Local Uniqueness for Nash Solutions of Multiparameter Singularly Perturbed Systems

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***Abstract*—In this brief, linear quadratic infinite-horizon Nash games for general multiparameter singularly perturbed systems are studied. The local uniqueness and the asymptotic structure of the solutions to the cross-coupled multiparameter algebraic Riccati equation (CMARE) are newly established. Utilizing the asymptotic structure of the solutions to the CMARE, the parameter-independent Nash strategy is established. A numerical example is given to demonstrate the efficiency and feasibility of the proposed analysis.**

***Index Terms*—Cross-coupled algebraic Riccati equation (CARE), general multiparameter singularly perturbed systems (GMSPS), local uniqueness, Nash games, parameter-independent Nash strategy.**

I. Introduction

LINEAR quadratic Nash games and their applications have been studied widely in many literatures (see, e.g., [1]). It is well known that in order to obtain a Nash equilibrium strategy, the cross-coupled algebraic Riccati equations (CARE) must be solved. In [2], the Newton-type algorithm for solving CARE has been applied. In [3], an algorithm that is called the Lyapunov iterations for solving CARE has been derived. However, these researches have concentrated on determining feedback gain matrices for two-player Nash games. When N-player Nash games are solved, it should be noted that it is extremely hard to find the solution of the N-coupled CARE (see, e.g., [4] and reference therein) because for the required computational workspace, the same dimension of the full systems is needed and the computations are highly expensive. Thus, it is very important to investigate the extension to N-player Nash games before applying it in practical plants.

The control problems for multiparameter singularly perturbed systems (MSPS) have been investigated extensively (see, e.g., [5], [6], and reference therein). Recent advances in the numerical computation approach for singularly perturbed systems (SPS) and the MSPS have allowed us to expand the study on Nash games [7], [9], [10]. The numerical computation approach seems to be a very powerful and reliable methodology. It can be utilized to find the feasible solutions with the adequately high-order accuracy of the Nash strategy. However, a limitation of these numerical approaches is that the small parameters are assumed to be known. Thus, it is not applicable to a large class of problems where the parameters represent small unknown perturbations whose values are not known exactly. When the perturbation parameters are unknown, the parameter-independent Nash strategies have been tackled [5], [6], [11]. However, these studies have also been focused on the decision of the strategies for two-player Nash games. As another serious disadvantage, the existence and the uniqueness for the solutions of the cross-coupled multiparameter algebraic Riccati equation (CMARE) have not been discussed in detail so far in these literatures.

In this brief, the linear quadratic infinite-horizon N-player Nash game for the general MSPS (GMSPS) is discussed. The main contribution is to newly show the local uniqueness for the solutions of CMARE in the neighborhood of reduced-order CARE. It may be noted that although the global uniqueness is not proved, the local uniqueness is useful to choose the appropriate Nash strategy. After proving the local uniqueness and the asymptotic structure of the solutions for CMARE, the reduced-order solutions of the parameter-independent CARE are formulated. Finally, utilizing the reduced-order solutions of CMARE, the parameter-independent Nash strategy is constructed. It is worth pointing out that the proposed strategies can be constructed without any information for the small parameters. Moreover, the required workspace to solve the reduced-order equations is the same as the reduced-order slow and fast subsystems that are smaller than the dimension of the full-order system. A numerical example is given to demonstrate the efficiency and feasibility of the proposed analysis.

Notation: The notations used in this brief are fairly standard. **block diag** denotes the block diagonal matrix. $\otimes$ denotes the Kronecker product. $I_n \in \mathbf{R}^{n\times n}$ denotes the identity matrix. $O_{p\times q} \in \mathbf{R}^{p\times q}$ denotes the zero matrix.

