Existence of tangential limits for α-harmonic functions on half spaces

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Abstract

Our aim in this paper is to prove the existence of tangential limits for Poisson integrals of the fractional order of functions in the L^p Hölder space on half spaces.

1 Introduction

For $0 < \alpha < 2$, Riesz [6] defined the notion of α-harmonic functions on a domain Ω in the n-dimensional Euclidean space $\mathbf{R}^n$, as solutions of the fractional Laplace operators (see also Itô [3] and the book by Landkof [4]).

In the half space $H = \{x = (x', x_n) \in \mathbf{R}^{n-1} \times \mathbf{R} : x_n > 0\}$, consider

$$P_\alpha f(x) = c_\alpha \int_{\mathbf{R}^n \setminus \overline{H}} \left(\frac{x_n}{|y_n|} \right)^{\alpha/2} |x - y|^{-n} f(y) dy$$

for a measurable function f on $\mathbf{R}^n$, where $c_\alpha = \Gamma(n/2)\pi^{-n/2-1} \sin(\pi\alpha/2)$. Here f satisfies

$$\int_{\mathbf{R}^n \setminus \overline{H}} |y_n|^{-\alpha/2} (|y| + 1)^{-n} |f(y)| dy < \infty, \tag{1}$$

which is equivalent to

$$P_\alpha |f| \not\equiv \infty.$$

If this is the case, then we see from Remarks 4 and 5 below that $P_\alpha f$ is α-harmonic in H.

Let $1 < p < \infty$ and $\beta > 1/p$. Recently, Bass and You [1] have shown the existence of nontangential limits for $P_\alpha f$ with $f \in \Lambda_\beta^{p,\infty}(\mathbf{R}^n)$, which is the space of L^p Hölder continuous functions of order β; more precisely, $f \in L^p(\mathbf{R}^n)$ and it

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satisfies L^p Hölder continuity of order β, that is, there exists a positive constant C such that

$$\left(\int |f(y+h) - f(y)|^p dy \right)^{1/p} \le C|h|^\beta \qquad \text{for all } h \in \mathbf{R}^n \ . \tag{2}$$

Our aim in this note is to extend their result on the existence of nontangential limits for $P_\alpha f$. For $\gamma > 0$ and $\xi \in \partial H$, our approach region is defined by

$$T_\gamma(\xi) = \{x \in H : |x - \xi|^{1+\gamma/(n-\alpha/2)} < x_n\},$$

which is tangential to the boundary ∂H at ξ .

THEOREM. *Let $0 < \alpha < 2(1 - 1/p)$ and $0 < \gamma < \beta - 1/p$. If f is a measurable function in $\Lambda_\beta^{p,\infty}(\mathbf{R}^n)$ satisfying* (1), *then there exists $E \subset \partial H$ of $(n-1)$-dimensional measure zero such that $P_\alpha f$ has a finite limit along $T_\gamma(\xi)$ for every $\xi \in \partial H \setminus E$.*

REMARK 1. Note that $|y_n|^{-\alpha/2}$ is locally $L^{p'}$ integrable on $\mathbf{R}^n$ where $p' = p/(p-1)$ if and only if $0 < \alpha < 2(1 - 1/p)$. But, in case $2(1 - 1/p) \le \alpha < 2$, we do not know whether $P_\alpha f$ has a tangential limit at almost every boundary point or not.

REMARK 2. Bass and You [2] have also obtained nontangential limit result for α-harmonic functions in Lipschitz domains. It will be expected that tangential convergence holds in a way similar to the proof of the theorem given later, because they have shown a good estimate for Poisson kernel near the boundary; but the discussions are left to be open.

Our method of proof is carried out directly, as in the discussions of harmonic case (see Landkof [4], Stein [7] and the author [5]).

2 Proof of the Theorem

Throughout this note, let C denote various constants independent of the variables in question. For a locally integrable function g on $\mathbf{R}^n$, we define its integral mean over the ball $B(x, r)$ centered at ξ of radius $r > 0$ by

$$⨍_{B(x,r)} g(y)dy = \frac{1}{\sigma_n r^n} \int_{B(x,r)} g(y)dy,$$

where σ_n denotes the n-dimensional Lebesgue measure of the unit ball.

Let f be a measurable function on $\mathbf{R}^n$ as in the theorem. First note that

$$\begin{aligned} & ⨍_{B(\xi,r)} |y_n|^{-\alpha/2} \left| f(y) - ⨍_{B(\xi,r)} f(z)dz \right| dy \\ \le \ & Cr^{-2n} \int_{B(0,r)} \int_{B(0,r)} |y_n|^{-\alpha/2} \left| f(\xi + y) - f(\xi + z) \right| dydz \end{aligned}$$

for $\xi \in \partial H$. Further, for $E \subset \partial H$, denote by $|E|$ the $(n-1)$-dimensional measure of E.

