugate prior for the binomial distribution. An application to mosquito data will be considered, where a comparison is made between West Nile virus (WNV) infection prevalences in field collected Culex nigripalpus mosquitoes trapped at different heights.

Keywords: Bayes factors, conjugate prior, group testing.

1. INTRODUCTION

In Bayesian terminology we are not testing, but doing model comparison. Jeffreys (1961) introduced and developed the Bayesian approach to hypothesis testing. See Kass and Raftery (1995) and Robert et al. (2009) for a detailed discussion and explanation of Bayes factors, where they emphasize different points on Bayes factors. In this paper we will focus on Bayes factors for grouped data, where model comparison will be made for two proportions from grouped data. Group testing is where units are pooled together and tested as a group rather than individually. Group testing is also known as pooled testing, where pooled testing was introduced by Dorfman (1943). Dorfman (1943) used group testing for medical screening purposes to identify infected individuals. Bayes factors will be applied to an example by Biggerstaff (2008), where a comparison was made between West Nile virus (WNV) infection prevalences in field collected Culex nigripalpus mosquitoes trapped at different heights.

Not much has been done in literature from a Bayesian point of view on group testing. Hanson et al. (2006) used a two-stage sampling procedure and developed a Bayesian method that allows for sampling multiple sites in a specific region. Gastwirth and Johnson (1994) used independent beta priors. Chick (1996) used the beta (α, ß). prior for obtaining posterior distributions of the unknown proportion p. The methods were applied to grouped test data for gene transfer experiments and limiting dilution assay data for immunocompetency studies. Raubenheimer and van der Merwe (2014) looked at estimation of binomial proportions from pooled samples using an objective prior.

Where point estimates and credibility intervals were calculated for a single proportion as well as the difference between two binomial proportions from pooled samples with unequal pool sizes, using the data from Biggerstaff (2008).

The importance of Bayes factors will be discussed in Section 2. Notation, the likelihood function and some theoretical aspects will be considered in Section 3. Bayes factors for grouped data from binomial distributions will be discussed and shown in Section 4, simulation studies will be considered in Section 5 and the application will be considered in Section 6. The discussion and conclusion will be given in Section 7.

2. THE IMPORTANCE OF BAYES FACTORS

Bayes factors can be used as an alternative to p-values. The editors of the journal Basic and Applied Social Psychology (BASP) banned null hypothesis significance testing procedures from their articles. The editors of BASP said that null hypothesis significance testing procedures are invalid, and manuscripts containing these will be highly scrutinized and authors will have to remove all vestiges of null hypothesis significance testing procedures. The editors of BASP have reserved the right to make case-by-case judgments with regards to manuscripts containing Bayesian methods. Jim Berger commented the following in the ISBA March 2015 bulletin: “Where I diverge with the editors is that they do not offer a viable alternative to the p-value; the solution is objective Bayesian alternatives, which are simultaneously Bayesian and frequentist.” Berger and Pericchi (2015) wrote the following in their paper “Science needs to abandon p-values and adopt Bayes factors”. This predicts exciting times for Bayesian statistics as a whole and Bayes factors in particular.

Kass and Raftery (1995) made the following comparison between Bayes factors and non-Bayesian significance testing:

- There is no reason to expect a p-value to be similar to the posterior probability that the null hypothesis is correct. But partly because this misinterpretation of the p-value is common among non-statisticians, it is of some interest to compare the results. There is a general feeling that Bayes factors are more conservative than p-values, mainly because when comparisons are made, it becomes clear that a p-value of 0.05 cannot represent much evidence against the null hypothesis.
- Frequentist tests tend to reject the null hypothesis almost systematically in very large samples, whereas Bayes factors do not.
- Bayes factors, like Bayesian procedures generally, follow the likelihood principle. As a result, in settings such as clinical trials where cases may accrue sequentially, Bayes factors may be applied without concerns about unscheduled analysis of the data.

- Bayes factors can be applied as easily to non-nested models as to nested ones.
- Non-Bayesian significance tests were developed for the comparison of two models, but practical data analysis often involves far more than two models, at least implicitly. In this case, carrying out multiple frequentist tests to guide a search for the best model can give very misleading results. By allowing us to take into account model uncertainty, Bayes factors can avoid this problem.

