AN EXPLORATION OF GRADE 12 LEARNERS' USE OF INAPPROPRIATE ALGORITHMS IN CALCULUS.

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Abstract

This study was conducted with 29 Grade 12 learners who were studying calculus. The purpose was to explore how the learners responded to questions based on the derivative and why they did so. Data was collected from the written responses of the learners to two assessments carried out over a six-month period as well as interviews with four of the learners. It was found that learners made extensive use of inappropriate formulae, drawn from other sections of the curriculum The study recommends that teachers should not focus solely on how to carry out procedures, but they should also pay attention to why certain procedures are appropriate or not.
Keywords: calculus; derivative, qualitative study, mathematics, misconceptions

1. INTRODUCTION

Research concerning basic calculus concepts has been well documented with many of these studies being conducted at universities with undergraduate students (Baker, Cooley & Trigueros 2000; Bezuidenhout, 2001; Bowie 2000; Gucler 2013; Jojo, Maharaj & Brijlall, 2013; Maharaj, 2013; Orton, 1983; Palmiter 1991; Parameswaran, 2007). In South African schools, calculus is encountered for the first time by learners who are in their Grade 12 year of study of mathematics, and research at this level is very limited. The study on which this article is based was targeted at this under-represented sector in order to investigate Grade 12 learners' understanding of calculus. The participants in the study were of average ability in mathematics, who are often not able to shift to more sophisticated ways of working with mathematical concepts. In this study we analyse their responses to questions based on the concept of the derivative in calculus in an attempt to understand why they responded in the manner that they did. In particular, the focus is on their use of inappropriate algorithms and procedures when responding to questions based on the derivative.

More than a decade ago, White and Mitchelmore (1996) expressed concern about the memorisation that takes place by the large number of students who take calculus, and they called for more research on this issue. Since then other authors have also called for more research in calculus in aspects such as the coordination of symbolic and graphical representations (Tall, 2010); cognitive difficulties in calculus (Tall, 2010), dynamic and procedural aspects of calculus (Guchler, 2012), as well as students' struggles with symbolisation in calculus (Maharaj, 2013).

These calls echo Tall's (1991: 12) contention that more studies are needed "on the difficulties encountered by students of differing abilities and experience, to obtain empirical evidence to build and test theories of learning to enable more fruitful learning experiences for students in calculus". Thus it is hoped that this study, which focused on the struggles of the average mathematics learner, will add to existing knowledge about how to construct successful learning experiences for such learners.

2. RELATED LITERATURE

In the last three decades research exploring student understanding of the derivative and the various aspects of the derivative has been well documented.

Ubuz's (2001) study consisted of 147 students enrolled in calculus courses in four different universities in England. The research results revealed that students have some misconceptions such as the following: the derivative at a point gives the function at a point; the tangent equation is the derivative function; the derivative at a point is the tangent equation and the derivative at a point is the value of the tangent equation at that point. Similarly, Maharaj (2013) in his study with first year university students, found in one item that a large number of students attempted to work out the value of a function at a point when asked to find the rate of change at that point. This suggests a difficulty with distinguishing between the value of the function and the value of the derivative of the function at a given point.

Studies conducted by Santos and Thomas (2005) and Judson and Nishimori (2005) involved students who were doing a basic calculus course in above-average high schools in Japan and the United States. The aim of the study was to determine any differences in students' conceptual understanding of calculus and their ability to use algebra to solve traditional calculus problems. The findings revealed that most Japanese and American high school calculus students had a solid grasp of the mechanics of calculus, and that they understood the derivative as the rate of change and how it can be used to sketch the graphs of functions. There was little difference in the students' conceptual understanding of calculus between the two groups of students, but the Japanese students demonstrated much stronger algebra skills than the American students. The American students lacked fluency in manipulating algebraic expressions containing radicals and had difficulty with problems the Japanese found to be straightforward.

White and Mitchelmore (1996) found that the students who participated in their study had a very primitive understanding of the variable. Their study involved first year university students who studied calculus in secondary schools. They were presented with word problems involving rate of change that could be solved using algebra and calculus.

