A Stochastic Model on An Additional Warranty Service Contract

KOICHIRO RINSAKA and HIROAKI SANDOH

Department of Information Engineering, Hiroshima University,

1-4-1, Kagamiyama, Higashi-Hiroshima 739–8527, Japan

E-mail: rinsaka@rel.hiroshima-u.ac.jp

Department of Business Administration,

Kobe Gakuin University,

518 Arise, Ikawadani-cho, Nishi-ku, Kobe 651–2180, Japan

E-mail: sandoh@ba.kobegakuin.ac.jp

Abstract– In general, a newly purchased item or system is warranted for a specific period. When the system fails during the warranty period, it is repaired free of charge. Even if the system is repairable, there exists some warranty services under which the manufacturer replaces the failed system during the warranty period. This study considers a case where a manufacturer offers an additional warranty service under which the failed system is replaced by a new one for its first failure, but minimal repairs are carried out to the system for its succeeding failures before the warranty expires. In this paper, we propose a mathematical model for setting a suitable charge of such an additional warranty service. Numerical examples assuming a personal computer are also presented.

Keywords– Warranty Contract, Replacement, Expected Utility, Expected Profit, Optimal Strategy.

1. INTRODUCTION

Studies on warranty have been conducted in these decades, and a large number of warranty policies have also been investigated [1–11]. Most of them analyze warranty cost after both the warranty policy and the reliability of the objective system are specified. There also exist a few studies, which deal with optimization problems on the warranty fee or the length of warranty period. Murthy and Asgharizadeh [12] and Asgharizadeh and Murthy [13] have treated the contract price optimization problem of a maintenance service contract. Sandoh and Rinsaka [14] have dealt with the same problem for the software system.

From the customer's viewpoint, Iskandar and Sandoh [15], and Iskandar, Klefsjö and Sandoh [16] have discussed an opportunistic preventive replacement policy under a general warranty. Rinsaka, Sandoh and Nakagawa [17] have considered a preventive replacement policy under the additional warranty service under which, throughout the warranty period, (1) the manufacturer copes with the first failure of the objective system by replacing it with a new one free of charge, but (2) he conducts minimal repairs to its succeeding failures without charge during the warranty period, and finally (3) he performs minimal repairs to the failures at a suitable fee after the warranty expires.

This paper deals with the same additional warranty service and discusses an optimal price setting of such an additional warranty service fee. The structure of this paper is as follows. Section **2** explains the options of the warranty proposed in this study, and the assumptions of the model. In Section **3**, the customer's monetary returns under each option are formulated to obtain his expected utilities. In Section **4**, the manufacturer's expected profits are formulated under each option. In Section **5**, after discussed is the customer's optimal strategy that maximizes his expected utility, the manufacturer's optimal strategy maximizing his expected profit is discussed. Section **6** discusses characteristics of proposed model, by showing numerical examples assuming a personal computer.

2. MODEL DESCRIPTION

Let us consider a case where a customer purchases a system at cost $P_s(> 0)$ and the system is accompanied by a base warranty with a warranty period $(0, \tau](\tau > 0)$. Under the base warranty, if the system fails during the warranty period, the manufacturer will conduct minimal repairs [18, 19] to the failed system without charge. After the warranty expires, he performs a minimal repair to the failed system by charging the customer for a fee $C_s(> 0)$. The average cost $C_r(0 < C_r < C_s)$ is incurred to the manufacturer per minimal repair activity.

We consider the two options as follows:

Option A_1 At an additional expense under this option, the customer can receive the warranty services as follows: Throughout the warranty period $(0, \tau]$, (1) the manufacturer copes with the first failure of the system by replacing it with a new one free of charge, but (2) he conducts minimal repairs to its succeeding failures without charge. After the warranty expires, the manufacturer carries out minimal repairs to the failed system and charges the customer for C_s per minimal repair activity.

Option A_2 Under this option, the customer pays no additional fee. He can receive the base warranty service mentioned above.