Bayes factors offer a way of evaluating evidence in favour of a null hypothesis, and they provide a way of including other information when assessing the evidence for a hypothesis.

3. NOTATION AND LIKELIHOOD FUNCTION FOR BINOMIAL PROPORTIONS FROM POOLED SAMPLES

Assume that the proportion of successes in a given population is p. We will refer to an infected individual as a success in a binomial trial. The following notation will be used in this paper:

N - number of individuals to be sampled independently from the population
m_i - the size of the pool where $i = 1, 2, ..., M^*$
M^* - the number of distinct pool sizes
n_i - the number of pools of size m_i
X_i - the number of the n_i pools that is positive.

In the case of grouped data assume that $X_1, X_2, \ldots X_{M^*}$ are independent binomial random variables with parameters n_i and $1-(1-p)^{mi}$, i.e. $X_i \sim bin(n_i, 1-(1-p)^{mi})$.

The likelihood function is given by

$$L(p|data) = \prod_{i=1}^{M^*} \binom{n_i}{x_i} \{[1-(1-p)^{m_i}]^{x_i}[(1-p)^{m_i}]^{n_i-x_i}\}$$

$$\propto \prod_{i=1}^{M^*} \{[1-(1-p)^{m_i}]^{x_i}[(1-p)^{m_i}]^{n_i-x_i}\}.$$

In this paper we are interested in comparing two proportions, say p_1 and p_2. The likelihood function will then be:

$$L(p_1, p_2|\underline{x}_1, \underline{x}_2) = \prod_{i=1}^{2}\prod_{j=1}^{M_i^*} \binom{n_{ij}}{x_{ij}} \{[1-(1-p_i)^{m_{ij}}]^{x_{ij}}[(1-p_i)^{m_{ij}}]^{n_{ij}-x_{ij}}\}$$

$$\propto \prod_{i=1}^{2}\prod_{j=1}^{M_i^*} \{[1-(1-p_i)^{m_{ij}}]^{x_{ij}}[(1-p_i)^{m_{ij}}]^{n_{ij}-x_{ij}}\}.$$

A conjugate prior to the binomial distribution is used. Conjugacy may be defined as a joint property of the prior and the likelihood function that provides a posterior from the same distribution family as the prior, (Robert, 2001). Statisticians make use of conjugate priors to be certain that the posterior is predictable in its form. Consider a beta prior, i.e. $p_i \sim beta(\alpha, ß)$ for the p's

$$\pi(p_1, p_2) \quad \propto \quad \prod_{i=1}^{2} p_i^{\alpha-1}(1-p_i)^{\beta-1}$$

4. BAYES FACTORS

The Bayes factor is the ratio of the posterior probabilities of the null and the alternative hypotheses divided by the ratio of the prior probabilities for the null and the alternative hypotheses (Robert, 2001). The classical approach to hypothesis testing is not probability based; one could not place a probability on a hypothesis because a hypothesis is not a random variable in the frequentist sense. Using a frequentist approach, one has to make do with quantities like the p - value where this is conditional on H_0 being true. We do not know if H_0 is true, the real question is actually $P(H_0 \text{ is true/data})$. The Bayesian wants to find a probability that H_0 is true. The Bayes factor is a summary of the evidence provided by the data in favour of a scientific theory, represented by a statistical model, as opposed to another (Kass and Raftery, 1995). In Bayesian terminology we are not testing as in the classical sense, but we are comparing two possible models. This is also known as model comparison or Bayes factor analysis. For example, comparing model $f(x/\Theta_0, y)$ with model $f(x/\Theta, y)$. Where Θ is unspecified parameter and y is a nuisance parameter. In this instance we are interested in testing $H_0: \Theta = \Theta_0$ against $H_1: \Theta \neq \Theta_0$, where H_0 is the null hypothesis and H_1 the alternative hypothesis. Instead of calling the two options hypotheses, we shall call them models M_0 and M_1, respectively. The probability that M_0 is the 'correct' model will then be calculated.

