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PH. D. THESIS

博士論文

# THE FINITE ELEMENT METHOD ON A RIEMANN SURFACE

# リーマン面上の有限要素法

1989

平成元年

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# The Finite Element Method on a Riemann Surface

by

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Thesis presented in fulfillment of the requirements
for the degree of Doctor of Philosophy of
Hiroshima University
1989

## Acknowledgements

I would like to express my gratitude to Professor Hisao Mizumoto, Hiroshima University, for his continued guidance and invaluable advice during the period of carrying out this research. He oriented firstly my interest to numerical analysis in my undergraduate study at Okayama University.

I would also like to express my gratitude to Professor O. C. Zienkiewicz and Professor Peter Bettess (currently the University of Newcastle upon Tyne) for their kind guidances during my study in the Master of Science course of the University of Wales, Swansea. They gave me the best education to assist the essential understanding of finite element method in engineering science.

Furthermore I would like to express my special gratitude to Professor Emeritus Tadahiko Kawai, the University of Tokyo, for his kind encouragement of my study in computational mechanics.

I am grateful to Dr. T. Watanabe, Dr. N. Tanaka, Dr. Y. Nagashima and Dr. T. Ohi who were heads of Tamano Laboratory, Mitsui E & S Co. Ltd. and gave me the opportunities of study in comfort. Special thanks should go to Dr. T. Iwaki, Dr. M. Kishi and Mr. A. Urushihara who have given me the fresh stimulations through their ways of lives as researchers in a private company.

Finally I thank my wife Tomoko, and sons Soichiro and Mikijiro for their help of giving me enough time at home to prepare this thesis.

Contents

# Introduction

In the present thesis we aim to establish a method of finite element approximations on a Riemann surface. Our method matches the abstract definition of Riemann surfaces, and also offer a new technique of high practical use in numerical calculation not only for the case of Riemann surfaces but also for the case of plane domains. It is characteristic of our method that we adopt ordinary triangular meshes and linear elements on a subregion of every fixed parametric disk, and thus our approximating differentials express singular property exactly near singularities. Hence the approximations of high precision of differentials are obtained. It should be noted that we do not adopt any so-called refined or curvilinear mesh near singularities.

Let $\Omega$ be a closed Riemann surface or a subdomain of a Riemann surface $W$ whose closure $\overline{\Omega}$ is a compact bordered subregion of $W$. We choose a fixed finite collection $\Phi = \{z = \varphi_j(p),\ p \in U_j;\ j = 1, \cdots, m\}$ of local parameters $z = \varphi_j(p)$ and parametric disks $U_j$ so that $\overline{\Omega} \subset \bigcup_{j=1}^{m} U_j$. Chapter 1 is devoted to construction of a triangulation $K$ of $\overline{\Omega}$ with width $h$ associated to $\Phi$ (cf. § 1.2), a normal subdivision of $K$ (cf. § 1.3), and a naturalized triangulation $K'$ associated to $K$ (cf. § 1.4). The triangulation $K$ of $\overline{\Omega}$ is constructed as the sum of subtriangulations $K_j$ $(j = 1, \cdots, m)$ in such a way that $|K_j| \subset U_j$, each 2-simplex $s$ of $K$ belongs to one and only one $K_j$, each $s \in K_j$ is natural (see § 1.2) at most except for the case when it has a common side with another $s' \in K_k$ $(k \neq j)$, and the diameter of $\varphi_j(s)$ is at most $h$ for each $s \in K_j$ $(j = 1, \cdots, m)$. Let $K'_j$ $(j = 1, \cdots, m)$ be triangulations consisting of all 2-simplices of $K_j$ which are not minor or major, and all naturalized simplices of $K_j$ (see § 1.4). Then the triangulation $K'$ is defined as the sum of $K'_j$ $(j = 1, \cdots, m)$.

In Chapter 2, we introduce and investigate two spaces $\Lambda = \Lambda(K)$ and $\Lambda' = \Lambda'(K')$ of differentials: the comparable space $\Lambda = \Lambda(K)$ (with $\omega$) and the computable space $\Lambda' = \Lambda'(K')$. The space $\Lambda$ consists of locally exact differentials $\sigma_h$ such that for each 2-simplex $s \in K_j$ $(j = 1, \cdots, m)$ the coefficients of $\sigma_h$ are constant on $\varphi_j(s)$ except that $\sigma_h$ is modified on all lunes of minor or major simplices (see § 1.4 and § 2.1). To each $\sigma_h \in \Lambda$, we associate a differential $\sigma'_h = F(\sigma_h)$ on $K'$ whose coefficients are constant on $\varphi_j(s)$ for each 2-simplex $s \in K'_j$ $(j = 1, \cdots, m)$ and which is equal to $\sigma_h$ on $\overline{\Omega}$ except for all lunes of $K$ (cf. § 2.2). The space $\Lambda'$ consists of all $\sigma'_h = F(\sigma_h)$ $(\sigma_h \in \Lambda)$. We shall investigate estimates of differences of Dirichlet norms $\|\sigma_h\|^2_\Omega$ and $\|\sigma'_h\|^2_{K'}$, (see Lemma 2.2).

Let $\Theta$ be a given closed differential on $\Omega$ with finite norm, and let $\Gamma_\Theta$ be a set of all closed differentials which have finite norms and satisfy same period conditions and boundary behaviors as $\Theta$. Then there exists a unique harmonic differential $\omega$ which satisfies the minimal property (see § 3.1):

$$\|\omega\| = \min_{\sigma \in \Gamma_\Theta} \|\sigma\|.$$

The finite element approximations $\psi_h$ and $\omega'_h$ of $\omega$ are defined in the spaces $\Lambda$ and $\Lambda'$ respectively (cf. § 3.2 and § 3.3 resp.). The differential $\omega'_h$ can be numerically calculated. Chapter 3 is devoted to error estimates of $\psi_h$ and $\omega_h$ for $\omega$, where $\omega_h = F^{-1}(\omega'_h)$. We shall make use of Bramble and Zlámal's lemma (see Lemma 3.5). In Theorems 3.1 and 3.2, we obtain error estimates:

$$\|\psi_h - \omega\|^2 \leqq Ch^2 \quad \text{and} \quad \|\omega_h - \omega\|^2 \leqq C'h^2,$$

where $C$ and $C'$ are constants which depend only on the differential $\omega$ and the smallest value of interior angles of triangles $\varphi_j(s)$ for all $s \in K'_j$ $(j = 1, \cdots, m)$. Further, in Theorem 3.2, we obtain an estimate for $\|\omega\|^2$:

$$\|\omega\|^2 \leqq \|\omega'_h\|^2 + \varepsilon(\omega'_h)$$

in a special case (see § 3.2), where $\varepsilon(\omega'_h)$ is a quantity of $O(h^2)$ which can be numerically calculated.

In Chapter 4 we apply our results to numerical calculation of periodicity moduli of closed and compact bordered Riemann surfaces, and we shall show that calculation results for some concrete Riemann surfaces of genus one are very good. Let $\{A, B\}$ be a canonical homology basis of $\overline{\Omega}$ such that $A \times B = 1$. Then there exists a unique system of harmonic differentials $\{\phi, \rho, \chi, \tau\}$ on $\Omega$ satisfying some period and boundary conditions (see (4.1) ~ (4.4)). The periodicity moduli $p_1$ and $p_2$ of $\Omega$ with respect to $A$ and $B$ respectively are determined by

$$p_1 \equiv \int_A *\phi = \|\phi\|^2 = \frac{1}{\|\tau\|^2} \quad \text{and} \quad p_2 \equiv \int_B *\rho = \|\rho\|^2 = \frac{1}{\|\chi\|^2}.$$

With respect to the problems of this type, there have been some investigations by means of finite-difference method (Gaier [11],[12], Mizumoto [14],[15],[16], Opfer [21],[22]).

Finally, in Chapter 5 we apply our results to numerical calculation of the modulus of quadrilaterals. Let $\Omega$ be a simply-connected subdomain of a Riemann surface whose closure $\overline{\Omega}$ is a compact bordered subregion. We assume that the boundary $\partial\Omega$ of $\Omega$ is a piecewise analytic curve. We assign four points $p_1$, $p_2$, $p_3$ and $p_4$ on $\partial\Omega$ (in positive orientation w.r.t. $\Omega$), and the two opposite arcs $c_0$ (from $p_1$ to $p_2$) and $c_1$ (from $p_3$ to $p_4$). Then we say that a quadrilateral $Q$ with opposite sides $c_0$ and $c_1$ is given.

We can conformally map the domain $\Omega$ onto a rectangular domain $R = \{ w \mid 0 < \mathrm{Re}\, w < 1,\ 0 < \mathrm{Im}\, w < M \}$ by a function $w = \mathfrak{f}(p)$ so that $p_1$, $p_2$, $p_3$ and $p_4$ are mapped to $iM$, $0$, $1$ and $1 + iM$ respectively. Let $\Theta$ be the differential in $\Gamma_c(\overline{\Omega})$ satisfying $\Theta = 0$ along $c_0 \cup c_1$ and $\int_\gamma \Theta = 1$ where $\gamma$ is a path from a point on $c_0$ to a point on $c_1$. Then the modulus $M(Q) = M$ of the quadrilateral $Q$ is uniquely determined by $Q$, and is given by

$$M(Q) = \min_{\sigma\in\Gamma_\Theta} \|\sigma\|^2.$$

Next we assign the two opposite arcs $\tilde{c}_0$ (from $p_2$ to $p_3$) and $\tilde{c}_1$ (from $p_4$ to $p_1$) on $\partial\Omega$. Then a quadrilateral $\tilde{Q}$ with the opposite sides $\tilde{c}_0$ and $\tilde{c}_1$ is defined. We can easily see that $M(Q) = 1/M(\tilde{Q})$. By making use of this relation Gaier [11] presented a method to obtain upper and lower bounds for the modulus $M(Q)$ in the case of some restricted domain $\Omega$ (e.g. a lattice domain, etc.) by the finite difference approximation which originates from Opfer [21], [22]. We shall present a method to obtain good upper and lower bounds for $M(Q)$ by our finite element approximation even in the case of a domain $\Omega$ with curvilinear boundary arcs, and with inner and corner singularities of high order. It should be noted that the approximating differentials satisfy the boundary conditions exactly in all cases of Chapters 4 and 5.

Our treatment at critical points of a Riemann surface is closely related to that at boundary singularities on a plane (cf. Akin [2], Babuška [3], Babuška and Rosenzweig [4], Babuška, Szabo and Katz [5], Barnhill and Whiteman [6], Blackburn [7], Craig, Zhu and Zienkiewicz [10], Opfer and Puri [23], Rivara [24], Schatz and Wahlbin [25], [26], Thatcher [29], Tsamasphyros [30], Weisel [31], Whiteman and Akin [32], and Yserentant [33]).

## Chapter 1. Triangulation.

§ 1.1. Collection $\Phi$ of local parameters. Let $\Omega$ be a closed Riemann surface or a subdomain of a Riemann surface $W$ whose closure $\overline{\Omega}$ is a compact bordered subregion of $W$. In the latter case, we assume that the boundary $\partial\Omega$ consists of a finite number of analytic arcs meeting at vertices $p'_k$ $(k = 1, \cdots, l)$, and there exist parametric disks $V_k$ $(k = 1, \cdots, l)$ with the centers $p'_k$ and local parameters $z = \psi_k(p)$ by which $V_k \cap \overline{\Omega}$ are mapped onto sectors $\{|z| \leqq r_k\} \cap \{0 \leqq \arg z \leqq \beta_k\}$ $(0 < \beta_k \leqq 2\pi,\ \beta_k \neq \pi)$. For conformity, if $\Omega$ is a closed Riemann surface, then we interpret that $\Omega = W$.

Let $\{C_1, C_2, C_3, C_4\}$ be a partition to four parts of the boundary $\partial\Omega$ such that each $C_j$ $(j = 1, \cdots, 4)$ is a sum of boundary components of $\partial\Omega$ and $C_4$ consists at most one boundary component. We assign $2n$ points $p_1, \cdots, p_{2n}$ $(n \geqq 1)$ on $C_4$ (in the positive orientation with respect to $\Omega$).

By $\Phi = \{z = \varphi_j(p),\ U_j;\ j = 1, \cdots, m\}$ we denote a finite collection of local parameters $z = \varphi_j(p)$ $(j = 1, \cdots, m)$ and parametric disks $U_j$ $(j = 1, \cdots, m)$ on $W$ which satisfies the following conditions ( i )~(iv):

( i ) By the mapping $z = \varphi_j(p)$ $(j = 1, \cdots, m)$, $U_j$ is mapped onto a disk $|z| < \rho_j$.

(ii) $\overline{\Omega}$ is covered by $\{U_j\}_{j=1}^{m}$.

(iii) If $U_j \cap U_k \neq \phi$, then there exists a constant $L$ ( $>1$ ) such that for the mapping $\zeta = f(z) \equiv \varphi_k \circ \varphi_j^{-1}(z)$, $1/L < |f'(z)| < L$ on $\varphi_j(U_j \cap U_k)$.

Let $p_k$ $(k = 2n + 1, \cdots, \nu)$ be the all vertices of $\partial\Omega$ which are defined as points of $\{p'_k\}_{k=1}^{l} - \{p_k\}_{k=1}^{2n}$.

(iv) Each $U_j$ $(j = 1, \cdots, m)$ contains at most one $p_k$ $(k = 1, \cdots, \nu)$ and if $p_k \in U_j$ then $\varphi_j(p_k) = 0$.

( v ) If $U_j \cap \partial\Omega \neq \phi$ and $U_j$ does not contain any $p_k$ $(k = 1, \cdots, \nu)$, then $\varphi_j(U_j \cap \Omega)$ is a half disk $\{|z| < \rho_j\} \cap \{\mathrm{Im}\, z > 0\}$. If $U_j$ contains some $p_k$ $(k = 1, \cdots, \nu)$, then $\varphi_j(U_j \cap \Omega)$ is a sector $\{|z| < \rho_j\} \cap \{0 < \arg z < \alpha_j\}$ $(0 < \alpha_j \leqq 2\pi)$.

In the latter case of ( v ) and the case of $p_k \neq p_1, \cdots, p_{2n}$, if $p_k \in C_1$, or $p_k \notin C_1$ and $\alpha_j > \pi/2$, then by the mapping $\zeta = (\varphi_j(p))^{\pi/\alpha_j}$, $U_j \cap \Omega$ is mapped onto a half disk $\{|\zeta| < \rho_j^{\pi/\alpha_j}\} \cap \{\mathrm{Im}\, \zeta > 0\}$. In this case we define anew $z = \varphi_j(p)$ and $\rho_j$ by $\zeta = (\varphi_j(p))^{\pi/\alpha_j}$ and $\rho_j^{\pi/\alpha_j}$ respectively. Further, in the case where $U_j$ contains some $p_k$ $(k = 1, \cdots, 2n)$, then by the mapping $\zeta = (\varphi_j(p))^{\pi/2\alpha_j}$, $U_j \cap \Omega$ is mapped onto a sector $\{|\zeta| < \rho_j^{\pi/2\alpha_j}\} \cap \{0 < \arg \zeta < \pi/2\}$. In this case we define anew $z = \varphi_j(p)$ and $\rho_j$ by $\zeta = (\varphi_j(p))^{\pi/2\alpha_j}$ and $\rho^{\pi/2\alpha_j}$ respectively. Then, in the case that $U_j$ contains some $p_k$ $(k = 1, \cdots, \nu)$ the local parameter $z = \varphi_j(p)$ is no longer conformal at the center of $U_j$ except for the case when $U_j$ contains some $p_k$ $(k = 1, \cdots, 2n)$ and $\alpha_j = \pi/2$.

§ 1.2. Triangulation K associated to $\Phi$. For the collection $\Phi$ of local parameters and parametric disks defined in § 1.1, and for a sufficiently small positive number $h$, we construct a triangulation $K = K^h$ of $\overline{\Omega}$ which satisfies the following conditions ( i )~( v ). This is called a triangulation of $\overline{\Omega}$ with width $h$ associated to $\Phi$.

( i ) The points $p_1, \cdots, p_\nu$ are carriers of some 0-simplices of K.

(ii) $K$ is the sum of subtriangulations $K_1, \cdots, K_m$ of $K$ such that each 2-simplex of $K$ belongs to one and only one $K_j$ ($j = 1, \cdots, m$), and the carrier $|s|$ of each 2-simplex $s$ of $K_j$ is contained in $U_j$.

If a 1-simplex $e \in K_j$ does not belong to another $K_k$ ($k \neq j$), or a 1-simplex $e$ belongs to $K_j \cap K_k$ ($j \neq k$) and the mapping $\varphi_k \circ \varphi_j^{-1}$ is an affine transformation, then $e$ is said to be <u>linear</u>. If each edge of a 2-simplex $s \in K_j$ is linear and $\varphi_j(s)$ is an ordinary triangle, then $s$ is called a <u>natural</u> <u>simplex</u>.

(iii) Each 2-simplex $s \in K_j$ which has not a common edge with any 2-simplex of another $K_k$ ($k \neq j$), is a natural simplex.

A 2-simplex of $K_k$ which has a common edge with a 2-simplex $s \in K_j$ ($j \neq k$), is said to be an <u>adjoint</u> (simplex) of $s$ and is denoted by $s'$.

(iv) For each pair of a 2-simplex $s \in K_j$ and its adjoint $s' \in K_k$ with a common edge $e$, either one of the following three cases ( a ), ( b ), ( c ) occurs.

( a ) Both $s$ and $s'$ are natural simplices.

( b ) $\varphi_j(s)$ is a curvilinear triangle such that $\varphi_j(e)$ is a strictly concave arc w.r.t. $\varphi_j(s)$, $\varphi_k(s')$ is an ordinary triangle, and all edges of $s$ and $s'$ except for $e$ are linear (cf. Fig.1).

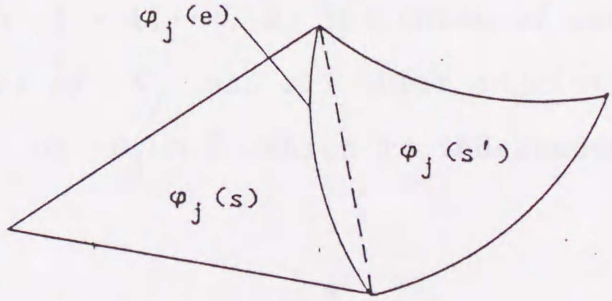



Fig. 1 Minor simplex $s$ and its adjoint $s'$

Then $s$ is called a minor simplex. The case where $s'$ is a minor simplex and $s$ is its adjoint may also occur.

( c ) $\varphi_j(s)$ is a curvilinear triangle such that $\varphi_j(e)$ is a strictly convex arc w.r.t. $\varphi_j(s)$, $\varphi_k(s')$ is an ordinary triangle, and all edges of $s$ and $s'$ except for $e$ are linear (cf. Fig.2).

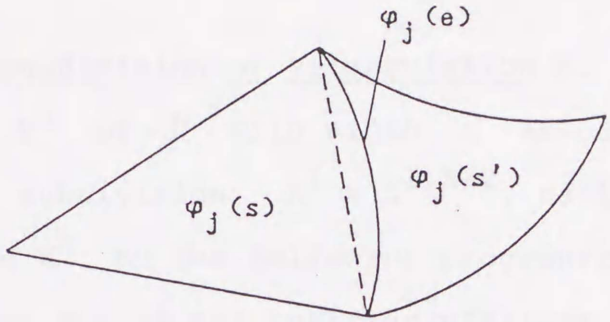



Fig. 2 Major simplex $s$ and its adjoint $s'$

Then $s$ is called a major simplex. The case where $s'$ is a major simplex and $s$ is its adjoint may also occur.

If $s$ is a minor or major simplex of $K_j$, then it is assumed that $|s'| \subset U_j$ for its adjoint $s'$.

( v ) For each 2-simplex $s \in K_j$ $(j = 1, \cdots, m)$, $d(\varphi_j(s)) \leqq h$, where throughout the present paper we denote the diameter of a region $G$ by $d(G)$.

Next, we assume that for the fixed $\Phi$ the class of the triangulations $K = K^h$ satisfies the following conditions ( i') and (ii'):

( i') For each $j = 1, \cdots, m$ the union of carriers of all minor and major simplices of $K_j$, and all their adjoints is contained in a closed subset $R_j$ of $U_j \cap \overline{\Omega}$ which is independent of the individual triangulation $K$.

(ii') The number $N$ of minor and major simplices of $K$ satisfies the inequality:

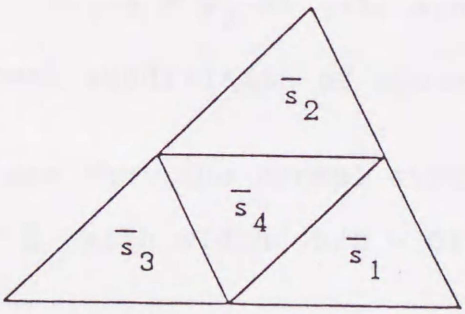
$$N \leqq M \cdot \frac{1}{h}, \tag{1.1}$$

where $M$ is a constant which is independent of the individual triangulation $K$.

§ 1.3. Normal subdivision of triangulation K. For a triangulation $K = K^h$ of $\overline{\Omega}$ with width $h$ associated to $\Phi$ we can construct a subdivision $K^1 = K^{1,h/2}$, called the normal subdivision of $K = K^h$ by the following procedure:

( i ) $K^1$ is the sum of the subtriangulations $K_1^1, \cdots, K_m^1$ which are the subdivisions of $K_1, \cdots, K_m$ respectively which are defined in the following (ii), (iii).

(ii) If $s \in K_j$ is a 2-simplex which is not minor or major, then $s$ is subdivided to four 2-simplices $s_1$, $s_2$, $s_3$ and $s_4$ of $K_j^1$ so that $\varphi_j(s_1)$, $\varphi_j(s_2)$, $\varphi_j(s_3)$ and $\varphi_j(s_4)$ are mutually congruent ordinary triangles as in Fig.3.




$\overline{a} = \varphi_j(a)$ (a: simplex)

Fig. 3 Normal subdivision of 2-simplex which is not minor or major

(iii) Let $s \in K_j$ and $s' \in K_k$ be a minor (or major) simplex and its adjoint, and let $e_1$, $e_2$ and $e_3$ be edges of $s$ such that $e_1$ is the common edge of $s$ and $s'$. We subdivide the edges $e_1$, $e_2$ and $e_3$ to two edges $e_{11}$ and $e_{12}$, $e_{21}$ and $e_{22}$, and $e_{31}$ and $e_{32}$ respectively so that $\varphi_k(e_{11})$ and $\varphi_k(e_{12})$, $\varphi_j(e_{21})$ and $\varphi_j(e_{22})$, and $\varphi_j(e_{31})$ and $\varphi_j(e_{32})$ have the same length respectively. Then we subdivide the simplex $s$ to two minor (or major resp.) simplices $s_1$ and $s_2$ of $K_j^1$ and, two natural simplices $s_3$ and $s_4$ of $K_j^1$ so that $e_{11}$, $e_{12}$, $e_{21}$, $e_{22}$, $e_{31}$ and $e_{32}$ are edges of $s_1$, $s_2$ and $s_3$ (cf. Fig.4). Here we note that such a subdivision is always possible if $h$ is sufficiently small.

