Near-Optimal Kalman Filters for Multiparameter Singularly Perturbed Linear Systems

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***Abstract*—In this brief, we study the near-optimal Kalman filtering problem for multiparameter singularly perturbed system (MSPS). The attention is focused on the design of the near-optimal Kalman filters. It is shown that the resulting filters in fact remove ill-conditioning of the original full-order singularly perturbed Kalman filters. In addition the resulting filters can be used compared with the previously proposed result even if the fast state matrices are singular.**

***Index Terms*—Multiparameter algebraic Riccati equations (MARE), multiparameter singularly perturbed system (MSPS), optimal Kalman filters.**

I. Introduction

Recently, filtering problems for the multiparameter singularly perturbed system (MSPS) have been investigated [1]–[3], [11]. Such problems arise in large-scale dynamic systems. For example, the MSPS in practice is illustrated by the passenger car model [3]. In order to obtain the optimal solution to the filtering problems, we must solve the multiparameter algebraic Riccati equation (MARE), which are parameterized by small positive same order parameters $\varepsilon_1, \varepsilon_2, \ldots$. Various reliable approaches to the theory of the algebraic Riccati equation (ARE) have been well documented in many literatures (see e.g., [5], [6]). One of the approaches is the invariant subspace approach which is based on the Hamiltonian matrix [5]. However, such an approach is not adequate to the MSPS since for the computed solution there is no guarantee of symmetry when the ARE is ill-conditioned [5]. In order to avoid the numerical stiffness, the exact slow-fast decomposition method for solving the MARE has been proposed in [3]. However, the dimension of the required workspace to carry out the calculations for the solution is the same dimension of the original full-system. Furthermore, only the Kalman filtering problem for MSPS with two fast subsystems has been considered.

A popular approach to deal with the MSPS is the two-time-scale design method [4], [7]. However, in order to obtain the slow subsystem, the nonsingularity of the fast state matrices are needed. Furthermore, the near-optimal Kalman filtering of the MSPS has not been investigated so far. In this brief, we study the near-optimal Kalman filtering problem for the MSPS. The results obtained are valid for steady state. Note that there exist several singular perturbation parameters $\varepsilon_1, \varepsilon_2, \ldots$ for the considered MSPS compared with the previous results [1]–[3], [11]. We first investigate the uniqueness and boundedness of the solution to such MARE and establish its asymptotic structure. The proof of the existence of the solution to the MARE with asymptotic expansion is obtained by an implicit function theorem [4]. The main contribution of this brief is to propose the near-optimal Kalman filters. As a result, we have only to solve the ARE with same order dimension of the reduced-order slow and fast systems which do not depend on the values of the small parameters. Furthermore, we claim that the proposed filters can be constructed even if the fast state matrices are singular.

In [2] the well-posedness of multimodel strategies for a linear-quadratic-Gaussian (LQG) optimal control problem has been studied. The local control problem of a control agent of the above paper is obtained by neglecting the fast dynamics of the other agent's subsystem and each agent uses the optimal solution of his local control problem. However, the nonsingularity assumptions for the fast state matrices A_{ii}, $i = 1, 2, \ldots, N$ are also needed. On the other hand, the proposed near-optimal Kalman filter can be obtained without such assumptions. Therefore, it is possible to construct the near-optimal Kalman filter for the wider class of the MSPS. Furthermore, the proposed design method has a feature, in which a new filter gain is obtained by neglecting all small perturbation parameters of the optimal Kalman filter and solving the reduced-order ARE's. Hence, compared with the existing results [2], since the proposed technique is the batch processing which is not based on the two-time-scale design method [4], [7], it is easy to design the near-optimal Kalman filters.

Manuscript received January 29, 2002; revised October 29, 2002. This paper was recommended by Associate Editor W.-X. Zheng.

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Digital Object Identifier 10.1109/TCSI.2003.811026

II. Optimal Kalman Filtering Problem

We consider the linear time-invariant MSPS [1], [2]

$$\dot{x}_0(t) = A_{00}x_0(t) + \sum_{j=1}^{N} A_{0j}x_j(t) + \sum_{j=1}^{N} D_{0j}w_j(t) \tag{1a}$$

$$\varepsilon_i\dot{x}_i(t) = A_{i0}x_0(t) + A_{ii}x_i(t) + D_{ii}w_i(t), \qquad i = 1, 2, \ldots N \tag{1b}$$

with the corresponding measurements

$$y_i(t) = C_{i0}x_0(t) + C_{ii}x_i(t) + v_i(t), \qquad i = 1, 2, \ldots N \tag{2}$$