4.1 Two Samples with $M^*_1 = M^*_2 = 1$

We first consider the simplest case where $n_1 = n_2 = n$ and $m_1 = m_2 = m$. The n's can be different, as long as the m's are the same. Then the equality of the p's is equivalent to the model $M_0: \Theta_1 = \Theta_2 = \Theta$ which will be compared to the model $M_1: \Theta_1 \neq \Theta_2$. Here we have $\Theta = (1-p)^m$. Under M_0 the prior on Θ is $beta(\alpha, ß)$, while under M_1 we have two independent $beta(\alpha, ß)$ priors.

The Bayes factor in favour of M_0 is given by

$$B_{01} = \frac{\int_0^1 L(\theta|x,M_0)\pi(\theta)d\theta}{\int_0^1\int_0^1 L(\theta_1,\theta_2|x_1,x_2,M_1)\pi(\theta_1)\pi(\theta_2)d\theta_1 d\theta_2} = \frac{f(x|M_0)}{f(x_1,x_2|M_1)}$$

$$= \frac{B(x+\alpha,2n-x+\beta)}{B(\alpha,\beta)} \div \frac{B(x_1+\alpha,n-x_1+\beta)B(x_2+\alpha,n-x_2+\beta)}{B(\alpha,\beta)^2}$$

$$= \frac{B(x+\alpha,2n-x+\beta)B(\alpha,\beta)}{B(x_1+\alpha,n-x_1+\beta)B(x_2+\alpha,n-x_2+\beta)}$$

where $x = x_1 + x_2$.

Another approach to calculate Bayes factors, is to use fractional Bayes factors. This was proposed by O'Hagan (1995). Here one uses part of the information from the data to create proper priors from improper priors. It uses a fraction of the likelihood to obtain proper priors. If we let $\alpha = \beta = ½$ i.e. considering a *beta(½, ½)* prior for the *p*'s we actually make use of the Jeffreys prior. In this case the Jeffreys prior is proper, and there is no need to make use of the fractional Bayes factor. If we let $\alpha = \beta = 0$ i.e considering a *beta(0, 0)* prior for the *p*'s, we actually make use of the Haldane prior. This prior was introduced by Haldane (1932). According to Zellner (1977) the Haldane prior is popular due to the posterior mean being equal to the maximum likelihood estimator. In this case the Haldane prior is improper, and we can't use the Bayes factor and therefore have to make use of partial Bayes factors, to be more specific the fractional Bayes factor. To create a proper prior for the parameters under the models, a fraction b of the likelihood should be used. For illustration and comparison purposes we will consider the fractional Bayes factor when using the Jeffreys and Haldane priors. The fractional Bayes factor in favour of model M_0 is given by

$$B_{01} = \frac{f^F(x|M_0)}{f^F(x_1,x_2|M_1)} = \frac{f(x|M_0)}{f_b(x|M_0)} \div \frac{f(x_1,x_2|M_1)}{f_b(x_1,x_2|M_1)}$$

When using the Jeffreys prior, the fractional Bayes factor in favour of model M_0 is given by

$$B_{01} = \frac{B(2n-x+{}^1/_2, x+{}^1/_2)B(b(n-x_1)+{}^1/_2, bx_1+{}^1/_2)B(b(n-x_2)+{}^1/_2, bx_2+{}^1/_2)}{B(b(2n-x)+{}^1/_2, bx+{}^1/_2)B(n-x_1+{}^1/_2, x_1+{}^1/_2)B(n-x_2+{}^1/_2, x_2+{}^1/_2)}.$$

When using the Haldane prior, the fractional Bayes factor in favour of model M_0 is given by

$$B_{01} = \frac{B(2n-x,x)B(b(n-x_1),bx_1)B(b(n-x_2),bx_2)}{B(b(2n-x),bx)B(n-x_1,x_1)B(n-x_2,x_2)}.$$

4.2 General Case for Two Samples

For the choice of prior given in the previous section, let $\alpha = \beta = ½$ i.e. considering a *beta(½, ½)* prior for the p's. Consider two models M_0: $p_1 = p_2 = p$ and M_1: $p_1 \neq p_2$