II. Problem Formulation

Consider a linear time-invariant GMSPS

$$\dot{x}_0 = \sum_{i=0}^{N} A_{0i}x_i + \sum_{i=1}^{N} B_{0i}u_i, \qquad x_0(0) = x_0^0 \tag{1a}$$

$$\varepsilon_i \dot{x}_i = A_{i0}x_0 + A_{ii}x_i + B_{ii}u_i, \qquad x_i(0) = x_i^0 \tag{1b}$$

with the quadratic cost functions

$$J_i(u_1,\dots,u_N) = \frac{1}{2}\int_0^\infty \left[y_i^T y_i + u_i^T R_{ii} u_i\right] dt \tag{2a}$$

$$y_i = C_{i0}x_0 + C_{ii}x_i = C_i x \tag{2b}$$

$$x = \left[x_0^T\ x_1^T \cdots x_N^T\right]^T \tag{2c}$$

where $R_{ii} > 0$, $i = 1,\dots,N$, $x_i \in \mathbf{R}^{n_i}$, $i = 0,1,\dots,N$ are the state vectors, $u_i \in \mathbf{R}^{m_i}$, $i = 1,\dots,N$ are the control inputs,

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and $y_i \in \mathbf{R}^{l_i}, i = 0, 1, \ldots, N$ are the outputs. It is supposed that the ratios of the small positive parameter $\varepsilon_i > 0, i = 1, \ldots, N$ are bounded by some positive constants $\underline{k}_{ij}, \bar{k}_{ij}$ (see, e.g., [5])

$$0 < \underline{k}_{ij} \le \alpha_{ij} \equiv \frac{\varepsilon_j}{\varepsilon_i} \le \bar{k}_{ij} < \infty. \tag{3}$$

Furthermore, it is also assumed that the limit of α_{ij} exists as ε_i and ε_j tend to zero, that is,

$$\bar{\alpha}_{ij} = \lim_{\substack{\varepsilon_j \to 0^+ \\ \varepsilon_i \to 0^+}} \alpha_{ij}. \tag{4}$$

Let us introduce the partitioned matrices

$$\begin{aligned}
A &:= \begin{bmatrix} A_{00} & A_{0f} \\ A_{f0} & A_f \end{bmatrix} \\
A_{0f} &:= [A_{01} \ \cdots \ A_{0N}] \\
A_{f0} &:= \left[A_{10}^T \ \cdots \ A_{N0}^T\right]^T \\
A_f &:= \mathbf{block\ diag}\,(A_{11} \cdots A_{NN}) \\
B_1 &:= \left[B_{10}^T \ B_{11}^T \ 0 \ \cdots \ 0\right]^T \\
B_i &:= \left[B_{i0}^T \ \cdots \ B_{ii}^T \ \cdots \ 0\right]^T \\
B_N &:= \left[B_{0N}^T \ 0 \ 0 \ \cdots \ B_{NN}^T\right]^T \\
S_i &:= B_i R_{ii}^{-1} B_i^T \\
&= \begin{bmatrix} S_{i00} & O & S_{i0i} & O \\ O & O & O & O \\ S_{i0i}^T & O & S_{iii} & O \\ O & O & O & O \end{bmatrix} \\
Q_i &:= C_i C_i^T \\
&= \begin{bmatrix} Q_{i00} & O & Q_{i0i} & O \\ O & O & O & O \\ Q_{i0i}^T & O & Q_{iii} & O \\ O & O & O & O \end{bmatrix}.
\end{aligned}$$

Without loss of generality, the following basic assumptions (see, e.g., [3] and [6]) are made.

Assumption 1:

1) The triples $(A, B_i, C_i), i = 1, \ldots, N$ are stabilizable and detectable.
2) The triples $(A_{ii}, B_{ii}, C_{ii}), i = 1, \ldots, N$ are stabilizable and detectable.

These conditions are quite natural since at least one control agent has to be able to control and observe unstable modes. Our purpose is to find a linear feedback strategy set $(u_1^*, \ldots, u_N^*)$ such that

$$J_i(u_1^*, \ldots, u_N^*) \le J_i\left(u_1^*, \ldots, u_{i-1}^*, u_i, u_{i+1}^*, \ldots, u_N^*\right), \quad i = 1, \ldots, N. \tag{5}$$

The decision makers are required to select the closed-loop strategy u_i^*, if they exist, such that (5) holds. Moreover, each player uses the strategy u_i^* such that the closed-loop system is asymptotically stable for sufficiently small ε_i. The following lemma is already known [1], [4], [9].