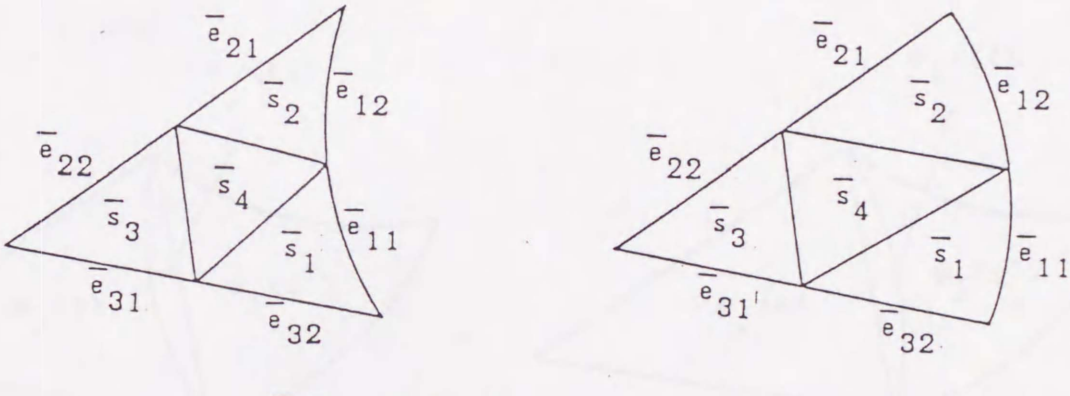



Fig. 4 Normal subdivision of minor and major simplices

We can easily see that the normal subdivision $K^1 = \Sigma_{j=1}^m K_j^1$ is a triangulation of $\bar{\Omega}$ with width $h/2 + O(h^2)$ associated to $\Phi$ (cf. (1.10)).

§ 1.4. Naturalized triangulation. For each minor (or major) simplex $s \in K_j$ we define the naturalized simplex $\natural s$ of $s$ as the

2-simplex such that $|s| \subset |\natural s|$ ($|\natural s| \subset |s|$ resp.) and $\varphi_j(\natural s)$ is the ordinary triangle which has two common sides with $\varphi_j(s)$. Further we define a 2-simplex $\flat\ell = \flat\ell(s)$ ($\sharp\ell = \sharp\ell(s)$ resp.) with two edges whose carrier is the closed region $\overline{|\natural s|-|s|}$ ($\overline{|s|-|\natural s|}$ resp.). $\flat\ell(s)$ ($\sharp\ell(s)$ resp.) is called the <u>deficient</u> (<u>excessive</u> resp.) <u>lune</u> of $s$.

Each triple of a minor (or major) simplex $s \in K_j$, its adjoint $s' \in K_k$ and its deficient lune $\flat\ell$ (excessive lune $\sharp\ell$ resp.) is denoted by $(s, s', \flat\ell)$ ($(s, s', \sharp\ell)$ resp.), and is called a <u>triple for a minor</u> (<u>major</u> resp.) <u>simplex</u> $s$ or <u>for a deficient</u> (<u>excessive</u> resp.) <u>lune</u> $\flat\ell$ ($\sharp\ell$ resp.) (cf. Fig.5), where it is always assumed that $|\flat\ell| \subset |s'|$ for each $(s, s', \flat\ell)$.

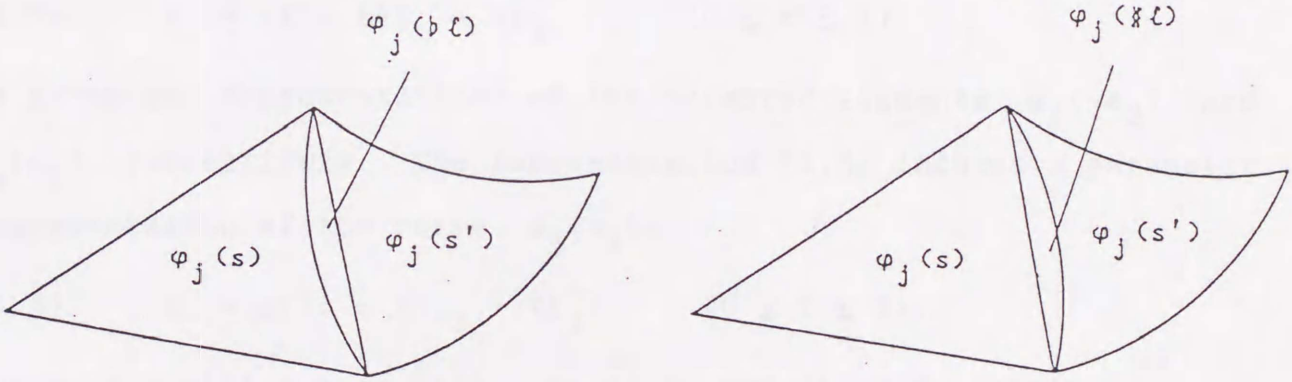



Fig. 5 Triple for a minor simplex $(s, s', \flat\ell)$ and triple for a major simplex $(s, s', \sharp\ell)$

For simplicity of notation, we also denote $\flat\ell = \flat\ell(s)$ or $\sharp\ell = \sharp\ell(s)$ by $\ell = \ell(s)$. If a minor or major simplex $s$ is in $K_j$, then we say that $\ell = \ell(s)$ is a <u>lune of</u> $K_j$ and write $\ell \in K_j$.

Now we shall define the <u>naturalized triangulation</u> $K'$ <u>associated to</u> $K$.

First, $K'_j$ $(j = 1, \cdots, m)$ are defined as triangulations such that the collection of all 2-simplices of $K'_j$ consists of all 2-simplices of $K_j$ which are not minor or major, and of all naturalized simplices of minor and major ones of $K_j$. Then the triangulation $K'$ is defined as the sum of $K'_j$ $(j = 1, \cdots, m)$. We should note that $K'$ is no longer a triangulation of $\bar{\Omega}$, and also is not an ordinary triangulation.

§ 1.5. Parametrization of lunar domains. Let $(s, s', \ell)$ be an arbitrary triple for a deficient or excessive lune $\ell$, and let $e_1$ and $e_2$ be two edges of $\ell$ such that $|e_1| \subset |\partial s|$. Further, let

$$(1.2) \qquad z' = (1 - t)z_1 + tz_2 \qquad (0 \leq t \leq 1)$$

and

$$(1.3) \qquad \zeta'' = (1 - t)\zeta_1 + t\zeta_2 \qquad (0 \leq t \leq 1)$$

be parameter representations of the oriented segments $\varphi_j(-e_2)$ and $\varphi_k(e_1)$ respectively. The representation (1.3) induces a parameter representation of the curve $\varphi_j(e_1)$:

$$(1.4) \qquad z'' = g((1 - t)\zeta_1 + t\zeta_2) \qquad (0 \leq t \leq 1),$$

where $z = g(\zeta) \equiv \varphi_j \circ \varphi_k^{-1}(\zeta)$. By (1.2) and (1.4) we obtain a parameter representation of the lunar domain $\varphi_j(\ell)$:

$$(1.5) \qquad z = z(t,\tau) \equiv (1 - \tau)z' + \tau z''$$
$$= (1 - \tau)((1 - t)z_1+tz_2) + \tau g((1 - t)\zeta_1+t\zeta_2)$$
$$(0 \leq t \leq 1,\ 0 \leq \tau \leq 1).$$

## § 1.6. Area of lune.

LEMMA 1.1. Let $(s, s', \ell)$ be a triple for an arbitrary deficient or excessive lune $\ell$. Then, the estimate

$$(1.6) \qquad A(\varphi_j(\ell)) \leqq \frac{h_1^3}{8}\left(\left|\frac{g''(\xi_1)}{g'(\xi_1)^2}\right| + O(h_1)\right)$$

holds, where throughout the present paper we denote the area of a region $G$ by $A(G)$, $z = g(\xi) \equiv \varphi_j \circ \varphi_k^{-1}(\xi)$, $h_1 = d(\varphi_j(\ell))$ and $\xi_1$ is one of the vertices of the lunar domain $\varphi_k(\ell)$.

PROOF. Here we shall preserve the notations in § 1.5. By Taylor's expansion we have

$$(1.7) \qquad z'' - z_1 = g'(\xi_1)(\xi_2 - \xi_1)t + \frac{1}{2} g''(\xi_1)(\xi_2 - \xi_1)^2 t^2 + \cdots$$

for the point $z''$ of (1.4) on $\varphi_j(e_1)$, and

$$(1.8) \qquad \begin{aligned} z' - z_1 &= t(z_2 - z_1) \\ &= g'(\xi_1)(\xi_2 - \xi_1)t + \frac{1}{2} g''(\xi_1)(\xi_2 - \xi_1)^2 t + \cdots \end{aligned}$$

for the point $z'$ of (1.2) on $\varphi_j(-e_2)$, where we assume that the triangulation $K$ is so chosen that $\varphi_k(e_1)$ is contained in a disk $V$ centered at $\xi_1$ such that $\varphi_k^{-1}(V) \subset U_j \cap U_k$. By (1.7) and (1.8) we find that the equality

$$(1.9) \qquad z'' - z' = (\xi_2 - \xi_1)^2 \cdot \frac{t(t-1)}{2} \cdot g''(\xi_1) + O((\xi_2 - \xi_1)^3)$$

holds for the point $z'$ of (1.2) on $\varphi_j(-e_2)$ and the point $z''$ of (1.4) on $\varphi_j(e_1)$ with common $t$.

Since $|\xi_2 - \xi_1| \leqq h_1(1/|g'(\xi_1)| + O(h_1))$, the equality (1.9) implies

$$(1.10)\qquad |z'' - z'| \leq \frac{h_1^2}{8}\left(\left|\frac{g''(\xi_1)}{g'(\xi_1)^2}\right| + O(h_1)\right).$$

Therefore we obtain the estimates

$$A(\varphi_j(\ell)) \leq |z_2 - z_1| \cdot \max_{0\leq t\leq 1} |z' - z''|$$

$$\leq \frac{h_1^3}{8}\left(\left|\frac{g''(\xi_1)}{g'(\xi_1)^2}\right| + O(h_1)\right).$$

## Chapter 2. Spaces of differentials.

§ 2.1. Subspace $\Lambda$ of $\Gamma_c$. Let $\Gamma_c^0 = \Gamma_c^0(\bar{\Omega})$ be the set of all locally exact differentials $\sigma$ in the class $C^0$ on $\bar{\Omega}$ with the finite Dirichlet norm

$$\|\sigma\|^2 = \|\sigma\|_\Omega^2 = \int_\Omega \sigma * \sigma < \infty,$$

where by $*\sigma$ we denote the conjugate differential of $\sigma$. Let $\Gamma_c = \Gamma_c(\bar{\Omega})$ be the completion of $\Gamma_c^0$. We should note that in Chapter V of Ahlfors and Sario [1], $\Gamma_c$ is defined as the completion of $\Gamma_c^1 \equiv \Gamma_c^0 \cap C^1$.

We define a subspace $\Lambda = \Lambda(K)$ of $\Gamma_c$ as the space of differentials $\sigma_h$ which satisfy the following conditions ( i )~(iv):

( i ) $\sigma_h \in \Gamma_c$.

(ii) If $s \in K_j$ $(j=1,\cdots,m)$ is a natural simplex, then

$$\sigma_h = a_0 dx + b_0 dy \qquad \text{on } \varphi_j(s) \quad (z = x + iy),$$

where $a_0$ and $b_0$ are constants.

(iii) Let $(s, s', \flat\ell)$ be a triple for a minor simplex $s$, and let $e_1$ and $e_2$ be two edges of $\flat\ell$ such that $-e_1 \subset \partial s$. Then

$$\sigma_h = a_0 dx + b_0 dy \qquad \text{on } \varphi_j(s),$$

$$\sigma_h = \alpha_0 d\xi + \beta_0 d\eta \qquad \text{on } \varphi_k(s') - \varphi_k(\flat\ell),$$

and $\sigma_h$ is a harmonic differential in $\flat\ell$ which satisfies the boundary conditions

$$\sigma_h = a_0 dx + b_0 dy \qquad \text{along } \varphi_j(e_1)$$

and

$$\sigma_h = \left(\alpha_0 \frac{\partial \xi}{\partial x} + \beta_0 \frac{\partial \eta}{\partial x}\right) dx + \left(\alpha_0 \frac{\partial \xi}{\partial y} + \beta_0 \frac{\partial \eta}{\partial y}\right) dy \quad \text{along} \quad \varphi_j(e_2),$$

where $a_0$, $b_0$, $\alpha_0$ and $\beta_0$ are constants, and

$$\zeta = f(z) \equiv \varphi_k \circ \varphi_j^{-1}(z) \quad (z = x + iy, \ \zeta = \xi + i\eta).$$

(iv) Let $(s, s', \sharp\ell)$ be a triple for a major simplex $s$, and let $e_1$ and $e_2$ be two edges of $\sharp\ell$ such that $e_1 \subset \partial s$. Then

$$\sigma_h = a_0 dx + b_0 dy \qquad \text{on} \quad \varphi_j(\natural s),$$

$$\sigma_h = \alpha_0 d\xi + \beta_0 d\eta \qquad \text{on} \quad \varphi_k(s'),$$

and $\sigma_h$ is a harmonic differential in $\sharp\ell$ which satisfies the boundary conditions

$$\sigma_h = a_0 dx + b_0 dy \qquad \text{along} \quad \varphi_j(e_2)$$

and

$$\sigma_h = \left(\alpha_0 \frac{\partial \xi}{\partial x} + \beta_0 \frac{\partial \eta}{\partial x}\right) dx + \left(\alpha_0 \frac{\partial \xi}{\partial y} + \beta_0 \frac{\partial \eta}{\partial y}\right) dy \quad \text{along} \quad \varphi_j(e_1),$$

where $a_0$, $b_0$, $\alpha_0$ and $\beta_0$ are constants, and $\zeta = \xi + i\eta$ is as in (iii).

We note that $\sigma_h \in \Lambda$ is generally discontinuous on each edge of 2-simplices of $K$.

§ <u>2</u>.<u>2</u>. <u>Space</u> $\Lambda'$. Let $K'$ be the naturalized triangulation associated to $K$. For each differential $\sigma_h \in \Lambda$, we define the <u>differential</u> $\sigma'_h$ <u>on</u> $K'$ <u>associated</u> <u>to</u> $\sigma_h$ as the differential $\sigma'_h$ which satisfies the following conditions ( i )~(iv):

( i ) For each 2-simplex $s \in K'_j$ $(j = 1, \cdots, m)$

$$\sigma'_h = a_0 dx + b_0 dy \qquad \text{on} \quad \varphi_j(s),$$

where $a_0$ and $b_0$ are constants.

(ii) If $s \in K$ is a natural simplex, then

$$\sigma'_h = \sigma_h \qquad \text{on } |s|.$$

(iii) If $(s, s', \flat\ell)$ is a triple for a minor simplex $s$, then

$$\sigma'_h = \sigma_h \qquad \text{on } |s| \cup |s'| - |\flat\ell|.$$

(iv) If $(s, s', \sharp\ell)$ is a triple for a major simplex $s$, then

$$\sigma'_h = \sigma_h \qquad \text{on } |\natural s| \cup |s'|.$$

We should note that the differential $\sigma'_h$ is defined just twice on each deficient lune $\flat\ell$, while it is never defined on any excessive lune $\sharp\ell$. In the former case, for each triple $(s, s', \flat\ell)$ we shall denote the differential $\sigma'_h$ on $\natural s \in K'_j$ and $s' \in K'_k$ by $\sigma'_{h,\natural s}$ and $\sigma'_{h,s'}$, respectively.

The space of all differentials $\sigma'_h$ associated to $\sigma_h \in \Lambda$ is denoted by $\Lambda' = \Lambda'(K')$. Let $\sigma'_h$ and $\chi'_h$ be two differentials of $\Lambda'$. Then the <u>inner product</u> $(\sigma'_h, \chi'_h)$ of $\sigma'_h$ and $\chi'_h$ is defined by

$$\begin{aligned}(\sigma'_h, \chi'_h) &= (\sigma'_h, \chi'_h)_{K'} \\ &= \sum_{s \in K'} \int_{|s|} \sigma'_h * \chi'_h,\end{aligned}$$

and the <u>norm</u> $\|\sigma'_h\|$ of $\sigma'_h$ is defined by

$$\|\sigma_h'\| = \|\sigma_h'\|_{K'} = \sqrt{(\sigma_h', \sigma_h')_{K'}}.$$ [1)]

We see that $\sigma_h' = F(\sigma_h)$ defines a one-to-one mapping of $\Lambda$ onto $\Lambda'$.

§ 2.3. Finite element interpolations. Let $\sigma$ be an element of $\Gamma_c$. We define the *finite element interpolation* $\hat{\sigma}$ *of* $\sigma$ *in the space* $\Lambda$ as the differential uniquely determined by the following conditions ( i ) and (ii):

( i ) $\hat{\sigma} \in \Lambda$;

(ii) For each 1-simplex $e \in K$,

$$\int_e \hat{\sigma} = \int_e \sigma.$$

§ 2.4. Harmonic differentials on a lune.

LEMMA 2.1. Let $\ell = \ell(s)$ be a deficient or excessive lune of $K_j$, let $e_1$ and $e_2$ be two edges of $\ell$, and let $\sigma_1$ and $\sigma_2$ be exact differentials in the class $C^0$ on $\ell$ which satisfy the condition

$$\int_{e_1} \sigma_1 = -\int_{e_2} \sigma_2.$$

Further, let $\chi$ be the differential harmonic in $\ell$ and continuous on $\ell$ which satisfies the boundary conditions

$$\chi = \sigma_i \qquad \text{along } e_i \ (i = 1, 2).$$

---

1) We shall use the common notations $(\ ,\ )$ and $\|\ \|$ for both inner products and both norms of differentials of the spaces $\Lambda$ and $\Lambda'$.

Then the inequalities

$$(2.1)\qquad \|\chi\|_{\ell}^{2} \leqq \iint_{\varphi_j(\ell)} \max\{(a_1{}^2+b_1{}^2),\ (a_2{}^2+b_2{}^2)\}dxdy$$

$$\leqq \|\sigma_1\|_{\ell}^{2} + \|\sigma_2\|_{\ell}^{2}$$

hold, where

$$\|\chi\|_{\ell}^{2} = \int_{|\ell|} \chi * \chi, \quad \text{etc.},$$

and

$$\sigma_1 = a_1 dx + b_1 dy \quad \text{and} \quad \sigma_2 = a_2 dx + b_2 dy \quad \text{on} \quad \varphi_j(\ell).$$

PROOF. By making use of the parameter representation (1.5) of the lunar domain $\varphi_j(\ell)$, we define a differential $\sigma$ on $\ell$ by

$$\sigma\circ\varphi_j^{-1}(z) = (1-\tau)\sigma_1\circ\varphi_j^{-1}(z) + \tau\sigma_2\circ\varphi_j^{-1}(z)$$

$$(z = z(t,\tau) \in \varphi_j(\ell)).$$

We note that $\sigma$ satisfies the same boundary conditions as $\chi$ on $\partial\ell$. Since $\chi$ is harmonic in $\ell$, the inequality

$$(2.2)\qquad \|\chi\|_{\ell}^{2} \leqq \|\sigma\|_{\ell}^{2}$$

holds. Further, the inequalities

$$(2.3)\qquad \|\sigma\|_{\ell}^{2} \leqq \iint_{\varphi_j(\ell)} \left((1-\tau)\sqrt{a_1{}^2+b_1{}^2} + \tau\sqrt{a_2{}^2+b_2{}^2}\right)^2 dxdy$$

$$\leqq \iint_{\varphi_j(\ell)} \max\{(a_1{}^2+b_1{}^2),\ (a_2{}^2+b_2{}^2)\}dxdy$$

hold. The inequalities (2.2) and (2.3) imply the inequality (2.1).

§ 2.5. Difference of norms of $\sigma_h$ and $\sigma_h'$.

LEMMA 2.2. Let $\sigma_h$ be an arbitrary differential of the space

$\Lambda$ and let $\sigma'_h = F(\sigma_h)$.

( i ) The inequalities

$$(2.4) \qquad \|\sigma_h\|^2 \leqq \|\sigma'_h\|^2 + \sum_{\sharp\ell \in K} \|\sigma_h\|^2_{\sharp\ell}$$
$$\leqq \|\sigma'_h\|^2 + \sum_{j=1}^{m} \sum_{\sharp\ell \in K_j} A(\varphi_j(\sharp\ell)) \cdot \left(\frac{1}{\lambda} \int_{e_2} \sigma'_h\right)^2 (1 + \kappa h)$$

hold, where $e_2$ is the edge of $\sharp\ell$ such that $\varphi_j(e_2)$ is a segment, $\lambda$ is the length of $\varphi_j(e_2)$ and $\kappa$ is a constant which depends only on the transformations $f(z) = \varphi_k \circ \varphi_j^{-1}(z)$.

(ii)

$$(2.5) \qquad \|\sigma'_h\|^2 \leqq \|\sigma_h\|^2 + \sum_{\flat\ell \in K} (\|\sigma'_{h,\natural s}\|^2_{\flat\ell} + \|\sigma'_{h,s'}\|^2_{\flat\ell})$$
$$= \|\sigma_h\|^2 + \sum_{j=1}^{m} \sum_{\flat\ell \in K_j} \{A(\varphi_j(\flat\ell)) \cdot (a_0^2 + b_0^2)$$
$$+ A(\varphi_k(\flat\ell)) \cdot (\alpha_0^2 + \beta_0^2)\},$$

where for each triple (s, s', $\flat\ell$) the notations in (iii) of § 2.1 are preserved.

PROOF. ( i ) By Lemma 2.1 we see that for each triple (s, s', $\flat\ell$)

$$(2.6) \qquad \|\sigma_h\|^2_{\flat\ell} \leqq \|\sigma'_{h,\natural s}\|^2_{\flat\ell} + \|\sigma'_{h,s'}\|^2_{\flat\ell}.$$

Hence the first inequality of (2.4) is obtained.

Let (s, s', $\sharp\ell$) be a triple for an excessive lune $\sharp\ell$. We preserve the notations in (iv) of § 2.1. We shall prove the inequality

$$(2.7) \qquad \|\sigma_h\|^2_{\sharp\ell} \leqq A(\varphi_j(\sharp\ell)) \cdot \left(\frac{1}{\lambda} \int_{e_2} \sigma'_h\right)^2 \cdot (1 + \kappa h),$$

from which the second inequality of (2.4) follows.

By $\gamma$ and $\delta$ we denote the arguments of the oriented segments $\varphi_j(-e_2)$ and $\varphi_k(e_1)$ respectively. By making use of the parameter representation (1.5) of the lunar domain $\varphi_j(\natural\ell)$, we define a differential $\sigma$ on $\natural\ell$ by

$$(2.8) \qquad \sigma = a\,dx + b\,dy$$

$$\equiv (1-\tau)(a_0\cos\gamma + b_0\sin\gamma)\cdot((\cos\gamma)dx + (\sin\gamma)dy)$$

$$+ \tau(\alpha_0\cos\delta + \beta_0\sin\delta)\cdot$$

$$\cdot\left((\cos\delta)\left(\frac{\partial\xi}{\partial x}dx + \frac{\partial\xi}{\partial y}dy\right) + (\sin\delta)\left(\frac{\partial\eta}{\partial x}dx + \frac{\partial\eta}{\partial y}dy\right)\right)$$

$$(z = z(t,\tau) \in \varphi_j(\natural\ell)).$$

We note that $\sigma$ satisfies the same boundary conditions as $\sigma_h$ on $\partial(\natural\ell)$. Hence

$$(2.9) \qquad \|\sigma_h\|^2_{\natural\ell} \leqq \|\sigma\|^2_{\natural\ell} \leqq A(\varphi_j(\natural\ell)) \max_{\varphi_j(\natural\ell)} (a^2 + b^2),$$

since $\sigma_h$ is harmonic in $\natural\ell$.