The customer's choice between Options A_1 and A_2 is influenced by the price structure and the attitude of the customer against risk. The customer would select an option yielding a larger value of his expected utility. If his expected utility should be negative under both Options A_1 and A_2, the customer would alternatively choose the following option.

Option A_0 The customer does not purchase the system under this option.

The optimal choice for the customer is based on maximizing the expected utility function. We assume that it is given by

$$U(\omega) = \frac{1 - e^{-\beta\omega}}{\beta}, \quad \beta > 0 \tag{1}$$

where $U(\omega)$ is the utility associated with a wealth of ω. The advantage of this utility function is that the initial wealth is of no importance. Note that this captures the attitude to risk. The risk aversion increases with β.

Let us consider the case where the customer chooses Option A_1. When the system fails for the first time during the warranty period, it is replaced by a new one by the manufacturer free of charge. In addition, if the replaced system fails again before the warranty expires, a minimal repair is conducted free of charge. Let us denote, by N_1, the number of failures after the warranty expires, and N_1 satisfies

$$\Pr\{N_1 = n\} = \begin{cases} \frac{[H(T-X_1)-H(\tau-X_1)]^n}{n!} e^{-[H(T-X_1)-H(\tau-X_1)]} & X_1 \le \tau,\ n = 0,1,2,\cdots \\ \frac{[H(T)-H(\tau)]^n}{n!} e^{-[H(T)-H(\tau)]} & X_1 > \tau,\ n = 0,1,2,\cdots \end{cases} \quad (2)$$

where $H(\cdot)$ is a mean value function of a nonhomogeneous Poisson process and $X_1(> 0)$ is a random variable expressing time to the first failure. It is assumed that both replace and repair time are negligible.

We consider the case when the customer chooses Option A_2. Whenever the system fails before the warranty expires, the base warranty service is applied to the failed system, that is, a minimal repair is executed to the failed system free of charge. Let N_2 denote the number of failures after the warranty expired, then N_2 satisfies

$$\Pr\{N_2 = n\} = \frac{[H(T) - H(\tau)]^n}{n!} e^{-[H(T)-H(\tau)]} \quad n = 0,1,2,\cdots. \quad (3)$$

In the following, we assume

$$[H'(t)]' = h'(t) > 0. \quad (4)$$

Inequality (4) signifies that the system becomes easy to fail with progress of time.

3. CUSTOMER'S EXPECTED UTILITY

Let $R(> 0)$ denote a revenue per unit of time the customer can receive by operating the system. The customer's monetary return under Option A_1 is given by

$$\omega(A_1) = RT - P_s - P_a - C_s N_1, \quad (5)$$

while the customer's monetary return under Option A_2 is given by

$$\omega(A_2) = RT - P_s - C_s N_2. \quad (6)$$

Under Option A_0, it is given by

$$\omega(A_0) = 0. \quad (7)$$

From Eqs.(1),(2) and (5), the customer's expected utility under Option A_1 becomes

$$\begin{aligned} E[U(A_1;P_a,C_s)] &= \frac{1}{\beta}\Bigg\{1-e^{-\beta(RT-P_s-P_a)}\Bigg[\int_0^{\tau} e^{-[H(T-x)-H(\tau-x)]\left(1-e^{\beta C_s}\right)}dF(x) \\ &\quad + e^{-[H(T)-H(\tau)]\left(1-e^{\beta C_s}\right)}\overline{F}(\tau)\Bigg]\Bigg\}, \end{aligned} \tag{8}$$

where

$$F(x) = 1-e^{-H(x)}, \tag{9}$$

$$\overline{F}(x) = 1-F(x), \tag{10}$$

$$f(x) = \frac{dF(x)}{dx}. \tag{11}$$

From Eqs.(1), (3), (6) and (7), likewise the customer's expected utility under Options A_2 and A_0 respectively become

$$E[U(A_2;P_a,C_s)] = \frac{1}{\beta}\left[1-e^{-\beta(RT-P_s)-[H(T)-H(\tau)]\left(1-e^{\beta C_s}\right)}\right] \tag{12}$$

and

$$E[U(A_0;P_a,C_s)] = 0. \tag{13}$$

4. MANUFACTURER'S EXPECTED PROFIT

This section formulates the manufacturer's expected profit which depends on both the manufacturer's decision and the customer's. It is assumed in the following that the manufacturer is risk neutral and is interested in maximizing his expected profit.