In solving these word problems, students had to identify the appropriate concepts needed to solve the problem as well as some algebraic relationships among the variables or the selection of some calculus concepts involving variables (such as the derivative) and their expression in symbolic form. White and Mitchelmore (1996) refer to the process of selecting a calculus concept and expressing it in symbolic form as the symbolisation process. Results of their analysis show that very few students were able to correctly symbolise at any one time with the more complex rate of change problems; those who did, were almost always correct. This study also revealed that some students tended to focus on manipulation in which they based their decisions about which procedure to apply on the given symbols and to ignore the meanings behind the symbols. This approach was highlighted during the interview comments, as students were actively "looking for symbols to which they could apply known manipulations" (White & Mitchelmore, 1996: 88).

White and Mitchelmore (1996: 91) identified three examples where the variables were treated as symbols to be manipulated rather than as quantities:

> "...failure to distinguish a general relationship from a specific value, searching for symbols to which known procedures are applied regardless of what the symbols refer to and remembering procedures solely in terms of the symbols used when they were first learned".

Their findings highlight that students who focus on manipulation have a concept of a variable that is limited to algebraic symbols, because they have learned to operate with symbols without any regard to their contextual meaning. Maharaj (2013) noted similarly that students perform poorly because they are unable to adequately handle information about objects that are given in symbolic form.

3. THEORETICAL FRAMEWORK

Sfard (1991: 5) argues that the ability of learners to see a mathematical concept both as a process and as an object "is indispensable for a deep understanding of mathematics, whatever the understanding of mathematics is". This dual nature of a mathematical construct is present in various kinds of symbolic representation and verbal descriptions of a mathematical concept. Sfard's process-object duality construct can be seen as a contraction of the more comprehensive APOS (action, process, object, schema) theory of Dubinsky (1991) in that the former does not include actions and schemas. According to the APOS theory (Dubinsky, Weller, Mcdonald & Brown, 2005) an individual deals with a mathematical situation by using certain mental mechanisms to build cognitive structures that are applied to the situation. The main mechanisms are called interiorisation and encapsulation and the related structures are actions, processes, objects and schemas (Dubinsky et al., 2005).

The structures are explained below.

- Action: An action is an externally driven, repeatable physical or mental manipulation that transforms objects.
- Process: A process is an action that takes place entirely in the mind.
- Object: The distinction between a process and an object is drawn by stating that a process becomes an object when it is perceived as an entity upon which actions and processes can be made, and such actions are made in the mind of the learner.
- Schema: A schema is a more or less coherent collection of cognitive objects and internal processes for manipulating these objects. A schema could help students to “understand, deal with, organise, or make sense out of a perceived problem situation” (Dubinsky, 1991: 102).

Tall (2010) sees mathematical thinking begin in conceptual embodiment based on human perception, action and reflection and broaden to include proceptual symbolism, in which dynamic actions, such as counting, are symbolised so that the symbols may be used dually as process and concept (procept). This is similar to Sfard's process-object duality concept and Dubinsky's APOS theory. De Lima and Tall (2008: 8) assert that the “symbolic development from process to object is mirrored in a shift in focus from the steps of an action to the effect of an action”. They see two distinct ways of introducing a concept through embodiment and symbolism, which enables a view of an embodied development that runs parallel to the construction of the symbolic process-object compression:

> “Starting from physical procedures that embody the symbolic procedures, then looking at different procedures that have the same effect, gives a parallel embodied development to the first part of the theoretical APOS construction from action to process. If that process can be embodied in a physical way, then this enables the further shift from embodied process to embodied object” (De Lima & Tall, 2008: 16)

All these theories emphasise the hierarchical nature of mathematical constructs in the development of a concept. The process conception precedes the object conception (“operational-structural” in Sfard's terms and “proceptual symbolism” in Tall's case); as in computational mathematics, a vast majority of the ideas originate in the process rather than in the objects. Sfard and Linchevski (1994) provide an explanation for what can happen if the desired operational-structural operation does not occur. Sfard (1991) uses the term “pseudo-structural conception” to describe this phenomenon when it occurs. Many learners are not able to see the mathematical object that they are required to see because it is not clear to them for a variety of reasons. However, these learners are now required to perform some complex operations on this virtually non-existing object.