where $x_i(t) \in \mathbf{R}^{n_i}$, $i = 0, 1, 2, \ldots, N$ are state vectors, $y_i(t) \in \mathbf{R}^{p_i}$, $i = 1, 2, \ldots, N$ are system measurements, $w_i(t) \in \mathbf{R}^{q_i}$, $i = 1, 2, \ldots, N$ and $v_i(t) \in \mathbf{R}^{r_i}$, $i = 1, 2, \ldots, N$ are zero-mean stationary, Gaussian, mutually uncorrelated, white-noise stochastic processes with intensities $W_i \geq 0$ and $V_i > 0$, respectively. All the matrices are constant matrices of appropriate dimensions. ε_i, ε_j, i, $j = 1, 2, \ldots, N$ are the small positive singular perturbation parameters of the same order of magnitude [1]–[4], [7], [11] such that

$$0 < \underline{k}_{ij} \leq \alpha_{ij} \equiv \frac{\varepsilon_j}{\varepsilon_i} \leq \bar{k}_{ij} < \infty. \tag{3}$$

That is, we assume that the ratio of ε_i and ε_j is bounded by some positive constants. In this brief we design the near-optimal Kalman filters to estimate system states x_i. The optimal Kalman filters of (1) and (2) are given by [3]

$$\dot{\xi}_0(t) = A_{00}\xi_0(t) + \sum_{j=1}^{N} A_{0j}\xi_j(t) + \sum_{j=1}^{N} K_{0j}\eta_j(t) \tag{4a}$$

$$\varepsilon_i\dot{\xi}_i(t) = A_{i0}\xi_0(t) + A_{ii}\xi_i(t) + \sum_{j=1}^{N} K_{ij}\eta_j(t) \tag{4b}$$

$$\eta_i(t) = y_i(t) - C_{i0}\xi_0(t) - C_{ii}\xi_i(t), \qquad i = 1, 2, \ldots, N \tag{4c}$$

where the filter gain K_{ij} are obtained from

$$\begin{aligned} K_e &= X_e C^T V^{-1} = \Phi_e^{-1} K^{\mathrm{opt}} \\ &= \Phi_e^{-1} X C^T V^{-1} = \Phi_e^{-1} \begin{bmatrix} K_{01} & \cdots & K_{0N} \\ \vdots & \ddots & \vdots \\ K_{N1} & \cdots & K_{NN} \end{bmatrix} \\ \Phi_e &:= \begin{bmatrix} I_{n_0} & 0 \\ 0 & \Pi_e \end{bmatrix} \\ \Pi_e &:= \mathbf{block-diag}\left(\varepsilon_1 I_{n_1} \ \cdots \ \varepsilon_N I_{n_N} \right) \end{aligned} \tag{5}$$

with matrix X_e representing the positive semidefinite stabilizing solution of the MARE

$$A_e X_e + X_e A_e^T - X_e S X_e + U_e = 0 \tag{6}$$

where

$$\begin{aligned}
A_e &:= \begin{bmatrix} A_{00} & A_{0f} \\ \Pi_e^{-1} A_{f0} & \Pi_e^{-1} A_f \end{bmatrix} \\
A_{0f} &:= [\, A_{01} \quad \cdots \quad A_{0N} \,] \\
A_{f0} &:= [\, A_{10}^T \quad \cdots \quad A_{N0}^T \,]^T \\
A_f &:= \mathbf{block-diag}\,(\, A_{11} \quad \cdots \quad A_{NN} \,) \\
C &:= [\, C_0 \quad C_f \,] \\
C_0 &:= [\, C_{10}^T \quad \cdots \quad C_{N0}^T \,]^T \\
C_f &:= \mathbf{block-diag}\,(\, C_{11} \quad \cdots \quad C_{NN} \,) \\
D_e &:= \begin{bmatrix} D_0 \\ \Pi_e^{-1} D_f \end{bmatrix} \\
D_0 &:= [\, D_{01} \quad \cdots \quad D_{0N} \,] \\
D_f &:= \mathbf{block-diag}\,(\, D_{11} \quad \cdots \quad D_{NN} \,) \\
W &:= \mathbf{block-diag}\,(\, W_1 \quad \cdots \quad W_N \,) \\
V &:= \mathbf{block-diag}\,(\, V_1 \quad \cdots \quad V_N \,) \\
S &:= C^T V^{-1} C = \begin{bmatrix} S_{00} & S_{0f} \\ S_{0f}^T & S_f \end{bmatrix} \\
S_{00} &:= \sum_{j=1}^{N} C_{j0}^T V_j^{-1} C_{j0} \\
S_{0f} &:= [\, S_{01} \quad \cdots \quad S_{0N} \,] \\
&= [\, C_{10}^T V_1^{-1} C_{11} \quad \cdots \quad C_{N0}^T V_N^{-1} C_{NN} \,] \\
S_f &:= \text{block} - \text{diag}\,(\, S_{11} \quad \cdots \quad S_{NN} \,) \\
&= \mathbf{block-diag}\,(\, C_{11}^T V_1^{-1} C_{11} \quad \cdots \quad C_{NN}^T V_N^{-1} C_{NN} \,) \\
U_e &:= D_e W D_e^T \\
&= \begin{bmatrix} U_{00} & U_{0f} \Pi_e^{-1} \\ \Pi_e^{-1} U_{0f}^T & \Pi_e^{-1} U_f \Pi_e^{-1} \end{bmatrix} \\
U_{00} &:= \sum_{j=1}^{N} D_{0j} W_j D_{0j}^T \\
U_{0f} &:= [\, U_{01} \quad \cdots \quad U_{0N} \,] \\
&= [\, D_{01} W_1 D_{11}^T \quad \cdots \quad D_{0N} W_N D_{NN}^T \,] \\
U_f &:= \text{block} - \text{diag}\,(\, U_{11} \quad \cdots \quad U_{NN} \,) \\
&= \mathbf{block-diag}\,(\, D_{11} W_1 D_{11}^T \quad \cdots \quad D_{NN} W_N D_{NN}^T \,).
\end{aligned}$$

