Title	CAN STUDENTS CONSTRUCT NON-CONSTRUCTIVE REASONING? IDENTIFYING FUNDAMENTAL SITUATIONS FOR PROOF BY CONTRADICTION
Auther(s)	Hayata, Toru; Uegatani, Yusuke; Hakamata, Ryoto
Citation	Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education , 3 : 43 - 50
Issue Date	2018-07-12
DOI	
Self DOI	
URL	http://ir.lib.hiroshima-u.ac.jp/00046414
Right	Copyright © 2018 left to the authors. All rights reserved.
Relation	

CAN STUDENTS CONSTRUCT NON-CONSTRUCTIVE REASONING? IDENTIFYING FUNDAMENTAL SITUATIONS FOR PROOF BY CONTRADICTION

Toru Hayata[1], Yusuke Uegatani[2], and Ryoto Hakamata[3]

[1]Naruto University of Education [2]Hiroshima University High School, Fukuyama
[3]Hiroshima University High School

The purpose of this study is to identify and empirically corroborate a fundamental situation (Brousseau, 1997) for constructing "proof by contradiction." We identified the four elements of a fundamental situation: i) obtaining strong conviction; ii) negating the given proposition naturally without being aware of the assumption; iii) finding a contradiction easily; and iv) noticing the origin of the contradiction. Based on this study, a new research question arises: How can students construct "proof by contradiction" using teacher support?

INTRODUCTION

"Proof by contradiction" (PbC) is one of the most valuable types of reasoning in mathematics and mathematics education. However, students have specific cognitive and didactic difficulties in negating propositions and using laws such as the excluded middle (Antonini & Mariotti, 2008). Thus, although some authors have proposed didactic suggestions to help students overcome PbC difficulties (e.g., Wu Yu Lin & Lee, 2003; Antonini & Mariotti, 2008), in our opinion, many students are still unable to resolve these difficulties. One possible reason for this may be an overlooked component in the studies of students. In other words, almost all students who are analyzed in studies of PbC are either supplied PbC by their teachers before they engage in constructing PbC for the first time, or they have already been taught PbC before they engage in research.

In contrast, we believe that in order to understand a concept, students must construct knowledge by themselves (with their teacher's support). We assume that students cannot fully understand a concept if teachers or others tell them about it beforehand. Therefore, suggestions provided by the previous studies are inadequate as they are derived from observations of students whose understanding of PbC is not sufficient. In clarifying the conditions that enable students to construct PbC by themselves with their teacher's support, findings of previous studies become more meaningful, paving the way for elaboration and further research. Thus, our study aims to do the following:

P1: To identify a fundamental situation (Brousseau, 1997) for constructing PbC

P2: To corroborate the identified fundamental situation empirically

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 43-50). Umeå, Sweden: PME.

THEORETICAL BACKGROUND AND METHODOLOGY

The theoretical background for this study is based on the Theory of Didactical Situations (TDS; Brousseau, 1997), and the methodology adopted is didactical engineering, particularly *a priori* and *a posteriori* analysis (Artigue, 1992) within the framework of TDS. We used TDS because it is one of the most scientific theories in the discipline. Learning is defined in TDS as follows: "The student learns by adapting herself to a *milieu* which generates contradictions, difficulties and disequilibria, rather as human society does. This knowledge, the result of the student's adaptation, manifests itself by new responses which provide evidence of learning" (Brousseau, 1997, p. 30, italics in the original). This definition aligns with our assumption that students must construct knowledge by themselves.

TDS assumes that students construct mathematical knowledge in didactical or adidactical situations. Since any mathematical knowledge has been historically incubated in some situation, there always exist situations wherein it can be constructed. Because not all situations are replicable in educational settings, TDS assumes that *all* mathematical knowledge has at least one fundamental situation (FS) that can become a didactical situation (Brousseau, 1997, p. 30). However, FSs are not always easily identified by mathematics educators, and PbC does not typically employ constructive reasoning (in the sense of *intuitionism*). Thus, an FS for constructing PbC has not yet been identified. In TDS, on identifying an FS based on theory, we corroborate it through *a priori* and *a posteriori* analyses: first, by designing a didactical situation based on the FS (*a priori* analysis); second, by trying to realize this situation in an actual mathematics classroom; and third, by corroborating our hypothesis about the FS underlying the design.