Lemma 1: There exists an admissible strategy such that the inequality (5) holds iff the generalized CMAREs (GCMAREs)

$$P_i^T\left(A - \sum_{j=1}^N S_j P_j\right) + \left(A - \sum_{j=1}^N S_j P_j\right)^T P_i + P_i^T S_i P_i + Q_i = 0 \tag{6}$$

have solutions $P_{ie} := \Phi_e P_i \ge 0$, where

$$\begin{aligned}
\Phi_e &:= \mathbf{block\ diag}\,(I_{n_0} \ \varepsilon_1 I_{n_1} \ \cdots \ \varepsilon_N I_{n_N}) \\
\Pi_e &:= \mathbf{block\ diag}\,(\varepsilon_1 I_{n_1} \ \cdots \ \varepsilon_N I_{n_N}) \\
P_i &:= \begin{bmatrix} P_{i00} & P_{if0}^T \Pi_e \\ P_{if0} & P_{if} \end{bmatrix} \\
P_{i00} &:= P_{i00}^T \\
P_{if0} &:= \left[P_{i10}^T \ \cdots \ P_{iN0}^T\right]^T \\
\Pi_e P_{if} &:= P_{if}^T \Pi_e
\end{aligned}$$

and P_{if} is expressed as shown in the equation at the bottom of the page. Then the closed-loop linear Nash equilibrium solutions to the full-order problem are given by

$$u_i^*(t) = -R_{ii}^{-1} B_i^T P_i x(t). \tag{7}$$

It should be noted that it is impossible to solve GCMARE (6) if the small perturbed parameters ε_i are unknown. In fact, it is well known that the small perturbed parameter ε_i are often not exactly known [5]. Thus, our purpose is to find the parameter-independent Nash strategies.

III. Local Uniqueness of the Solution for GCMARE

The parameter-independent Nash strategies for the GMSPS will be studied under the following basic assumption.

$$P_{if} := \begin{bmatrix} P_{i11} & \alpha_{i12} P_{i21}^T & \alpha_{i13} P_{i31}^T & \cdots & \alpha_{i1N} P_{iN1}^T \\ P_{i21} & P_{i22} & \alpha_{i23} P_{i32}^T & \cdots & \alpha_{i2N} P_{iN2}^T \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P_{i(N-1)1} & P_{i(N-1)2} & P_{i(N-1)3} & \cdots & \alpha_{i(N-1)N} P_{iN(N-1)}^T \\ P_{iN1} & P_{iN2} & P_{iN3} & \cdots & P_{iNN} \end{bmatrix}$$

Assumption 2: The Hamiltonian matrices $T_{iii}, i = 1, \ldots, N$, are nonsingular, where

$$T_{iii} := \begin{bmatrix} A_{ii} & -S_{iii} \\ -Q_{iii} & -A_{ii}^T \end{bmatrix}. \tag{8}$$

Under Assumptions 1 and 2, the following zeroth-order equations of GCMARE (6) are given as $\|\mu\| := \sqrt{\varepsilon_1^2 + \cdots + \varepsilon_N^2} \to 0^+$, where $\mu := [\varepsilon_1 \ \cdots \ \varepsilon_N]^T$:

$$\bar{P}_{i00}\left(A_s - \sum_{j=1}^{N} S_{s_j}\bar{P}_{j00}\right) + \left(A_s - \sum_{j=1}^{N} S_{s_j}\bar{P}_{j00}\right)^T \bar{P}_{i00} + \bar{P}_{i00}S_{s_i}\bar{P}_{i00} + Q_{s_i} = 0 \tag{9a}$$

$$A_{ii}^T\bar{P}_{iii} + \bar{P}_{iii}A_{ii} - \bar{P}_{iii}S_{iii}\bar{P}_{iii} + Q_{iii} = 0 \tag{9b}$$

$$\bar{P}_{ikl} = 0, \quad k > l, \quad \bar{P}_{ijj} = 0, \quad i \neq j \tag{9c}$$