If the customer selects Option A_1, the expected number of minimal repairs which the manufacturer carries out free of charge during the warranty period is given by

$$\int_0^{\tau} H(\tau-x)dF(x). \tag{14}$$

The expected number of minimal repair after the warranty period is given by

$$\int_0^{\tau}[H(T-x)-H(\tau-x)]dF(x)+[H(T)-H(\tau)]\overline{F}(\tau). \tag{15}$$

Hence, the manufacturer's expected profit under Option A_1 is written as

$E[\pi(P_a,C_s;A_1)]$

$$
\begin{aligned}
&= P_s + P_a - P_s' - (P_s' - P_v)\, F(\tau) - C_r \int_0^{\tau} H(\tau - x) dF(x) \\
&\quad + (C_s - C_r) \left\{ \int_0^{\tau} [H(T - x) - H(\tau - x)] dF(x) + [H(T) - H(\tau)] \overline{F}(\tau) \right\}, \quad (16)
\end{aligned}
$$

where $P_s'(> 0)$ is the prime cost, and $P_v(> 0)$ signifies the salvage value at the time of replacement. Parameter P_v is introduced because the manufacturer sells each individual failed system as a second-hand product after repair .

On the other hand, if the customer chooses Option A_2, the manufacturer's expected profit is given by

$$
E[\pi(P_a, C_s; A_2)] = P_s - P_s' - C_r H(\tau) + (C_s - C_r)\, [H(T) - H(\tau)]. \quad (17)
$$

If the customer chooses Option A_0, the manufacturer's expected profit becomes

$$
E[\pi(P_a, C_s; A_0)] = 0. \quad (18)
$$

In the above we have derived the manufacturer's expected profit for each case where the customer chooses Option A_k for $k = 0, 1, 2$.

5. OPTIMAL STRATEGY

This section discusses the optimal strategy of the customer by maximizing his expected utility in Eqs.(8), (12) and (13), and then we seek for the optimal strategy for the manufacturer.

5.1 Customer's Optimal Strategy

We first compare Option A_1 with A_2. Option A_1 is preferred to Option A_2 if $E[U(A_1; P_a, C_s)] > E[U(A_2; P_a, C_s)]$, and if $E[U(A_1; P_a, C_s)] < E[U(A_2; P_a, C_s)]$, A_2 is preferred to A_1. The customer is indifferent between two options if $E[U(A_1; P_a, C_s)] = E[U(A_2; P_a, C_s)]$, which is equivalent to

$$
P_a = \frac{1}{\beta} \left\{ -\xi \left(1 - e^{\beta C_s}\right) - \ln \left[\int_0^{\tau} e^{-\rho(x)\left(1 - e^{\beta C_s}\right)} dF(x) + e^{-\xi\left(1 - e^{\beta C_s}\right)} \overline{F}(\tau) \right] \right\}, \quad (19)
$$

where

$$\rho(x) \equiv H(T-x) - H(\tau - x) \tag{20}$$

and

$$\xi \equiv H(T) - H(\tau). \tag{21}$$

Let $\Psi_1(C_s)$ express the right-hand-side of Eq.(19).