Under model M_0 the likelihood will be

$$L(p|\underline{x}, M_0) \propto \prod_{i=1}^{2}\prod_{j=1}^{M_i^*}\{[1-(1-p)^{m_{ij}}]^{x_{ij}}[(1-p)^{m_{ij}}]^{n_{ij}-x_{ij}}\},$$

and the marginal likelihood is then

$$f(\underline{x}|M_0) = \frac{1}{\pi}\int_0^1 p^{-\frac{1}{2}}(1-p)^{-\frac{1}{2}}L(p|\underline{x}, M_0)dp.$$

Under model M_1 the likelihood will be

$$L(p_1, p_2|\underline{x}_1, \underline{x}_2, M_1) \propto \prod_{i=1}^{2}\prod_{j=1}^{M_i^*}\{[1-(1-p_i)^{m_{ij}}]^{x_{ij}}[(1-p_i)^{m_{ij}}]^{n_{ij}-x_{ij}}\},$$

and the marginal likelihood is then

$$f(\underline{x}_1, \underline{x}_2|M_1) = \frac{1}{\pi^2}\int_0^1\int_0^1 p_1^{-\frac{1}{2}}(1-p_1)^{-\frac{1}{2}}p_2^{-\frac{1}{2}}(1-p_2)^{-\frac{1}{2}}L(p_1, p_2|\underline{x}_1, \underline{x}_2, M_1)dp_1dp_2.$$

The Bayes factor in favour of M_0 is given by

$$B_{01} = \frac{\frac{1}{\pi}\int_0^1 p^{-\frac{1}{2}}(1-p)^{-\frac{1}{2}}L(p|\underline{x}, M_0)dp}{\frac{1}{\pi^2}\int_0^1\int_0^1 p_1^{-\frac{1}{2}}(1-p_1)^{-\frac{1}{2}}p_2^{-\frac{1}{2}}(1-p_2)^{-\frac{1}{2}}L(p_1, p_2|\underline{x}_1, \underline{x}_2, M_1)dp_1dp_2}.$$

If one assumes that the two models are equally likely beforehand, i.e. $P(M_0) = P(M_1)$ the posterior probability of model M_0 is

$$P(M_0|\underline{x}) = \left(1+\frac{1}{B_{01}}\right)^{-1}.$$

5. SIMULATION RESULTS FOR TWO SAMPLES WITH $M_1^* = M_2^* = 1$

Here we consider the simplest case where $n_1 = n_2 = n$ and $m_1 = m_2 = m$. Then the equality of the p's is equivalent to the model M_0:$\theta_1 = \theta_2 = \theta$, which will be compared to the model M_1:$\theta_1 \neq \theta_2$. Here we have $\theta=(1-p)^m$. Under M_0 the prior on θ is *beta(α, β)*, while under M_1 we have two independent *beta(α, β)* priors. We consider two different priors here, one where $\alpha = \beta = ½$, the Jeffreys prior, and one where $\alpha = \beta = 1$, the uniform prior. Figures 1, 2, 3 and 4 show the posterior probabilities for M_0 when $\alpha = \beta = ½$ as well as when $\alpha = \beta = 1$. This is for the selected value of x_1 and a range of outcomes for x_2 when n= 20, 50, 100 and 200. In general the results look reasonable, with probabilities usually lower with the smaller values of α and β, except when n is small.