They develop a way of dealing with them by creating their own meaning and these meanings may not be appropriate at all. The mathematical object is now identified with its representation. A symbol, formula or graph becomes the object that is dealt with and this new "knowledge remains detached from its operational underpinnings and from the previously developed system of concepts. In these circumstances, the secondary processes must seem totally arbitrary" (Sfard & Linchevski, 1994: 117). Learners may still be able to perform these routines, but their understanding may remain at an action level, because as Sfard and Linchevski (1994) argue, meaningfulness comes when the learner is able to see the abstract ideas hidden behind the symbols. In this regard De Lima and Tall (2008) claim that meaningfulness is achieved when the focus of the learner can shift from the steps of an action to the effect of an action.

The result of learners adopting a pseudo-structural approach may lead to their developing a conception of mathematics that is not coherent and lacks rich relationships. Learners who adopt the pseudo-structural approach and confuse the abstract objects with their representations "do not realise that the symbols cannot perform the magic their referents are able to do: they cannot glue together lots of detailed pieces of knowledge into one powerful whole" (Sfard &Linchevski, 1994: 75). De Lima and Tall (2008: 3) similarly draw attention to the practice of students from their study, of engaging in symbol-shifting and performing 'magic'. They write: "[T]hey build their own ways of working based on the embodied actions they perform on the symbols, mentally picking them up and moving them around, with the added 'magic' of [particular] rules." De Lima and Tall (2008: 10) found that students were using their own ways of working by drawing on specific techniques they had encountered before rather than using general principles. Not a single student was able to explain any understanding of the mathematical reasons for performing certain actions. The students were simply shifting symbols around, which seemed to be like physical entities to them, and which they could move around in ways they judged to be appropriate. These authors attributed the resulting widespread errors that they observed to the fragility of the students' knowledge.

A "met-before" (De Lima & Tall, 2008: 6) is a "mental construct that an individual uses at a given time based in experiences they have met before" and it may influence current learning both negatively or positively. Met-befores can cause problems when they are used outside their domain of validity.

These authors comment further that new experiences can also affect the way in which we conceive old knowledge. For example, when a learner who has learnt how to draw a parabola using the formula for the turning point, may find she has more options when she learns how to find the turning point of a cubic graph by differentiating the cubic function and then equating it to zero and solving for the unknown. In such a case a learner may use the algorithm for the turning point of a cubic graph to find the turning point of a parabola.

De Lima and Tall (2008: 6) use the term “met-after” to denote an experience met at a later time that affects the memories of previous knowledge. In the preceding discussion, the calculus algorithm to find the turning point of a cubic graph can be considered as a met-after for the experience of learning about quadratic functions before calculus is learnt.

4. METHODOLOGY

The study method was that of a naturalistic inquiry with particular reference to the interpretive approach, as the main goal of this study was to understand the learners' interpretations of reality (Cohen, Manion & Morrison, 2007). The participants of the study comprised one class of 29 Grade 12 learners. Data for the study was generated from students' written responses to two assessments. In addition to the document analysis of their written responses, four of the learners were interviewed using a semi-structured interview schedule. The purpose of the interview was to find out more about their understanding of the derivative as well as to identify issues that influenced their responses. One of the issues that were identified was the frequency with which students used inappropriate formulae that had been learnt prior to the teaching of calculus, and some of which had been learnt after. The research questions that are addressed in this article are: 1. What are the previously learnt[1] concepts and newly introduced concepts[2] that influenced this group of learners when answering questions based on the derivative? 2. How can these trends be accounted for? Answers to the first research question are considered in Section 5.1, while the second research question is addressed in Section 6.

5. RESULTS

In this section we first present the results pertaining to errors related to the use of previously learnt concepts ('met-befores') in Section 5.1 followed by the results pertaining to errors arising from newly introduced concepts ('met-afters') in Section 5.2. Thereafter the results from the interviews are presented in Section 5.3 revealing their low levels of understanding the concept of the derivative.