Secondly, Option A_1 is compared with A_0. Option A_1 is better than Option A_0 for the customer if $E[U(A_1; P_a, C_s)] > 0$, while if $E[U(A_1; P_a, C_s)] < 0$, Option A_0 is preferred. By solving $E[U(A_1; P_a, C_s)] = 0$ with respect to P_a and letting $P_a = \Psi_2(C_s)$ denote its solution, we have, as reservation price for the system,

$$\Psi_2(C_s) = RT - P_s - \frac{1}{\beta} \ln \left[\int_0^{\tau} e^{-\rho(x)\left(1-e^{\beta C_s}\right)} dF(x) + e^{-\xi\left(1-e^{\beta C_s}\right)} \overline{F}(\tau) \right]. \tag{22}$$

Thirdly, we make a comparison of Options A_2 and A_0. Between Options A_0 and A_2, the solution to $E[U(A_2; P_a, C_s)] = 0$ with respect to C_s is given by

$$\overline{C_s} = \frac{1}{\beta} \ln \left[\frac{\beta(RT - P_s)}{\xi} + 1 \right] \tag{23}$$

as a reservation price for C_s.

Let $\Omega_i (i = 0, 1, 2)$ be defined by

$$\Omega_0 = \left\{ (P_a, C_s); P_a \geq \Psi_2(C_s), C_s \geq \overline{C_s} \right\}, \tag{24}$$

$$\Omega_1 = \left\{ (P_a, C_s); P_a < \Psi_1(C_s), P_a < \Psi_2(C_s) \right\}, \tag{25}$$

$$\Omega_2 = \left\{ (P_a, C_s); P_a \geq \Psi_1(C_s), C_s < \overline{C_s} \right\}, \tag{26}$$

and the optimal strategy of the customer becomes

$$A^*(P_a, C_s) = \begin{cases} A_0, & \text{if } (P_a, C_s) \in \Omega_0 \\ A_1, & \text{if } (P_a, C_s) \in \Omega_1 \\ A_2, & \text{if } (P_a, C_s) \in \Omega_2 \end{cases}. \tag{27}$$

Figure 1 shows the characterization of customer's optimal actions.

5.2 MANUFACTURER'S OPTIMAL STRATEGY

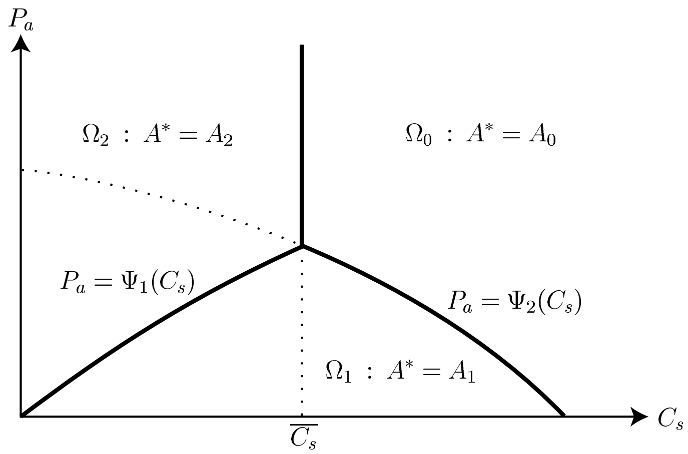

Figure 1: Characterization of customer's optimal actions.

The manufacturer's optimal strategy for P_a and C_s is obtained by maximizing his expected profit considering the customer's optimal strategy $A^*(P_a, C_s)$.

For $(P_a, C_s) \in \Omega_1$, the customer's optimal option is A_1. In this case, the manufacturer's expected profit is given by Eq.(16). Since $\partial E[\pi(P_a, C_s; A_1)]/\partial P_a > 0$ and $\partial E[\pi(P_a, C_s; A_1)]/\partial C_s > 0$, the manufacturer's expected profit under Option A_1 becomes the maximum by a certain point on the curve $P_a = \Psi_2(C_s)$(See Fig.1). By substituting $P_a = \Psi_2(C_s)$ given by Eq.(22) for Eq.(16), the expected profit $\Pi(C_s)$ on $P_a = \Psi_2(C_s)$ becomes

$$\begin{aligned}
&\Pi(C_s) \\
&\quad = \; RT - P_s' - (P_s' - P_v)\, F(\tau) - \frac{1}{\beta} \ln \left[\int_0^\tau e^{-\rho(x)\left(1-e^{\beta C_s}\right)} dF(x) + e^{-\xi\left(1-e^{\beta C_s}\right)} \overline{F}(\tau) \right] \\
&\qquad - C_r \int_0^\tau H(\tau - x) dF(x) + (C_s - C_r) \left[\int_0^\tau \rho(x) dF(x) + \xi \overline{F}(\tau) \right].
\end{aligned} \tag{28}$$