From the equation (2.8) it follows that

$$(2.10) \qquad \max_{\varphi_j(\natural\ell)} (a^2 + b^2) \leqq \max\{(a_0\cos\gamma + b_0\sin\gamma)^2,$$

$$(\alpha_0\cos\delta + \beta_0\sin\delta)^2 \max_{\varphi_j(\natural\ell)} |f'(z)|^2\}.$$

Further we note that

$$(2.11) \qquad a_0\cos\gamma + b_0\sin\gamma = \frac{1}{\lambda}\int_{-e_2} \sigma_h'$$

and

$$(2.12) \qquad \alpha_0\cos\delta + \beta_0\sin\delta = \frac{1}{\mu}\int_{e_1} \sigma_h' = \frac{1}{\mu}\int_{-e_2} \sigma_h',$$

where

(2.13) $\lambda = \int_{\varphi_j(e_2)} |dz|$ and $\mu = \int_{\varphi_j(e_1)} |f'(z)dz|$.

By making use of the power series expansion of $f'$ around a vertex $z_1$ of the lunar domain $\varphi_j(\sharp\ell)$, we see that

$$(2.14) \qquad \max_{\varphi_j(\sharp\ell)} |f'(z)|^2 \leqq |f'(z_1)|^2 (1 + \kappa_1 h)$$

and

$$(2.15) \qquad \mu \geqq (|f'(z_1)| - \kappa_2 h) \int_{\varphi_j(e_2)} |dz| = \lambda(|f'(z_1)| - \kappa_2 h)$$

with constants $\kappa_1, \kappa_2 > 0$ depending only on $f$. Then the estimate (2.7) follows from (2.9)~(2.15).

(ii) The inequality (2.5) is obvious from the definition of $\sigma'_h$.

## Chapter 3. Finite element approximations.

§ 3.1. Formulation of problems. Let $\gamma_k^2$ $(k = 1, \cdots, \kappa)$ be the boundary components of $C_2$. Let $\gamma_k^4$ $(k = 1, \cdots, 2n)$ be the arcs on $C_4$ from $p_k$ to $p_{k+1}$ $(k = 1, \cdots, 2n;\ p_{2n+1} = p_1)$ and let $C_4' = \Sigma_{k=1}^n \gamma_{2k-1}^4$, $C_4'' = \Sigma_{k=1}^n \gamma_{2k}^4$, where $\{p_k\}_{k=1}^{2n}$ are the assigned 2n points on $C_4$ defined in § 1.1.

Let $\Theta$ be a differential in $\Gamma_c$ which satisfies the following conditions ( i ), (ii) and (iii):

( i ) If $U_j \cap C_1 \neq \phi$, then $\Theta \circ \varphi_j^{-1}$ is harmonic on a neighborhood of $\varphi_j(U_j \cap C_1)$;

(ii) $\Theta = 0$ along $C_2 \cup C_4'$;

(iii) $\Theta$ is exact on a neighborhood of each boundary component of $C_3$, where the conditions ( i ), (ii) and (iii) may be ignored if $\partial\Omega = \phi$.

By $\Gamma_\Theta$ we denote the subspace of $\Gamma_c$ consisting of all differentials $\sigma$ for which there exists a function $v$ on $\bar{\Omega}$ such that

$$dv = \Theta - \sigma \quad \text{on } \bar{\Omega},$$
$$v = 0 \quad \text{on } C_1 \cup C_4',$$
$$v = \text{const.} \quad \text{on } \gamma_k^2\ (k = 1, \cdots, \kappa).$$

By $\omega$ we denote the harmonic differential in $\Gamma_\Theta$ uniquely determined by the conditions

$$(3.1) \qquad \int_{\gamma_k^2} {*\omega} = 0 \qquad (k = 1, \cdots, \kappa)$$

and

$$(3.2) \qquad {*\omega} = 0 \qquad \text{along } C_3 \cup C_4''.$$

The differential $\omega$ can be constructed by the following procedure. Let $\chi$ be the harmonic component of $\theta$ in the orthogonal decomposition of $\Gamma_c$ (cf. Chapter V of Ahlfors and Sario [1]), and let $u$ be the solution of the boundary value problem:

$$\begin{aligned} &u \text{ is a harmonic function on } \Omega, \\ &u = 0 \qquad \text{on } C_1 \cup C_4', \\ &u = \text{const.} \qquad \text{on } \gamma_k^2, \\ &\int_{\gamma_k^2} *du = \int_{\gamma_k^2} *\chi \qquad (k = 1, \cdots, \kappa) \end{aligned}$$

and

$$*du = *\chi \qquad \text{along } C_3 \cup C_4''.$$

Then, $\omega = \chi - du$. We note that the differential $\omega$ is harmonic on the closure $\overline{\Omega}$.[1)]

LEMMA 3.1. The harmonic differential $\omega$ satisfies the minimal property

$$(3.3) \qquad \|\omega\| = \min_{\sigma\in\Gamma_\theta} \|\sigma\|.$$

In the equality (3.3), the minimum of the right hand side is attained if and only if $\sigma = \omega$.

PROOF. For each $\sigma \in \Gamma_\theta$ there exists a function $v$ such that

$$(3.4) \qquad \begin{cases} dv = \sigma - \omega, \\ v = 0 \qquad \text{on } C_1 \cup C_4', \\ v = \text{const.} \qquad \text{on } \gamma_k^2 \ (k = 1, \cdots, \kappa). \end{cases}$$

From (3.1), (3.2) and (3.4) it follows that

1) It is sufficient for our purpose that $\omega$ is of the class $C^1$ on the closure $\overline{\Omega}$ and hence we can weaken the assumption ( i ) for $\theta$.

$$(3.5) \qquad (\sigma - \omega, \omega) = \int_{\partial\Omega} v*\omega$$

$$= \int_{C_1} v*\omega + \sum_{k=1}^{\kappa} \int_{\gamma_k^2} v*\omega + \int_{C_3} v*\omega + \int_{C_4'} v*\omega + \int_{C_4''} v*\omega = 0,$$

where

$$(\sigma, \tau) = (\sigma, \tau)_\Omega = \int_\Omega \sigma*\tau.$$

The equality (3.5) implies that

$$\|\sigma\|^2 = \|\omega\|^2 + \|\sigma - \omega\|^2 \geq \|\omega\|^2.$$

In the last inequality, the equality holds if and only if $\sigma = \omega$.

The unique harmonic differential $\omega$ in $\Gamma_\Theta$ is called the <u>harmonic</u> <u>solution</u> <u>in</u> $\Gamma_\Theta$.

Our aim is to obtain finite element approximations of $\omega$ in the spaces $\Lambda$ and $\Lambda'$, and error estimates between them and $\omega$.

§ <u>3</u>.<u>2</u>. <u>Finite element approximation</u> $\psi_h$ <u>in</u> $\Lambda$. Let $\hat{\Theta}$ be the finite element interpolation of $\Theta$ in the space $\Lambda$. By $\Lambda_\Theta$ we denote the subspace of $\Lambda$ consisting of all differentials $\sigma_h \in \Lambda$ for which there exists a function $v$ on $\overline{\Omega}$ such that

$$dv = \hat{\Theta} - \sigma_h,$$

$$v = 0 \qquad \text{on } C_1 \cup C_4',$$

$$v = \text{const.} \qquad \text{on } \gamma_k^2 \ (k = 1, \cdots, \kappa).$$

By $\psi_h$ we denote the differential of $\Lambda_\Theta$ such that

$$(3.6) \qquad \|\psi_h\| = \min_{\sigma_h \in \Lambda_\Theta} \|\sigma_h\|.$$

We call $\psi_h$ the finite element approximation of $\omega$ in the space $\Lambda$.

Next, we consider the special case where the differential $\Theta$ satisfies the condition:

$$\Theta = 0 \qquad \text{along } C_1.$$

We denote such a differential $\Theta$ by $\Theta_0$. Since $\Lambda_{\Theta_0} \subset \Gamma_{\Theta_0}$, we see that

(3.7) $$\|\omega\| \leq \|\psi_h\|.$$

LEMMA 3.2. ( i ) In the case of general $\Theta$, the equality

(3.8) $$\|\psi_h - \omega\| = \min_{\sigma_h \in \Lambda_\Theta} \|\sigma_h - \omega\|$$

holds, where the minimum is attained if and only if $\sigma_h = \psi_h$.

(ii) In the case of $\Theta = \Theta_0$, the equality

(3.9) $$\|\psi_h - \omega\|^2 = \|\psi_h\|^2 - \|\omega\|^2$$

holds.

PROOF. ( i ) First, by a method similar to (3.5), it is shown that

(3.10) $$(\omega, \sigma_h - \psi_h) = 0 \qquad \text{for each } \sigma_h \in \Lambda_\Theta.$$

By (3.6), standard arguments imply that

(3.11) $$(\psi_h, \sigma_h - \psi_h) = 0 \qquad \text{for each } \sigma_h \in \Lambda_\Theta.$$

From (3.10) and (3.11), it follows that

$$\|\omega - \sigma_h\|^2 = \|\omega - \psi_h\|^2 + \|\sigma_h - \psi_h\|^2 \geq \|\omega - \psi_h\|^2.$$

In the last inequality, the equality holds if and only if $\sigma_h = \psi_h$.

(ii) Since $\Lambda_{\Theta_0} \subset \Gamma_{\Theta_0}$, both $\psi_h$ and $\omega$ are elements of $\Gamma_{\Theta_0}$. Hence, by (3.5) $(\omega, \psi_h - \omega) = 0$ and thus

$$\|\psi_h - \omega\|^2 = \|\psi_h\|^2 - \|\omega\|^2.$$

From (3.11) the following lemma immediately follows.

LEMMA 3.3. In the case of general $\Theta$, the equality

$$(3.12) \qquad \|\sigma_h - \psi_h\|^2 = \|\sigma_h\|^2 - \|\psi_h\|^2$$

holds for each $\sigma_h \in \Lambda_\Theta$.

§ <u>3.3</u>. <u>Finite element approximation</u> $\omega_h'$ <u>in</u> $\Lambda'$. Let $\Lambda_\Theta' = \{\sigma_h' \mid \sigma_h' = F(\sigma_h), \ \sigma_h \in \Lambda_\Theta\}$. By $\omega_h'$ we denote the differential of $\Lambda_\Theta'$ such that

$$(3.13) \qquad \|\omega_h'\| = \min_{\sigma_h' \in \Lambda_\Theta'} \|\sigma_h'\|.$$

We call $\omega_h'$ the <u>finite element approximation of</u> $\omega$ <u>in the space</u> $\Lambda'$.

LEMMA 3.4. The equality

$$(3.14) \qquad \|\sigma_h' - \omega_h'\|^2 = \|\sigma_h'\|^2 - \|\omega_h'\|^2$$

holds for each $\sigma_h' \in \Lambda_\Theta'$.

PROOF. By a method similar to the proof of (3.11), it is shown that the equality

$$(3.15) \qquad (\omega_h', \sigma_h' - \omega_h') = 0$$

holds for each $\sigma_h' \in \Lambda_\Theta'$. This implies (3.14).

§ 3.4. Lemma of Bramble and Zlámal. The following lemma is due to J.H. Bramble and M. Zlámal (cf. [9]).

LEMMA 3.5. Let $\Delta$ be a closed triangle on the z-plane ($z = x + iy$) with $d(\Delta) \leqq h$, let $v$ be a function of the class $C^2$ defined on $\Delta$ such that $v = 0$ at each vertex of $\Delta$. Then, the inequality

$$(3.16) \quad \iint_\Delta \left( \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 \right) dxdy \leqq \frac{B}{\sin^2\theta} h^2 \iint_\Delta \left( \left( \frac{\partial^2 v}{\partial x^2} \right)^2 + 2\left( \frac{\partial^2 v}{\partial x \partial y} \right)^2 + \left( \frac{\partial^2 v}{\partial y^2} \right)^2 \right) dxdy$$

holds, where $B$ is an absolute constant and $\theta$ is the smallest interior angle of the triangle $\Delta$.

§ 3.5. Pointwise estimate.

LEMMA 3.6. Let $\Delta$ be a closed curvilinear triangle on the z-plane ($z = x + iy$) with $d(\Delta) \leqq h$ which is the image of some 2-simplex $s \in K_j$ ($j = 1, \cdots, m$) by $z = \varphi_j(p)$, and let $v$ be a function of the class $C^2$ defined on $\Delta$ such that $v = 0$ at each vertex of $\Delta$. Then,

$$\left|\frac{\partial v}{\partial x}\right|, \left|\frac{\partial v}{\partial y}\right| \leqq h \cdot \frac{4}{\sin\theta} \max_{z \in \Delta} \left( \left|\frac{\partial^2 v}{\partial x^2}\right| + 2\left|\frac{\partial^2 v}{\partial x \partial y}\right| + \left|\frac{\partial^2 v}{\partial y^2}\right| \right) (1 + \kappa h)$$

on $\Delta$, where $\theta$ is the smallest interior angle of the ordinary triangle which has common vertices with $\Delta$, and $\kappa$ is a constant which depends only on $f(z) = \varphi_k \circ \varphi_j^{-1}(z)$.

PROOF. (Cf. Theorem 3.1 of Strang and Fix [27].) Let $z_0 =$

$x_0 + iy_0$ be a fixed point and $z = x + iy$ an arbitrary point in $\Delta$, and let $k = x - x_0$ and $\ell = y - y_0$. Here we choose the point $z_0$ so that for each $z \in \Delta$ the segment between $z_0$ and $z$ is contained in $\Delta$.

By Taylor's theorem we have that

$$v(z) = P(z) + r(z),$$

where

$$P(z) = v(z_0) + \left(k \frac{\partial}{\partial x} + \ell \frac{\partial}{\partial y}\right) v(z_0),$$

$$r(z) = \frac{1}{2!} \left(k \frac{\partial}{\partial x} + \ell \frac{\partial}{\partial y}\right)^2 v(z') \tag{3.17}$$

with some point $z'$ on the segment between $z_0$ and $z$. First, from (3.17) the estimate

$$|r(z)| \leq \frac{h^2}{2} \max_{z \in \Delta} \left( \left|\frac{\partial^2 v}{\partial x^2}\right| + 2\left|\frac{\partial^2 v}{\partial x \partial y}\right| + \left|\frac{\partial^2 v}{\partial y^2}\right| \right) \quad (z \in \Delta) \tag{3.18}$$

immediately follows. Let $z_j$ $(j = 1, 2, 3)$ be the vertices of $\Delta$. Then, by the assumption of the lemma

$$v(z_j) = P(z_j) + r(z_j) = 0 \quad (j = 1, 2, 3). \tag{3.19}$$

Since $P(z)$ is a linear function of $x$ and $y$, by (3.19) we have the expression

$$P(z) = -r(z_1)\phi_1(z) - r(z_2)\phi_2(z) - r(z_3)\phi_3(z), \tag{3.20}$$

where $\phi_j$ $(j = 1, 2, 3)$ are linear functions of $x$ and $y$ such that

$$\phi_j(z_k) = \delta_{jk} \quad (j, k = 1, 2, 3)$$

with Kronecker's symbol $\delta_{jk}$. (3.18) and (3.20) imply the estimate

$$\left|\frac{\partial P}{\partial x}\right| \leq |r(z_1)| \left|\frac{\partial \phi_1}{\partial x}\right| + |r(z_2)| \left|\frac{\partial \phi_2}{\partial x}\right| + |r(z_3)| \left|\frac{\partial \phi_3}{\partial x}\right| \tag{3.21}$$

$$\leqq \frac{3}{2} h^2 \max_{z\in\Delta} \left(\left|\frac{\partial^2 v}{\partial x^2}\right| + 2\left|\frac{\partial^2 v}{\partial x\partial y}\right| + \left|\frac{\partial^2 v}{\partial y^2}\right|\right) \cdot \max_{1\leqq j\leqq 3} \left|\frac{\partial\phi_j}{\partial x}\right| .$$

Here we can easily verify that

$$(3.22) \qquad \left|\frac{\partial\phi_j}{\partial x}\right| \leqq \frac{1}{h_1} \cdot \frac{2}{\sin\theta} \qquad (j = 1, 2, 3),$$

where $h_1$ is the diameter of the ordinary triangle which has common vertices with $\Delta$. From (3.21) and (3.22) it follows that

$$(3.23) \qquad \left|\frac{\partial P}{\partial x}\right| \leqq 3h \cdot \frac{1}{\sin\theta} \max_{z\in\Delta} \left(\left|\frac{\partial^2 v}{\partial x^2}\right| + 2\left|\frac{\partial^2 v}{\partial x\partial y}\right| + \left|\frac{\partial^2 v}{\partial y^2}\right|\right)(1 + \kappa h).$$

By Taylor's theorem we have that

$$\frac{\partial v(z)}{\partial x} = \frac{\partial v(z_0)}{\partial x} + \left(k \frac{\partial}{\partial x} + \ell \frac{\partial}{\partial y}\right) \frac{\partial}{\partial x} v(z'')$$

with some point $z''$ on the segment between $z_0$ and $z$. Since $\partial v(z_0)/\partial x = \partial P(z_0)/\partial x$ and

$$\left|\left(k \frac{\partial}{\partial x} + \ell \frac{\partial}{\partial y}\right)\frac{\partial}{\partial x} v(z'')\right| \leqq h \max_{z\in\Delta} \left(\left|\frac{\partial^2 v}{\partial x^2}\right| + \left|\frac{\partial^2 v}{\partial x\partial y}\right|\right),$$

by (3.23) we obtain the estimate

$$\left|\frac{\partial v(z)}{\partial x}\right| \leqq \frac{4h}{\sin\theta} \max_{z\in\Delta} \left(\left|\frac{\partial^2 v}{\partial x^2}\right| + 2\left|\frac{\partial^2 v}{\partial x\partial y}\right| + \left|\frac{\partial^2 v}{\partial y^2}\right|\right)(1 + \kappa h).$$

Analogously the estimate for $\left|\frac{\partial v}{\partial y}\right|$ is obtained.

## § 3.6. Smoothness of $\omega$ on $\overline{\Omega}$.

LEMMA 3.7. Let $\omega$ be the harmonic solution in $\Gamma_\theta$. Then $\omega\circ\varphi_j^{-1}$ $(j = 1, \cdots, m)$ are of the class $C^1$ on $\varphi_j(U_j \cap \overline{\Omega})$ respectively.

PROOF. ( i ) The case where $U_j$ contains some $p_k$ $(k = 1, \cdots, 2n)$.

Let us assume that $U_j$ contains $p_1$. The other cases are also similar. Then, $\varphi_j(p_1) = 0$, $\varphi_j(U_j \cap \overline{\Omega}) = \{|z| < \rho_j\} \cap \{0 \leqq \arg z \leqq \pi/2\}$, and there exists a harmonic function $u$ on $U_j \cap \overline{\Omega}$ such that $\omega = du$,

$$(3.24) \qquad u\circ\varphi_j^{-1} = 0 \qquad \text{on } \{z \mid \operatorname{Im} z = 0,\ 0 \leqq \operatorname{Re} z \leqq \rho_j\}$$

and

$$(3.25) \qquad \frac{\partial}{\partial n} u\circ\varphi_j^{-1} = 0 \qquad \text{on } \{z \mid \operatorname{Re} z = 0,\ 0 < \operatorname{Im} z \leqq \rho_j\},$$

where by $\partial/\partial n$ we denote the inner normal derivative. By (3.24) and (3.25) we see that $u\circ\varphi_j^{-1}$ can be harmonically continued to $\varphi_j(U_j) = \{|z| < \rho_j\}$ and thus especially is of the class $C^2$ on $\varphi_j(U_j \cap \overline{\Omega})$.

(ii) The case where $\varphi_j(U_j \cap \overline{\Omega}) = \{|z| < \rho_j\} \cap \{0 \leqq \arg z \leqq \alpha_j\}$ and $\alpha_j \leqq \pi/2$.

There exists an analytic function $\mathfrak{f}$ on $U_j \cap \overline{\Omega}$ such that $d(\operatorname{Re} \mathfrak{f}) = \omega$. Let $\mathfrak{g}$ be the function defined on $D = \{\operatorname{Im} \xi > 0\} \cap \{|\xi| < \rho_j^{\pi/\alpha_j}\}$ by $\mathfrak{g}(\xi) \equiv \mathfrak{f}\circ\varphi_j^{-1}(\xi^{\alpha_j/\pi})$. Since $\operatorname{Re} \mathfrak{g} = \text{const.}$ or $\operatorname{Im} \mathfrak{g} = \text{const.}$ on $\{\operatorname{Im} \xi = 0\} \cap \{|\xi| < \rho_j^{\pi/\alpha_j}\}$, $\mathfrak{g}$ is analytic on the closure $\overline{D}$. Then

$$\frac{d\mathfrak{f}\circ\varphi_j^{-1}(z)}{dz} = \frac{d\mathfrak{g}}{d\xi}(z^{\pi/\alpha_j}) \cdot \frac{\pi}{\alpha_j} z^{\pi/\alpha_j - 1}$$

and

$$\frac{d^2\mathfrak{f}\circ\varphi_j^{-1}(z)}{dz^2} = \frac{d^2\mathfrak{g}}{d\xi^2}(z^{\pi/\alpha_j}) \cdot \left(\frac{\pi}{\alpha_j}\right)^2 z^{2(\pi/\alpha_j - 1)}$$
$$+ \frac{d\mathfrak{g}}{d\xi}(z^{\pi/\alpha_j}) \cdot \frac{\pi}{\alpha_j} \left(\frac{\pi}{\alpha_j} - 1\right) z^{\pi/\alpha_j - 2}$$

on $\varphi_j(U_j \cap \overline{\Omega})$. Hence, $\alpha_j \leqq \pi/2$ implies that $d^2\mathfrak{f}\circ\varphi_j^{-1}(z)/dz^2$

is continuous on $\varphi_j(U_j \cap \overline{\Omega})$ and thus $u\circ\varphi_j^{-1} = \mathrm{Re}\, f\circ\varphi_j^{-1}$ is of the class $C^2$ on $\varphi_j(U_j \cap \overline{\Omega})$.

(iii) The cases except ( i ) and (ii).

Since $u\circ\varphi_j^{-1} = \mathrm{const.}$, $\partial u\circ\varphi_j^{-1}/\partial n = 0$ or $u\circ\varphi_j^{-1}$ is harmonic on $\varphi_j(U_j \cap \partial\Omega) = \{|z| < \rho_j\} \cap \{\mathrm{Im}\, z = 0\}$, or $\varphi_j(U_j \cap \partial\Omega) = \phi$, $u\circ\varphi_j^{-1}$ is harmonic on $\varphi_j(U_j \cap \overline{\Omega})$.

§ 3.7. Approximation by $\psi_h$.