5.1 Previously learnt concepts and newly introduced concepts that influenced learners' response

One of the findings that emerged from the analysis of the learners' written responses was the extensive use of quadratic theory in providing answers to questions based on the derivative. In this paper we report on 41 instances where learners made inappropriate use of the quadratic function in the examination items.

[1]De Lima and Tall (2008) refer to these as “met-befores”.
[2]De Lima and Tall (2008) refer to these as “met-afters”.

For example, many learners transformed a given quadratic function into an equation by equating the expression to 0, and then solving the equation when asked to find the derivative of the function. Other examples included using the properties of a quadratic function when asked to sketch the graph of a cubic function. On reason for the extensive use of quadratic theory could be because in Grade 11, learners spend much of their time studying quadratic theory and the graph of the quadratic function, and this topic influenced their responses to questions on the derivative. Table 1 presents a quantitative summary of some of these concepts that emerged in the learners' responses.

Use of 'met befores ' related to quadratic functions	**Frequency**
Solving a quadratic equation instead of finding the derivative	5
Solving a quadratic equation instead of finding the value of a function	4
Evaluating a quadratic expression instead of evaluating the derivative	16
Finding the axis of symmetry instead of finding the derivative	2
Confusing the properties of the graph of a cubic function with those of a parabola	14

Table 1. Learner errors related to inappropriate use of concepts from quadratic theory

We now provide the qualitative details of how learners used each of these inappropriate algorithms in sections 5.1.1 to 5.1.6. The learners are referred to as S_1, S_2, etc.

5.1.1 Presentation of the solution to $f(x)=0$ instead of finding $f'(x)$

The first example of this inappropriate algorithm appeared in the May assessment where learners were required to find $\frac{dy}{dx}$ if $y=(x+1)(2-3x)$. Because the product rule for differentiation is not introduced at Grade 12 level, learners apply the power rule for finding $f'(x)$ when given $f(x)=x^n$. The rule is $f'(x)=nx^{n-1}$. Some learners changed the expression $y=(x+1)(2-3x)$ to the quadratic equation $(x-1)(2-3x)=0$, and 'solved' for $\frac{dy}{dx}$ as they would for x. For example S_2, S_9 and S_{23} 'concluded' that $\frac{dy}{dx}=-1$ and $\frac{dy}{dx}=\frac{2}{3}$ while S28 expanded the brackets incorrectly and thereafter incorrectly factorised, leading to the answer of $x=-\frac{1}{3}$ and $x=2$.

Another question that elicited the response of solving for x, was when learners were required to find $\frac{dy}{dx}$ if $y = \frac{x^2+6x-4}{x}$

S_3 S_9 and S_{26} incorrectly factorised the numerator and 'solved' for x. S_2 used the quadratic formula $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ to find the roots of the algebraic expression in the numerator of $y = \frac{x^2+6x-4}{x}$

5.1.2 Presentation of the solution to *B(t)=0*, instead of the calculation of *B(0)*

In the May assessment learners were given a word problem that was accompanied by a formula, $B(t)=-3t^2+30t+1500$ where $B(t)$ representing the number of bacteria present, t hours after an observation started. Learners were asked to find the amount of bacteria present at the beginning of the observation. Instead of calculating $B(0)$ S_{12}, S_{19}, S_{25} and S_{28} equated the expression $-3t^2+30t+1500$ to zero and then incorrectly factorised and solved for t. One response was as follows:

$$\begin{aligned} B(t) &= -3t^2 + 30t + 1500 \\ &= -3(t+30)(t+50) \\ &= -30 \quad or \quad -50. \end{aligned}$$

Here it seems as if the quadratic form of the function may have been a cue for them to factorise the expression.

5.1.3 Calculation of $B(10)$ instead of $B'(10)$

A further question based on the function discussed above required learners to calculate $B'(10)$, the rate of change of the levels of bacteria at 10 hours. Fourteen of the 27 learners' responses were as follows:

$$\begin{aligned} B(10) &= -3(10)^2 + 30(10) + 1500 \\ &= -300 + 300 + 1500 \\ &= 1500 \end{aligned}$$

A further two learners S_{18} and S_{19} also calculated $B(10)$, but their answer contained computational errors as well. Thus these 16 learners amounting to 59% of the class opted to substitute $t=10$ into the function instead of differentiating the function and then evaluating the derivative function at $t=10$.