By differentiating $\Pi(C_s)$ with respect to C_s, $\Pi'(C_s) \le 0$ agrees with

$$e^{\beta C_s} \frac{\int_0^\tau \rho(x) e^{-\rho(x)\left(1-e^{\beta C_s}\right)} dF(x) + \xi e^{-\xi\left(1-e^{\beta C_s}\right)} \overline{F}(\tau)}{\int_0^\tau e^{-\rho(x)\left(1-e^{\beta C_s}\right)} dF(x) + e^{-\xi\left(1-e^{\beta C_s}\right)} \overline{F}(\tau)} \ge \int_0^\tau \rho(x) dF(x) + \xi \overline{F}(\tau). \tag{29}$$

Let $L(C_s)$ denote the left-hand-side of Inequality(29). Then, we have

$$L(0) = \int_0^\tau \rho(x) dF(x) + \xi \overline{F}(\tau), \tag{30}$$

and $L'(C_s) > 0$ from Appendix **A**. It follows that $\Pi(C_s)$ is strictly decreasing in C_s. The maximum expected profit is, therefore, obtained for $C_s^* \to \overline{C_s} + 0$ and $P_a^* \to \Psi_2(C_s^*) - 0$.

For $(P_a, C_s) \in \Omega_2$, the customer's optimal option is A_2. In this case, the manufacturer's expected profit is given by Eq.(17). The maximum expected profit is obtained for $C_s^* \to \overline{C_s} - 0$ and $P_a^* > \Psi_2(C_s^*)$.

Finally, for $(P_a, C_s) \in \Omega_0$, the customer's optimal option is A_0 and the manufacturer's expected profit is given by Eq.(18). In this case, the manufacturer cannot control his own expected profit.

It can easily be shown that the manufacturer must select between either $[C_s^* \to \overline{C_s} + 0$ and $P_a^* \to \Psi_2(C_s^*) - 0]$ or $[C_s^* \to \overline{C_s} - 0$ and $P_a^* > \Psi_2(C_s^*)]$ to maximize his expected profit if one or both of these provide a positive expected profit. The optimal choice is the one that gives a positive larger value for the expected profit. If both are negative, then the best strategy is to have $C_s^* > \overline{C_s}$ and $P_a^* > \Psi_2(C_s^*)$ so that the customer may choose Option A_0 and the manufacturer's expected profit is given by Eq.(18).

6. NUMERICAL EXAMPLES

In the above, the customer's and the manufacturer's optimal strategies were discussed for the additional warranty service. This section examines the characteristics of the proposed model through the numerical examples assuming personal computer systems. We apply the following function as the mean value function of nonhomogeneous Poisson process:

$$H(t) = \lambda t^m, \quad t \geq 0, \ \lambda > 0, \ m > 1. \tag{31}$$

This function is introduced due to its simple structure satisfying Inequality (4).

In the following, we set up the parameters considering personal computers as an objective system. The case of $\tau = 1$ (year., e.g.) is considered. Table 1 shows the case considered here, and Fig.2 reveals the characterization of customer's optimal actions.

Table 2 indicates the optimal strategies for the customer and the manufacturer. We can observe in Table 2 that the reservation price $\overline{C_s}$ of C_s decreases with increasing λ, which is also obtained by differentiation $\overline{C_s}$ in reference to λ through ξ. It is also observed in Table 2 that the manufacturer's maximum expected profit $MEP(A_i)$ decreases with increasing λ, where $MEP(A_i)$ is given by

$$MEP(A_i) \equiv \max_{(P_a, C_s) \in \Omega_i} E[\pi(P_a, C_s; A_i)], \quad i = 1, 2. \tag{32}$$

These observations signify that the manufacturer should decrease the charge of each minimal repair activity along with his maximum expected profit when the reliability of his system is low.