Since the matrices A_e and D_e contain the term of ε_i^{-1} order, a solution X_e of the MARE (6), if it exists, must contain terms of order ε_i. Taking this fact into consideration, we look for a solution X_e to the MARE (6) with the structure

$$\begin{aligned}
X_e &:= \begin{bmatrix} X_{00} & X_{0f} \\ X_{0f}^T & \Pi_e^{-1} X_f \end{bmatrix} \\
X_{00} &= X_{00}^T, \ \Pi_e^{-1} X_f = X_f^T \Pi_e^{-1} \\
X_{0f} &:= \begin{bmatrix} X_{01}^T \\ \vdots \\ X_{0N}^T \end{bmatrix}^T
\end{aligned}$$

and

$$X_f := \begin{bmatrix} X_{11} & \alpha_{21} X_{12} & \alpha_{31} X_{13} & \cdots & \alpha_{N1} X_{1N} \\ X_{12}^T & X_{22} & \alpha_{32} X_{23} & \cdots & \alpha_{N2} X_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ X_{1(N-1)}^T & X_{2(N-1)}^T & X_{3(N-1)}^T & \cdots & \alpha_{N(N-1)} X_{(N-1)N} \\ X_{1N}^T & X_{2N}^T & X_{3N}^T & \cdots & X_{NN} \end{bmatrix}.$$

In the following analysis, we need some assumptions.

Assumption 1: The pairs (A_{ii}, C_{ii}), $i = 1, 2, \ldots, N$ are detectable.

Assumption 2: The Hamiltonian matrices T_{ii}, $i = 1, 2, \ldots, N$ have not eigenvalues on the imaginary axis, where $T_{ii} := \begin{bmatrix} A_{ii}^T & -S_{ii} \\ -U_{ii} & -A_{ii} \end{bmatrix}$.

Assumption 3:

$$\text{rank} \begin{bmatrix} sI_{n_0} - A_{00}^T & -A_{f0}^T & C_0^T \\ -A_{0f}^T & -A_f^T & C_f^{Tt} \end{bmatrix} = \bar{n} \tag{7a}$$

$$\text{rank} \begin{bmatrix} sI_{n_0} - A_{00} & -A_{0f} & D_0 \\ -A_{f0} & -A_f & D_f \end{bmatrix} = \bar{n} \tag{7b}$$

with $\forall s \in \mathbf{C}$, Re$[s] \geq 0$ and $\bar{n} := \sum_{j=0}^{N} n_j$.

Before investigating the optimal Kalman filtering problem, we investigate the asymptotic structure of the MARE (6). In order to avoid the ill condition caused by the large parameter ε_i^{-1} which is included in the MARE (6), we introduce the following useful lemma [11].

Lemma 1: The MARE (6) is equivalent to the following generalized multiparameter algebraic Riccati equation (GMARE) (8)

$$\mathcal{F}(X) = AX^T + XA^T - XSX^T + U = 0 \tag{8}$$

where $A = \Phi_e A_e$, $U = \Phi_e U_e \Phi_e$ and $X = \Phi_e X_e$.