FUNDAMENTAL SITUATION OF PROOF BY CONTRADICTION

Indirect argumentation seems to be a natural way of thinking (Freudenthal, 1973, p. 629). Thus, an FS for constructing PbC should enable students to employ indirect argumentation and develop this into a PbC. However, previous research suggests that ruptures between indirect argumentation and PbC may occur. Mathematicians and mathematics educators have pointed out the specific difficulties of PbC (e.g., Wu Yu Lin & Lee, 2003; Antonini & Mariotti, 2008); we distinguish between three types here in order to identify our FS.

D-I: Difficulties in considering PbC as an option and in carrying out the method of PbC

When students try to prove a proposition, they usually do not consider using indirect proof, including PbC, as an option. Although they may consider PbC suitable for proving a given proposition, they often give up constructing PbC mid-way. Several difficulties in the process have been reported: negating the proposition, formalizing and interpreting the negation (Wu Yu, Lin & Lee, 2003), finding a contradiction, and so on.

D-II: Difficulties in accepting the result of a PbC

Even if one is able to prove a proposition using PbC, the result may not seem acceptable: "I think this is one source of frustration, of the feeling that we have been cheated, that nothing has been really proved, that it is merely some sort of a trick—a sorcery—that has been played on us" (Leron 1985, p. 323).

D-III: Difficulties in grasping the very structure of PbC

PbC has a specific structure, that is, when one assumes the negation of a true proposition *P*, a contradiction comes into being implying that the negation is false and *P* is true. Thus, one needs to know the *theory* and the *meta-theory* (Antonini & Mariotti, 2008) of PbC.

In Japan, students engage in PbC in mathematics when they are in the 9^{th} grade and learn that the square root of 2 is irrational. However, since they have not been introduced to PbC until then, they face D-I, D-II, and D-III all at once. This confuses them. Additionally, knowing the structure of PbC is necessary for overcoming D-I and D-II, that is, students must have already overcome D-III to resolve D-I and D-II. Therefore, before students engage in PbC, they should engage in PbC in FSs in which they are required to face and overcome only D-III.

In this study, we focus on an insight from Dawkins & Karunakaran (2016), according to which, research on student learning of mathematical proofs should pay greater attention to the role of mathematical content. Thus, in order to avoid D-II, FSs for PbC should enable students to surmise that the proposition to be proved is true. For example, students who have already accepted that the square root of 2 is irrational have less trouble accepting the PbC in order to prove it (Antonini & Mariotti, 2008, p.407). In addition, in order to avoid the emergence of D-I, an FS should enable students to negate the sentence naturally and formalize the proposition to be proved. Such situations enable students to find a contradiction easily because they autonomously begin to enquire into what statements can hold in the false world. Items (i) – (iii) (Figure 1) are a summary of the above consideration.

A fundamental situation (FS) for constructing proof by contradiction is one in which students must do the following four things:

(i) Be strongly convinced that the proposition to be proved is true

(ii) In investigating the *milieu*, they must construct a false world by naturally assuming the negation of the proposition (without being aware of the assumption).

(iii) Easily find a contradiction in the false world

(iv) Notice that they make the assumption themselves and that this is the origin of the contradiction

Figure 1: A fundamental situation for constructing proof by contradiction.

However, even if a student is able to find the contradiction and conclude that a proposition is true, s/he may still reason this using indirect argumentation rather than indirect proof. Because the core of PbC lies explicitly in assuming the negation of a true proposition, students must make such assumptions after they negate and formalize propositions. In order to do this, students must identify the origins of a contradiction. Thus, we have added (iv) to Figure 1.

Figure 1 is our proposal for a possible fundamental situation for constructing PbC. In the next section, we corroborate this by *a priori* and *a posteriori* analysis.