5.1.4 Calculation of the axis of symmetry instead of the derivative

Another response drawn from quadratic theory was the use of the well-known formula $x = \frac{-b}{2a}$ to find the axis of symmetry of a quadratic function, $f(x) = ax^2 + bx + c$ which learners encounter in their Grade 11 mathematics. When asked for $\frac{dy}{dx}$ if $y=(x+1)(2-3x)$, S20 expanded the expression to $3x^2 - x + 2$ and thereafter proceeded to find $\Delta = \frac{-b}{2a} = \frac{-(-1)}{2(3)} = \frac{1}{6}$.

Another student, S6, worked similarly in the August assessment where learners were required to find $\frac{dy}{dx}$ if $y=-7x(x-2)$. S_6 expanded the expression by multiplying out the brackets and proceeded to find $\Delta = \frac{-b}{2a} = \frac{-(14)}{2(-7)} = 1$

5.1.5 Calculation of the turning point of a graph of a quadratic function instead of the turning point of a cubic graph

Other questions that cued the use of the $x = \frac{-b}{2a}$ formula were the following which appeared in the May and August assessments. In the May assessment, learners were required to prove that one of the turning points of the cubic function $h(x) = -x^3 + 3x + 2 = (x+1)^2(2-x)$ was (1;4) and in the August assessment, learners had to find the co-ordinates of the turning point of the graph of the function $f(x) = -x^3 + 3x + 2 = -(x+1)^2(x-2)$.

In the May assessment, S_{10} S_{20} S_{29} began their solution with the formula $x = \frac{-b}{2a}$ Thereafter S10 substituted $b=-2$ and $a=1$ and got the answer 1. S_{10} then took this value as the x co-ordinate of C, the turning point of the cubic function. S_{20} 'substituted' $b=3$ and $a=-1$ into the formula and simplified this to $\frac{3}{2}$. S_{29} substituted $b=3$ and $a=2$ into the formula, obtained $x = \frac{-3}{4}$ and 'simplified' further to $x=4-3=1$. S_7 began the solution with an incorrect formula, $\frac{b^2}{2a}$ 'substituted' $b=3$ and $a=-1$ and simplified this to $\frac{-9}{2}$. S15 merely wrote the formula $\frac{-b}{2a}$ with no further working details.

5.1.6 Sketching a quadratic graph instead of a cubic graph

The fact that graphs of parabolas have just one turning point seemed to have influenced learners when asked to find the turning point of the cubic function $f(x)=-x^3+3x+2$ S_9 and S_{16} found one turning point and then sketched a parabola. S_{20} also found one turning point but did not provide a sketch. S_1 S_2 S_3 S_{12} and S_{22} calculated only one turning point, which was incorrect, and drew a graph of the parabola. S_5 also sketched the graph of the parabola but did not use the above-mentioned learners' approach. Instead this learner used the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to find the roots of the cubic function and substituted $a=1, b=3$ and $c=2$. Two roots were obtained and the learner drew a parabola with these two roots. In these cases the learners ignored the fact that the highest power of the polynomial was 3, making it cubic and not quadratic. Graphs of cubic functions usually have three x-intercepts and two turning points, whereas a graph of a quadratic function usually has one turning point and two x-intercepts.[3]

[3]It is possible that a cubic function may have only 1 or 2 intercepts and a point of inflection, and a parabola may have no x-intercepts.

5.2 Reference to concepts which were encountered after the study of calculus

There were also instances where learners used formulae from concepts that they learnt in Grade 12 after they were taught calculus. Learners used the midpoint formula as well as the distance formula from the co-ordinate geometry section inappropriately. Interestingly, the use of these formulae was only evident in questions that made reference to a point shown on the graph. For example, the May assessment included the graph of a cubic function with points A, C and D marked on the graph. Learners were required to find the co-ordinates of A and D and to prove that the co-ordinates of C (1; 4).