THEOREM 3.1. Let $\omega$ be the harmonic solution in $\Gamma_\Theta$ defined in § 3.1 and let $\psi_h$ be the finite element approximation of $\omega$ in the space $\Lambda$. Then,

$$(3.26) \quad \|\psi_h - \omega\|^2 \le \frac{h^2}{\sin^2\theta}\left(B\sum_{j=1}^{m}\iint_{\varphi_j(K'_j)}\left(\left(\frac{\partial a}{\partial x}\right)^2 + \left(\frac{\partial a}{\partial y}\right)^2 + \left(\frac{\partial b}{\partial x}\right)^2 + \left(\frac{\partial b}{\partial y}\right)^2\right)dxdy + C h^2 \sum_{j=1}^{m}\max_{\varphi_j(R_j)}\left(\left(\frac{\partial a}{\partial x}\right)^2 + \left(\frac{\partial a}{\partial y}\right)^2 + \left(\frac{\partial b}{\partial x}\right)^2 + \left(\frac{\partial b}{\partial y}\right)^2\right)\right),$$

where B and C are constants independent of the triangulation K and the differential $\theta$, $\theta$ is the smallest value of interior angles of all triangles $\varphi_j(s)$ $(s \in K'_j;\ j = 1, \cdots, m)$,

$$\omega = a\,dx + b\,dy \qquad \text{on } \varphi_j(U_j \cap \overline{\Omega}) \ (j = 1, \cdots, m),$$

by $\varphi_j(K'_j)$ we denote the image set by $\varphi_j$ of the carrier of $K'_j$, and $R_j$ $(j = 1, \cdots, m)$ are the closed subsets of $U_j \cap \overline{\Omega}$ defined in ( i′ ) of § 1.2.

PROOF. First, by ( i ) of Lemma 3.2,

(3.27) $\|\psi_h - \omega\| \leqq \|\hat{\omega} - \omega\|.$

Hence it is sufficient to estimate $\|\hat{\omega} - \omega\|$.

We have

$$(3.28) \qquad \|\hat{\omega} - \omega\|_{\Omega}^2 = \sum_{j=1}^{m} \sum_{s\in K_j} \|\hat{\omega} - \omega\|_s^2.$$

Here we note that $\omega\circ\varphi_j^{-1}$ $(j = 1, \cdots, m)$ is of the class $C^1$ on $\varphi_j(U_j \cap \overline{\Omega})$. Then, by Lemma 3.5,

$$(3.29) \qquad \|\hat{\omega} - \omega\|_s^2 \leqq \frac{B}{\sin^2\theta} h^2 \iint_{\varphi_j(s)} \left(\left(\frac{\partial a}{\partial x}\right)^2 + \left(\frac{\partial a}{\partial y}\right)^2 + \left(\frac{\partial b}{\partial x}\right)^2 + \left(\frac{\partial b}{\partial y}\right)^2\right) dxdy$$

for each natural simplex $s$ of $K_j$. For simplicity, we denote the right hand side of (3.29) by $I[\varphi_j(s)]$.

For a triple $(s, s', \ell)$ for a minor simplex $s$, we denote the differential $\hat{\omega}'$ on $\natural s \in K_j'$ and $s' \in K_k'$ by $\hat{\omega}_{\natural s}'$ and $\hat{\omega}_{s'}'$, respectively. Then, by Lemma 2.1

$$(3.30) \qquad \|\hat{\omega} - \omega\|_{\ell}^2 \leqq \|\hat{\omega}_{\natural s}' - \omega\|_{\ell}^2 + \|\hat{\omega}_{s'}' - \omega\|_{\ell}^2.$$

This inequality and Lemma 3.5 imply that

$$(3.31) \qquad \|\hat{\omega} - \omega\|_{s+s'}^2 \leqq \|\hat{\omega}_{\natural s}' - \omega\|_{\natural s}^2 + \|\hat{\omega}_{s'}' - \omega\|_{s'}^2$$
$$\leqq I[\varphi_j(\natural s)] + I[\varphi_k(s')].$$

Let $(s, s', \ell)$ be a triple for a major simplex $s$. Then, by Lemma 3.5

$$(3.32) \qquad \|\hat{\omega} - \omega\|_s^2 \leqq I[\varphi_j(\natural s)] + \|\hat{\omega} - \omega\|_{\ell}^2$$

and

$$(3.33) \qquad \|\hat{\omega} - \omega\|_{s'}^2 \leqq I[\varphi_k(s')].$$

Let

$$\hat{\omega} = a_0 dx + b_0 dy \qquad \text{on } \varphi_j(s), \text{ and}$$

$$\hat{\omega} = \alpha_0 d\xi + \beta_0 d\eta \qquad \text{on } \varphi_k(s'),$$

where $a_0$, $b_0$, $\alpha_0$ and $\beta_0$ are constants. Then we define differentials $\hat{\omega}_s$ and $\hat{\omega}_{s'+\ell}$ on $s$ and $s'+\ell$ respectively by

$$\hat{\omega}_s = a_0 dx + b_0 dy \qquad \text{on } \varphi_j(s), \text{ and}$$

$$\hat{\omega}_{s'+\ell} = \alpha_0 d\xi + \beta_0 d\eta \qquad \text{on } \varphi_k(s'+\ell).$$

Then, by Lemma 2.1

$$(3.34) \qquad \|\hat{\omega} - \omega\|_\ell^2 \leqq \|\hat{\omega}_s - \omega\|_\ell^2 + \|\hat{\omega}_{s'+\ell} - \omega\|_\ell^2.$$

Further, by Lemma 3.6

$$(3.35) \qquad \|\hat{\omega}_s - \omega\|_\ell^2 \leqq A(\varphi_j(\ell)) \cdot \frac{32h^2}{\sin^2\theta} \cdot \max_{\varphi_j(s)} \left( \left|\frac{\partial a}{\partial x}\right| + \left|\frac{\partial a}{\partial y}\right| + \left|\frac{\partial b}{\partial x}\right| + \left|\frac{\partial b}{\partial y}\right| \right)^2 (1 + \kappa h)^2$$

and

$$(3.36) \qquad \|\hat{\omega}_{s'+\ell} - \omega\|_\ell^2 \leqq A(\varphi_k(\ell)) \cdot \frac{32h^2}{\sin^2\theta} \cdot \max_{\varphi_k(s'+\ell)} \left( \left|\frac{\partial \alpha}{\partial \xi}\right| + \left|\frac{\partial \alpha}{\partial \eta}\right| + \left|\frac{\partial \beta}{\partial \xi}\right| + \left|\frac{\partial \beta}{\partial \eta}\right| \right)^2 (1 + \kappa h)^2,$$

where $\omega = a\, dx + b\, dy$ on $\varphi_j(s)$ and $\omega = \alpha\, d\xi + \beta\, d\eta$ on $\varphi_k(s'+\ell)$.

By (3.27)~(3.36), Lemma 1.1 and (1.1), the estimate (3.26) is obtained.

§ 3.8. Approximation by $\omega'_h$.

THEOREM 3.2. (i) Let $\omega$ be the harmonic solution in $\Gamma_\theta$ defined in § 3.1, let $\omega'_h$ be the finite element approximation of $\omega$ in the space $\Lambda'$ and let $\omega_h = F^{-1}(\omega'_h)$. Then

$$(3.37)\qquad \|\omega_h - \omega\|^2$$
$$\le \frac{h^2}{\sin^2\theta}\Bigg(A'\sum_{j=1}^{m}\iint_{\varphi_j(K'_j)}\left(\left(\frac{\partial a}{\partial x}\right)^2+\left(\frac{\partial a}{\partial y}\right)^2+\left(\frac{\partial b}{\partial x}\right)^2+\left(\frac{\partial b}{\partial y}\right)^2\right)dxdy$$
$$+B'h^2\sum_{j=1}^{m}\max_{\varphi_j(R_j)}\left(\left(\frac{\partial a}{\partial x}\right)^2+\left(\frac{\partial a}{\partial y}\right)^2+\left(\frac{\partial b}{\partial x}\right)^2+\left(\frac{\partial b}{\partial y}\right)^2\right)\Bigg)$$
$$+C'h^2\sum_{j=1}^{m}\max_{\varphi_j(R_j)}(a^2+b^2),$$

where $A'$, $B'$ and $C'$ are constants independent of the triangulation $K$ and the differential $\theta$, and other notations are the same as in Theorem 3.1.

(ii) Let $\theta_0$ be the differential defined in § 3.2, let $\omega$ be the harmonic solution in $\Gamma_{\theta_0}$ and let $\omega'_h$ be the finite element approximation of $\omega$ in the space $\Lambda'$. Then the estimate

$$(3.38)\qquad \|\omega\|^2 \le \|\omega'_h\|^2 + \varepsilon(\omega'_h)$$

holds with

$$(3.39)\qquad \varepsilon(\omega'_h) \equiv \sum_{j=1}^{m}\sum_{\sharp\ell\in K_j} A(\varphi_j(\sharp\ell))\cdot\left(\frac{1}{\lambda}\int_{e_2}\omega'_h\right)^2$$
$$\cdot\max\left\{1,\ \left(\frac{\lambda}{\mu}\right)^2\cdot\max_{\varphi_j(\sharp\ell)}|f'(z)|^2\right\},$$

where $e_1$ and $e_2$ are the edges of $\sharp\ell$ such that $\varphi_j(e_2)$ is a straight segment, $\lambda$ and $\mu$ are the lengths of the segments $\varphi_j(e_2)$ and $\varphi_k(e_1)$ resp., and $f(z) \equiv \varphi_k\circ\varphi_j^{-1}(z)$.

PROOF. ( i ) First, note that

$$(3.40)\qquad \|\omega_h - \omega\|^2 \leqq 2\|\psi_h - \omega\|^2 + 2\|\omega_h - \psi_h\|^2.$$

From Lemmas 2.1, 2.2 and 3.3, and (3.13), it follows that

$$(3.41)\qquad \|\omega_h - \psi_h\|^2 = \|\omega_h\|^2 - \|\psi_h\|^2$$
$$\leqq \|\omega_h'\|^2 - \|\psi_h\|^2 + \sum_{\sharp\ell\in K} \|\omega_h\|^2_{\sharp\ell}$$
$$\leqq \|\psi_h'\|^2 - \|\psi_h\|^2 + \sum_{\sharp\ell\in K} \|\omega_h\|^2_{\sharp\ell}$$
$$\leqq \sum_{j=1}^{m} \sum_{\flat\ell\in K_j} \Big(A(\varphi_j(\flat\ell))\cdot(a_0'^2+b_0'^2) + A(\varphi_k(\flat\ell))\cdot(\alpha_0'^2+\beta_0'^2)\Big)$$
$$+ \sum_{j=1}^{m} \sum_{\sharp\ell\in K_j} \Big(A(\varphi_j(\sharp\ell))\cdot(a_0^2+b_0^2) + A(\varphi_k(\sharp\ell))\cdot(\alpha_0^2+\beta_0^2)\Big),$$

where for each triple $(s, s', \flat\ell)$ for $\flat\ell \in K_j$

$$\psi_h' = a_0'dx + b_0'dy \qquad \text{on } \varphi_j(\natural s), \text{ and}$$

$$\psi_h' = \alpha_0'd\xi + \beta_0'd\eta \qquad \text{on } \varphi_k(s'),$$

for each triple $(s, s', \sharp\ell)$ for $\sharp\ell \in K_j$

$$\omega_h = a_0dx + b_0dy \qquad \text{on } \varphi_j(\natural s), \text{ and}$$

$$\omega_h = \alpha_0d\xi + \beta_0d\eta \qquad \text{on } \varphi_k(s')$$

with constants $a_0'$, $b_0'$, $\alpha_0'$, $\beta_0'$, $a_0$, $b_0$, $\alpha_0$ and $\beta_0$.

In the inequality (3.41), we have

$$(3.42)\qquad A(\varphi_j(\flat\ell))\cdot(a_0'^2 + b_0'^2)$$
$$= \frac{A(\varphi_j(\flat\ell))}{A(\varphi_j(s))} \|\psi_h\|^2_s$$

$$\leqq 2\,\frac{A(\varphi_j(\flat\ell))}{A(\varphi_j(s))}\,(\|\psi_h - \omega\|_s^2 + \|\omega\|_s^2)$$

$$\leqq 2\,\frac{A(\varphi_j(\flat\ell))}{A(\varphi_j(s))}\,\|\psi_h - \omega\|_s^2 + 2\,A(\varphi_j(\flat\ell))\cdot \max_{\varphi_j(s)} (a^2 + b^2).$$

Since we can easily verify that

$$A(\varphi_j(\natural s)) > \frac{h_1{}^2}{4}\sin\theta \qquad (h_1 = d(\varphi_j(\natural s))),$$

by Lemma 1.1 we have

$$(3.43)\qquad \frac{A(\varphi_j(\flat\ell))}{A(\varphi_j(s))} = \frac{A(\varphi_j(\flat\ell))}{A(\varphi_j(\natural s)) - A(\varphi_j(\flat\ell))}$$

$$\leqq \frac{h}{2\sin\theta}\left(\left|\frac{g''(\xi_1)}{g'(\xi_1)^2}\right| + O(h)\right)$$

with the notations in Lemma 1.1. (3.42) and (3.43) imply

$$(3.44)\qquad \sum_{j=1}^{m}\sum_{\flat\ell\in K_j} A(\varphi_j(\flat\ell))\cdot(a_0'^2 + b_0'^2)$$

$$\leqq \frac{Ch}{\sin\theta}\sum_{j=1}^{m}\sum_{\flat\ell\in K_j}\|\psi_h - \omega\|_s^2 + 2\sum_{j=1}^{m}\sum_{\flat\ell\in K_j} A(\varphi_j(\flat\ell))\max_{\varphi_j(s)}(a^2 + b^2),$$

where $C$ is a constant depending only on the transformations of local parameters. Since similar estimates for other terms of the right hand side of (3.41) are obtained, from (3.41) it follows that

$$(3.45)\qquad \|\omega_h - \psi_h\|^2$$

$$\leqq \frac{Ch}{\sin\theta}\|\omega_h - \omega\|^2 + \frac{Ch}{\sin\theta}\|\psi_h - \omega\|^2$$

$$+ 2\sum_{j=1}^{m}\sum_{\ell\in K_j}\left(A(\varphi_j(\ell))\max_{\varphi_j(s)}(a^2 + b^2) + A(\varphi_k(\ell))\max_{\varphi_k(s')}(\alpha^2 + \beta^2)\right),$$

where for each triple $(s, s', \ell)$ for $\ell \in K_j$

$$\omega = a\,dx + b\,dy \quad \text{on} \quad \varphi_j(s), \text{ and}$$
$$\omega = \alpha\,d\xi + \beta\,d\eta \quad \text{on} \quad \varphi_k(s').$$

(3.40), (3.45), Theorem 3.1, Lemma 1.1 and (1.1) imply the the estimate (3.37).

(ii) (3.7) and Lemma 3.3 and the proof of Lemma 2.2( i ) imply the inequalities

$$\|\omega\|^2 \leqq \|\psi_h\|^2 \leqq \|\omega_h\|^2$$
$$\leqq \|\omega_h'\|^2 + \sum_{j=1}^{m} \sum_{\natural\ell\in K_j} A(\varphi_j(\natural\ell))\left(\frac{1}{\lambda}\int_{e_2}\omega_h'\right)^2$$
$$\cdot\max\left\{1, \left(\frac{\lambda}{\mu}\right)^2 \cdot \max_{\varphi_j(\natural\ell)} |f'(z)|^2\right\}.$$

§ 3.9. Estimate of $\|\omega_h' - \hat{\omega}'\|$.

COROLLARY 3.1. Let $\omega$ and $\omega_h'$ be the same as in Theorem 3.2, $\hat{\omega}$ be the finite element interpolation of $\omega$ in the space $\Lambda$, and $\hat{\omega}' = F(\hat{\omega})$. Then, the estimate

$$\|\omega_h' - \hat{\omega}'\| \leqq A''h \tag{3.46}$$

holds, where $A''$ is a constant dependent only on $\omega$ and $\theta$ in Theorem 3.1.

PROOF. First, by Lemma 2.2(ii) and (3.43) we have

$$\|\omega_h' - \hat{\omega}'\|^2$$
$$\leqq \|\omega_h - \hat{\omega}\|^2 + \sum_{\flat\ell\in K} (\|\omega_{h,\natural s}' - \hat{\omega}_{\natural s}'\|_{\flat\ell}^2 + \|\omega_{h,s'}' - \hat{\omega}_{s'}'\|_{\flat\ell}^2)$$

$$\leqq \|\omega_h - \hat{\omega}\|^2 + \sum_{j=1}^{m} \sum_{\flat\ell\in K_j} \left( \frac{A(\varphi_j(\flat\ell))}{A(\varphi_j(s))} \|\omega_h - \hat{\omega}\|_s^2 \right.$$

$$\left. + \frac{A(\varphi_k(\flat\ell))}{A(\varphi_k(s')) - A(\varphi_k(\flat\ell))} \|\omega_h - \hat{\omega}\|_{s'}^2 \right)$$

$$\leqq \|\omega_h - \hat{\omega}\|^2 + \frac{Ch}{\sin\theta} \sum_{j=1}^{m} \sum_{\flat\ell\in K_j} (\|\omega_h - \hat{\omega}\|_s^2 + \|\omega_h - \hat{\omega}\|_{s'}^2)$$

$$\leqq \left(1 + \frac{Ch}{\sin\theta}\right) \|\omega_h - \hat{\omega}\|^2$$

$$\leqq 2\left(1 + \frac{Ch}{\sin\theta}\right) (\|\omega_h - \omega\|^2 + \|\omega - \hat{\omega}\|^2),$$

where $C$ is the same constant as in (3.44). Then, the proof of Theorem 3.1 and Theorem 3.2 imply (3.46).

## Chapter 4. Determination of the periodicty moduli of Riemann surfaces.

§ 4.1. Periodicity moduli of Riemann surfaces. Let $\overline{\Omega}$ be a closed or compact bordered Riemann surface of genus 1 with no or one boundary component. Let $\{A, B\}$ be a canonical homology basis of $\overline{\Omega}$ such that $A \times B = 1$. Then there exists a unique system of harmonic differentials $\{\phi, \rho, \chi, \tau\}$ on $\Omega$ satisfying the period and boundary conditions:

$$\int_B \phi = \int_B \chi = 1, \quad \int_A \phi = \int_A \chi = 0, \tag{4.1}$$

$$\int_A \rho = \int_A \tau = -1, \quad \int_B \rho = \int_B \tau = 0, \tag{4.2}$$

$$\phi = \rho = *\chi = *\tau = 0 \qquad \text{along } \partial\Omega \tag{4.3}$$

and

$$\int_{\partial\Omega} *\phi = \int_{\partial\Omega} *\rho = \int_{\partial\Omega} \chi = \int_{\partial\Omega} \tau = 0, \tag{4.4}$$

where the conditions (4.3) and (4.4) may be ignored if $\partial\Omega = \phi$. If $\partial\Omega = \phi$, then $\phi = \chi$ and $\rho = \tau$.

We can easily see that

$$\begin{cases} \|\phi\|^2 = \int_A *\phi, \quad \|\rho\|^2 = \int_B *\rho, \quad \text{and} \\ (\phi, \rho) = \int_B *\phi = \int_A *\rho = 0. \end{cases} \tag{4.5}$$

We call

$$p_1 = \int_A *\phi \quad \text{and} \quad p_2 = \int_B *\rho$$

periodicity moduli of $\Omega$ with respect to A and B respectively, which are the quantities determining the conformal structure of $\Omega$. By (4.1)~(4.5) we see that

$$\tau = -\frac{*\phi}{\|\phi\|^2} \quad \text{and} \quad \chi = \frac{*\rho}{\|\rho\|^2} .$$

These relations imply that

$$(4.6) \qquad p_1 = \|\phi\|^2 = \frac{1}{\|\tau\|^2} \quad \text{and} \quad p_2 = \|\rho\|^2 = \frac{1}{\|\chi\|^2} .$$

If $\partial\Omega = \phi$, then

$$(4.7) \qquad p_1 = \|\phi\|^2 = \frac{1}{\|\rho\|^2} = \frac{1}{p_2} .$$

By making use of a relation analogous to (4.7) for the modulus of quadrilaterals on the complex plane, Gaier [11] presented a method to obtain upper and lower bounds for the modulus by the finite difference approximation.

§ 4.2. Calculation of periodicity moduli. Let $\{\theta_1, \theta_2, \theta_3, \theta_4\}$ be a system of differentials in $\Gamma_c(\bar{\Omega})$ satisfying the period and boundary conditions:

$$\int_B \theta_1 = \int_B \theta_3 = 1, \quad \int_A \theta_1 = \int_A \theta_3 = 0,$$

$$\int_A \theta_2 = \int_A \theta_4 = -1, \quad \int_B \theta_2 = \int_B \theta_4 = 0,$$

$$\theta_1 = \theta_2 = 0 \quad \text{along} \quad \partial\Omega,$$

and $\theta_3$ and $\theta_4$ are exact on a neighborhood of $\partial\Omega$. Here we interpret that $\partial\Omega = C_2$ for $\theta_1$ and $\theta_2$, and $\partial\Omega = C_3$ for $\theta_3$ and $\theta_4$ in the notations in § 3.1. We note that $\theta_1$, $\theta_2$, $\theta_3$ and $\theta_4$ satisfy the conditions for the differential $\theta_0$ in § 3.2. Then we can easily see that $\phi$, $\rho$, $\chi$ and $\tau$ are the harmonic solutions in $\Gamma_{\theta_1}$, $\Gamma_{\theta_2}$, $\Gamma_{\theta_3}$ and $\Gamma_{\theta_4}$, respectively. Let $\phi_h'$, $\rho_h'$, $\chi_h'$ and $\tau_h'$ be the finite element approximations of $\phi$, $\rho$, $\chi$ and $\tau$ in the space $\Lambda'$ respectively. Then by (ii) of Theorem 3.2 and (4.6), we obtain upper and lower bounds for $p_1$ and $p_2$:

$$\frac{1}{\|\tau_h'\|^2 + \varepsilon(\tau_h')} \leqq p_1 \leqq \|\phi_h'\|^2 + \varepsilon(\phi_h') \tag{4.8}$$

and

$$\frac{1}{\|\chi_h'\|^2 + \varepsilon(\chi_h')} \leqq p_2 \leqq \|\rho_h'\|^2 + \varepsilon(\rho_h'). \tag{4.9}$$

If $\partial\Omega = \phi$, then $\phi = \chi$ and $\rho = \tau$, and thus (4.8) and (4.9) imply the inequalities

$$\frac{1}{\|\rho_h'\|^2 + \varepsilon(\rho_h')} \leqq p_1 = \frac{1}{p_2} \leqq \|\phi_h'\|^2 + \varepsilon(\phi_h').$$

§ <u>4.3</u>. <u>Numerical</u> <u>example</u> <u>1</u> (the case of a closed Riemann surface). Let $\Omega$ be the two-sheeted covering surface with four branch points $z = -3, -1, 1, 3$ over the extended z-plane. Then $\Omega$ is a closed Riemann surface of genus one. A canonical homology basis $\{A, B\}$ of $\Omega$ is chosen as in Fig.6. We aim to obtain good upper and lower approximate values of the periodicity moduli $p_1$ and $p_2$ of $\Omega$ with respect to A and B respectively.