DESIGN AND *A PRIORI* ANALYSIS

The subjects of our analysis are 9th grade students who come across PbC for the first time (as mentioned earlier). These students have already learned basic direct proofs in geometry and algebra, algebraic skills and concepts, and the notion of irrational numbers. They have also learned—but not proven—that the square root of 2 cannot be represented as p/q (where p and q are disjointed integers and q is not equal to 0). In their textbook, PbC is introduced in order to prove this. We thus designed a mathematics lesson as shown in Figure 3. The teaching protocol employed in this lesson followed the "problem-solving lesson" model presented in Figure 2.

Our experimental lesson was conducted in June 2016 in a junior high school attached to a national university. This experiment was conducted during one lesson (50 minutes) on 40 students (20 males/20 females). The teacher was the students' regular mathematics teacher, and is one of the authors of this study as well. We did not investigate students' pre-conceptions, because such an investigation may affect students' performance in the study. However, our reflection on the experiment revealed that none of the students seemed to know PbC well before the experiment; even after students found a contradiction, they did not to try to construct PbC by themselves. Instead, they all needed the teacher's support to shift from indirect argumentation to indirect proof.

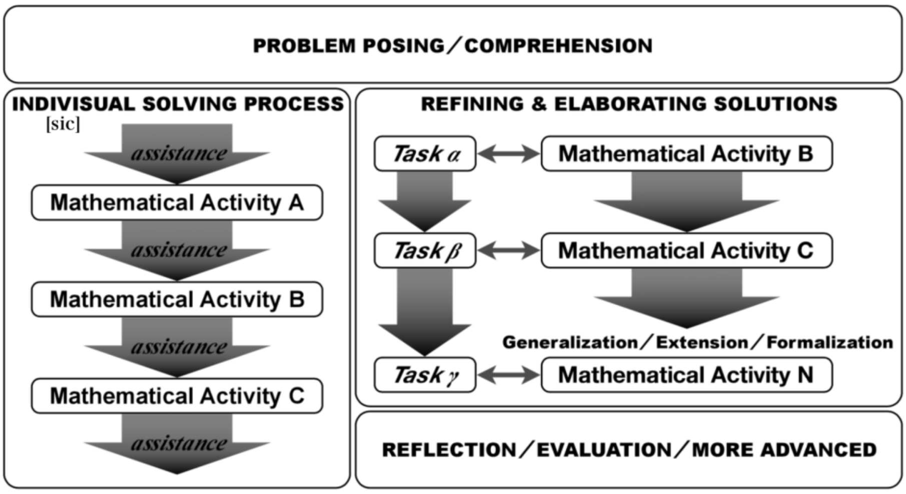

Figure 2: Lesson model (Mizoguchi, 2015, p. 627; reprinted with permission).

TS: Teacher's support

Problem

Let a,b be rational numbers. Do there exist a,b such that $a+b\sqrt{2}=0$? If these do exist, show all a,b and explain why there are no other. If these do not exist, explain the reason. (It is known that $\sqrt{2}$ is irrational number.)

Mathematical Activity C

Students infer the answer is only $a=b=0$ by inserting any value into a,b.

TS1: Is it just that you cannot find it?

TS2: Can you explain the reason ?

Mathematical Activity B-1

Students observe $a=-b\sqrt{2}$, and they become aware of the fact that the right side is a rational number, and the left side is an irrational number.

TS1: Can you show that $-b\sqrt{2}$ is irrational number ?

TS2: If $b=0$, so? Can you use known knowledge by using deformation of the formula?

Mathematical Activity B-2

Students observe $-\frac{a}{b}=\sqrt{2}$, and they become curious about the fact that the right side is a rational number, and the left side is an irrational number.

TS1: Can you explain your curiousness around the inference?

TS2: Can you find the root of the curiousness?

TS3: If $a=b=0$, can you obtain $-\frac{a}{b}=\sqrt{2}$?

Mathematical Activity A

Students become aware that, if one assumes $b\neq 0$, there appears the curiousness. For this reason, they conclude that the assumption is not correct thus the answer is only $a=b=0$.

TS1: Can you explain why the answer is only $a=b=0$?

TS2: What is the structure of your explanation?

Figure 3: Lesson designed to corroborate the FS identified in this study[1].