The GMARE (8) can be partitioned into

$$\begin{aligned}
f_1 = {} & A_{00} X_{00} + X_{00} A_{00}^T + A_{0f} X_{0f}^T + X_{0f} A_{0f}^T \\
& - X_{00} S_{00} X_{00} - X_{0f} S_f X_{0f}^T - X_{00} S_{0f} X_{0f}^T \\
& - X_{0f} S_{0f}^T X_{00} + U_{00} = 0
\end{aligned} \tag{9a}$$

$$\begin{aligned}
f_2 = {} & A_{0f} X_f^T + A_{00} X_{0f} \Pi_e + X_{00} A_{f0}^T + X_{0f} A_f^T \\
& - X_{00} S_{00} X_{0f} \Pi_e - X_{0f} S_{0f}^T X_{0f} \Pi_e - X_{00} S_{0f} X_f^T \\
& - X_{0f} S_f X_f^T + U_{0f} = 0
\end{aligned} \tag{9b}$$

$$\begin{aligned}
f_3 = {} & A_f X_f^T + X_f A_f^T + A_{f0} X_{0f} \Pi_e + \Pi_e X_{0f}^T A_{f0}^T \\
& - X_f S_f X_f^T - \Pi_e X_{0f}^T S_{0f} X_f^T - X_f S_{0f}^T X_{0f} \Pi_e \\
& - \Pi_e X_{0f}^T S_{00} X_{0f} \Pi_e + U_f = 0.
\end{aligned} \tag{9c}$$

It is assumed that the limit of α_{ij} exists as ε_i and ε_j tend to zero (see e.g., [4], [7]), that is

$$\bar{\alpha}_{ij} = \lim_{\substack{\varepsilon_j \to 0^+ \\ \varepsilon_i \to 0^+}} \alpha_{ij}. \tag{10}$$

Let $\bar{X}_{00}$, $\bar{X}_{f0}$, and $\bar{X}_f$ be the limiting solutions of the above (9) as $\varepsilon_i \to 0^+$, $\varepsilon_j \to 0^+$, $i, j = 1, \ldots, N$, then we obtain the following :

$$\begin{aligned}
& A_{00} \bar{X}_{00} + \bar{X}_{00} A_{00}^T + A_{0f} \bar{X}_{0f}^T + \bar{X}_{0f} A_{0f}^T \\
& \quad - \bar{X}_{00} S_{00} \bar{X}_{00} - \bar{X}_{0f} S_f \bar{X}_{0f}^T \\
& \quad - \bar{X}_{00} S_{0f} \bar{X}_{0f}^T - \bar{X}_{0f} S_{0f}^T \bar{X}_{00} + U_{00} = 0
\end{aligned} \tag{11a}$$

$$\begin{aligned}
& A_{0f} \bar{X}_f^T + \bar{X}_{00} A_{f0}^T + \bar{X}_{0f} A_f - \bar{X}_{00} S_{0f} \bar{X}_f^T \\
& \quad - \bar{X}_{0f} S_f \bar{X}_f^T + U_{0f} = 0
\end{aligned} \tag{11b}$$

$$A_f \bar{X}_f^T + \bar{X}_f A_f^T - \bar{X}_f S_f \bar{X}_f^T + U_f = 0 \tag{11c}$$

where

$$\bar{X}_f := \begin{bmatrix} \bar{X}_{11} & \bar{\alpha}_{21}\bar{X}_{12} & \bar{\alpha}_{31}\bar{X}_{13} & \cdots & \bar{\alpha}_{N1}\bar{X}_{1N} \\ \bar{X}_{12}^T & \bar{X}_{22} & \bar{\alpha}_{32}\bar{X}_{23} & \cdots & \bar{\alpha}_{N2}\bar{X}_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{X}_{1(N-1)}^T & \bar{X}_{2(N-1)}^T & \bar{X}_{3(N-1)}^T & \cdots & \bar{\alpha}_{N(N-1)}\bar{X}_{(N-1)N} \\ \bar{X}_{1N}^T & \bar{X}_{2N}^T & \bar{X}_{3N}^T & \cdots & \bar{X}_{NN} \end{bmatrix}$$

and

$$\bar{X}_{ii} = \bar{X}_{ii}^T, \qquad i = 0, 1, 2, \ldots, N. \tag{12}$$

Note that the ARE (11c) is asymmetric. However, it can be seen that the ARE (11c) admits at least a symmetric positive semidefinite stabilizing solution as follows.

Theorem 1: Under Assumptions 1 and 2, the ARE (11c) admits a unique symmetric positive semidefinite stabilizing solution $\bar{X}_f$ which can be written as

$$\bar{X}_f^* := \mathbf{block-diag}\,(\,\bar{X}_{11}^* \quad \cdots \quad \bar{X}_{NN}^*\,) \tag{13}$$

where $\bar{X}_{ii}^*$ is a unique symmetric positive semidefinite stabilizing solution, respectively, for the following AREs:

$$A_{ii}\bar{X}_{ii}^* + \bar{X}_{ii}^* A_{ii}^T - \bar{X}_{ii}^* S_{ii}\bar{X}_{ii}^* + U_{ii} = 0, \ i = 1, 2, \ldots, N. \tag{14}$$