To complete a proof of the theorem, we first prepare two lemmas.

LEMMA 1. *Let $0 \le \alpha < 2(1-1/p)$ and $0 < \gamma < \beta - 1/p$. For $f \in \Lambda_\beta^{p,\infty}(\mathbf{R}^n)$, set*

$$\begin{aligned} E(\gamma) = \{\xi \in \partial H : & \limsup_{r \to 0+} r^{\alpha/2-\gamma-2n} \\ & \times \int_{B(0,r)} \int_{B(0,r)} |y_n|^{-\alpha/2} |f(\xi+y) - f(\xi+z)| \, dy dz > 0\}. \end{aligned}$$

Then $|E(\gamma)| = 0$.

PROOF. For $M > 0$ and a positive integer j, set

$$\begin{aligned} E(\gamma, M, j) = \{\xi \in \partial H : & \, 2^{j(\gamma+2n-\alpha/2)} \\ & \times \int_{B(0,2^{-j})} \int_{B(0,2^{-j})} |y_n|^{-\alpha/2} |f(\xi+y) - f(\xi+z)| \, dy dz > 1/M\}. \end{aligned}$$

Let $a > 0$. Then we have

$$\begin{aligned} |E(\gamma, M, j) \cap B(0,a)| \le & \, M 2^{j(\gamma+2n-\alpha/2)} \\ & \times \int_{B(0,a)\cap \partial H} \left(\int_{B(0,2^{-j})} \int_{B(0,2^{-j})} |y_n|^{-\alpha/2} |f(\xi+y) - f(\xi+z)| \, dy dz \right) d\xi. \end{aligned}$$

Letting $w = \xi + y = \xi + (y', y_n)$, we see from Hölder's inequality and (2) that

$$\begin{aligned} & \int_{\{y_n : |y_n| < 2^{-j}\}} \left(\int_{B(0,a)\cap \partial H} |y_n|^{-\alpha/2} |f(\xi+y) - f(\xi+y+(0,z_n))| \, d\xi \right) dy_n \\ \le & \int_{\{w \in B(0,a+1) : |w_n| < 2^{-j}\}} |w_n|^{-\alpha/2} |f(w) - f(w+(0,z_n))| \, dw \\ \le & \, C (2^{-j})^{-\alpha/2+1-1/p+\beta} \end{aligned}$$

for $y, z \in B(0, 2^{-j})$. Similarly, letting $w = \xi + z = \xi + (z', z_n)$, we obtain

$$\begin{aligned} & \int_{\{z_n : |z_n| < 2^{-j}\}} \left(\int_{B(0,a)\cap \partial H} |f(\xi+y+(0,z_n)) - f(\xi+z)| \, d\xi \right) dz_n \\ \le & \int_{\{w \in B(0,a+1) : |w_n| < 2^{-j}\}} |f(w+(y'-z', y_n)) - f(w)| \, dw \\ \le & \, C (2^{-j})^{1-1/p+\beta} \end{aligned}$$

for $y, z \in B(0, 2^{-j})$. Hence it follows that

$$\begin{aligned} & |E(\gamma, M, j) \cap B(0,a)| \\ \le & \, CM 2^{j(\gamma+2n-\alpha/2)} (2^{-j})^{-\alpha/2+1-1/p+\beta} 2^{-j(n-1)} 2^{-jn} \\ & + CM 2^{j(\gamma+2n-\alpha/2)} (2^{-j})^{1-1/p+\beta} (2^{-j})^{-\alpha/2+n} 2^{-j(n-1)} \\ \le & \, CM 2^{-j(\beta-1/p-\gamma)}. \end{aligned}$$

Define $E(\gamma, M) = \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} E(\gamma, M, j)$. Then, since $\beta - 1/p - \gamma > 0$, we find

$$|E(\gamma, M) \cap B(0, a)| = 0$$

for all $a > 0$, which implies that $|E(\gamma, M)| = 0$. Noting that $E(\gamma) = \bigcup_{M=1}^{\infty} E(\gamma, M)$, we see that $|E(\gamma)| = 0$ and

$$\lim_{j \to \infty} 2^{j(\gamma + 2n - \alpha/2)} \int_{B(0,2^{-j})} \int_{B(0,2^{-j})} |y_n|^{-\alpha/2} \, |f(\xi + y) - f(\xi + z)| \, dydz = 0$$

for every $\xi \in \partial H \setminus E(\gamma)$ ■

Lemma 2. *Let f be a function in $L^p(\mathbf{R}^n)$ satisfying (2). If $0 < \gamma < \beta - 1/p$, then*

$$A(\xi) = \lim_{r \to 0+} \fint_{B(\xi,r)} f(y)dy$$

exists and is finite for almost every $\xi \in \partial H$. Further,

$$\lim_{r \to 0+} r^{\alpha/2 - \gamma - n} \int_{B(\xi,r)} |y_n|^{-\alpha/2} |f(y) - A(\xi)| dy = 0$$

for almost every $\xi \in \partial H$.