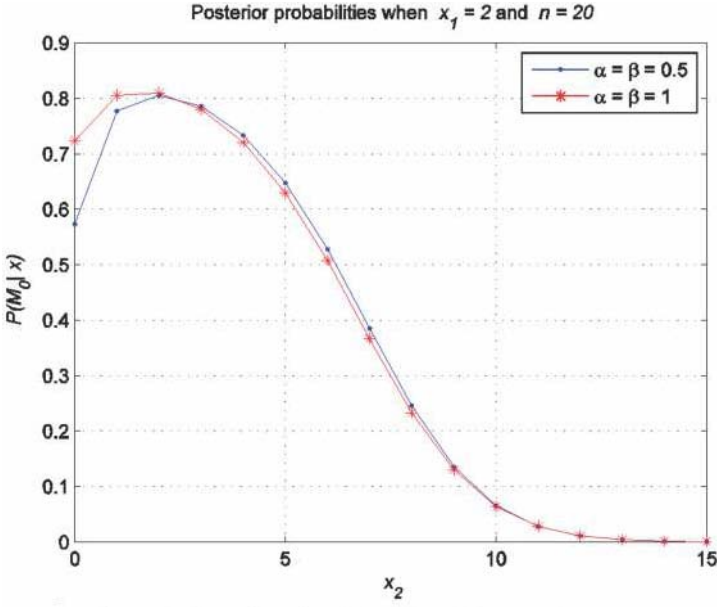


Figure 1: Posterior probabilities, given that $x_1 = 2$ for $n = 20$.

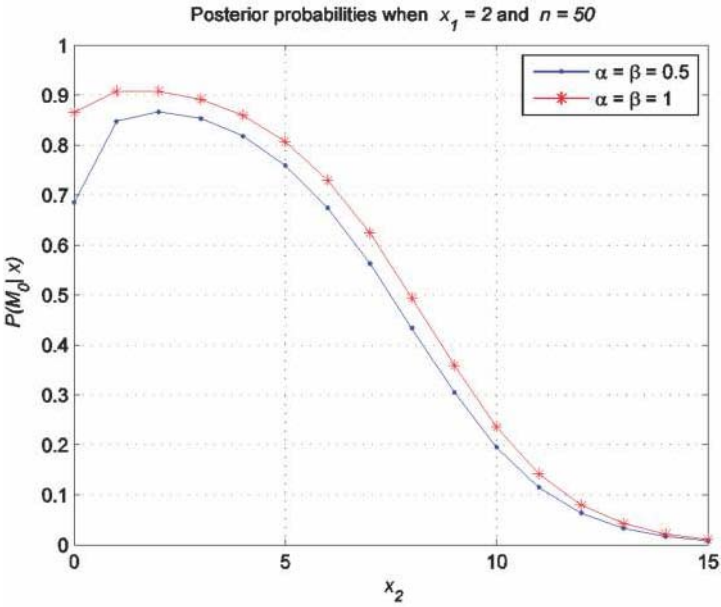


Figure 2: Posterior probabilities, given that $x_1 = 2$ for $n = 50$.

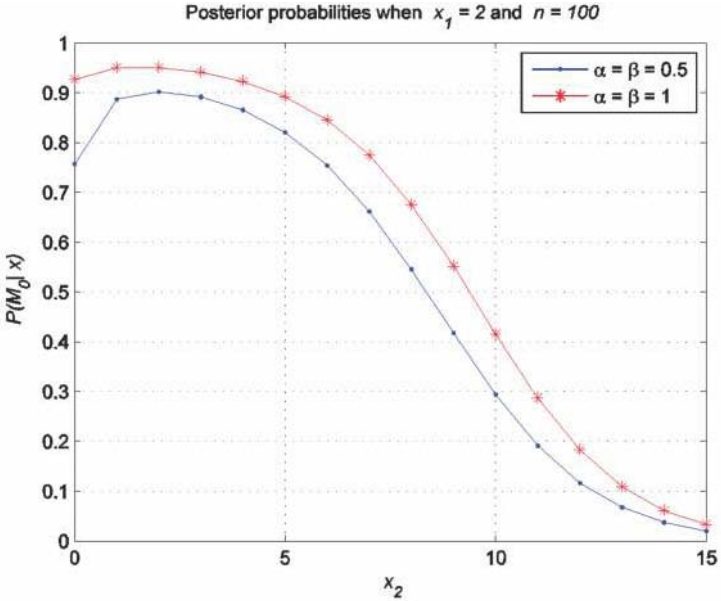


Figure 3: Posterior probabilities, given that $x_1 = 2$ for $n = 100$.