Proof: Substituting (13) into the ARE (11c) as $\bar{X}_f \to \bar{X}_f^*$, it is easy to verify that $A_f\bar{X}_f^* + \bar{X}_f^* A_f^T - \bar{X}_f^* S_f \bar{X}_f^* + U_f = 0$. Furthermore, it can be seen that $\bar{X}_f^* = \bar{X}_f^{*T} \geq 0$ and the matrix

$$A_f - \bar{X}_f^* S_f = \mathbf{block-diag}\,(\,A_{11} - \bar{X}_{11}^* S_{11} \cdots A_{NN} - \bar{X}_{NN}^* S_{NN}\,)$$

is stable because $\bar{X}_{ii}^*$ is a unique symmetric positive semidefinite stabilizing solution under Assumptions 1 and 2. Consequently, there exists a unique solution of the ARE (11c) and its solution is (13) itself. ■

Remark 1: Since by Assumptions 1 and 2 the matrices $A_{ii} - \bar{X}_{ii}^* S_{ii}$ are stable, the unique solution of the ARE (11c) is given by (13) under $\bar{X}_{ij} \equiv 0$, $i = 1, 2, \ldots, N-1$, $j = 2, 3, \ldots, N$, $i < j$. Thus the parameters $\bar{\alpha}_{ij}$, $i = 1, 2, \ldots, N-1$, $j = 2, 3, \ldots, N$, $i < j$ do not appear in the (11). In other words, α_{ij} do not affect the (9) in the limit when ε_i tends to zero. In addition, by Assumption 3 the unique solution of the (11a) is guaranteed. Therefore, there exist the unique limit points of the solutions X_{00}, X_{0f} and X_f as the parameter ε_i tends to zero. Finally, the existence of the limits $\bar{X}_{00}, \bar{X}_{0f}, \bar{X}_f$ is guaranteed. For the basic idea of the proof, see for example [4].

Substituting the solution of (11c) into (11b) and substituting $\bar{X}_{0f}^*$ into (11a) and making some lengthy calculations (the detail is omitted for brevity), we obtain the following zeroth-order equations

$$\mathcal{A}\bar{X}_{00}^* + \bar{X}_{00}^*\mathcal{A}^T - \bar{X}_{00}^*\mathcal{S}\bar{X}_{00}^* + \mathcal{U} = 0 \tag{15a}$$

$$\bar{X}_{0f}^* = [\,-\bar{X}_{00} \quad I_{n_0}\,]\,H_2 H_4^{-1}\begin{bmatrix} I_{\bar{n}} \\ \bar{X}_f \end{bmatrix} \tag{15b}$$

$$A_f\bar{X}_f^* + \bar{X}_f^* A_f^T - \bar{X}_f^* S_f \bar{X}_f^* + U_f = 0 \tag{15c}$$

where

$$H_1\,(=T_{00}) := \begin{bmatrix} A_{00}^T & -S_{00} \\ -U_{00} & -A_{00} \end{bmatrix}$$

$$H_2 := \begin{bmatrix} A_{f0}^T & -S_{0f} \\ -U_{0f} & -A_{0f} \end{bmatrix}$$

$$H_3 := \begin{bmatrix} A_{0f}^T & -S_{0f}^T \\ -U_{0f}^T & -A_{f0} \end{bmatrix}$$

$$H_4 := \begin{bmatrix} A_f^T & -S_f \\ -U_f & -A_f \end{bmatrix}$$

$$H_0 := \begin{bmatrix} \mathcal{A}^T & -\mathcal{S} \\ -\mathcal{U} & -\mathcal{A} \end{bmatrix} = H_1 - H_2 H_4^{-1} H_3.$$

Note that Assumptions 1 and 2 ensures that the matrix $A_f - \bar{X}_f^* S_f$ is nonsingular because the matrices $A_{ii} - \bar{X}_{ii}^* S_{ii}$, $i = 1, 2, \ldots, N$ are nonsingular. Moreover, Assumptions 1 and 2 ensures that H_4 are also nonsingular because $\Omega^T H_4 \Omega = \mathbf{block-diag}\,(\,T_{11} \quad \cdots \quad T_{NN}\,)$, where

$$\Omega = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_{n_2} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{n_3} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & I_{n_N} & 0 \\ 0 & I_{n_1} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & I_{n_2} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{n_3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & I_{n_N} \end{bmatrix}. \tag{16}$$

It should be noted that if (14) has a stabilizing solution, then the Hamiltonian matrices T_{ii} have not eigenvalues on the imaginary axis and therefore these matrices are nonsingular.

In the following we established the relation between the GMARE (8) and the zeroth-order equations (15). Before doing that, we give the results for the ARE (15a).