$$\begin{aligned}
&[\bar{P}_{110} \ \bar{P}_{210} \ \cdots \ \bar{P}_{N10}] \\
&\quad = \begin{bmatrix} \bar{P}_{111} \\ -I_{n_1} \end{bmatrix}^T T_{111}^{-1}T_{110} \begin{bmatrix} I_{n_0} & 0 & \cdots & 0 \\ \bar{P}_{100} & \bar{P}_{200} & \cdots & \bar{P}_{N00} \end{bmatrix} \\
&[\bar{P}_{120} \ \bar{P}_{220} \ \cdots \ \bar{P}_{N20}] \\
&\quad = \begin{bmatrix} \bar{P}_{222} \\ -I_{n_2} \end{bmatrix}^T T_{222}^{-1}T_{220} \begin{bmatrix} 0 & I_{n_0} & \cdots & 0 \\ \bar{P}_{100} & \bar{P}_{200} & \cdots & \bar{P}_{N00} \end{bmatrix} \\
&[\bar{P}_{1N0} \ \bar{P}_{2N0} \ \cdots \bar{P}_{NN0}] \\
&\quad = \begin{bmatrix} \bar{P}_{NNN} \\ -I_{n_N} \end{bmatrix}^T T_{NNN}^{-1}T_{NN0} \\
&\qquad \times \begin{bmatrix} 0 & 0 & \cdots & I_{n_0} \\ \bar{P}_{100} & \bar{P}_{200} & \cdots & \bar{P}_{N00} \end{bmatrix}
\end{aligned} \tag{9d}$$

where

$$\begin{bmatrix} A_s & * \\ * & -A_s^T \end{bmatrix} = \begin{bmatrix} A_{00} & * \\ * & -A_{00}^T \end{bmatrix} - \sum_{i=1}^{N} T_{i0i}T_{iii}^{-1}T_{ii0}$$

$$\begin{bmatrix} * & -S_{s_i} \\ -Q_{s_i} & * \end{bmatrix} = T_{i00} - T_{i0i}T_{iii}^{-1}T_{ii0}$$

$$T_{i00} = \begin{bmatrix} A_{00} & -S_{i00} \\ -Q_{i00} & -A_{00}^T \end{bmatrix}$$

$$T_{i0i} = \begin{bmatrix} A_{0i} & -S_{i0i} \\ -Q_{i0i} & -A_{i0}^T \end{bmatrix}$$

$$T_{ii0} = \begin{bmatrix} A_{i0} & -S_{i0i}^T \\ -Q_{i0i}^T & -A_{0i}^T \end{bmatrix}, \qquad i = 1, \ldots, N.$$

The following theorem shows the relation between the solutions P_i and the zeroth-order solutions $\bar{P}_{ikl} i = 1, \ldots, N$, $k \geq l$, $0 \leq k, l \leq N$.

Theorem 1: Suppose that the condition shown in (10) at the bottom of the page holds, where $\hat{A}_s := A_s - \sum_{j=1}^{N} S_{s_j}\bar{P}_{j00}$ and $\hat{A}_s$ are stable matrices. Under Assumptions 1 and 2, there is a neighborhood $\mathcal{V}(0)$ of $\|\mu\| = 0$ such that for all $\|\mu\| \in \mathcal{V}(0)$ there exists a solution $P_i = P_i(\varepsilon_1, \ldots, \varepsilon_N)$. These solutions are unique in the neighborhood of $\bar{P}_i = P_i(0, \ldots, 0)$. Then, GCMARE (6) possess the power series expansion at $\|\mu\| = 0$. That is, the following form is satisfied:

$$\begin{aligned}
P_i &= \bar{P}_i + O(\|\mu\|) \\
&= \begin{bmatrix}
\bar{P}_{i00} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\bar{P}_{i10} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\bar{P}_{ii0} & 0 & \cdots & 0 & \bar{P}_{iii} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\bar{P}_{iN0} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0
\end{bmatrix} + O(\|\mu\|).
\end{aligned} \tag{11}$$