To find the x-intercepts of the graph of the cubic function S_2 and S_{20} responded to this question by starting with $\left(\frac{x_1+x_2}{2};\frac{y_1+y_2}{2}\right)$. S_2 continued to substitute values for the variables and obtained incorrect co-ordinates for A and D while S_{20} did not proceed after the formula. S_9 began her response with $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ the distance formula, continued to substitute values for the variables and obtained incorrect co-ordinates for A and D. To prove that the co-ordinates of the turning point were (1;4), S_{12} began his response with the midpoint formula $\left(\frac{x_1+x_2}{2};\frac{y_1+y_2}{2}\right)$ and then continued to substitute some chosen values for the variables such that when simplified, the correct ordered pair (1;4) was obtained. The assessment also included a word problem based on the function $B(t) = -3t^2+30t+1500$ where $B(t)$ represents the number of bacteria present, t hours after an observation started. Learners were required to calculate $B'(10)$, which is the rate of change at 10 hours. Some learners used an inappropriate formula. S2 began the response with the formula $S_n=\frac{n}{2}(a+l)$ while S5 wrote $T_n= a(n-1)d$. It is pertinent to note that this question appeared in the May examination, directly after the learners had completed a section on sequences and series which includes the study and application of the formulae $S_n=\frac{n}{2}(a+l)$ and $T_n= a(n-1)d$.

Furthermore, formulae from analytical geometry and sequences and series were prevalent in some learners' responses. Both these sections were undertaken by the learners just prior to the May examinations and they used them inappropriately. These are examples of what De Lima and Tall (2008) have named “met-afters”.

5.3 Results from learners' interviews

The responses of the learners during the interview revealed low levels of understanding of the concept of a derivative. Only one learner came close to an appropriate description of a derivative. In response to the question, “What is your understanding of a derivative?” the learner replied, “to find the average gradient”.

Although this is a very limited definition, she was the only person who made any link to the concept of gradient when talking about the derivative. Other learners claimed openly that they did not know what it was.

One learner's explanation of finding the derivative from first principles is, “It's $f'(x)$ limits, $\frac{f(x+h)-x}{h}$, whatever”. His casual dismissal of the expression, illustrated by his use of the word “whatever”, clearly conveys the impression that he does not consider the details to be very important. When asked about the meaning of the derivative, one boy laughed and said, “not much”.

Another said, “Honestly ma'am, I don't know what the derivative is but it's easy enough to work out, once you're taught how to do it. I don't know what it actually means.” This comment shows that his belief was that the derivative is something you can “work out” without having to understand what it means. In fact, some learners associated the derivative with the process of solving an equation. This perspective was evident in the written responses which were analysed in the beginning of this section, where some learners associated x with $\frac{dy}{dx}$.

During the interview one learner described the study of calculus as “just simply finding x”. He elaborated further: “Ma'am, from what I know, from what I've been taught to do is, calculus is just finding x. Finding all forms of x and what x is unknown.” This comment and the written responses seem to suggest that the learners confuse finding the derivative of a function with solving given equations.

Some learners thought of calculus as finding formulae into which one could make substitutions. A comment by a learner in the interview follows:

> “[Calculus is] finding all forms of x and what x is unknown and just different forms and different like methods... there's many, many things like things in calculus and you just sub[stitute].”

His opinion is that there are various formulae and a person makes substitutions into these formulae or equations. Another learner found calculus to be fun because of the different formulae he could use to try to get an answer:

> “Calculus was definitely the most fun I had with maths, everything else not very much, but calculus because there's always a way to find out you can try with many different ways with one equation and finally you'll find your answer.”

When asked why she preferred working with rules (finding the derivative from first principles), one girl said:

> "I don't know, I just like to be... there is something to do and there you have to just do it that way. I don't like finding any roundabout way of doing it."

Some learners' comments revealed a procedurally embodied perspective (De Lima & Tall, 2008) of finding the derivative of a function. A learner described her method of differentiating as follows:

> "You take the number that is on top and you and you multiply it by the number … and you minus 1 from the exponent."