Lemma 2: Under Assumptions 1–3, there exists a matrix $\mathcal{C} \in \mathbf{R}^{\bar{p} \times n_0}$, $\bar{p} := \sum_{j=1}^N p_j$, and a matrix $\mathcal{D} \in \mathbf{R}^{n_0 \times \bar{q}}$, $\bar{q} := \sum_{j=1}^N q_j$ such that $\mathcal{S} = \mathcal{C}^T V^{-1} \mathcal{C}$, $\mathcal{U} = \mathcal{D} W \mathcal{D}^T$. Moreover, the triple ($\mathcal{A}^T$, $\mathcal{C}^T$, $\mathcal{D}^T$) is stabilizable and detectable.

Proof: By using the similar technique done by the proof of [10], we give the proof for Lemma 2. It is easy to verify that $\mathcal{S} = (C_0 + C_f N_1^T)^T V^{-1}(C_0 + C_f N_1^T)$, where $N_1^T = -\Gamma_4^{-1}\Gamma_2$, $\Gamma_4 = A_f - \bar{X}_f^* S_f$, $\Gamma_2 = A_{f0} - \bar{X}_f^* S_{0f}^T$. Thus, we have $\mathcal{C} = C_0 + C_f N_1^T$. However, it seems difficult to find $\mathcal{D}$ from (15a). In order to do that, we introduce a dual ARE

$$A_f^T \hat{X}_f^* + \hat{X}_f^* A_f - \hat{X}_f^* U_f \hat{X}_f^* + S_f = 0 \tag{17}$$

where

$$\hat{X}_f^* := \mathbf{block-diag}\,(\,\hat{X}_{11}^* \quad \cdots \quad \hat{X}_{NN}^*\,)$$
$$A_{ii}^T \hat{X}_{ii}^* + \hat{X}_{ii}^* A_{ii} - \hat{X}_{ii}^* S_{ii} \hat{X}_{ii}^* + U_{ii} = 0, \qquad i = 1, 2, \ldots, N.$$

Note that there exists a symmetric positive semidefinite solution $\hat{X}_f^*$ under Assumptions 1 and 2. Using the above ARE (17), we find that

$$H_4 = \begin{bmatrix} I_{\hat{n}} & -\hat{X}_f^* \\ 0 & I_{\hat{n}} \end{bmatrix}\begin{bmatrix} \hat{A}_f & 0 \\ -U_f & -\hat{A}_f^T \end{bmatrix}\begin{bmatrix} I_{\hat{n}} & \hat{X}_f^* \\ 0 & I_{\hat{n}} \end{bmatrix} \quad \hat{n} = \sum_{j=1}^N n_j$$

where $\Xi_4 = A_f - U_f \bar{X}_f^*$ is stable under Assumptions 1 and 2. After the calculation of H_0, we arrive at another expression for $\mathcal{U}$, that is, $\mathcal{U} = U_{00} + M_1 U_{0f}^T + U_{0f} M_1^T + M_1 U_f M_1^T$, where $M_1 = -\Xi_2 \Xi_4^{-1}$, $\Xi_2 = A_{0f} - U_{0f}\bar{W}_f^*$. Hence, it is easy to find that $\mathcal{D} = D_0 + M_1 D_f$ because $\mathcal{U} = (D_0 + M_1 D_f) W (D_0 + M_1 D_f)^T = \mathcal{D} W \mathcal{D}^T$.

Let us now prove the stabilizability and detectability of the triple ($\mathcal{A}^T$, $\mathcal{C}^T$, $\mathcal{D}^T$). Note the relation

$$\begin{aligned} &\begin{bmatrix} I_{n_0} & -\Gamma_2^T \Gamma_4^{-T} \\ 0 & -\Gamma_4^{-T} \end{bmatrix}\begin{bmatrix} sI_{n_0} - A_{00}^T & -A_{f0}^T & C_0^T \\ -A_{0f}^T & -A_f^T & C_f^T \end{bmatrix} \\ &\cdot \begin{bmatrix} I_{n_0} & 0 & 0 \\ -\Gamma_4^{-T} A_{0f}^T & I_{\hat{n}} & \Gamma_4^{-T} C_f^T \\ \Lambda_1 & \Lambda_2 & \Lambda_3 \end{bmatrix} \\ &= \begin{bmatrix} sI_{n_0} - A_{00}^T - N_1 A_{0f}^T & 0 & \mathcal{C}^T \\ 0 & I_{\hat{n}} & 0 \end{bmatrix} \end{aligned} \tag{18}$$

where $\Lambda_1 = -V^{-1} C_f \bar{X}_f^* \Gamma_4^{-T} A_{0f}^T$, $\Lambda_2 = V^{-1} C_f \bar{X}_f^*$, $\Lambda_3 = I_{\bar{p}} + V^{-1} C_f \bar{X}_f^* \Gamma_4^{-T} C_f^T$.