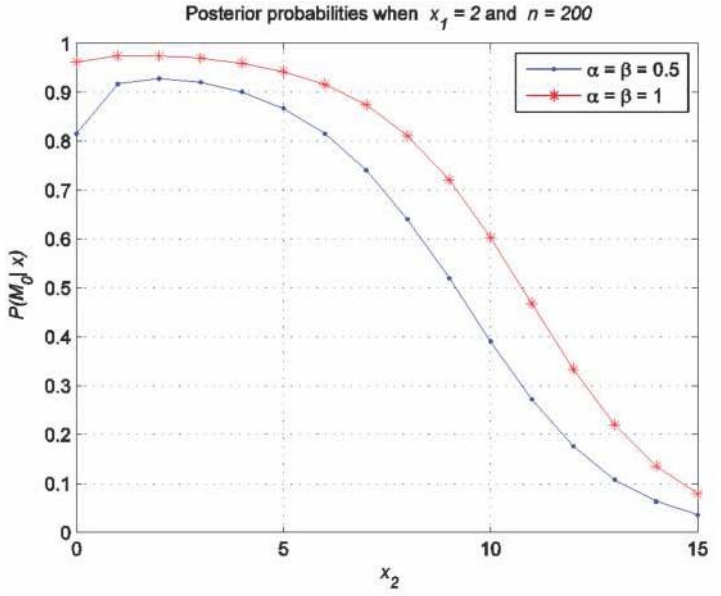


Figure 4: Posterior probabilities, given that $x_1 = 2$ for $n = 200$.

We will now apply the fractional Bayes factor using the Jeffreys and Haldane priors when b = 0.01, x_1 = 2 and n = 20, 50, 100 and 200. The results are displayed in Table 1. In this case the Jeffreys prior is proper, and there is no need to make use of the fractional Bayes factor. It is used here just for comparison purposes.

Table 1: Posterior probabilities, given that b = 0.01, x_1 = 2 and n = 20, 50, 100 and 200.

	Jeffreys	**Haldane**	**Jeffreys**	**Haldane**	**Jeffreys**	**Haldane**	**Jeffreys**	**Haldane**
x_2	*n = 20*		*n = 50*		*n = 100*		*n = 200*	
0	0.5592		0.6414		0.6774		0.6971	
1	0.7679	0.9748	0.8202	0.9746	0.8418	0.9745	0.8531	0.9745
2	0.7971	0.9755	0.8432	0.9749	0.8622	0.9748	0.8722	0.9747
3	0.7777	0.9708	0.8288	0.9701	0.8496	0.9700	0.8606	0.9700
4	0.7251	0.9604	0.7900	0.9603	0.8159	0.9604	0.8294	0.6905
5	0.6390	0.9412	0.7267	0.9433	0.7606	0.9440	0.7782	0.9445
6	0.5195	0.9068	0.6369	0.9150	0.6815	0.9175	0.7047	0.9188
7	0.3783	0.8457	0.5231	0.8692	0.5789	0.8756	0.6081	0.8787
8	0.2408	0.7422	0.3962	0.7975	0.4597	0.8115	0.4936	0.8180
9	0.1330	0.5851	0.2744	0.6928	0.3381	0.7190	0.3734	0.7312
10	0.0643	0.3914	0.1740	0.5558	0.2299	0.5977	0.2625	0.6171
11	0.0276	0.2139	0.1023	0.4029	0.1459	0.4579	0.1725	0.4841
12	0.0106	0.0959	0.0565	0.2616	0.0874	0.3204	0.1073	0.3496
13	0.0037	0.0363	0.0298	0.1535	0.0501	0.2055	0.0639	0.2331
14	0.0011	0.0120	0.0151	0.0830	0.0278	0.1226	0.0368	0.1450
15	0.0003	0.0034	0.0074	0.0423	0.0150	0.0692	0.0208	0.0857

The probabilities when using the Haldane prior are considerably higher than those from the Jeffreys prior. In the case of the Haldane prior all x's must be larger than zero, and $b>0$. One of the main questions is: What should the value of b be? We know that $P(M_0 \mid \underline{x}) \to 1$ when b→0 and $P(M_0 \mid \underline{x}) \to 0.5$ when $b \to 1$, so the posterior probability can be manipulated by the choice of b. The usual practice is to choose $b \propto n^{-1}$, and O'Hagan (1995) suggested $b = q/n$, where q is the minimal sample size.