Her explanation suggests that she associated a physical symbol-shifting movement in carrying out the power rule for differentiation. Another learner said he found it easy to find the derivative because he followed the rule:

> L: I just, eh, like say I just multiplied by, what you call it?
> T: The exponent.
> L: The exponent. Ja that's it, and then subtract 1 from the exponent.

Again, this learner's casual description of the procedure, while having forgotten the term "exponent" shows that he was not overly concerned with the details, but just performed the algorithm.

It is important to note that none of these learners attributed any importance to when and why they applied the formulae. Some were able to casually describe how they carried out the procedure, but did not articulate when they used the formula. It is clear that their first step was to carry out a procedure. If it was clear what the procedure was, they then completed it by picking out information that was provided, without considering whether the procedure was appropriate or not. In some cases, they were unable to distinguish between solving an equation and differentiating a function, and some opted to use quadratic theory to solve the quadratic equations they created from the functions that were given. In other cases, they retrieved formulae from quadratic theory, such as the axis of symmetry, or the quadratic formula, and carried out these procedures. The words of one student can be used to summarise this approach: "You can try with many different ways with one equation and finally you'll find your answer."

6. DISCUSSION AND CONCLUDING REMARKS

In trying to address the second research question, there are three themes which we pursue in this final section. These are:

- Poor understanding of the meaning of the derivative of a function
- Performing irrelevant procedures without questioning the relevance
- Non-encapsulation of concepts

6.1 Poor understanding of the concept of derivative

The analysis of the interview transcripts revealed that none of the students were able to provide any sensible meaning for the derivative – they associated the computation of the derivative with following certain rules or procedures. Learners' comments suggest that they felt that they had to carry out the rules in a specific manner, e.g. “There is something to do and there you have to just do it that way.”

One learner was able to link the concept with the gradient of a function but not in a clear manner. Other learners admitted that they had no idea what the derivative of a function was. One learner tried unsuccessfully to recall the formula for finding the derivative of a function using first principles and then dismissed it with a “whatever”. The written responses revealed that many learners solved a quadratic equation when asked to find the derivative. The interview responses of two learners confirmed that they misunderstood a derivative to be the solving of an equation. One learner, in fact, thought that the whole aim of calculus was to identify various formulae into which a person would make substitutions. Similar to the case of the students in De Lima and Tall's (2008) study, no student was able to provide a mathematical explanation for performing certain actions.

6.2 Performing irrelevant procedures without questioning the relevance

In this situation, many of the students seemed to have limited mathematical proficiency, especially with respect to the conceptual understanding and procedural fluency strands (Kilpatrick, Swafford & Findell, 2001). Their responses indicate that they had little understanding of how the various mathematical concepts fit together. In terms of procedural fluency, some learners were able to carry out some procedures accurately but were unable to judge whether the procedure they chose was appropriate or not.

Thus many learners in this study are familiar with the 'how' of a routine but not the 'when'. This leads to the question: How can students recognise whether a procedure is the relevant one? Carrying out a procedure in the form in which it is presented by substituting in values and computing the output, is not as challenging as questioning whether the procedure is appropriate to the question. Sfard (2008: 259) remarks that the how of a routine is usually individualised well before the when. Sfard (2008: 223) further cautions that when “teaching focuses on the issue of how the routine should be performed to the almost total neglect of the question of when the routine is most appropriate”, there will be little space for learners to extend their thinking. Doing the 'how' requires an action level understanding – an external carrying out of the routine (Dubinsky et al., 2005). A process level understanding of the routine would require an interiorisation of the action into a process, where a person does not have to externally consider each step separately.

However, an object-level understanding is what will enable a learner to shift his/her attention beyond just carrying out the steps of a routine. This study shows that many of these learners were not able to shift their understanding of the derivative beyond an action level.

6.3 Non-encapsulation of concepts

An object-level understanding of a concept requires an encapsulation of the concept from seeing it as an action or process to seeing it as a whole upon which further transformations are possible (Dubinsky et al., 2005). Being able to recognise the appropriateness of a procedure requires a shift of attention from seeing the procedure as an action/process to seeing it as an object, and requires an encapsulation of the procedure into an object. This calls for the development of an insight that allows an individual to see it in a different manner. De Lima and Tall (2008) write that "an expert shifts the focus from the steps of the procedures concerned to the overall effect of that procedure". Encapsulation theory would suggest that the learners whose responses have been presented were not able to encapsulate the power rule for differentiation $(f'(x) = nx^{n-1}$, when $f(x) = x^n)$ into an object.