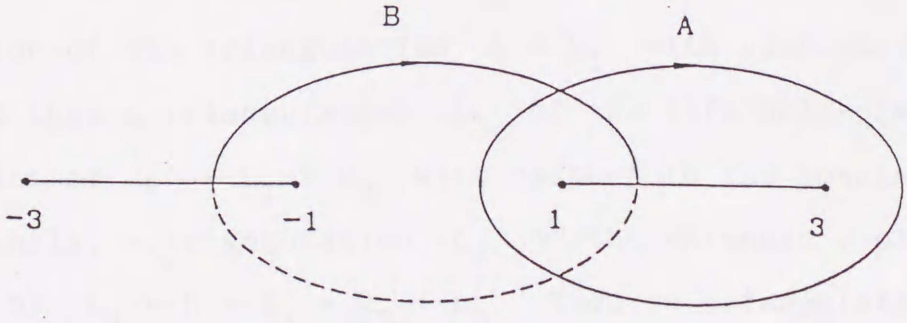



Fig. 6 Numerical example 1 (the case of a closed Riemann surface)

First, we construct a triangulation of the closed region:

$$\overline{D} = \{z \mid |z| \leqq \sqrt{3},\ \ \mathrm{Re}\, z \geqq 0,\ \ \mathrm{Im}\, z \geqq 0\}$$

as in Fig.7. The closed regions $G_2$ and $G_3$ are mapped onto the regions $G_2^*$ and $G_3^*$ resp. by the local parameters $\xi = \varphi_2(z) = a\sqrt{z - 1}$ and $w = \varphi_3(z) = b \log z$ ($a = 2(\sqrt{3} - 1)^{1/2}$ and $b = \sqrt{3}$) respectively, where $a$ and $b$ are so determined that $|d\xi/dz| = 1$ and $|dw/dz| = 1$ on $|z - 1| = \sqrt{3} - 1$ and $|z| = \sqrt{3}$ respectively. We construct ordinary triangulations $K_2^*$ and $K_3^*$ of $G_2^*$ and $G_3^*$ as in Fig.7 respectively. By $K_2$ and $K_3$ we denote the image triangulations of $K_2^*$ and $K_3^*$ by the mappings $\varphi_2^{-1}$ and $\varphi_3^{-1}$ respectively. The triangulation $K_1$ of the region $G_1 = \overline{D - (G_2 \cup G_3)}$ in Fig.7 is so constructed that each 2-simplex $s$ of $K_1$ is natural, minor or major according as $|s| \cap |K_2 + K_3| = \phi$, $|s| \cap |K_2| \neq \phi$ or $|s| \cap |K_3| \neq \phi$, where if some intersection is a point then it is interpreted to be vacuous, and the local parameter $\varphi_1(z)$ of $K_1$ is the identity mapping $\varphi_1(z) \equiv z$.

A triangulation $L_1$ of the region $\overline{D}_1 = \{z \mid |z| \geqq \sqrt{3},\ \mathrm{Re}\, z \geqq 0,\ \mathrm{Im}\, z \geqq 0\}$ is defined by the reflection of the triangulation $L \equiv K_1 + K_2 + K_3$ with respect to the circle $|z| = \sqrt{3}$ (cf. Fig.8). Next we define a triangulation $L_2$ of the fourth quadrant by the reflection of the triangulation $L + L_1$ with respect to the real axis and then a triangulation $L_3$ of the left half-plane by the reflection of $L + L_1 + L_2$ with respect to the imaginary axis. Consequently, a triangulation $L_4$ of the extended z-plane is defined by $L_4 = L + L_1 + L_2 + L_3$. Then, a triangulation $K$ of the covering surface $\Omega$ is so constructed that the projection $T$ of $K$ onto the extended z-plane is the triangulation $L_4$. We see that

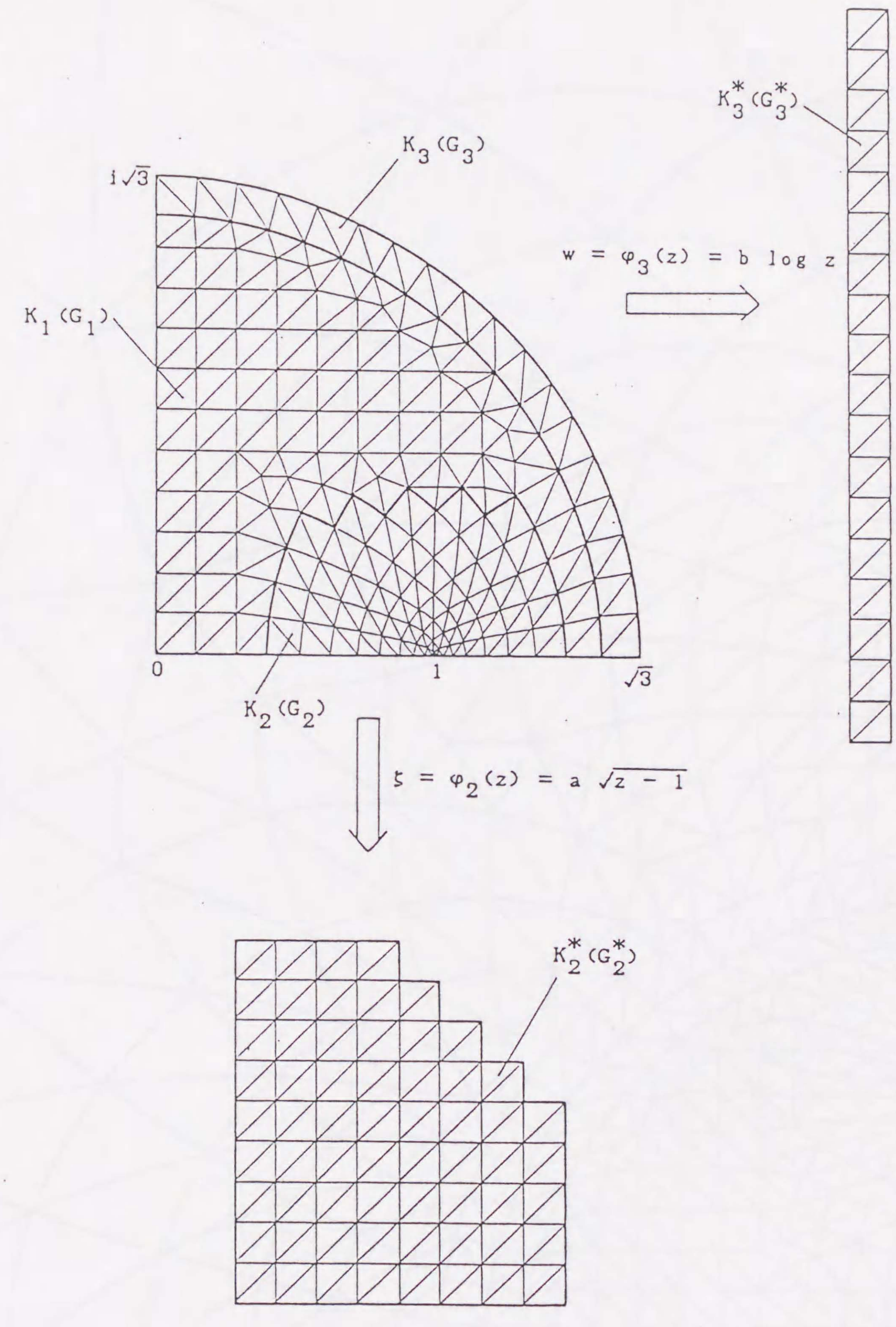



Fig. 7 Triangulation L of example 1

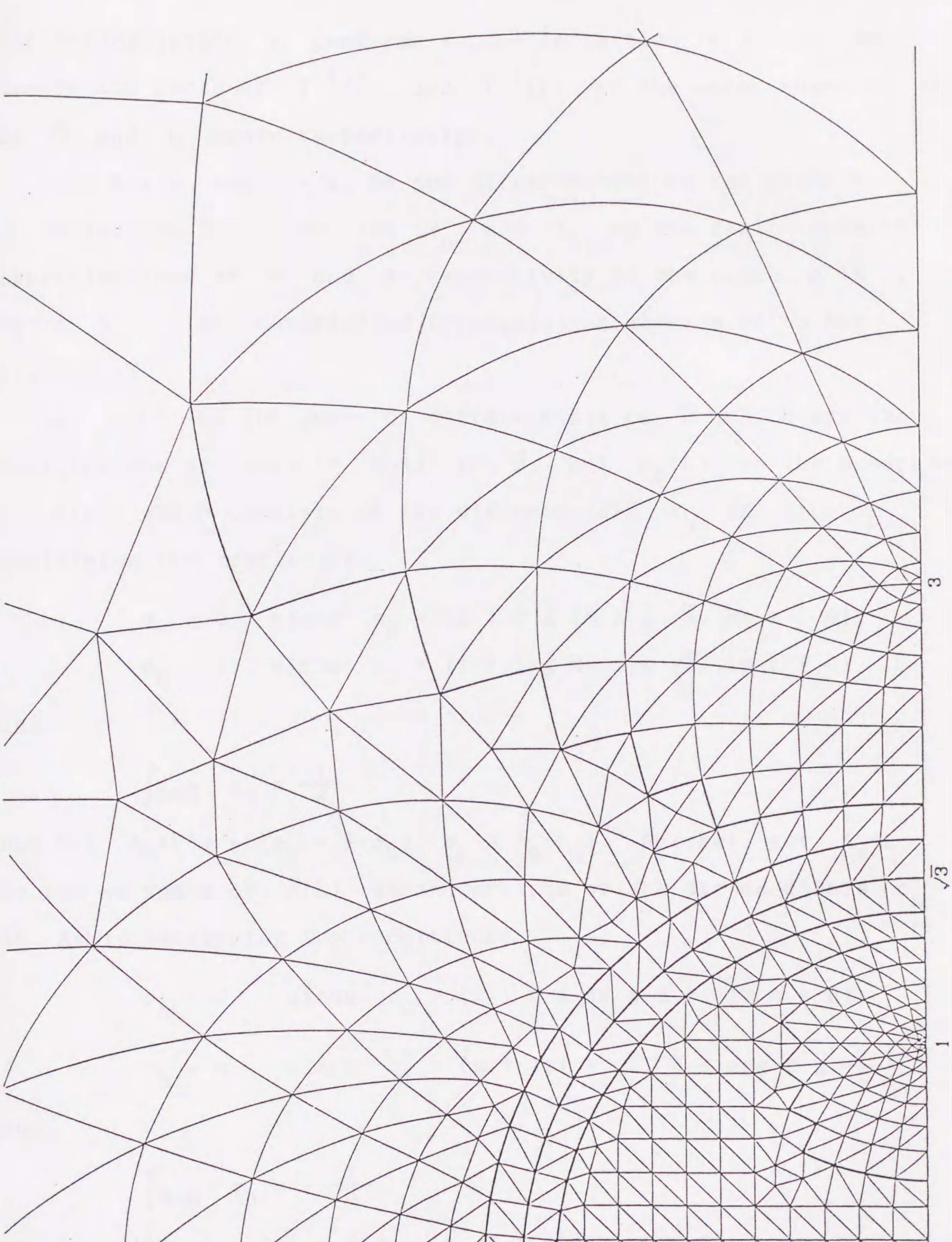



Fig. 8 Triangulation $L + L_1$ of example 1

the triangulation K conforms to the definition in § 1.2. We denote the parts of $T^{-1}(\overline{D})$ and $T^{-1}(L)$ on the upper sheet of $\Omega$ by $\overline{D}$ and $L$ again respectively.

Let $\phi = \chi$ and $\rho = \tau$ be the differentials on the present $\Omega$ defined in § 4.1, and let $\phi'_h$ and $\rho'_h$ be the finite element approximations of $\phi$ and $\rho$ respectively in the space $\Lambda'(K')$, where $K'$ is the naturalized triangulation associated to the present $K$.

Let $\Lambda(L)$ be the space of differentials on $\overline{D}$ which are the restrictions of those in $\Lambda(K)$ to $\overline{D}$. Let $\Lambda_\phi(L)$ be the subspace of $\Lambda(L)$ which consists of the differentials $\sigma_h$ in $\Lambda(L)$ satisfying the conditions:

$$\sigma_h = 0 \quad \text{along} \quad c_0 = \{z \mid 0 \leqq \operatorname{Im} z \leqq \sqrt{3},\ \operatorname{Re} z = 0\},$$

$$\sigma_h = 0 \quad \text{along} \quad c_1 = \{z \mid 1 \leqq \operatorname{Re} z \leqq \sqrt{3},\ \operatorname{Im} z = 0\}$$

and

$$\int_{B\cap\overline{D}} \sigma_h = \frac{1}{4},$$

and let $\Lambda'_\phi(L') = \{\sigma'_h = F(\sigma_h),\ \sigma_h \in \Lambda_\phi(L)\}$. Further, let $\Lambda_\rho(L)$ be the subspace of $\Lambda(L)$ which consists of the differentials $\sigma_h$ in $\Lambda(L)$ satisfying the conditions:

$$\sigma_h = 0 \quad \text{along} \quad c_0^* = \{z \mid 0 \leqq \operatorname{Re} z \leqq 1,\ \operatorname{Im} z = 0\},$$

$$\sigma_h = 0 \quad \text{along} \quad c_1^* = \left\{z \mid |z| = \sqrt{3},\ 0 \leqq \arg z \leqq \frac{\pi}{2}\right\}$$

and

$$\int_{A\cap\overline{D}} \sigma_h = -\frac{1}{4},$$

and let $\Lambda'_\rho(L') = \{\sigma'_h = F(\sigma_h),\ \sigma_h \in \Lambda_\rho(L)\}$. By $\phi'_{h,L}$ and $\rho'_{h,L}$ we denote the differentials in $\Lambda'_{\phi}(L')$ and $\Lambda'_{\rho}(L')$ respectively which minimize norms $\|\sigma'_h\|_{L'}$ in $\Lambda'_{\phi}(L')$ and $\Lambda'_{\rho}(L')$ respectively. Then, by making use of the symmetricity of $K'$, the period and boundary conditions of $\phi'_h$, $\rho'_h$, $\phi'_{h,L}$ and $\rho'_{h,L'}$, and their minimality w.r.t. norm, we can verify that $\phi'_{h,L}$ and $\rho'_{h,L}$ are the restrictions of $\phi'_h$ and $\rho'_h$ to $L'$ respectively, and $\|\phi'_h\|^2_{K'} = 16\ \|\phi'_{h,L}\|^2_{L'}$ and $\|\rho'_h\|^2_{K'} = 16\ \|\rho'_{h,L}\|^2_{L'}$. Consequently, to attain our aim it is sufficient to make numerical calculations of $\phi'_{h,L}$ and $\rho'_{h,L}$ (cf. Mizumoto and Hara [17],[18] for the calculation method).

We should note that the symmetricity of $\phi$ and $\rho$ on $\Omega$ has not been used and thus our method does not reject an application to the differentials which do not have symmetricity on $\Omega$.

Table 1 shows the exact value of the periodicity moduli $p_1$ which can be calculated by making use of a complete elliptic integral, and the values of our finite element approximations. Furthermore, computational results for the normal subdivision $K^1$ (see Fig.9) of the present $K$ are shown. It can be said that the both of upper and lower bounds of $p_1$ are close to the exact value.

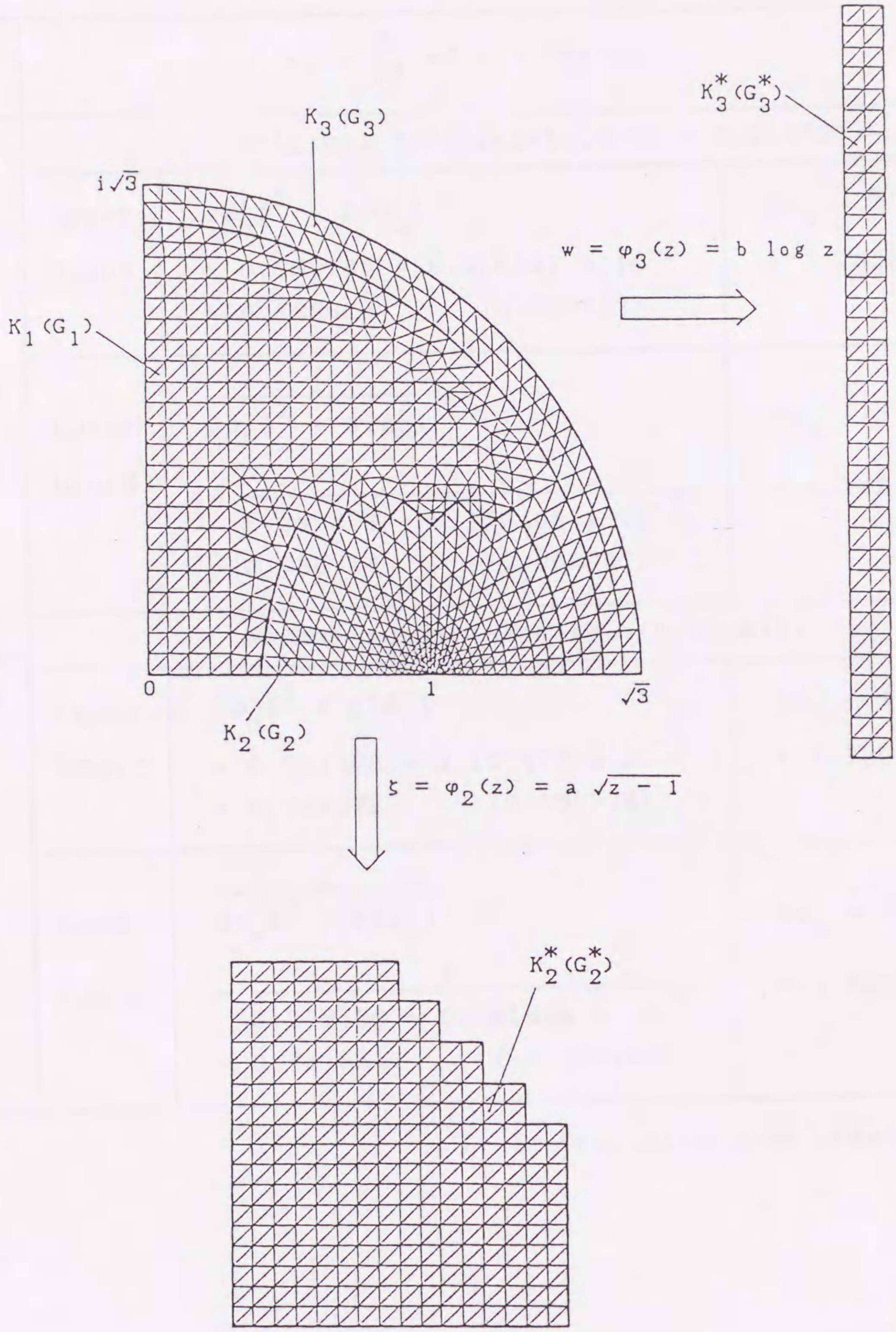



Fig. 9 Normal subdivision of example 1

Table 1 Periodicity moduli $p_1$ of example 1 (closed Riemann surface)

| | | | |
|---|---|---|---|
| Exact value | $p_1 = \int_A *\phi = 0.781701$ | | |
| Finite element approximations | Original triangulation ($h = 0.213758$) | | |
| | Upper bound | $\|\phi_h'\|^2 + \varepsilon(\phi_h')$ $= 0.782184 + 0.429347 \times 10^{-3}$ $= 0.782613$ (0.000912) | $\|\phi_h' - \hat{\phi}'\|$ $= 3.76256 \times 10^{-3}$ |
| | Lower bound | $\dfrac{1}{\|\rho_h'\|^2 + \varepsilon(\rho_h')}$ $= \dfrac{1}{1.280878 + 0.150405 \times 10^{-5}}$ $= 0.780714$ (-0.000987) | $\|\rho_h' - \hat{\rho}'\|$ $= 6.14254 \times 10^{-3}$ |
| | Normal subdivision ($h = 0.106879$) | | |
| | Upper bound | $\|\phi_h'\|^2 + \varepsilon(\phi_h')$ $= 0.781968 + 0.107413 \times 10^{-3}$ $= 0.782075$ (0.000374) | $\|\phi_h' - \hat{\phi}'\|$ $= 1.12050 \times 10^{-3}$ |
| | Lower bound | $\dfrac{1}{\|\rho_h'\|^2 + \varepsilon(\rho_h')}$ $= \dfrac{1}{1.279506 + 0.381486 \times 10^{-6}}$ $= 0.781551$ (-0.000150) | $\|\rho_h' - \hat{\rho}'\|$ $= 1.83821 \times 10^{-3}$ |

( ): Deviation from exact value.

§ 4.4. Numerical example 2 (the case of a compact bordered Riemann surface). Let $\overline{\Omega}$ be a two-sheeted compact bordered covering surface with three branch points $z = -1, 1, 3$ over the ellipse:

$$E = \left\{ z = x + iy \;\middle|\; \frac{x^2}{16} + \frac{y^2}{15} \leqq 1 \right\}.$$

Then $\overline{\Omega}$ is a compact bordered Riemann surface of genus one with one boundary component. A canonical homology basis $\{A, B\}$ of $\overline{\Omega}$ is chosen as in Fig.10. We aim to obtain good upper and lower

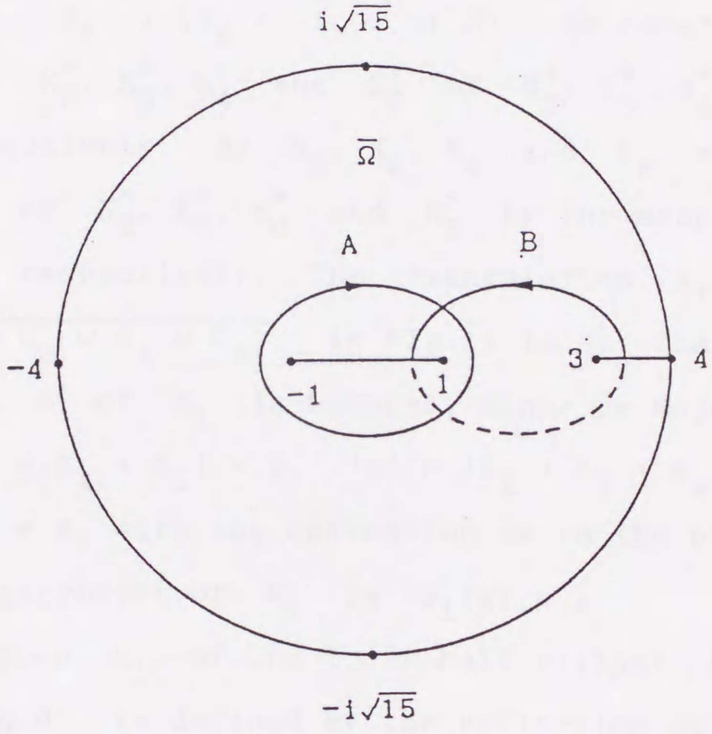



Fig. 10 Numerical example 2
(the case of a compact bordered Riemann surface)

approximate values of the periodicity moduli $p_1$ and $p_2$ of $\bar{\Omega}$ with respect to A and B respectively.

First, we construct a triangulation of the upper half ellipse $\bar{D} = E \cap \{z \mid \mathrm{Im}\, z \geq 0\}$ as in Fig.11. The closed regions $G_2$, $G_3$, $G_4$ and $G_5$ are mapped onto the regions $G_2^*$, $G_3^*$, $G_4^*$ and $G_5^*$ resp. by the local parameters $\zeta = \varphi_2(z) = a\sqrt{z + 1}$, $\zeta = \varphi_3(z) = a\sqrt{z - 1}$, $\zeta = \varphi_4(z) = b\sqrt{z - 3}$ and $w = \varphi_5(z) = \cosh^{-1} z$ ($a = 2/\ 5^{1/4}$ and $b = 2/\ 85^{1/4}$) respectively, where $a$ and $b$ are so determined that $|d\zeta/dz|$ are equal to $|dw/dz|$ at $z = z_0 + i$ ($z_0 = -1$, 1 or 3). We construct ordinary triangulations $K_2^*$, $K_3^*$, $K_4^*$ and $K_5^*$ of $G_2^*$, $G_3^*$, $G_4^*$ and $G_5^*$ as in Fig.11 respectively. By $K_2$, $K_3$, $K_4$ and $K_5$ we denote the image triangulations of $K_2^*$, $K_3^*$, $K_4^*$ and $K_5^*$ by the mappings $\varphi_2^{-1}$, $\varphi_3^{-1}$, $\varphi_4^{-1}$ and $\varphi_5^{-1}$ respectively. The triangulation $K_1$ of the region $G_1 = \overline{\Omega - (G_2 \cup G_3 \cup G_4 \cup G_5)}$ in Fig.11 is so constructed that each 2-simplex $s$ of $K_1$ is natural, minor or major according as $|s| \cap |K_2 + K_3 + K_4 + K_5| = \phi$, $|s| \cap |K_2 + K_3 + K_4| \neq \phi$ or $|s| \cap |K_5| \neq \phi$, with the convention as in the previous section, and the local parameter of $K_1$ is $\varphi_1(z) \equiv z$.