It is well known that if $f \in L^p(\mathbf{R}^n)$, then $A(x)$ exists and is finite for almost every $x \in \mathbf{R}^n$. Since ∂H is of n-dimensional measure zero, we need an additional condition like L^p Hölder continuity for f in the present lemma.

Proof of Lemma 2. For simplicity, set $A(\xi, r) = \fint_{B(\xi,r)} f(y)dy$. Then, for $r \le t \le 2r$, note that

$$|A(\xi, t) - A(\xi, r)| \;\; \le \;\; Cr^{-2n} \int_{B(0,2r)} \int_{B(0,2r)} |f(\xi + y) - f(\xi + z)| \, dydz,$$

so that Lemma 1 (with $\alpha = 0$) yields

$$\lim_{r \to 0+} r^{-\gamma} \, |A(\xi, 2r) - A(\xi, r)| = 0$$

for almost every $\xi \in \partial H$. In this case,

$$\lim_{k \to \infty} 2^{k\gamma} \sum_{j=k}^{\infty} |A(\xi, 2^{-j+1}) - A(\xi, 2^{-j})| = 0,$$

which implies that $A(\xi)$ exists and

$$\lim_{r \to 0+} r^{-\gamma} \, |A(\xi) - A(\xi, r)| = 0.$$

This proves the first part of the result.

Similarly, we see from Lemma 1 that

$$\lim_{r\to 0+} r^{\alpha/2-\gamma-n}\int_{B(\xi,r)} |y_n|^{-\alpha/2}|f(y)-A(\xi,2r)|dy = 0$$

for almost every $\xi \in \partial H$. Note here that

$$\begin{aligned}
&\int_{B(\xi,r)} |y_n|^{-\alpha/2}|f(y)-A(\xi)|dy \\
\le\ &\int_{B(\xi,r)} |y_n|^{-\alpha/2}|f(y)-A(\xi,2r)|dy + Cr^{n-\alpha/2}|A(\xi)-A(\xi,2r)|.
\end{aligned}$$

This yields the second result. ■

Proof of the Theorem. Let f be as in the theorem. Then Lemma 2 implies that

$$A(\xi) = \lim_{r\to 0+} ⨍_{B(\xi,r)} f(y)dy$$

exists and is finite for almost every $\xi \in \partial H$. Moreover we see from Remark 4 below that

$$P_\alpha f(x) - A(\xi) = c_\alpha \int_{\mathbf{R}^n\setminus\overline{H}} \left(\frac{x_n}{|y_n|}\right)^{\alpha/2} |x-y|^{-n}\{f(y)-A(\xi)\}dy.$$

In view of Lemma 2 we see that

$$\lim_{r\to 0+} r^{\alpha/2-\gamma-n}\int_{B(\xi,r)} |y_n|^{-\alpha/2}|f(y)-A(\xi)|dy = 0$$

holds for almost every $\xi \in \partial H$. Hence, for given $\varepsilon > 0$ and $\xi \in \partial H$, we assume that

$$r^{\alpha/2-\gamma-n}\int_{B(\xi,r)} |y_n|^{-\alpha/2}|f(y)-A(\xi)|dy < \varepsilon \tag{3}$$

whenever $0 < r < r_0$. Since $0 < \alpha/2 < 1-1/p$, we see that

$$\lim_{x\to\xi}\int_{\mathbf{R}^n\setminus\{\overline{H}\cup B(\xi,r_0)\}} \left(\frac{x_n}{|y_n|}\right)^{\alpha/2} |x-y|^{-n}|f(y)-A(\xi)|dy = 0. \tag{4}$$

Letting $r = 2|x-\xi| < r_0$, we obtain by (3)

$$\begin{aligned}
&\int_{B(\xi,r)\setminus\overline{H}} \left(\frac{x_n}{|y_n|}\right)^{\alpha/2} |x-y|^{-n}|f(y)-A(\xi)|dy \\
\le\ & Cx_n^{\alpha/2-n}\int_{B(\xi,r)\setminus\overline{H}} |y_n|^{-\alpha/2}|f(y)-A(\xi)|dy \\
\le\ & C\varepsilon x_n^{\alpha/2-n} r^{\gamma-\alpha/2+n}
\end{aligned}$$

and

$$
\begin{aligned}
&\int_{B(\xi,r_0)\setminus\{\overline{H}\cup B(\xi,r)\}} \left(\frac{x_n}{|y_n|}\right)^{\alpha/2} |x-y|^{-n}|f(y)-A(\xi)|dy \\
\le\ & C\int_{B(\xi,r_0)\setminus\{\overline{H}\cup B(\xi,r)\}} \left(\frac{x_n}{|y_n|}\right)^{\alpha/2} |\xi-y|^{-n}|f(y)-A(\xi)|dy \\
\le\ & Cx_n^{\alpha/2}\int_r^{r_0}\left(\int_{B(\xi,t)\setminus\overline{H}} |y_n|^{-\alpha/2}|f(y)-A(\xi)|dy\right)t^{-n-1}dt.
\end{aligned}
$$