RESULTS AND *A POSTERIORI* ANALYSIS

In the lesson, the teacher posed the problem to the students and shared with them the property that the square root of 2 cannot be represented as a common fraction. We obtained data from video recordings and the students' worksheets. Only the problem and name fields are written in their worksheets. We banned eraser use so that we could examine all the ideas that students produced. During the "individual solving process" phase (Figure 2), students tried to solve the problem on their worksheets, and the teacher supported them verbally and individually, following the plan in Figure 3. The teacher was careful to align his support appropriately in keeping with the students' levels of progress. In the "refining and elaborating solutions" phase, the teacher picked students to present their own solutions (in the order of the mathematical activities C, B-1, B-2, and A) and all the students refined and elaborated their own solutions through discussion involving the entire class.

(a) When we solve $a + b\sqrt{2}$,
$b\sqrt{2} = -a$, then $\sqrt{2} = -\frac{a}{b}$
Both a and b are rational numbers…

(b) If there are any a and b that satisfy $a + b\sqrt{2}$
When we solve $a + b\sqrt{2}$,
$b\sqrt{2} = -a$
$\sqrt{2} = -\frac{a}{b}$
Both a and b are rational numbers. So $-\frac{a}{b}$ is a rational number too. Thus, $\sqrt{2}$ is a rational number too; however this contradicts the fact that $\sqrt{2}$ is an irrational number, so there are no a and b that satisfy $a + b\sqrt{2}$
$(a, b) = (0,0)$

(c) If there are any a and b that satisfy $a + b\sqrt{2}$ $(\underline{a \neq 0, b \neq 0})$
When we solve $a + b\sqrt{2}$,
$b\sqrt{2} = -a$
$\sqrt{2} = -\frac{a}{b}$ If $b = 0$, I can not divide both sides, then $\sqrt{2} = -\frac{a}{b}$ by b, so I assume $\underline{b \neq 0}$.
Both a and b are rational numbers.
So $-\frac{a}{b}$ is a rational number too. Thus, $\sqrt{2}$ is a rational number too; however this contradicts the fact that $\sqrt{2}$ is an irrational number, so there are no a and b that satisfy $a + b\sqrt{2}$ when $a \neq 0, b \neq 0$.
Next, I insert $a = 0$ into $a + b\sqrt{2}$, so $0 + b\sqrt{2}$.
Thus, $b\sqrt{2} = 0$, so $\underline{b = 0}$.
From this result, if we insert $b = 0$ into $a + b\sqrt{2}$, it becomes $(a, b) = (0,0)$ too.
For above reasons, the answer is only $(a, b) = (0,0)$.

Figure 4: Male student Y's worksheet (translated into English by the authors, underlined by the student; (a), (b), and (c) added by the authors for convenience).

In the experimental lesson, all the students completed mathematical activity C successfully, and almost all the students completed B-1 or B-2 successfully in the first phase, that is, they found a contradiction (although some students described it as "strange"). Student Y (male) is one of the students who successfully constructed PbC. Figures 4 is an example of students' answers (translated here from their native language). In this example, the teacher supported him in constructing PbC (activity A), but PbC seemed difficult for him. In the "refining and elaborating solutions" phase, Student Y's presentation was mathematically sound and hence was accepted by the other students (See Figure 4 (c)). Next, the teacher presented: "When we need to prove a supposition, if we assume the opposite to be true and derive a contradiction, then, the initial supposition to be proved is considered true. We call this method 'proof by contradiction.'"

Here, let us focus on Student Y's problem-solving process. As soon as the "individual problem-solving process" phase began, Student Y thought the answer was only $(a, b) = (0,0)$ and that $\sqrt{2} = -\frac{a}{b}$ was contradictive. To indicate this, he wrote (a), as shown in Figure 4. However, he was puzzled by the contradiction and wrote, "Both a and b are rational numbers…" Thus, the teacher supported him by following TS-1 for B-2 in

Figure 3. Five minutes later, he finished writing indirect argumentation (b). Although it was a persuasive argument, he did not pay attention to his implicit assumption that ($a \neq 0, b \neq 0$). Hence, the teacher supported him by following TS-2 for B-2 in Figure 3. Ten minutes later, he finished writing a mathematically acceptable PbC (c). While in the "refining and elaborating solutions" phase, Student Y explained (c) to the other students after another student had explained B-2. However, some students could not find the essential difference between these two explanations. Thus, the teacher asked all the students, "The explanation by Y is very similar to another explanation (B-2). What is the important difference between them?" and asked Student Y to explain it. Student Y said, "Umm… $-\frac{a}{b}$, oh, sorry. Well… there is $b\sqrt{2} = -a$ in my explanation, well… we cannot divide $b\sqrt{2}$ by b" (the original was spoken in his native language), and Student Y pointed out that the assumption $b \neq 0$ is important. This showed that he noticed the importance of assuming negation of the proposition to be proved.