Learners such as those whose verbal description of the rule has been given have shown that they saw the rule in a disconnected manner, and had not yet interiorised this into a process. This would not allow them to proceed to an encapsulation. Their description also echoes De Lima and Tall's (2008) description of the instances of procedural embodiment (instead of the necessary conceptual embodiment) they observed with their sample of learners who were working on solving linear equations. De Lima and Tall (2008: 14) remark that the students in their study "'shift the symbols around' following their natural human inclinations to embody their actions with physical meaning". They write that the students engage in practice such as "picking up a term" and "shifting it to the other side". The learners in the study that is reported in this article also demonstrated this procedurally embodied manner of working, as shown in the one learner's description of the power rule as "You take the number that is on top and you multiply it by the number … and you minus 1 from the exponent." Her use of the words "on the top" suggests that she is bringing that number down and multiplying it by the number in front of the variable. The comment by a learner that you multiply by "the exponent" and then subtract one from "the exponent" also suggests a procedurally embodied manner of working with the exponent – shifting it around and forming a product with it; subtracting a number from it. Similar to the students in the study carried out by De Lima and Tall (2008), these learners seemed to be simply shifting symbols around (which seemed to be like physical entities to them) according to their own rules. Based on their observations, De Lima and Tall (2008) caution that although mathematical teaching requires valuable practice in carrying out specific algorithms which is useful in routinising actions so that they can be carried out automatically, teachers should beware that such practice can lead to a fragile form of procedural embodiment which is evidenced by the shifting of symbols without meaning.

An expert is able to shift "the focus from the steps of the procedures concerned to the overall effect of that procedure to compress knowledge and make it more easy to manipulate in a sophisticated way" (De Lima & Tall, 2008: 16). However, many academically weaker children do not make that shift so easily and "remain with the comfort of learned procedures and progress less easily to the more flexible use of symbols and process and concept" (De Lima & Tall, 2008: 16). In this case the procedures remain disconnected and external because the action-process-object compression has not occurred. Consequently, for these learners the cognitive load is heavier (than for those learners who have been able to shift to an object- level understanding) when having to identify, remember and use the appropriate routines accurately.

An object-level understanding would give learners a vantage point that would allow them to choose the correct procedures and to carry them out fluently and accurately. Another implication from the process-object encapsulation theory is that the concepts comprising the met-befores and met-afters that were identified could not have been encapsulated into objects as well. The met-befores of the quadratic theory were used inappropriately, showing that the learners knew the 'how' of these procedures but not the 'when'. Similarly their use of the previously encountered concepts such as formulae from sequences and series and coordinate geometry illustrates that they had not acquired an object-level understanding of these prerequisite concepts because they did recognise that the formulae were not appropriate to the situation.

How could teachers help learners move to an object-level understanding of concepts? Should learners be discouraged from practising rules and procedures that they have learnt? De Lima and Tall (2008) comment that practice in carrying out specific algorithms is useful in routinising actions so that they can be carried out automatically. These authors explain that when a sequence of actions is repeated until it becomes automatic it can lead to learning because it "relegates the routine to subconscious activity leaving conscious thought to think about important issues in any given context" (De Lima & Tall, 2008:4). The authors contend that compression of knowledge is at the heart of mathematical thinking and practice of new routines is therefore essential in the learning process. However, they caution that teachers should beware that such practice can lead to fragile understanding if little attention is paid to helping learners move beyond the external action-level ways of working with routines.

It is crucial that teachers pay attention to helping learners interiorise an action into process and then to encapsulate it into an object. This suggests that different types of practice are necessary at different stages. At first the aim is to routinise and familiarise an individual with the how of a routine. Once that is accomplished, attention must be paid to providing learning opportunities that can lead to encapsulation which will allow learners to deal with the when of routines and thus to recognise appropriate routines and when they should be used.

7. REFERENCES

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