If $\gamma-\alpha/2<0$, then

$$
\int_{B(\xi,r_0)\setminus\{\overline{H}\cup B(\xi,r)\}} \left(\frac{x_n}{|y_n|}\right)^{\alpha/2} |x-y|^{-n}|f(y)-A(\xi)|dy \le C\varepsilon x_n^{\alpha/2}r^{\gamma-\alpha/2},
$$

which together with (4) implies that

$$
\limsup_{x\to\xi,x\in T_\gamma(\xi)} |P_\alpha f(x)-A(\xi)| \le C\varepsilon.
$$

If $\gamma-\alpha/2\ge 0$, then

$$
\int_{B(\xi,r_0)\setminus\{\overline{H}\cup B(\xi,r)\}} \left(\frac{x_n}{|y_n|}\right)^{\alpha/2} |x-y|^{-n}|f(y)-A(\xi)|dy \le C\varepsilon x_n^{\alpha/2}\log(r_0/r),
$$

so that

$$
\limsup_{x\to\xi,x\in T_\gamma(\xi)} |P_\alpha f(x)-A(\xi)| \le C\varepsilon.
$$

Hence it follows that

$$
\lim_{x\to\xi,x\in T_\gamma(\xi)} |P_\alpha f(x)-A(\xi)| = 0,
$$

as required. ■

3 Further remarks

Remark 3. Let us consider the Riesz potential

$$
U_\alpha\mu(x)=\int |x-y|^{\alpha-n}d\mu(y)
$$

for a nonnegative measure μ on $\mathbf{R}^n$, where $0<\alpha\le 2$ and $U_\alpha\mu\not\equiv\infty$. Then it is known that $U_\alpha\mu$ is α-superharmonic in $\mathbf{R}^n$ and α-harmonic outside the support of μ; for this, see Riesz [6] and Landkof [4].

Remark 4. Let

$$
P_\alpha(x,y)=c_\alpha\left(\frac{x_n}{|y_n|}\right)^{\alpha/2}|x-y|^{-n}
$$

for $x \in H$ and $y \in \mathbf{R}^n \setminus \overline{H}$. For $x_0 \in H$, we consider the inversion with respect to the ball $B_0 = B(x_0, R)$ with $R = (x_0)_n$. For $x \in \mathbf{R}^n$, write

$$x^* = x_0 + R^2 \frac{x - x_0}{|x - x_0|^2}.$$

Set $\tilde{x}_0 = (x_0', R/2)$ and $\tilde{B}(x_0) = B(\tilde{x}_0, R/2)$. Then, for $x \in B(x_0, R)$ and $y \in \mathbf{R}^n \setminus \overline{H}$, we have

$$\begin{aligned} P_\alpha(x,y) &= c_\alpha R^{-2n} |x_0 - x^*|^{n-\alpha} \left(\frac{|x^* - \tilde{x}_0|^2 - (R/2)^2}{(R/2)^2 - |y^* - \tilde{x}_0|^2} \right)^{\alpha/2} \\ &\quad \times |x^* - y^*|^{-n} |x_0 - y^*|^{n+\alpha}, \end{aligned}$$

so that (A.1) of [4, Appendix] gives

$$\int_{\mathbf{R}^n \setminus \overline{H}} P_\alpha(x,y) dy = 1.$$

Remark 5. In the same way as above, we find

$$\int_{\mathbf{R}^n \setminus \overline{H}} P_\alpha(x,y) |x_0 - y|^{\alpha - n} dy = |x - x_0|^{\alpha - n}$$

for $x_0 \in H$. This is the key result to discuss the α-harmonicity of $P_\alpha f$; see Itô [3, Lemma 1] and Landkof [4, Lemma 1.13 in Chap.1].

Remark 6. Let $0 < \alpha < 2(1 - 1/p)$ and $\beta p > 1$. Suppose f is a measurable function in $\Lambda_\beta^{p,\infty}(\mathbf{R}^n)$. Then we can find a set $E \subset \partial H$ of measure zero such that $P_\alpha f$ has a finite limit along $T_\gamma(\xi)$ for every $\xi \in \partial H \setminus E$ and γ with $0 < \gamma < \beta - 1/p$.

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