A triangulation $L_1$ of the lower half ellipse $\bar{D}_1 = E \cap \{z \mid \mathrm{Im}\, z \leq 0\}$ is defined by the reflection of the triangulation $L \equiv K_1 + K_2 + K_3 + K_4 + K_5$ with respect to the real axis and a triangulation $L_2$ of $E$ is defined by $L_2 = L + L_1$. Then, a triangulation $K$ of the covering surface $\bar{\Omega}$ is so constructed that the projection $T$ of $K$ onto the z-plane is the triangulation $L_2$. We see that the triangulation $K$ conforms to the definition in § 1.2. We denote the parts of $T^{-1}(\bar{D})$ and $T^{-1}(L)$ on the upper

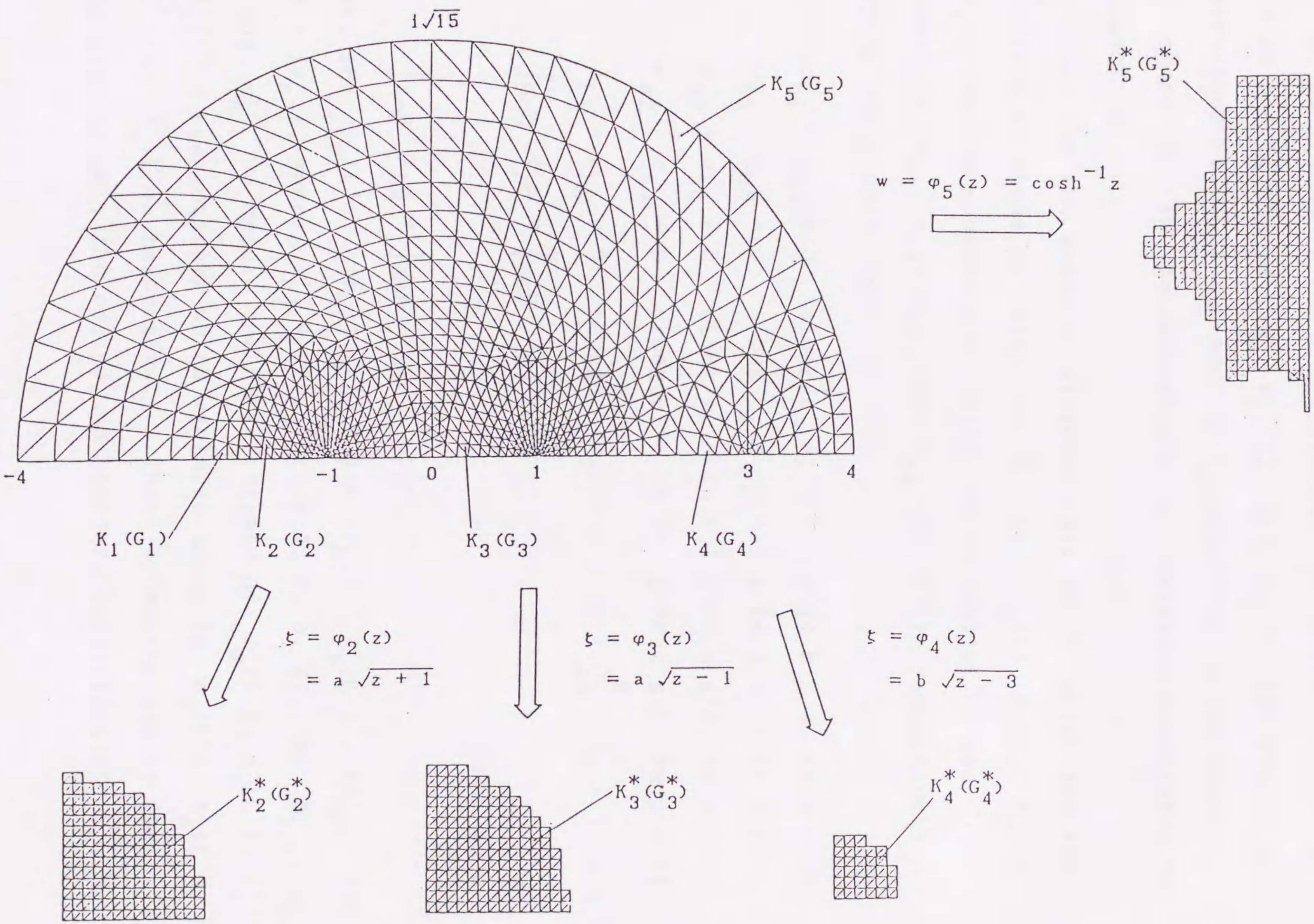



Fig. 11 Triangulation of example 2

sheet of $\overline{\Omega}$ by $\overline{D}$ and L again respectively.

Let $\phi$, $\rho$, $\chi$ and $\tau$ be the differentials on the present $\overline{\Omega}$ defined in § 4.1, and let $\phi'_h$, $\rho'_h$, $\chi'_h$ and $\tau'_h$ be the finite element approximations of $\phi$, $\rho$, $\chi$ and $\tau$ respectively in the space $\Lambda'(K')$, where $K'$ is the naturalized triangulation associated to the present K.

Let $\Lambda(L)$ be the space of differentials on $\overline{D}$ which are the restrictions of those in $\Lambda(K)$ to $\overline{D}$. Let $\Lambda_\phi(L)$, $\Lambda_\rho(L)$, $\Lambda_\chi(L)$ and $\Lambda_\tau(L)$ be the subspaces of $\Lambda(L)$ which consist of the differentials $\sigma_{h1}$, $\sigma_{h2}$, $\sigma_{h3}$ and $\sigma_{h4}$ in $\Lambda(L)$ respectively satisfying the conditions:

$$\sigma_{h1} = \sigma_{h3} = 0 \quad \text{along } c_0 = \{z \mid 3 \leqq \operatorname{Re} z \leqq 4,\ \operatorname{Im} z = 0\},$$
$$\sigma_{h1} = \sigma_{h3} = 0 \quad \text{along } c_1 = \{z \mid -1 \leqq \operatorname{Re} z \leqq 1,\ \operatorname{Im} z = 0\},$$
$$\sigma_{h2} = \sigma_{h4} = 0 \quad \text{along } c_0^* = \{z \mid 1 \leqq \operatorname{Re} z \leqq 3,\ \operatorname{Im} z = 0\},$$
$$\sigma_{h2} = \sigma_{h4} = 0 \quad \text{along } c_1^* = \{z \mid -4 \leqq \operatorname{Re} z \leqq -1,\ \operatorname{Im} z = 0\},$$
$$\sigma_{h1} = \sigma_{h2} = 0 \quad \text{along } c = \left\{z = x + iy \;\middle|\; \frac{x^2}{16} + \frac{y^2}{15} = 1,\ y \geqq 0\right\},$$
$$\int_{B\cap\overline{D}} \sigma_{h1} = \int_{B\cap\overline{D}} \sigma_{h3} = \frac{1}{2}$$

and

$$\int_{A\cap\overline{D}} \sigma_{h2} = \int_{A\cap\overline{D}} \sigma_{h4} = -\frac{1}{2}.$$

Further, let $\Lambda'_\phi(L') = \{\sigma'_{h1}\}$, $\Lambda'_\rho(L') = \{\sigma'_{h2}\}$, $\Lambda'_\chi(L') = \{\sigma'_{h3}\}$ and $\Lambda'_\tau(L') = \{\sigma'_{h4}\}$, where $\sigma'_{hj} = F(\sigma_{hj})$ $(j = 1, 2, 3, 4)$. By $\phi'_{h,L}$, $\rho'_{h,L}$, $\chi'_{h,L}$ and $\tau'_{h,L}$ we denote the differentials of $\Lambda'_\phi(L')$, $\Lambda'_\rho(L')$, $\Lambda'_\chi(L')$ and $\Lambda'_\tau(L')$ respectively which minimize norms in $\Lambda'_\phi(L')$, $\Lambda'_\rho(L')$, $\Lambda'_\chi(L')$ and $\Lambda'_\tau(L')$ respectively. Then, by making use of the symmetricity of $K'$, the period and boundary conditions of $\phi'_h$, $\rho'_h$,

$\chi'_h$, $\tau'_h$, $\phi'_{h,L}$, $\rho'_{h,L}$, $\chi'_{h,L}$ and $\tau'_{h,L}$, and their minimality w.r.t. norm, we can verify that $\phi'_{h,L}$, $\rho'_{h,L}$, $\chi'_{h,L}$ and $\tau'_{h,L}$ are the restrictions of $\phi'_h$, $\rho'_h$, $\chi'_h$ and $\tau'_h$ to $L'$ respectively, and $\|\phi'_h\|^2_{K'} = 4\ \|\phi'_{h,L}\|^2_{L'}$, $\|\rho'_h\|^2_{K'} = 4\ \|\rho'_{h,L}\|^2_{L'}$, $\|\chi'_h\|^2_{K'} = 4\ \|\chi'_{h,L}\|^2_{L'}$ and $\|\tau'_h\|^2_{K'} = 4\ \|\tau'_{h,L}\|^2_{L'}$. Consequently, to attain our aim it is sufficient to make numerical calculations of $\phi'_{h,L}$, $\rho'_{h,L}$, $\chi'_{h,L}$ and $\tau'_{h,L}$.

The exact values of the periodicity moduli $p_1$ and $p_2$ can be calculated by the following procedure.

Let $\tilde{c}_0$ and $\tilde{c}_1$ be the boundary parts of the upper half ellipse domain $D$ defined by

$$\tilde{c}_0 = \{z \mid 3 \leqq \operatorname{Re} z \leqq 4,\ \operatorname{Im} z = 0\} \cup \left\{z = x + iy \;\middle|\; \frac{x^2}{16} + \frac{y^2}{15} = 1,\ y \geqq 0\right\}$$

and

$$\tilde{c}_1 = \{z \mid -1 \leqq \operatorname{Re} z \leqq 1,\ \operatorname{Im} z = 0\}.$$

Let $\Delta$ be the rectangular domain

$$\Delta = \{W \mid 0 < \operatorname{Re} W < 1,\ \ 0 < \operatorname{Im} W < \tau\},$$

and let $\gamma_0$ and $\gamma_1$ be the boundary parts of $\Delta$ defined by

$$\gamma_0 = \{W \mid 0 \leqq \operatorname{Im} W \leqq \tau,\ \ \operatorname{Re} W = 0\}$$

and

$$\gamma_1 = \{W \mid 0 \leqq \operatorname{Im} W \leqq \tau,\ \ \operatorname{Re} W = 1\}.$$

If $D$ is conformally mapped onto $\Delta$ so that $\tilde{c}_0$ and $\tilde{c}_1$ are mapped onto $\gamma_0$ and $\gamma_1$ respectively, then the periodicity moduli

$p_1$ is equal to $\tau$. The conformal map $W = f(z): D \to \Delta$ is constructed by the composition of the following mappings:

(i) $w = \dfrac{2}{\cosh^{-1}4} \cdot \cosh^{-1}z - 1;$

(ii) $\xi = \mathrm{sn}(K(k) \cdot w)$, where $\dfrac{K'(k)}{K(k)} = \dfrac{2\pi}{\cosh^{-1}4};$

(iii) $\dfrac{Z - Z_1}{Z - Z_2} \cdot \dfrac{Z_3 - Z_2}{Z_3 - Z_1} = \dfrac{\xi - \xi_1}{\xi - \xi_2} \cdot \dfrac{\xi_3 - \xi_2}{\xi_3 - \xi_1},$

where $\xi_j = \mathrm{sn}(K(k) \cdot w_j)$ $(j = 1, 2, 3, 4)$ with $w_1 = -1 + i(2\pi / \cosh^{-1}4)$, $w_2 = -1$, $w_3 = 2\cosh^{-1}3 / \cosh^{-1}4 - 1$, $w_4 = 1 + i(2\pi / \cosh^{-1}4)$, and $Z_1 = -1/\kappa$, $Z_2 = -1$, $Z_3 = 1$, $Z_4 = 1/\kappa$ with $\kappa = (\sqrt{1/c} - \sqrt{1/c - 1})^2$, $c = ((\xi_4 - \xi_1)/(\xi_4 - \xi_2)) \cdot ((\xi_3 - \xi_2)/(\xi_3 - \xi_1));$

(iv) $W = -\dfrac{1}{2}\left(\dfrac{1}{K(\kappa)} \displaystyle\int_0^Z \dfrac{dZ}{\sqrt{(1 - Z^2)(1 - \kappa^2 Z^2)}} - \left(1 + i\dfrac{K'(\kappa)}{K(\kappa)}\right)\right).$

Then we see that

$$p_1 = \tau = \frac{K'(\kappa)}{2K(\kappa)}.$$

Next, let $\tilde{c}'_0$ and $\tilde{c}'_1$ be the boundary parts of $D$ given by

$$\tilde{c}'_0 = \{z \mid 1 \leqq \mathrm{Re}\, z \leqq 3,\ \mathrm{Im}\, z = 0\}$$

and

$$\tilde{c}'_1 = \left\{z = x + iy \;\middle|\; \frac{x^2}{16} + \frac{y^2}{15} = 1,\ y \geqq 0\right\} \cup \{z \mid -4 \leqq \mathrm{Re}\, z \leqq -1,\ \mathrm{Im}\, z = 0\}.$$

Let $\Delta$, $\gamma_0$ and $\gamma_1$ be as above. If the domain $D$ is conformally mapped onto the domain $\Delta$ so that $\tilde{c}'_0$ and $\tilde{c}'_1$ are mapped onto

$\gamma_0$ and $\gamma_1$ respectively, then the periodicity moduli $p_2$ is equal to $\tau$. The conformal map $W = f(p): D \to \Delta$ is constructed similarly to the case of periodicity moduli $p_1$.

Tables 2 and 3 show the exact values of the periodicity moduli $p_1$ and $p_2$, and the values of our finite element approximations. Furthermore, computation results for the normal subdivision $K^1$ of the present $K$ are shown. It can be said that the both of upper and lower bounds of $p_1$ and $p_2$ are close to the exact values.

Table 2 Periodicity moduli $p_1$ of example 2
(compact bordered Riemann surface)

| | | | |
|---|---|---|---|
| Exact value | $p_1 = \int_A *\Phi = 1.539330$ | | |
| Finite element approximations | Original triangulation ($h = 0.138840$) | | |
| | Upper bound | $\|\phi_h'\|^2 + \varepsilon(\phi_h')$<br>$= 1.540588 + 0.572262 \times 10^{-4}$<br>$= 1.540645$ (0.00132) | $\|\phi_h' - \hat{\Phi}'\|$<br>$= 1.15335 \times 10^{-2}$ |
| | Lower bound | $\dfrac{1}{\|\tau_h'\|^2 + \varepsilon(\tau_h')}$<br>$= \dfrac{1}{0.649700 + 0.225117 \times 10^{-3}}$<br>$= 1.538639$ (-0.00069) | $\|\tau_h' - \hat{\tau}'\|$<br>$= 3.74131 \times 10^{-3}$ |
| | Normal subdivision ($h = 0.069420$) | | |
| | Upper bound | $\|\phi_h'\|^2 + \varepsilon(\phi_h')$<br>$= 1.539652 + 0.142916 \times 10^{-4}$<br>$= 1.539666$ (0.00034) | $\|\phi_h' - \hat{\Phi}'\|$<br>$= 5.89447 \times 10^{-3}$ |
| | Lower bound | $\dfrac{1}{\|\tau_h'\|^2 + \varepsilon(\tau_h')}$<br>$= \dfrac{1}{0.649652 + 0.558093 \times 10^{-4}}$<br>$= 1.539153$ (-0.00018) | $\|\tau_h' - \hat{\tau}'\|$<br>$= 1.09209 \times 10^{-3}$ |

( ): Deviation from exact value.

Table 3 Periodicity moduli $p_2$ of example 2
(compact bordered Riemann surface)

| | | | |
|---|---|---|---|
| Exact value | $p_2 = \int_B *\rho = 1.839350$ | | |
| Finite element approximations | Original triangulation (h = 0.138840) | | |
| | Upper bound | $\Vert\rho'_h\Vert^2 + \varepsilon(\rho'_h)$<br>$= 1.841976 + 0.351532 \times 10^{-3}$<br>$= 1.842328$ (0.00298) | $\Vert\rho'_h - \hat{\rho}'\Vert$<br>$= 7.65797 \times 10^{-3}$ |
| | Lower bound | $\dfrac{1}{\Vert\chi'_h\Vert^2 + \varepsilon(\chi'_h)}$<br>$= \dfrac{1}{0.544588 + 0.145580 \times 10^{-3}}$<br>$= 1.835760$ (-0.00359) | $\Vert\chi'_h - \hat{\chi}'\Vert$<br>$= 5.22574 \times 10^{-3}$ |
| | Normal subdivision (h = 0.069420) | | |
| | Upper bound | $\Vert\rho'_h\Vert^2 + \varepsilon(\rho'_h)$<br>$= 1.840016 + 0.875764 \times 10^{-4}$<br>$= 1.840104$ (0.00075) | $\Vert\rho'_h - \hat{\rho}'\Vert$<br>$= 2.28613 \times 10^{-3}$ |
| | Lower bound | $\dfrac{1}{\Vert\chi'_h\Vert^2 + \varepsilon(\chi'_h)}$<br>$= \dfrac{1}{0.543904 + 0.361871 \times 10^{-4}}$<br>$= 1.838437$ (-0.00091) | $\Vert\chi'_h - \hat{\chi}'\Vert$<br>$= 1.73332 \times 10^{-3}$ |

( ): Deviation from exact value.

## Chapter 5. Determination of the modulus of quadrilaterals.

§ 5.1. Quadrilateral on a Riemann surface. Let $\Omega$ be a simply-connected subdomain of a Riemann surface $W$ whose closure $\bar{\Omega}$ is a compact bordered subregion. We consider the case of $C_1 = C_2 = C_3 = \phi$, $C_4 = \partial\Omega$ and $n = 2$ for the notations defined in § 1.1. We assume that $\partial\Omega$ satisfies the conditions in § 1.1. And thus four points $p_1$, $p_2$, $p_3$ and $p_4$ on $\partial\Omega$, and the two opposite arcs $c_0 = \gamma_1^4$ (from $p_1$ to $p_2$) and $c_1 = \gamma_3^4$ (from $p_3$ to $p_4$) are assigned. Then we say that a quadrilateral $Q$ with opposite sides $c_0$ and $c_1$ is given.

§ 5.2. Formulation of problems. We can conformally map the domain $\Omega$ defined in § 5.1 onto a rectangular domain

$$R = \{w \mid 0 < \operatorname{Re} w < 1,\ 0 < \operatorname{Im} w < M\}$$

by a function $w = \mathfrak{f}(p)$ so that $p_1$, $p_2$, $p_3$ and $p_4$ are mapped to $iM$, $0$, $1$ and $1 + iM$ respectively. Then the modulus of the quadrilateral $Q$:

$$M(Q) = M$$

is uniquely determined by $Q$. Our aim is to determine $M(Q)$ by finite element method.

Now we assign the two opposite arcs $\tilde{c}_0$ (from $p_2$ to $p_3$) and $\tilde{c}_1$ (from $p_4$ to $p_1$) on $\partial\Omega$. Then a quadrilateral $\tilde{Q}$ with opposite sides $\tilde{c}_0$ and $\tilde{c}_1$ is defined. We see that the domain $\Omega$ can be conformally mapped onto a rectangular domain

$$\tilde{R} = \{w \mid 0 < \mathrm{Re}\, w < 1,\ 0 < \mathrm{Im}\, w < 1/M\}$$

by a function $w = \tilde{f}(p)$ so that $p_2$, $p_3$, $p_4$ and $p_1$ are mapped to $i/M$, $0$, $1$ and $1 + i/M$ respectively. Hence

$$M(\tilde{Q}) = \frac{1}{M(Q)}. \tag{5.1}$$

We characterize $M(Q)$ by a minimal property.

Let $\gamma$ $(\tilde{\gamma})$ be a curve which connects a point on $c_0$ $(\tilde{c}_0)$ to a point on $c_1$ $(\tilde{c}_1)$. Let $\{\Theta, \tilde{\Theta}\}$ be a system of differentials in $\Gamma_c(\bar{\Omega})$ satisfying the conditions

$$\Theta = 0 \quad \text{along } c_0 \cup c_1,$$

$$\tilde{\Theta} = 0 \quad \text{along } \tilde{c}_0 \cup \tilde{c}_1,$$

$$\int_\gamma \Theta = \int_{\tilde{\gamma}} \tilde{\Theta} = 1.$$

Let $\psi$ $(\tilde{\psi})$ be the harmonic solution in $\Gamma_\Theta$ $(\Gamma_{\tilde{\Theta}})$. Then $\psi$ $(\tilde{\psi})$ satisfies the condition $*\psi = 0$ $(*\tilde{\psi} = 0)$ along $\partial\Omega - c_0 \cup c_1$ $(\partial\Omega - \tilde{c}_0 \cup \tilde{c}_1)$. We can easily see that $\psi = d(\mathrm{Re}\, f)$ $(\tilde{\psi} = d(\mathrm{Re}\, \tilde{f}))$. Then by Lemma 3.1 the equalities

$$M(Q) = \|\psi\|^2 = \min_{\sigma \in \Gamma_\Theta} \|\sigma\|^2, \tag{5.2}$$

$$M(\tilde{Q}) = \|\tilde{\psi}\|^2 = \min_{\sigma \in \Gamma_{\tilde{\Theta}}} \|\sigma\|^2 \tag{5.3}$$

hold.

Let $\Lambda_\psi(K)$ be the subspace of $\Lambda(K)$ which consists of the differentials $\sigma_h$ in $\Lambda(K)$ satisfying the conditions

$$\sigma_h = 0 \qquad \text{along } c_0 \cup c_1,$$

$$\int_\gamma \sigma_h = 1$$

and let $\Lambda'_\psi(K') = \{\sigma'_h = F(\sigma_h),\ \sigma_h \in \Lambda_\psi(K)\}$. Further $\Lambda_{\tilde{\psi}}(K)$ be the subspace of $\Lambda(K)$ which consists of the differentials $\sigma_h$ in $\Lambda(K)$ satisfying the conditions

$$\sigma_h = 0 \qquad \text{along } \tilde{c}_0 \cup \tilde{c}_1,$$
$$\int_{\tilde{\gamma}} \sigma_h = 1$$

and let $\Lambda_{\tilde{\Psi}}'(K') = \{\sigma_h' = F(\sigma_h),\ \sigma_h \in \Lambda_{\tilde{\Psi}}(K)\}$.