6. APPLICATION

The Bayes factors discussed in the previous section will be applied to an example considered by Biggerstaff (2008). Godsey et al. (2005) and Godsey et al. (2013) studied the West Nile virus (WNV) infection prevalences in Culex nigripalpus mosquitoes in Louisiana in 2002 and 2002 – 2004, respectively. The West Nile virus is transmitted by mosquitoes and can cause in humans ranging from simple fevers to encephalitis, Marfin and Gubler (2001). Biggerstaff (2008) considered an example where a comparison is made between West Nile virus (WNV) infection prevalences in field collected Culex nigripalpus mosquitoes trapped at different heights. Table 2 summarises the data used by Biggerstaff (2008).

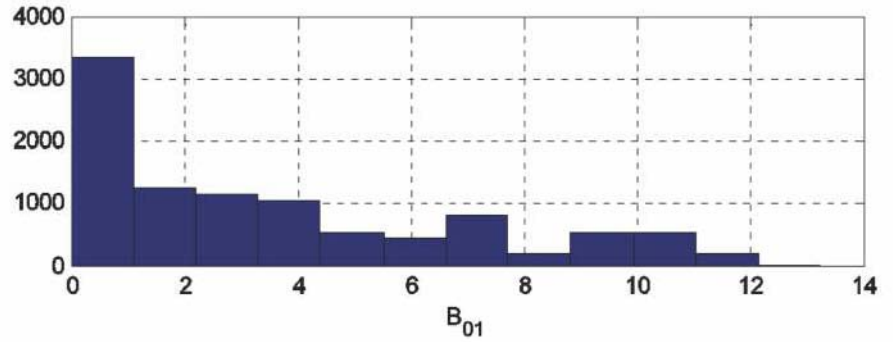
The general case for two samples will be considered here.

Table 2: Summary of Culex nigripalpus mosquitoes trapped at different heights of 6m and 1.5m

	Sample 1	**Sample 2**
	height = *6m*	**height = *1.5m***
Total	2 021	1 324
Number of pools	53	31
Average pool size	38.1321	42.7097
Minimum pool size	1	5
Maximum pool size	50	100
Number of positive pools	7	1

Using numerical integration, the Biggerstaff data yielded $B_{01}=2.3331$ with corresponding posterior probability of $P(M_0 \mid \underline{x})=0.7000$. This is moderate evidence in favour of model M0. Using the sample and pool sizes as given in Biggerstaff (2008) where $M_1^*=19$, with $p_1=0.004$, we simulated 10 000 outcomes of the 19×1 vector $\underline{x}_1$. By simulating 19 binomial observations, each with a sample size and a different probability. This was done since the pool sizes differ. The same was done with the second sample where $M_2^*=16$, with $p_2=0.001$. Using numerical integration, the Bayes factors and posterior probabilities were calculated and the histograms are shown in Figure 5. The mean of B_{01} is 3.6241 and the mean posterior probability is 0.6202, still favouring a single p slightly.

It is interesting to note that 626 of the 10 000 simulations gave the same result as the Biggerstaff (2008) data, 7 positives from the samples with p_1 and one positive from the samples with p_2, although not necessarily from samples with the same pool sizes. The range of posterior probabilities for the 626 simulations is (0.6925; 0.7208), with mean of 0.7030. So the pools from which the positive observations come do not have a large effect on the posterior.