Figure 3 shows sensitivities of the manufacturer's maximum expected profit under Options A_1 and A_2 when λ increases. In Fig.3, $MEP(A_2)$ is slightly larger than $MEP(A_1)$ for small values of λ, but $MEP(A_2)$ becomes smaller than $MEP(A_1)$ for large values of λ although $MEP(A_i)$ for $i = 1, 2$ turns to be negative as λ becomes large.

These tendencies can be explained as follows: For the high-reliability system, it is difficult for the manufacturer to raise his own expected profit even if he provides the customer with Options A_1 and A_2. However, he can raise his own expected profit by providing Options A_1 and A_2 if the system has a suitably low reliability.

Table 1: Case.

Case	a	b	c
λ	0.15	0.25	0.35
β	0.1		
m	2.0		
τ	1		
T	5		
R	7		
P_s	15		
P'_s	10		
P_v	4.3		
C_r	3		

7. CONCLUDING REMARKS

This study discussed an additional warranty service where the manufacturer copes with the first failure of the system by replacing it with a new one, but he conducts minimal repairs to its succeeding failures before the warranty expires. For such a service, we

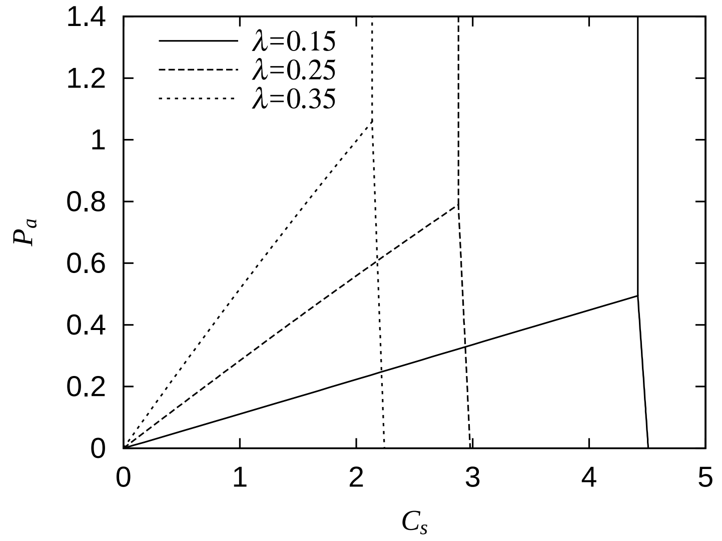

Figure 2: Characterization of customer's optimal actions.

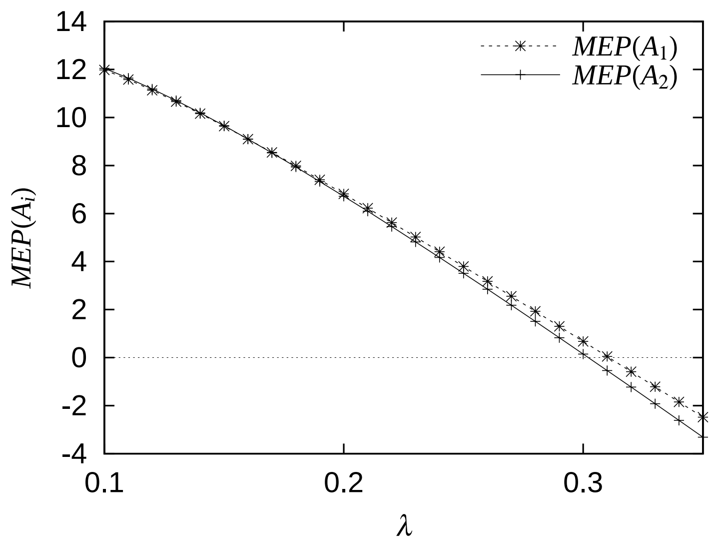

Figure 3: Manufacturer's expected profits.