Proof: First, the zeroth-order solutions for the asymptotic structure of GCMARE (6) are established. Under Assumption 2, the following equality holds:

$$\begin{bmatrix} A_{ii} & -S_{iii} \\ -Q_{iii} & -A_{ii}^T \end{bmatrix} = \begin{bmatrix} I_{n_i} & 0 \\ \bar{P}_{iii} & I_{n_i} \end{bmatrix} \begin{bmatrix} \hat{A}_{ii} & -S_{ii} \\ 0 & -\hat{A}_{ii}^T \end{bmatrix} \begin{bmatrix} I_{n_i} & 0 \\ -\bar{P}_{iii} & I_{n_i} \end{bmatrix} \tag{12}$$

where $\hat{A}_{ii} := A_{ii} - S_{iii}\bar{P}_{iii}$. Since T_{iii} is nonsingular, $\hat{A}_{ii}$ is also nonsingular. This means that T_{iii}^{-1} can be expressed explicitly in terms of $\hat{A}_{ii}^{-1}$. Therefore, using the above result, the formulations (9) are obtained. These transformations can be done by lengthy but direct algebraic manipulations [10], [11], which are omitted here.

For the local uniqueness of the solutions $P_i = P_i(\varepsilon_1, \ldots, \varepsilon_N)$, it is enough to verify that the corresponding Jacobian is nonsingular at $\|\mu\| = 0$. Formally calculating the derivative of GCMARE (6) and after some tedious algebra, the left-hand side of (10) is obtained. Setting $\|\mu\| = 0$ and using (9), the condition (10) is obtained. Finally, the implicit function theorem implies that there is a unique solutions map $P_i = P_i(\varepsilon_1, \ldots, \varepsilon_N)$ and a neighborhood $\mathcal{V}(0)$ of $\|\mu\| = 0$ because condition (10) is equivalent to the corresponding Jacobian at $\|\mu\| = 0$. ■

It is noteworthy that the local uniqueness is newly shown compared with the existing results [5], [6], [9]–[11]. Moreover, it may be noted that the formulas under (9) have been used to simplify the expressions for the first time.

According to the implicit function theorem, condition (10) comes into the picture for the existence of the local unique solution. That is, if condition (10) holds, the local uniqueness of Nash solutions is guaranteed by means of the implicit function theorem because the corresponding Jacobian is nonsingular at

$$\det\begin{bmatrix}
\hat{A}_s^T \otimes I_{n_0} + I_{n_0} \otimes \hat{A}_s^T & -(S_{s_2}\bar{P}_{100}) \otimes I_{n_0} - I_{n_0} \otimes (S_{s_2}\bar{P}_{100}) & \cdots & -(S_{s_N}\bar{P}_{100}) \otimes I_{n_0} - I_{n_0} \otimes (S_{s_N}\bar{P}_{100}) \\
-(S_{s_1}\bar{P}_{200}) \otimes I_{n_0} - I_{n_0} \otimes (S_{s_1}\bar{P}_{200}) & \hat{A}_s^T \otimes I_{n_0} + I_{n_0} \otimes \hat{A}_s^T & \cdots & -(S_{s_N}\bar{P}_{200}) \otimes I_{n_0} - I_{n_0} \otimes (S_{s_N}\bar{P}_{200}) \\
\vdots & \vdots & \ddots & \vdots \\
-(S_{s_1}\bar{P}_{N00}) \otimes I_{n_0} - I_{n_0} \otimes (S_{s_1}\bar{P}_{N00}) & -(S_{s_2}\bar{P}_{N00}) \otimes I_{n_0} - I_{n_0} \otimes (S_{s_2}\bar{P}_{N00}) & \cdots & \hat{A}_s^T \otimes I_{n_0} + I_{n_0} \otimes \hat{A}_s^T
\end{bmatrix} \neq 0 \tag{10}$$

$\|\mu\| = 0$. On the other hand, it is well known that CMARE (9a) could have several positive definite solutions and even some indefinite solutions [8]. However, the implicit function theorem admits the local uniqueness for each several solutions $\bar{P}_{i00}$ at the neighborhood of $\|\mu\| = 0$. It should be noted that this result has novelty compared with the existing results [5], [6], [9]–[11]. As another important feature, although the closed-loop solution of the reduced Nash problem depends on the path along which $\varepsilon_j/\varepsilon_i$ as $\varepsilon_i \to 0^+$, $\varepsilon_j \to 0^+$, generally, it can be concluded that the closed-loop solution of the full problem converges to the closed-loop solution of the reduced problem as the special case that is considered in this brief [5].