It should be noted that the rank condition (7a) is satisfied if and only if the following condition holds rank $[\, sI_{n_0} - A_{00}^T - N_1 A_{0f}^T \quad \mathcal{C}^T \,] = n_0, \forall s \in \mathbf{C}$ with Re$[s] \geq 0$. Hence, let us prove the stabilizability of $(A_{00}^T + N_1 A_{0f}^T, \mathcal{C}^T)$. Since $\mathcal{A}^T = A_{00}^T + N_1 A_{0f}^T + \mathcal{C}^T V^{-1} C_f N_2^T = A_{00}^T + N_1 A_{0f}^T + \mathcal{C}^T \mathcal{K}$ and the feedback $\mathcal{K}$ does not change the stabilizability property of $(A_{00}^T + N_1 A_{0f}^T, \mathcal{C}^T)$, we arrive at the conclusion that the matrix pair $(\mathcal{A}^T, \mathcal{C}^T)$ is also stabilizable. Similarly, we can prove that $(\mathcal{A}^T, \mathcal{D}^T)$ is detectable if and only if rank condition (7b) is satisfied. The detail is omitted for brevity because of the duality arguments. Thereby, we have finished the proof of Lemma 2. ■

Since the triple $(\mathcal{A}^T, \mathcal{C}^T, \mathcal{D}^T)$ is stabilizable and detectable, the ARE (15a) admits a unique stabilizing positive semidefinite symmetric solution, denoted by $\bar{X}_{00}^*$ and $\mathcal{A} - \bar{X}_{00}^* \mathcal{S}$ is stable.

The limiting behavior of X_e as the parameter $\|\mu\| := \sqrt[N]{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_N} \to 0^+$ is described by the following lemma.

Lemma 3: Under Assumptions 1–3, there exists a small σ^* such that for all $\|\mu\| \in (0, \sigma^*)$ the MARE (6) admits a symmetric positive semidefinite stabilizing solution X_e which can be written as

$$X_e = \Phi_e^{-1} \begin{bmatrix} \bar{X}_{00}^* + O(\|\mu\|) & \bar{X}_{0f}^* + O(\|\mu\|) \\ \Pi_e \left\{ \bar{X}_{0f}^* + O(\|\mu\|) \right\}^T & \bar{X}_f^* + O(\|\mu\|) \end{bmatrix} = \begin{bmatrix} \bar{X}_{00}^* + O(\|\mu\|) & \bar{X}_{0f}^* + O(\|\mu\|) \\ \left\{ \bar{X}_{0f}^* + O(\|\mu\|) \right\}^T & \Pi_e^{-1} \left\{ \bar{X}_f^* + O(\|\mu\|) \right\} \end{bmatrix}. \tag{19}$$

Proof: We apply the implicit function theorem [4] to (9). To do so, it is enough to show that the corresponding Jacobian is nonsingular at $\|\mu\| = 0$. It can be shown, after some algebra, that the Jacobian of (9) in the limit as $\|\mu\| \to 0$ is given by(20), shown at the bottom of the page, where vec denotes an ordered stack of the columns of its matrix [8] and

$$\begin{aligned} \mathbf{J}_{00} &= \Gamma_1 \otimes I_{n_0} + I_{n_0} \otimes \Gamma_1, \mathbf{J}_{01} = \Gamma_3 \otimes I_{n_0} + (I_{n_0} \otimes \Gamma_3)\mathcal{U}_{n_0 \hat{n}} \\ \mathbf{J}_{10} &= \Gamma_2 \otimes I_{n_0} = (\Gamma_2 \otimes I_{n_0})\mathcal{U}_{n_0 n_0}, \mathbf{J}_{11} = \Gamma_4 \otimes I_{n_0} \\ \mathbf{J}_{12} &= (I_{\hat{n}} \otimes \Gamma_3)\mathcal{U}_{\hat{n}\hat{n}}, \ \mathbf{J}_{22} = \Gamma_4 \otimes I_{\hat{n}} + I_{\hat{n}} \otimes \Gamma_4 \\ \Gamma_1 &= A_{00} - \bar{X}_{00}^* S_{00} - \bar{X}_{0f}^* S_{0f}^T \Gamma_3 = A_{0f} - \bar{X}_{00}^* S_{0f} - \bar{X}_{0f}^* S_f \end{aligned}$$

where $\otimes$ denotes Kronecker products and $\mathcal{U}_{n_0 n_0}$ is the permutation matrix in the Kronecker matrix sense [8]. The Jacobian (20) can be expressed as

$$\det \mathbf{J} = \det \mathbf{J}_{22} \cdot \det \mathbf{J}_{11} \cdot \det \left[\Gamma_0 \otimes I_{n_0} + I_{n_0} \otimes \Gamma_0 \right] \tag{21}$$

where $\Gamma_0 := \Gamma_1 - \Gamma_2 \Gamma_4^{-1} \Gamma_3$. Obviously, $\mathbf{J}_{ii}, j = 1, 2$ are nonsingular because the matrix $\Gamma_4 = A_f - \bar{X}_f^* S_f$ is stable under Assumptions 1 and 2. After some straightforward but tedious algebra, we see that $\mathcal{A} - \bar{X}_{00}^* \mathcal{S} = \Gamma_1 - \Gamma_2 \Gamma_4^{-1} \Gamma_3 = \Gamma_0$. Therefore, the matrix Γ_0 is also stable if Assumption 3 holds. Thus, $\det \mathbf{J} \neq 0$, i.e., $\mathbf{J}$ is nonsingular at $\|\mu\| = 0$. The conclusion of Lemma 3 is obtained directly by using the implicit function theorem.