Table 2: Optimal strategies.

Case	a	b	c
λ	0.15	0.25	0.35
$\overline{C_s}$	4.41833	2.87682	2.13574
$\Psi_2(\overline{C_s})$	0.49421	0.78948	1.05872
$MEP(A_1)$	9.63963	3.79523	-2.48191
$MEP(A_2)$	9.65598	3.51092	-3.30978
P_a^* , C_s^*	$C_s^* \to \overline{C_s} - 0$ & $P_a^* > \Psi_2(C_s^*)$	$C_s^* \to \overline{C_s} + 0$ & $P_a^* \to \Psi_2(C_s^*) - 0$	$C_s^* > \overline{C_s}$ & $P_a^* > \Psi_2(C_s^*)$
A^*	A_2	A_1	A_0
$E[\pi(P_a^*, C_s^*; A^*)]$	9.65598	3.79523	0.00000

proposed a mathematical model to determine optimal strategies of the manufacturer and the customer.

In this paper, we considered a warranty service that the manufacturer provides his customer with a service of replacing the first system failure by a new one and carrying out minimal repairs to the succeeding failures before the warranty expires. We can, however, extend our model so that the manufacturer replaces the system with a new one for its first k failures and conducts minimal repairs to the succeeding failures, although the analysis becomes very complicated.

In recent years, retailers also provide customers with warranty services which are slightly different from those by manufacturers. Mathematical models to deal with these problems are under investigation.

REFERENCES

[1] W.R. Blischke and D.N.P. Murthy, *Warranty cost analysis*, Marcel Dekker, New York (1994).

[2] H.N. Amato and E.E. Anderson, Determination of warranty reserves: An extension, *Manage. Sci.*, **22**, 1391–1394 (1976).

[3] H.N. Amato, E.E. Anderson, and D.W. Harvey, A general model of future period warranty costs, *The Accounting Review*, **51**, 854–862 (1976).

[4] Y. Balcer and I. Sahin, Replacement costs under warranty: Cost moments and time variability, *Oper. Res.*, **34**, 554–559 (1986).

[5] W.R. Blischke and E.M. Scheuer, Calculation of the cost of warranty policies as a function of estimated life distributions, *Nav. Res. Logist. Q.*, **22**, 681–696 (1975).

[6] W.R. Blischke and E.M. Scheuer, Applications of renewal theory in analysis of the free-replacement warranty, *Nav. Res. Logist. Q.*, **28**, 193–205 (1981).

[7] J.W. Mamer, Cost analysis of pro rata and free-replacement warranties, *Nav. Res. Logist. Q.*, **29**, 345–356 (1982).

[8] J.W. Mamer, Discounted and per unit costs of product warranty, *Manage. Sci.*, **33**, 916–930 (1987).

[9] W.W. Menke, Determination of warranty reserves, *Manage. Sci.*, **15**, B542–B549 (1969).

[10] D.G. Nguyen and D.N.P. Murthy, An optimal policy for servicing warranty, *Journal of the Operational Research Society*, **37**, 1081–1088 (1986).

[11] J.G. Patankar and G.H. Worm, Prediction intervals for warranty reserves and cash flows, *Manage. Sci.*, **27**, 237–241 (1981).

[12] D.N.P. Murthy and E. Asgharizadeh, Optimal decision making in a maintenance service operation, *European Journal of Operational Research*, **116**, 259–273 (1999).

[13] E. Asgharizadeh and D.N.P. Murthy, Service contracts: A stochastic model, *Mathematical and Computer Modelling*, **31**, 11–20 (2000).

[14] H. Sandoh and K. Rinsaka, Maintenance service contract model for software, *Proc. of the First Western Pacific and Third Australia-Japan Workshop on Stochastic Models in Engineering, Technology and Management*, ed. R.J. Wilson, S. Osaki, and M.J. Faddy, 466–475, Christchurch, New Zealand (1999).