IV. Parameter-Independent Strategy

Using the result (11), the parameter-independent Nash strategy is given as

$$\bar{u}_i(t) := -R_{ii}^{-1} B_i^T \bar{P}_i x(t), \quad i = 1, \dots, N. \tag{13}$$

Theorem 2: Under Assumptions 1 and 2, the use of the parameter-independent Nash strategy (13) results in $J_i(\bar{u}_1, \dots, \bar{u}_N)$ satisfying

$$J_i(\bar{u}_1, \dots, \bar{u}_N) = J_i\left(u_1^*, \dots, u_N^*\right) + O\left(\|\mu\|\right) \tag{14}$$

where $J_i(u_1^*, \dots, u_N^*)$ are the exact equilibrium values of the cost functions (2a).

Proof: When $\bar{u}_i(t)$ is used, the equilibrium value of the cost performances is

$$J_i(\bar{u}_1, \dots, \bar{u}_N) = x^T(0) X_{ie} x(0) \tag{15}$$

where X_{ie} is the positive semidefinite solution of the multiparameter algebraic Lyapunov equation (MALE)

$$X_{ie}\left(A_e - \sum_{j=1}^{N} S_{je}\bar{P}_{je}\right) + \left(A_e - \sum_{j=1}^{N} S_{je}\bar{P}_{je}\right)^T X_{ie} + Q_i + \bar{P}_{ie} S_{ie} \bar{P}_{ie} = 0 \tag{16}$$

with $A_e = \Phi_e^{-1}A$, $S_{ie} = \Phi_e^{-1} S_i \Phi_e^{-1}$, and $P_{ie} = \Phi_e P_{ie}$.

Subtracting (6) from (16), it is easy to verify that $V_{ie} = X_{ie} - P_{ie}$, $P_{ie} := \Phi_e P_i$ satisfies the MALE

$$V_{ie}\left(A_e - \sum_{j=1}^{N} S_{je}\bar{P}_{je}\right) + \left(A_e - \sum_{j=1}^{N} S_{je}\bar{P}_{je}\right)^T V_{ie} + (\bar{P}_{ie} - P_{ie}) S_{ie} (\bar{P}_{ie} - P_{ie}) + \sum_{j=1, j\neq i}^{N} P_{ie} S_{je} (\bar{P}_{je} - P_{je}) + \sum_{j=1, j\neq i}^{N} (\bar{P}_{je} - P_{je}) S_{je} P_{ie} = 0. \tag{17}$$

Using the relation $\bar{P}_{ie} - P_{ie} = O(\|\mu\|)$, the following MALE holds:

$$V_{ie}\left(A_e - \sum_{j=1}^{N} S_{je}\bar{P}_{je}\right) + \left(A_e - \sum_{j=1}^{N} S_{je}\bar{P}_{je}\right)^T V_{ie} = O\left(\|\mu\|\right). \tag{18}$$

Thus, it is easy to verify that $V_{ie} = O(\|\mu\|)$ because $A_e - \sum_{j=1}^{N} S_{je}\bar{P}_{je}$ is stable by using the standard Lyapunov theorem [12]. Consequently, the equality (14) holds. ■

Although ε_i is unknown, it is possible to design the parameter-independent strategy that achieves the $O(\|\mu\|)$ approximation for the equilibrium value of the cost functional.

Using the same technique as the proof of Theorem 2, the following conditions are satisfied.

Theorem 3: Under Assumptions 1 and 2, the following result holds:

$$J_i(\bar{u}_1, \dots, \bar{u}_{i-1}, u_i, \bar{u}_{i+1}, \dots, \bar{u}_N) = J_i\left(u_1^*, \dots, u_{i-1}^*, u_i, u_{i+1}^*, \dots, u_N^*\right) + O\left(\|\mu\|\right). \tag{19}$$

Proof: Since the proof can be done by using the above technique, it is omitted. ■

Finally, by using the same manner that has been established in [6], the main result is easily derived.

Theorem 4: Under Assumptions 1 and 2, the use of the parameter-independent strategy (13) results in

$$J_i(\bar{u}_1, \dots, \bar{u}_N) \le J_i(\bar{u}_1, \dots, \bar{u}_{i-1}, u_i, \bar{u}_{i+1}, \dots, \bar{u}_N) + O\left(\|\mu\|\right). \tag{20}$$

Proof: Using (14), (5), and (19), the proof of (20) completes. The other cases are similar. ■

It should be noted that our results include the existing strategies that have been introduced in [6] as a special case.