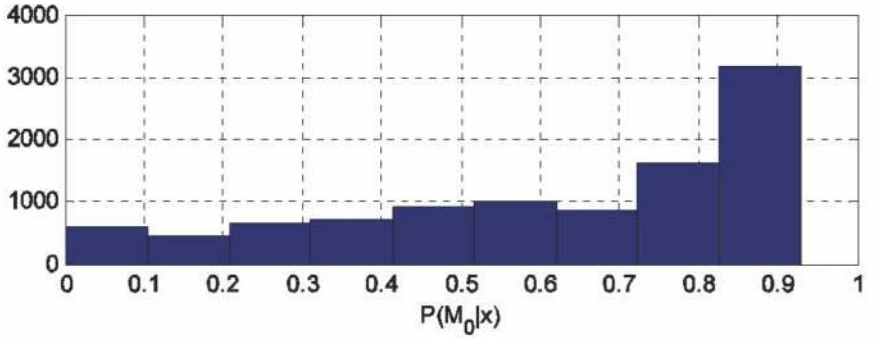


Figure 5: Histograms of the Bayes factor and posterior probabilities.

Kass and Raftery (1995) gave the following categories for interpreting the Bayes factor, B_{10}:

$log_{10}(B_{10})$	B_{10}	Evidence against H_0
0 to 0.5	1 to 3.2	Not worth more than a bare mention
0.5 to 1	3.2 to 10	Substantial
1 to 2	10 to 100	Strong
> 2	> 100	Decisive

Using these scales and categories to judge the evidence against M_0 for B_{01}, we obtain the following results:

- 85.12% of the time, the evidence was poor;
- 9.06% of the time, it was substantial;
- 5.49% of the time, it was strong;
- 0.33% of the time, it was decisive.

The impact of the West Nile virus on the economy

Zohrabian et al. (2004) stated that in 2002, an epidemic of WNV illness focused in the mid-western United States resulted in 4,156 reported cases; 2,942 cases had central nervous system (CNS) illness (meningitis, encephalitis, or acute flaccid paralysis), and 284 died. A total of 329 persons with WNV disease were reported in Louisiana, with illness onsets from June to November. Economic data about epidemics are essential for estimating the costs and benefits of strengthening and maintaining prevention and control programs, improving existing surveillance systems, and introducing other proposed interventions, such as vaccines.

Zohrabian et al. (2004) calculated the costs of the WNV epidemic as the sum of

1. The medical costs (inpatient and outpatient).
2. Non-medical costs, such as productivity losses caused by illness and premature death.
3. The costs incurred by public health and other government agencies for epidemic control.

Where the costs were estimated from June 2002, when the epidemic was first recognized, until February 2003, three months after the onset of illness of the last reported patient. Zohrabian et al. (2004) estimated that the costs from June 2002 to February 2003 attributable to the 2002 WNV epidemic in Louisiana were $20.1 million, including a $10.9 million cost of illness and a $9.2 million cost of public health response. The costs associated with WNV epidemics can be used to evaluate the economics of WNV prevention and control programs.

7. CONCLUSION

In this paper we looked at the Bayes factor for grouped data. We also considered fractional Bayes factors. The Bayes factor was applied to an example considered in Biggerstaff (2008), where a comparison was made between West Nile virus (WNV) infection prevalences in field collected Culex nigripalpus mosquitoes trapped at different heights. The two sample case with $M_1^*=M_2^*=1$ was first considered, where two priors were used *beta(α=1/2,β=1/2)* and *beta(α=1,β=1)*. The posterior probabilities were usually lower with the smaller values of α and *β*, except for small *n*. For the fractional Bayes factor two priors were considered a *beta(α=1/2,β=1/2)*, Jeffreys prior, and a *beta(α=0,β=0)*, Haldane prior. The probabilities when using the Haldane prior are considerably higher than those from the Jeffreys prior.

For the general case a *beta(1/2,1/2)* prior was used for the Bayes factor. Using numerical integration, the Bayes factors and posterior probabilities were calculated. The mean of B_{01} is 3.6241 and the mean posterior probability is 0.6202, favouring a single p slightly. As far as we know, Bayes factors have not been applied to this type of problem, where one deals with pooled samples. The Bayes factor offers an alternative to the classical method where one uses *p*-values. From our results we found that a single proportion is favoured, implying that the pools from which the positive observations come do not have a large effect on the posterior. With the recent developments in some journals, where frequentists' inferential procedures are banned, more researchers should consult the Bayesian method. This predicts exciting times for Bayesian statistics as a whole and Bayes factors in particular.

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