Student Y's problem-solving process (shown by (a), (b) and (c)) was in accordance with our design. Three observations support this claim: first, in (a), he surmised that the solution was only $(a, b) = (0,0)$ and found a contradiction in a false world, where the negation of the proposition to be proved was assumed; second, he made an indirect argument (b); and finally, he developed (b) into (c), that is, PbC, by detecting the origin of the contradiction and noticing that the negation of the true proposition was implicitly assumed. Thus, these empirical observations corroborate the fact that our designed lesson can produce a didactical situation and that our proposed situation in Figure 1 is an FS for constructing PbC.

IMPLICATION

The purpose of this study was not to design a "good" lesson, but to identify an FS for constructing PbC, and to corroborate it. Therefore, although not all the students were able to construct PbC by themselves in this lesson, the value of our findings cannot be undermined. Given the fact that Student Y (and some other students) constructed PbC by themselves (with the teacher's support), we may conclude that Figure 1 is valid as an FS. Designing a "good" lesson according to Figure 1 is thus a future task for mathematics teachers rather than for researchers. Our findings also imply a new research question: How can students construct PbC by themselves with their teacher's support? Future researchers investigating students' cognitive and didactical difficulties with PbC should expand their foci to the processes of construction of PbC by learners. Researchers should also investigate the differences between the processes underlying success and failure in constructing PbC.

We have three future tasks. First, we must investigate the processes of students who construct PbC by themselves, especially to examine whether or not they are able to use PbC by themselves, with their teacher's support (D-I), and whether or not they accept the results of PbC (D-II). Second, we must identify fundamental situations for overcoming D-I and D-II. In other words, we must design curriculum for understanding

PbC. Third, we must investigate the effects of applying previous studies' didactical suggestions to our teaching practices.

Notes

[1] They do not know that $-b\sqrt{2}$ is irrational. Thus, when students solved it in accordance with B-1, we supported their shift to B-2.

Acknowledgments

This work was partially supported by JSPS KAKENHI Grant Number 17H00147.

References

Antonini, S., & Mariotti, M. A. (2008). Indirect proof: what is specific to this way of proving?. *ZDM*, *40*(3), 401-412.

Artigue, M. (1992). Didactical engineering. In R. Douady & A. Mercier (Eds.), *Research in Didactique of Mathematics: Selected papers* (pp. 41-65). Grenoble: La Pansée Sauvage.

Brousseau, G. (1997). *Theory of Didactical Situations in Mathematics* (V. Warfield, N. Balacheff, M. Cooper & R. Sutherland, Trans.). Berlin, German: Kluwer.

Dawkins, P. C., & Karunakaran, S. S. (2016). Why research on proof-oriented mathematical behavior should attend to the role of particular mathematical content. *The Journal of Mathematical Behavior*, *44*, 65–75.

Freudenthal, H. (1973). *Mathematics as an educational task*. Dordrecht: Reidel.

Leron, U. (1985). A Direct approach to indirect proofs. *Educational Studies in Mathematics*, *16*(3), 321- 325.

Mizoguchi, T. (2015). Functions and equations: Developing an integrated curriculum with the required mathematical activities, *Proceedings of the 7th ICMI-East Asia Regional Conference on Mathematics Education*, 625-637.

Wu Yu, J.-Y., Lin, F.-L., & Lee, Y.-S. (2003). Students' understanding of proof by contradiction. In N.A. Pateman, B. J. Dougherty, & J. Zilliox (Eds.), *Proceedings of the 2003 Joint Meeting of PME and PMENA* (Vol. 4, pp. 443–449).