Let $\Psi_h'$ and $\tilde{\Psi}_h'$ be the finite element approximations of $\Psi$ and $\tilde{\Psi}$ in the space $\Lambda_{\Psi}'(K')$ and $\Lambda_{\tilde{\Psi}}'(K')$ respectively. Then by (ii) of Theorem 3.2 we have the estimates

$$\|\Psi\|^2 \leqq \|\Psi_h'\|^2 + \varepsilon(\Psi_h') \tag{5.4}$$

and

$$\|\tilde{\Psi}\|^2 \leqq \|\tilde{\Psi}_h'\|^2 + \varepsilon(\tilde{\Psi}_h'). \tag{5.5}$$

By (5.1)~(5.5) we have upper and lower bounds for the modulus $M(Q)$:

$$\frac{1}{\|\tilde{\Psi}_h'\|^2 + \varepsilon(\tilde{\Psi}_h')} \leqq M(Q) \leqq \|\Psi_h'\|^2 + \varepsilon(\Psi_h'). \tag{5.6}$$

§ 5.3. Numerical example 3 (the case of Gaier's example [11]).

Let $\Omega$ be the simply-connected domain on the z-plane defined by

$$\Omega = \{z \mid 0 < x < 1,\ 0 < y < 1\}$$
$$- \left\{z \mid \tfrac{1}{2} \leqq x < 1,\ \tfrac{1}{2} \leqq y < 1\right\},$$

and let $c_0$ and $c_1$ be the boundary parts of $\Omega$ defined by

$$c_0 = \left\{z \mid 0 \leqq x \leqq \tfrac{1}{2},\ y = 0\right\} \cup \{z \mid x = 0,\ 0 \leqq y \leqq 1\}$$
$$\cup \left\{z \mid 0 \leqq x \leqq \tfrac{1}{2},\ y = 1\right\}$$

and

$$c_1 = \left\{ z \mid \frac{1}{2} \leqq x \leqq 1,\ y = \frac{1}{2} \right\}$$

respectively, where $z = x + iy$. Let $Q$ be the quadrilateral with the two opposite sides $c_0$ and $c_1$ (cf. Fig. 12). We aim to obtain

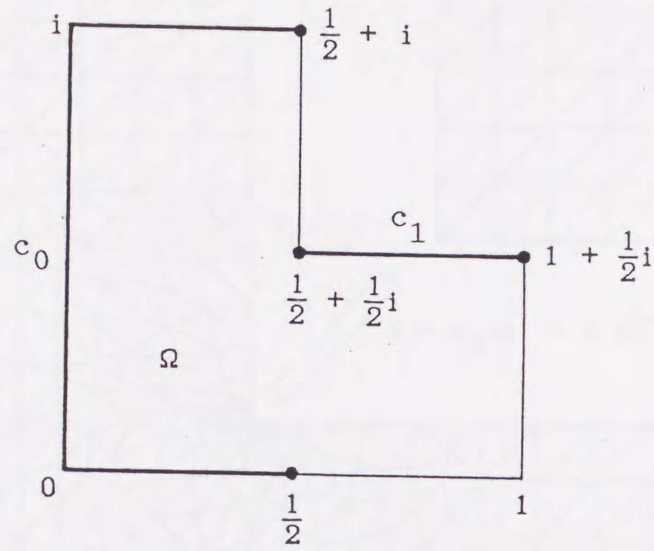



Fig. 12 Numerical example 3 (the example of Gaier)

good upper and lower approximate values of the modulus of $Q$.

We construct a triangulation of the closed region $\overline{\Omega}$ as in Fig. 13 . The closed regions $G_2$ and $G_3$ are mapped onto the regions $G_2^*$ and $G_3^*$ respectively by the local parameters $\zeta = \varphi_2(z) = a\sqrt{z - 1/2}$ and $\zeta = \varphi_3(z) = b\sqrt[3]{z - (1 + i)/2}$ ($a = 1$ and $b = e^{-\pi i/6}$) respectively, where $a$ and $b$ are so determined that $|d\zeta/dz| = 1$ on $|z - 1/2| = 1/4$ and $|z - (1 + i)/2| = 1/\sqrt{27}$ respectively. We construct ordinary triangulations $K_2^*$ and $K_3^*$ of $G_2^*$ and $G_3^*$ as in Fig. 13 respectively. By $K_2$ and $K_3$ we denote the image triangulations of $K_2^*$ and $K_3^*$ by the

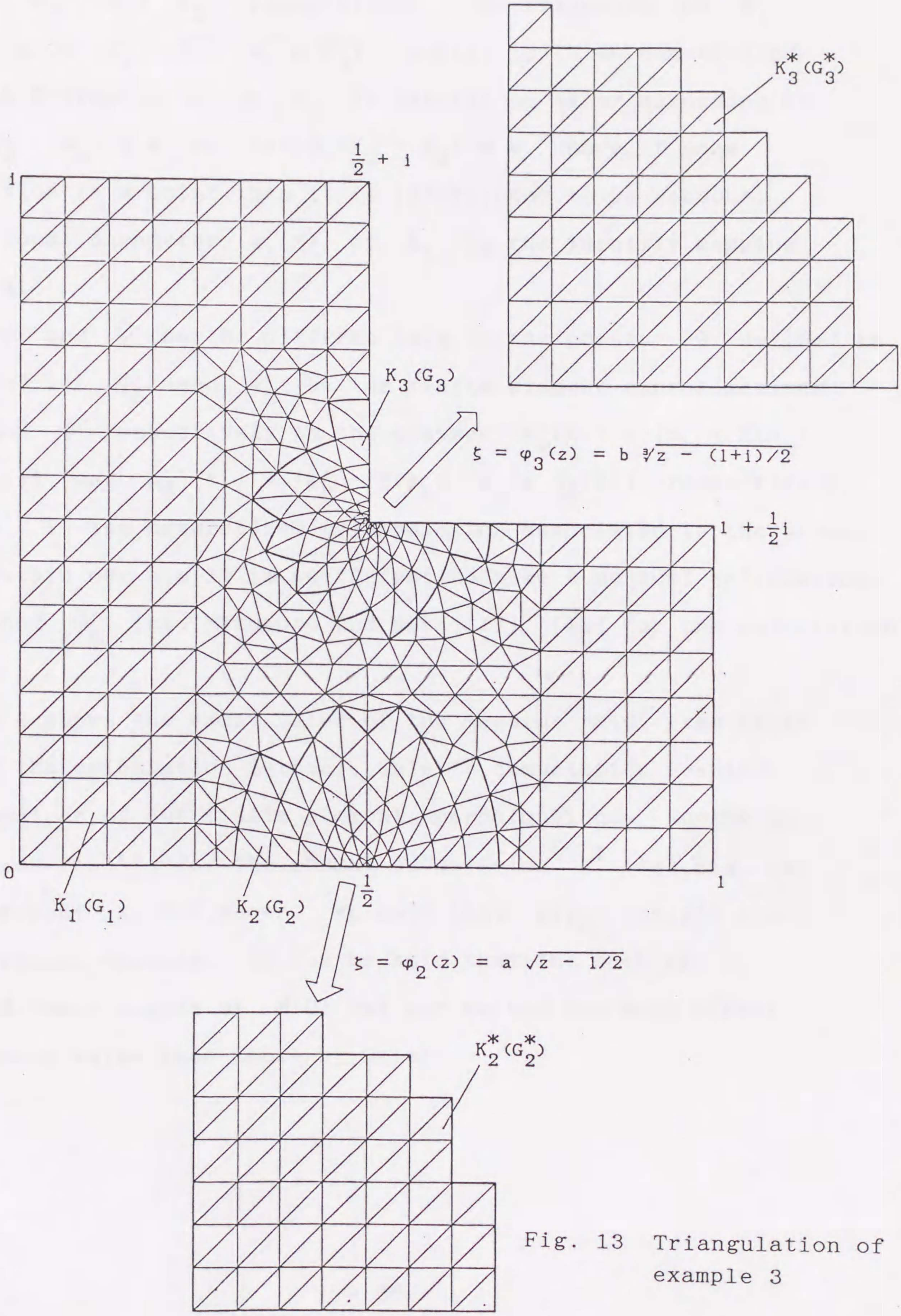



Fig. 13 Triangulation of example 3

mappings $\varphi_2^{-1}$ and $\varphi_3^{-1}$ respectively. The triangulation $K_1$ of the region $G_1 = \overline{\Omega - (G_2 \cup G_3)}$ in Fig. 13 is so constructed that each 2-simplex $s$ of $K_1$ is natural or minor according as $|s| \cap |K_2 + K_3| = \phi$ or $|s| \cap |K_2 + K_3| \neq \phi$, where if some intersection is a point then it is interpreted to be vacuous, and the local parameter $\varphi_1(z)$ of $K_1$ is the identity mapping $\varphi_1(z) \equiv z$.

Let $\psi$ and $\tilde{\psi}$ be the differentials on the present $\Omega$ defined in § 5.2, and let $\psi_h'$ and $\tilde{\psi}_h'$ be the finite element approximations of $\psi$ and $\tilde{\psi}$ respectively in the classes $\Lambda_\psi'(K') = \{\sigma_h' = F(\sigma_h),\ \sigma_h \in \Lambda_\psi(K)\}$ and $\Lambda_{\tilde{\psi}}'(K') = \{\sigma_h' = F(\sigma_h),\ \sigma_h \in \Lambda_{\tilde{\psi}}(K)\}$ respectively, where $K'$ is the naturalized triangulation associated to the present $K$. To attain our aim it is sufficient to make numerical calculations of $\psi_h'$ and $\tilde{\psi}_h'$ (cf. Mizumoto and Hara [17], [18] for the calculation method).

Table 4 shows the exact value of the modulus $M(Q)$ (see Gaier [11] for the calculation method), Gaier's computation results and the values of our finite element approximations. Furthermore, computation results for the normal subdivision $K^1$ (see Fig. 14) of the present $K$ are shown. We note that $\varepsilon(\psi_h') = \varepsilon(\tilde{\psi}_h') = 0$ in the present example. It can be said that the both of upper and lower bounds of $M(Q)$ by our method are much closer to the exact value than those by Gaier.

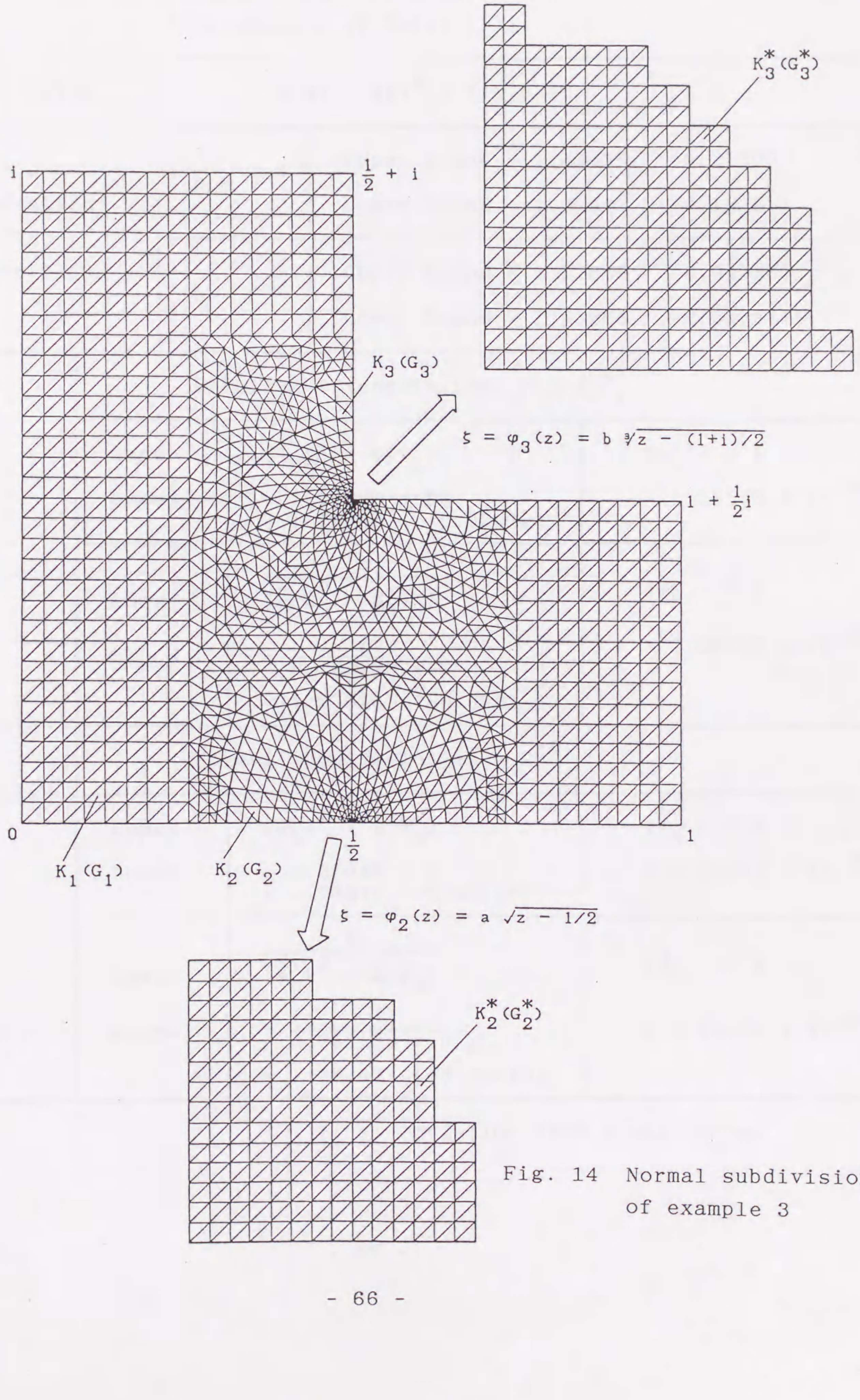



Fig. 14 Normal subdivision of example 3

Table 4 Modulus $M(Q)$ of example 3 (the example of Gaier [11])

| | | | |
|---|---|---|---|
| Exact value | $M(Q) = \|\psi\|^2 = 1.279262$ | | |
| Gaier's computation results (Gaier[11]) | $h = 2^{-4}$ | Upper bound = 1.49435 (0.21509) | |
| | | Lower bound = 1.09543 (-0.18383) | |
| | $h = 2^{-7}$ | Upper bound = 1.32659 (0.04733) | |
| | | Lower bound = 1.23368 (-0.04558) | |
| our computation results | Original triangulation $\left(h = 2^{-4}\right)$ | | |
| | Upper bound | $\|\psi_h'\|^2 + \varepsilon(\psi_h')$ = 1.28396 + 0 = 1.28396 (0.00470) | $\|\psi_h' - \hat{\Phi}'\|$ = $1.28545 \times 10^{-2}$ |
| | Lower bound | $\dfrac{1}{\|\tilde{\Phi}_h'\|^2 + \varepsilon(\tilde{\Phi}_h')} = \dfrac{1}{0.783599 + 0}$ = 1.27616 (-0.00310) | $\|\tilde{\Phi}_h' - \hat{\tilde{\Phi}}'\|$ = $7.25518 \times 10^{-3}$ |
| | Normal subdivision $\left(h = 2^{-5}\right)$ | | |
| | Upper bound | $\|\psi_h'\|^2 + \varepsilon(\psi_h')$ = 1.28046 + 0 = 1.28046 (0.00120) | $\|\psi_h' - \hat{\Phi}'\|$ = $3.89364 \times 10^{-3}$ |
| | Lower bound | $\dfrac{1}{\|\tilde{\Phi}_h'\|^2 + \varepsilon(\tilde{\Phi}_h')} = \dfrac{1}{0.782185 + 0}$ = 1.27847 (-0.00079) | $\|\tilde{\Phi}_h' - \hat{\tilde{\Phi}}'\|$ = $2.18573 \times 10^{-3}$ |

( ): Deviation from exact value.

§ 5.4. Numerical example 4 (the case of a Riemann surface). Let $D_1 = \{z \mid |z| < \infty\} - \{z \mid 0 \leqq x < \infty, y = 0\}$ and $c_0$ be the upper boundary part of $D_1$ lying on $\{z \mid 1 \leqq x < \infty, y = 0\}$, where $z = x + iy$. Let $D_2 = \{z \mid |z| < 1\} - \{z \mid 0 \leqq x < 1, y = 0\}$ and let $c_1$ be the boundary part of $D_2$ defined by $c_1 = \{z \mid |z| = 1, y \geqq 0\}$. Let $\Omega$ be the simply-connected covering surface obtained by connecting $D_1$ and $D_2$ crosswise along the segment $\{z \mid 0 \leqq x < 1, y = 0\}$ (cf. Fig. 15). Let Q be the quadrilateral with the

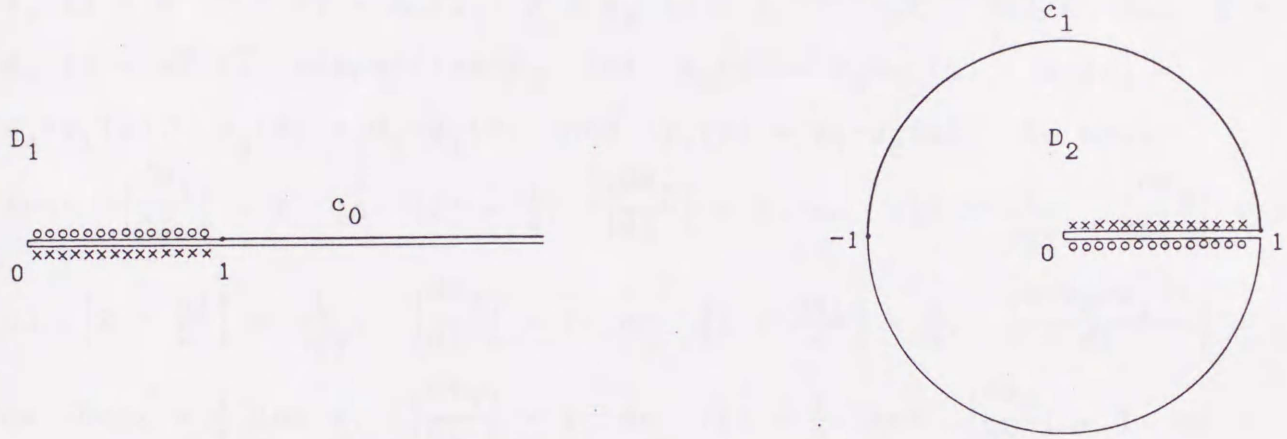



Fig. 15 Numerical example 4 (the case of a Riemann surface)

opposite sides $c_0$ and $c_1$. By symmetricity of Q we immediately see that $M(Q) = 1$. We aim to obtain good upper and lower approximate values of $M(Q)$. The present example is one which exhibits remarkable validity of our method. Namely, it is shown that an unbounded covering surface over the z-plane with many inner and corner singularities of high order, and with a curvilinear boundary is dealt with by our local treatment method without use of any global conformal mapping.

We construct a triangulation of the bordered region $\overline{\Omega}$ as in Figs. 16 and 17. In Fig. 16, the closed regions $G_1 \cup G_2 \cup \cdots \cup G_5$, $G_6 \cup G_7$ and $G_9$ are mapped onto the regions $G_1^* \cup G_2^* \cup \cdots \cup G_5^*$, $G_6^* \cup G_7^*$ and $G_9^*$ respectively by the mappings $\xi = \varphi_1(z) = (1/4)\cdot \log z$, $\xi = \varphi_6(z) = 1/z$ and $\xi = \varphi_9(z) = \sqrt{z}$ respectively. Further, the regions $G_3^*$, $G_4^*$, $G_5^*$ and $G_7^*$ are mapped onto the regions $G_3^{**}$, $G_4^{**}$, $G_5^{**}$ and $G_7^{**}$ respectively by the mappings $Z = \psi_3(\xi) = \sqrt[3]{\xi}$, $Z = \psi_4(\xi) = e^{-\pi i/6}\cdot\sqrt[3]{\xi - \pi i/2}$, $Z = \psi_5(\xi) = e^{-\pi i/4}\cdot\sqrt{\xi - 3\pi i/4}$ and $Z = \psi_7(\xi) = \sqrt{2}\,\sqrt[4]{\xi}$ respectively. Let $\varphi_3(z) = \psi_3\circ\varphi_1(z)$, $\varphi_4(z) = \psi_4\circ\varphi_1(z)$, $\varphi_5(z) = \psi_5\circ\varphi_1(z)$ and $\varphi_7(z) = \psi_7\circ\varphi_6(z)$. We note that $\left|\frac{d\varphi_1}{dz}\right| = 1$ on $|z| = \frac{1}{4}$, $\left|\frac{d\psi_3}{d\xi}\right| = 1$ on $|\xi| = \frac{1}{\sqrt{27}}$, $\left|\frac{d\psi_4}{d\xi}\right| = 1$ on $\left|\xi - \frac{\pi i}{2}\right| = \frac{1}{\sqrt{27}}$, $\left|\frac{d\psi_5}{d\xi}\right| = 1$ on $\left|\xi - \frac{3\pi i}{4}\right| = \frac{1}{4}$, $\left|\frac{d(\varphi_6\circ\varphi_1^{-1})}{d\xi}\right| = 1$ on $\mathrm{Re}\,\xi = \frac{1}{4}\log 4$, $\left|\frac{d\psi_7}{d\xi}\right| = 1$ on $|\xi| = \frac{1}{4}$ and $\left|\frac{d\varphi_9}{dz}\right| = 1$ on $|z| = \frac{1}{4}$. We construct ordinary triangulations $K_3^{**}$, $K_4^{**}$, $K_5^{**}$, $K_7^{**}$ and $K_9^*$ of $G_3^{**}$, $G_4^{**}$, $G_5^{**}$, $G_7^{**}$ and $G_9^*$ as in Fig. 17 respectively. By $K_3$, $K_4$, $K_5$, $K_7$ and $K_9$ we denote the image triangulations of $K_3^{**}$, $K_4^{**}$, $K_5^{**}$, $K_7^{**}$ and $K_9^*$ by the mappings $\varphi_3^{-1}$, $\varphi_4^{-1}$, $\varphi_5^{-1}$, $\varphi_7^{-1}$ and $\varphi_9^{-1}$ respectively, and the local parameters of $K_3$, $K_4$, $K_5$, $K_7$ and $K_9$ are $Z = \varphi_3(z)$, $Z = \varphi_4(z)$, $Z = \varphi_5(z)$, $Z = \varphi_7(z)$ and $\xi = \varphi_9(z)$ respectively. The triangulations $K_1$ and $K_2$ of $G_1$ and $G_2$ respectively in Fig. 17 are so constructed that each 2-simplex $s$ of $K_1$ and $K_2$ is natural or minor according as $|s| \cap |K_3 + K_4 + K_5| = \phi$ or $|s| \cap |K_3 + K_4 +$