V. Numerical Example

In order to demonstrate the efficiency of our proposed algorithm, a numerical example is given. The systems matrices are given by

$$A_{00} = \begin{bmatrix} -1 & 0 & 0 & 4.5 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 4.5 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 4.5 & -1 \\ 0 & 0 & 0 & -0.05 & 0 & 0 & -0.1 \\ 0 & 0 & 0 & 0 & -0.05 & 0 & 0.1 \\ 0 & 0 & 0 & 0 & 0 & -0.05 & 0.1 \\ 0 & 0 & 0 & 32.7 & -32.7 & -32.7 & 0 \end{bmatrix} \in \mathbf{R}^{7\times 7}$$

$$A_{01} = \begin{bmatrix} O_{2\times 3} & [0.1\ \ 0]^T & O_{2\times 3} \end{bmatrix}^T \in \mathbf{R}^{7\times 2}$$

$$A_{02} = \begin{bmatrix} O_{2\times 4} & [0.1\ \ 0]^T & O_{2\times 2} \end{bmatrix}^T \in \mathbf{R}^{7\times 2}$$

$$A_{03} = \begin{bmatrix} O_{2\times 5} & [0.1\ \ 0]^T & O_{2\times 1} \end{bmatrix}^T \in \mathbf{R}^{7\times 2}$$

$$A_{10} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.4 & 0 & 0 & 0 \end{bmatrix} \in \mathbf{R}^{2\times 7}$$

$$A_{20} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.4 & 0 & 0 \end{bmatrix} \in \mathbf{R}^{2\times 7}$$

$$A_{30} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.4 & 0 \end{bmatrix} \in \mathbf{R}^{2\times 7}$$

$$A_{11} = A_{22} = A_{33} = \begin{bmatrix} -0.05 & 0.05 \\ 0 & -0.1 \end{bmatrix} \in \mathbf{R}^{2\times 2}$$

$$B_{10} = B_{20} = B_{30} = [O_{7\times 1}] \in \mathbf{R}^{7}$$

$$B_{11} = B_{22} = B_{33} = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} \in \mathbf{R}^{2}$$

$$Q_1 = \mathbf{diag}(1,1,1,1,1,1,1,1,1,0,0,0,0) \in \mathbf{R}^{13\times 13}$$

$$Q_2 = \mathbf{diag}(1,1,1,1,1,1,1,0,0,1,1,0,0) \in \mathbf{R}^{13\times 13}$$

$$Q_3 = \mathbf{diag}(1,1,1,1,1,1,1,0,0,0,0,1,1) \in \mathbf{R}^{13\times 13}$$

$$R_1 = R_2 = R_3 = 20 \in \mathbf{R}.$$

It is easy to verify that for this example, Assumptions 1 and 2 and condition (10) are satisfied. Using the proposed technique, the parameter-independent strategies are given by (21), shown

$$\bar{u}_1(t) = [-1.1952e{-2} \quad 4.4208e{-3} \quad 4.4208e{-3} \quad -6.1183e{-1} \quad 6.0144e{-1} \quad 6.0144e{-1} \quad -9.2935e{-3} \quad -1.6226e{-2} \quad -3.2582e{-2} \quad 0 \quad 0 \quad 0 \quad 0]x(t) \tag{21a}$$

$$\bar{u}_2(t) = [4.4208e{-3} \quad -1.1952e{-2} \quad -4.4208e{-3} \quad 6.0144e-1 \quad -6.1183e{-1} \quad -6.0144e{-1} \quad 9.2935e{-3} \quad 0 \quad 0 \quad -1.6226e{-2} \quad -3.2582e{-2} \quad 0 \quad 0]x(t) \tag{21b}$$

$$\bar{u}_3(t) = [4.4208e{-3} \quad -4.4208e{-3} \quad -1.1952e{-2} \quad 6.0144e{-1} \quad -6.0144e{-1} \quad -6.1183e{-1} \quad 9.2935e{-3} \quad 0 \quad 0 \quad 0 \quad 0 \quad -1.6226e{-2} \quad -3.2582e{-2}]x(t) \tag{21c}$$