[15] B.P. Iskandar and H. Sandoh, An opportunity-based age replacement policy considering warranty, *International Journal of Reliability, Quality and Safety Engineering*, **6**, 229–236 (1999).

[16] B.P. Iskandar, B. Klefsjö and H. Sandoh, An opportunity-based age replacement policy with warranty analyzed by using TTT-transforms, *International Journal of Reliability and Applications*, **1**, 27–38 (2000).

[17] K. Rinsaka, H. Sandoh, and T. Nakagawa, Preventive replacement policy under warranty with one corrective replacement and minimal repairs, *Proc. of 10th International Symposium on Applied Stochastic Models and Data Analysis*, G. Govaert, J. Janssen, N. Limnios, eds., **2**, 887–892, Compiégne, France (2001).

[18] R.E. Barlow and L.C. Hunter, Optimum preventive maintenance policies, *Oper. Res.*, **8**, 90–100 (1960).

[19] R.E. Barlow and F. Proshchan, *Mathematical Theory of Reliability*, John Wiley & Sons, New York (1965).

A. Proof of $L'(C_s) > 0$

By differentiating $L(C_s)$ in Eq.(28) with respect to C_s, we have

$$L'(C_s) = \beta e^{\beta C_s} \frac{\left\{ \begin{array}{l} [\int_0^\tau \rho(x) e^{-\rho(x)\left(1-e^{\beta C_s}\right)} dF(x) + \xi e^{-\xi\left(1-e^{\beta C_s}\right)} \overline{F}(\tau)] \\ \quad \times [\int_0^\tau e^{-\rho(x)\left(1-e^{\beta C_s}\right)} dF(x) + e^{-\xi\left(1-e^{\beta C_s}\right)} \overline{F}(\tau)] \\ +e^{\beta C_s} \Big\{ e^{-\xi\left(1-e^{\beta C_s}\right)} \overline{F}(\tau) \times \int_0^\tau [\xi - \rho(x)]^2 e^{-\rho(x)\left(1-e^{\beta C_s}\right)} dF(x) \\ \quad + \int_0^\tau \rho^2(x) e^{-\rho(x)\left(1-e^{\beta C_s}\right)} dF(x) \times \int_0^\tau e^{-\rho(x)\left(1-e^{\beta C_s}\right)} dF(x) \\ \quad - [\int_0^\tau \rho(x) e^{-\rho(x)\left(1-e^{\beta C_s}\right)} dF(x)]^2 \Big\} \end{array} \right\}}{\left[\int_0^\tau e^{-\rho(x)\left(1-e^{\beta C_s}\right)} dF(x) + e^{-\xi\left(1-e^{\beta C_s}\right)} \overline{F}(\tau)\right]^2}. \quad \text{(A1)}$$

Let $Q(\tau)$ be defined by

$$Q(\tau) \equiv \int_0^\tau \rho^2(x) D(x) dx \times \int_0^\tau D(x) dx - \left[\int_0^\tau \rho(x) D(x) dx\right]^2, \quad \text{(A2)}$$

where

$$D(x) \equiv e^{-\rho(x)\left(1-e^{\beta C_s}\right)} f(x), \quad \text{(A3)}$$

and we clearly have $Q(0) = 0$. We also have

$$\begin{aligned} Q'(\tau) &= D(\tau) \int_0^\tau D(x) \left[\rho^2(x) + \rho^2(\tau) - 2\rho(x)\rho(\tau)\right] dx \\ &= D(\tau) \int_0^\tau D(x) \left[\rho(x) - \rho(\tau)\right]^2 dx \\ &\geq 0. \end{aligned} \tag{A4}$$

Since $Q(\tau)$ is an increasing function of τ, we have $Q(\tau) \geq 0$ and thus $L'(C_s) \geq 0$. Consequently $L(C_s)$ is increasing in C_s.