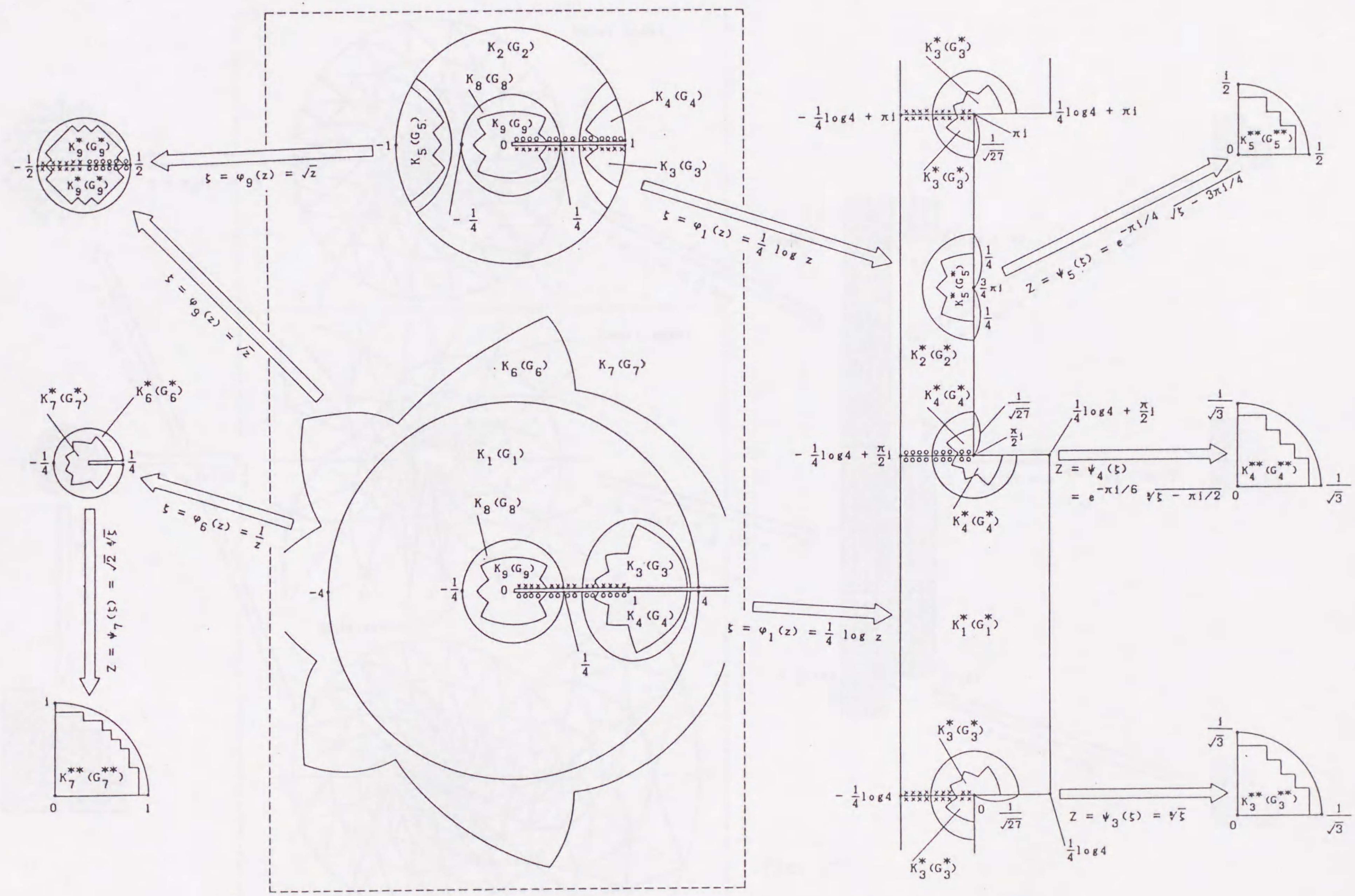



Fig. 16 Local parameters of example 4

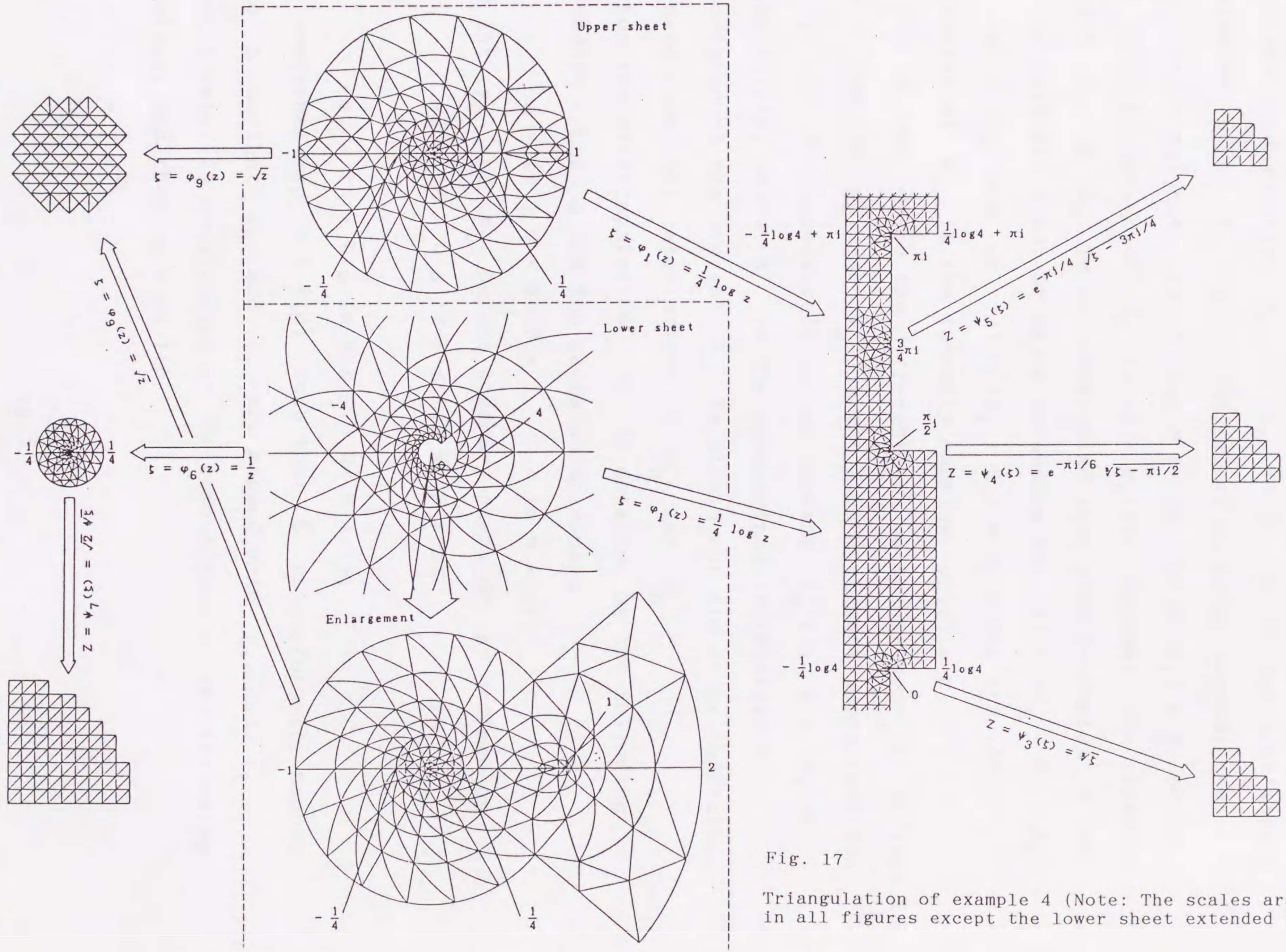



Fig. 17

Triangulation of example 4 (Note: The scales are identical in all figures except the lower sheet extended to infinity)

$K_5| \neq \phi$, where the local parameter of $K_1 + K_2$ is $\zeta = \varphi_1(z)$. Also the triangulation $K_6$ of $G_6$ is so constructed that each 2-simplex $s$ of $K_6$ is natural, minor or major according as $|s| \cap |K_1 + K_7| = \phi$, $|s| \cap |K_7| \neq \phi$ or $|s| \cap |K_1| \neq \phi$, where the local parameter of $K_6$ is $\zeta = \varphi_6(z)$. Further, the triangulation $K_8$ of $G_8$ is so constructed that each 2-simplex $s$ of $K_8$ is natural, minor or major according as $|s| \cap |K_1 + K_2 + K_9| = \phi$, $|s| \cap |K_9| \neq \phi$ or $|s| \cap |K_1 + K_2| \neq \phi$, where the local parameter of $K_8$ is the identity mapping $\varphi_8(z) \equiv z$.

Let $\Psi$ and $\tilde{\Psi}$ be the differentials on the present $\Omega$ defined in § 5.2, and let $\Psi_h'$ and $\tilde{\Psi}_h'$ be the finite element approximations of $\Psi$ and $\tilde{\Psi}$ respectively in the classes $\Lambda_\Psi'(K')$ and $\Lambda_{\tilde{\Psi}}'(K')$ respectively, where $K'$ is the naturalized triangulation associated to the present $K$. To attain our aim it is sufficient to make numerical calculations of $\Psi_h'$ and $\tilde{\Psi}_h'$.

Now the differential $\Psi = du$ is obtained by the following procedure. Let $\Delta$ be the rectangular domain

$$\Delta = \{W \mid 0 < \mathrm{Re}\, W < 1, \quad 0 < \mathrm{Im}\, W < 1\},$$

and let $\gamma_0$ and $\gamma_1$ be the boundary parts of $\Delta$ defined by

$$\gamma_0 = \{W \mid 0 \leqq \mathrm{Im}\, W \leqq 1, \quad \mathrm{Re}\, W = 0\}$$

and

$$\gamma_1 = \{W \mid 0 \leqq \mathrm{Im}\, W \leqq 1, \quad \mathrm{Re}\, W = 1\}.$$

The conformal map $W = f(p)$ such that $\Omega$ is conformally mapped onto $\Delta$ so that $c_0$ and $c_1$ are mapped onto $\gamma_0$ and $\gamma_1$ respectively, is constructed by the composition of the following mappings, and then $u = \mathrm{Re}\, f(p)$:

( i ) $w = \sqrt{z}$;

(ii) $\quad \xi = \left(\dfrac{w - 1}{w + 1}\right)^{2/3};$

(iii) $\quad \dfrac{Z - Z_1}{Z - Z_2} \cdot \dfrac{Z_3 - Z_2}{Z_3 - Z_1} = \dfrac{\xi - \xi_1}{\xi - \xi_2} \cdot \dfrac{\xi_3 - \xi_2}{\xi_3 - \xi_1},$

where $\xi_1 = 0$, $\xi_2 = -1$, $\xi_3 = 1$, $Z_1 = 1$, $Z_2 = -1$ and $Z_3 = 1/k$ with $1/k = 3 + 2\sqrt{2}$;

(iv) $\quad W = -\dfrac{1}{2K}\left(\displaystyle\int_0^Z \frac{dZ}{\sqrt{(1 - Z^2)(1 - k^2 Z^2)}} - (K + iK')\right),$

where $K = K(k)$ and $K' = K'(k)$ are the complete elliptic integrals.

Table 5 shows the values of our finite element approximations. Furthermore, computation results for the normal subdivision $K^1$ of the present $K$ are shown. It can be said that the both of upper and lower bounds of $M(Q)$ are close to the exact values.

Table 5 Modulus M(Q) of example 4
(the case of a Riemann surface)

| | | | |
|---|---|---|---|
| Exact value | $M(Q) = \|\psi\|^2 = 1.0$ | | |
| Finite element approximations | Original triangulation (h = 0.141421) | | |
| | Upper bound | $\|\psi'_h\|^2 + \varepsilon(\psi'_h)$ <br> $= 1.00484 + 0.103287 \times 10^{-2}$ <br> $= 1.00587$ (0.00587) | $\|\psi'_h - \hat{\psi}'\|$ <br> $= 1.88104 \times 10^{-2}$ |
| | Lower bound | $\dfrac{1}{\|\tilde{\psi}'_h\|^2 + \varepsilon(\tilde{\psi}'_h)}$ <br> $= \dfrac{1}{1.00484 + 0.103287 \times 10^{-2}}$ <br> $= 0.994164$ (-0.005836) | $\|\tilde{\psi}'_h - \hat{\tilde{\psi}}'\|$ <br> $= 1.88102 \times 10^{-2}$ |
| | Normal subdivision (h = 0.0707107) | | |
| | Upper bound | $\|\psi'_h\|^2 + \varepsilon(\psi'_h)$ <br> $= 1.00128 + 0.255952 \times 10^{-3}$ <br> $= 1.00154$ (0.00154) | $\|\psi'_h - \hat{\psi}'\|$ <br> $= 5.84884 \times 10^{-3}$ |
| | Lower bound | $\dfrac{1}{\|\tilde{\psi}'_h\|^2 + \varepsilon(\tilde{\psi}'_h)}$ <br> $= \dfrac{1}{1.00128 + 0.255957 \times 10^{-3}}$ <br> $= 0.998466$ (-0.001534) | $\|\tilde{\psi}'_h - \hat{\tilde{\psi}}'\|$ <br> $= 5.85420 \times 10^{-3}$ |

( ): Deviation from exact value.

§ 5.5. Numerical example 5 (the case of an unbounded domain; cf. example 1). Let $\Omega = \{z \mid y > 0\}$, and let $c_0$ and $c_1$ be the boundary parts of $\Omega$ defined by $c_0 = \{z \mid -3 \leqq x \leqq -1,\ y = 0\}$ and $c_1 = \{z \mid 1 \leqq x \leqq 3,\ y = 0\}$ respectively, where $z = x + iy$. Let $Q$ be the quadrilateral with the two opposite sides $c_0$ and $c_1$ (cf. Fig. 18). We obtain good upper and lower approximate values of the modulus of $Q$. See example 1 for the details. Table 6 shows

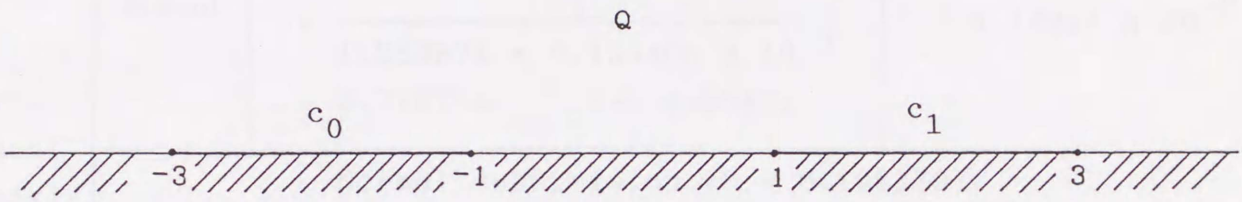



Fig. 18 Numerical example 5 (the case of an unbounded domain)

the exact value of the modulus $M(Q)$ which can be calculated by making use of a complete elliptic integral, and the values of our finite element approximations.

Table 6  Modulus $M(Q)$ of example 5
(the case of an unbounded domain)

| | | | |
|---|---|---|---|
| Exact value | $M(Q) = \|\psi\|^2 = 0.781701$ | | |
| Finite element approximations | Original triangulation ($h = 0.213758$) | | |
| | Upper bound | $\|\psi'_h\|^2 + \varepsilon(\psi'_h)$ <br> $= 0.782184 + 0.429347 \times 10^{-3}$ <br> $= 0.782613$ (0.000912) | $\|\psi'_h - \hat{\psi}'\|$ <br> $= 3.76256 \times 10^{-3}$ |
| | Lower bound | $\dfrac{1}{\|\tilde{\varphi}'_h\|^2 + \varepsilon(\tilde{\varphi}'_h)}$ <br> $= \dfrac{1}{1.280878 + 0.150405 \times 10^{-5}}$ <br> $= 0.780714$ (-0.000987) | $\|\tilde{\varphi}'_h - \hat{\tilde{\varphi}}'\|$ <br> $= 6.14254 \times 10^{-3}$ |
| | Normal subdivision ($h = 0.106879$) | | |
| | Upper bound | $\|\psi'_h\|^2 + \varepsilon(\psi'_h)$ <br> $= 0.781968 + 0.107413 \times 10^{-3}$ <br> $= 0.782075$ (0.000374) | $\|\psi'_h - \hat{\psi}'\|$ <br> $= 1.12050 \times 10^{-3}$ |
| | Lower bound | $\dfrac{1}{\|\tilde{\varphi}'_h\|^2 + \varepsilon(\tilde{\varphi}'_h)}$ <br> $= \dfrac{1}{1.279506 + 0.381486 \times 10^{-6}}$ <br> $= 0.781551$ (-0.000150) | $\|\tilde{\varphi}'_h - \hat{\tilde{\varphi}}'\|$ <br> $= 1.83821 \times 10^{-3}$ |

( ): Deviation from exact value.

§ 5.6. Numerical example 6 (the case of a curvilinear domain; cf. example 2). Let

$$\Omega = \left\{ z \;\middle|\; \frac{x^2}{16} + \frac{y^2}{15} < 1,\ y > 0 \right\},$$

and let $c_0$ and $c_1$ be the boundary parts of $\Omega$ defined by

$$c_0 = \{z \mid 3 \leqq x \leqq 4,\ y = 0\} \cup \left\{ z \;\middle|\; \frac{x^2}{16} + \frac{y^2}{15} = 1,\ y \geqq 0 \right\}$$

and

$$c_1 = \{z \mid -1 \leqq x \leqq 1,\ y = 0\}$$

respectively, where $z = x + iy$. Let $Q$ be the quadrilateral with the opposite sides $c_0$ and $c_1$ (cf. Fig. 19).

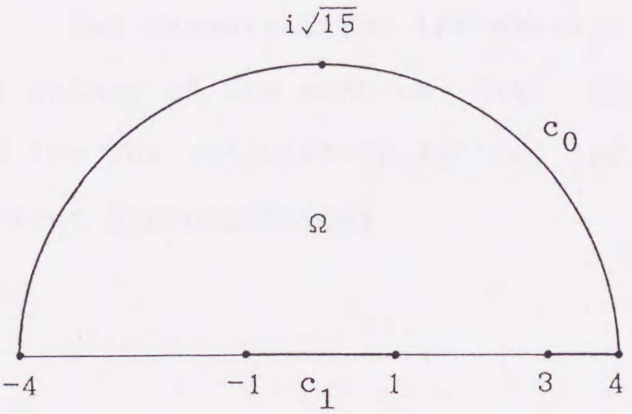



Fig. 19 Numerical example 6 (the case of a curvilinear domain: quadrilateral Q)

Further, let $c_0'$ and $c_1'$ be the boundary parts of $\Omega$ defined by

$$c_0' = \{z \mid 1 \leqq x \leqq 3,\ y = 0\}$$

and

$$c_1' = \{z \mid -4 \leqq x \leqq -1,\ y = 0\} \cup \left\{ z \;\middle|\; \frac{x^2}{16} + \frac{y^2}{15} = 1,\ y \geqq 0 \right\}$$

respectively, where $z = x + iy$. Let $Q'$ be the quadrilateral with the opposite sides $c'_0$ and $c'_1$ (cf. Fig. 20).

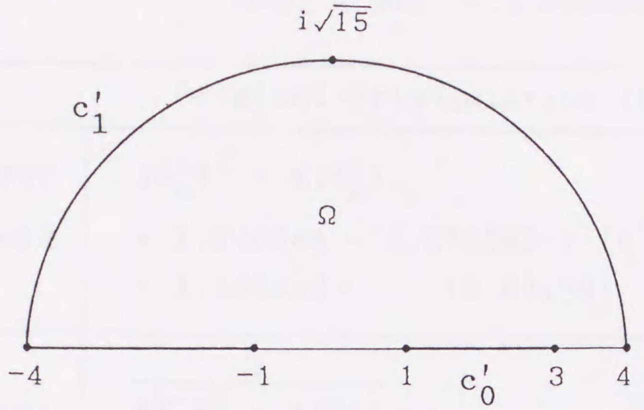



Fig. 20 Numerical example 6 (the case of a curvilinear domain: quadrilateral $Q'$)

We obtain good upper and lower approximate values of the modulus of $Q$ and $Q'$. See example 2 for the details. Tables 7 and 8 show the exact values of the modulus $M(Q)$ and $M(Q')$ respectively (see example 2 for the calculation method) and the values of our finite element approximations.

Table 7 Modulus M(Q) of example 6
(the case of a curvilinear domain)

| | | | |
|---|---|---|---|
| Exact value | $M(Q) = \|\psi\|^2 = 1.539330$ | | |
| Finite element approximations | Original triangulation (h = 0.138840) | | |
| | Upper bound | $\|\psi'_h\|^2 + \varepsilon(\psi'_h)$<br>$= 1.540588 + 0.572262 \times 10^{-4}$<br>$= 1.540645$ (0.00132) | $\|\psi'_h - \hat{\varphi}'\|$<br>$= 1.15335 \times 10^{-2}$ |
| | Lower bound | $\dfrac{1}{\|\tilde{\varphi}'_h\|^2 + \varepsilon(\tilde{\varphi}'_h)}$<br>$= \dfrac{1}{0.649700 + 0.225117 \times 10^{-3}}$<br>$= 1.538639$ (-0.00069) | $\|\tilde{\varphi}'_h - \hat{\tilde{\varphi}}'\|$<br>$= 3.74131 \times 10^{-3}$ |
| | Normal subdivision (h = 0.069420) | | |
| | Upper bound | $\|\psi'_h\|^2 + \varepsilon(\psi'_h)$<br>$= 1.539652 + 0.142916 \times 10^{-4}$<br>$= 1.539666$ (0.00034) | $\|\psi'_h - \hat{\varphi}'\|$<br>$= 5.89447 \times 10^{-3}$ |
| | Lower bound | $\dfrac{1}{\|\tilde{\varphi}'_h\|^2 + \varepsilon(\tilde{\varphi}'_h)}$<br>$= \dfrac{1}{0.649652 + 0.558093 \times 10^{-4}}$<br>$= 1.539153$ (-0.00018) | $\|\tilde{\varphi}'_h - \hat{\tilde{\varphi}}'\|$<br>$= 1.09209 \times 10^{-3}$ |

( ): Deviation from exact value.

Table 8 Modulus $M(Q')$ of example 6
(the case of a curvilinear domain)

| | | | |
|---|---|---|---|
| Exact value | $M(Q') = \|\psi\|^2 = 1.839350$ | | |
| | Original triangulation ($h = 0.138840$) | | |
| | Upper bound | $\|\psi_h'\|^2 + \varepsilon(\psi_h')$ <br> $= 1.841976 + 0.351532 \times 10^{-3}$ <br> $= 1.842328$ (0.00298) | $\|\psi_h' - \hat{\varphi}'\|$ <br> $= 7.65797 \times 10^{-3}$ |
| | Lower bound | $\dfrac{1}{\|\tilde{\varphi}_h'\|^2 + \varepsilon(\tilde{\varphi}_h')}$ <br> $= \dfrac{1}{0.544588 + 0.145580 \times 10^{-3}}$ <br> $= 1.835760$ (-0.00359) | $\|\tilde{\varphi}_h' - \hat{\varphi}'\|$ <br> $= 5.22574 \times 10^{-3}$ |
| Finite element approximations | Normal subdivision ($h = 0.069420$) | | |
| | Upper bound | $\|\psi_h'\|^2 + \varepsilon(\psi_h')$ <br> $= 1.840016 + 0.875764 \times 10^{-4}$ <br> $= 1.840104$ (0.00075) | $\|\psi_h' - \hat{\varphi}'\|$ <br> $= 2.28613 \times 10^{-3}$ |
| | Lower bound | $\dfrac{1}{\|\tilde{\varphi}_h'\|^2 + \varepsilon(\tilde{\varphi}_h')}$ <br> $= \dfrac{1}{0.543904 + 0.361871 \times 10^{-4}}$ <br> $= 1.838437$ (-0.00091) | $\|\tilde{\varphi}_h' - \hat{\varphi}'\|$ <br> $= 1.73332 \times 10^{-3}$ |

( ): Deviation from exact value.

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