$$u_1^*(t) = [-1.1952e{-2} \quad 4.4207e{-3} \quad 4.4207e{-3} \quad -6.1183e{-1} \quad 6.0144e{-1} \quad 6.0144e{-1} \quad -9.2886e{-3} \quad -1.6228e{-2} \quad -3.2583e{-2} \quad 1.1843e{-6} \quad 5.7345e{-7} \quad 1.1843e{-6} \quad 5.7345e{-7}]x(t) \tag{22a}$$

$$u_2^*(t) = [4.4207e{-3} \quad -1.1952e{-2} \quad -4.4207e{-3} \quad 6.0144e{-1} \quad -6.1183e{-1} \quad -6.0144e{-1} \quad 9.2886e{-3} \quad 1.1843e{-6} \quad 5.7345e{-7} \quad -1.6228e{-2} \quad -3.2583e{-2} \quad -1.1843e{-6} \quad -5.7345e{-7}]x(t) \tag{22b}$$

$$u_3^*(t) = [4.4207e{-3} \quad -4.4207e{-3} \quad -1.1952e{-2} \quad 6.0144e{-1} \quad -6.0144e{-1} \quad -6.1183e{-1} \quad 9.2886e{-3} \quad 1.1843e{-6} \quad 5.7345e{-7} \quad -1.1843e{-6} \quad -5.7345e{-7} \quad -1.6228e{-2} \quad -3.2583e{-2}]x(t) \tag{22c}$$

TABLE I
COST FUNCTIONAL-TO-PERTURBATION RATIO FOR VARIOUS ε

ε	ϕ_1	ϕ_2	ϕ_3
$1.0e-01$	$1.8470e+04$	$1.8470e+04$	$1.8470e+04$
$1.0e-02$	$6.6328e+03$	$6.6328e+03$	$6.6328e+03$
$1.0e-03$	$1.0465e+03$	$1.0465e+03$	$1.0465e+03$
$1.0e-04$	$1.3714e+03$	$1.3714e+03$	$1.3714e+03$
$1.0e-05$	$1.4024e+03$	$1.4024e+03$	$1.4024e+03$
$1.0e-06$	$1.4055e+03$	$1.4055e+03$	$1.4055e+03$

at the top of the page. On the other hand, when the small parameters are chosen as $\varepsilon_j = 1.0e-6$, $j = 1, 2, 3$, the exact Nash strategies are also given by (22) at the top of the page. After comparing these strategies, since the proposed parameter-independent strategies (21) are very close to the exact one (22) under the small parameters ε_j, the proposed approach is very reliable.

Next, the costs of using the approximate strategies (13) are evaluated. It is assumed that the initial conditions are zero mean independent random vector with the covariance matrix $E[x(0)x(0)^T] = I_{13}$. The values of the cost functionals for various $\varepsilon := \varepsilon_1 = \varepsilon_2 = \varepsilon_3$ are given in Table I, where $\phi_i := |E[J_i(\bar{u}_1, \ldots, \bar{u}_N] - E[J_i(u_1^*, \ldots, u_N^*)]|/\|\mu\|$. It is easy to verify that $E[J_i(\bar{u}_1, \ldots, \bar{u}_N] = E[J_i(u_1^*, \ldots, u_N^*)] + O(\|\mu\|)$ because $\phi_i < \infty$ is of the same order.

It is worth pointing out that the proposed strategy can be constructed without any information of the small parameters. As another important feature, the required workspace dimension to compute the strategy is small compared with the dimension of the full-order system. In this example, the dimension for the calculation is seven smaller than 13. Therefore, it is very useful for the construction of the strategy of the practical system.

VI. CONCLUSION

In this brief, the linear quadratic infinite-horizon Nash game for GMSPS has been investigated. The main contribution is that the local uniqueness and the boundedness of the solutions for GCMARE have been established for the first time. As another important feature, the sufficient conditions for the validity of the GMSPS of the approximate Nash strategies have been derived. The numerical examples have shown the validity of the proposed strategies for the unknown sufficiently small parameter.

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