The remainder of the proof is to show that X_e is the positive semidefinite stabilizing solution. Firstly, let us prove the stability of the matrix $A_e - X_e S$. Using (19), we obtain $A_e - X_e S = \Phi_e^{-1} \left(\begin{bmatrix} \Gamma_1 & \Gamma_2 \\ \Gamma_3 & \Gamma_4 \end{bmatrix} + O(\|\mu\|) \right)$. The matrices Γ_4 and Γ_0 are stable since Assumptions 1–3 hold. Therefore, if parameter $\|\mu\|$ is very small, $A_e - X_e S$ is stable by applying the Theorem 1 in [7]. Secondly, from MARE (6), we get

$$(A_e - X_e S) X_e + X_e (A_e - X_e S)^T + X_e S X_e + U_e = 0.$$

Noting that $A_e - X_e S$ is stable, we can change the form of the above MARE.

$$X_e = \int_0^\infty \exp\left[(A_e - X_e S)\, t\right] (X_e S X_e + U_e) \cdot \exp\left[(A_e - X_e S)^T t \right] dt.$$

Therefore, $X_e \geq 0$ because the matrix $X_e S X_e + U_e$ is the positive semidefinite. This completes the proof of Lemma 3. ■

Remark 2: It seems that the result in Lemma 3 is typical for the case with one small parameter [13]. However, when the MARE (6) contains the several small singular perturbation parameters, there has been no result of the detailed structure for the MARE (6). Therefore, it is worth pointing out that the existing result has been extended to wider class.

III. Near-Optimal Kalman Filters for the Nonstandard MSPS

The required solution of the MARE (6) exists under Assumptions 1–3. Our attention is focused on the design of the near-optimal Kalman filters. Such the filters are obtained by eliminating $O(\|\mu\|)$ item of the filter gain matrix (5). If $\|\mu\|$ is very small, it is obvious that the near-optimal Kalman filters (5) can be approximated as

$$K^{\mathrm{app}} = X^{\mathrm{app}} C^T V^{-1} = \begin{bmatrix} \bar{X}_{00}^* & \bar{X}_{01}^* & \cdots & \bar{X}_{0N}^* \\ 0 & \bar{X}_{11}^* & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \bar{X}_{NN}^* \end{bmatrix} C^T V^{-1} \tag{22}$$

where

$$\bar{X}_{0i}^* := [\, -\bar{X}_{00} \quad I_{n_0} \,] T_{0i} T_{ii}^{-1} \begin{bmatrix} I_{\bar{n}} \\ \bar{X}_f \end{bmatrix}, T_{0i} := \begin{bmatrix} A_{i0}^T & -S_{0i} \\ -U_{0i} & -A_{0i} \end{bmatrix}.$$

When $\|\mu\|$ is sufficiently small, we know from Lemma 3 that the resulting filter gain (22) will be close to the optimal Kalman filter gain K^{opt} of the (5).

Theorem 2: Under Assumptions 1–3, the use of the near-optimal Kalman filter gain (22) results in

$$\text{Trace } W_e = \text{Trace } X_e + O(\|\mu\|) \tag{23}$$

where Trace X_e is the optimal steady-state mean-square error, while Trace W_e is the near-optimal steady-state mean-square error and W_e is a positive semidefinite solution of the following multiparameter algebraic Lyapunov equation (MALE):

$$(A_e - X_e^{\mathrm{app}} S) W_e + W_e (A_e - X_e^{\mathrm{app}} S)^T + X_e^{\mathrm{app}} S X_e^{\mathrm{app}} + U_e = 0 \tag{24}$$

with $X_e^{\mathrm{app}} := \Phi_e^{-1} X^{\mathrm{app}}$.

$$\mathbf{J} = \nabla \mathcal{F} = \left. \frac{\partial\,(\mathrm{vec} f_1, \mathrm{vec} f_2,\ \mathrm{vec} f_3)}{\partial\,(\mathrm{vec} X_{00}, \mathrm{vec} X_{0f}, \mathrm{vec} X_f)^T} \right|_{\|\mu\|=0,\ X_{00}=\bar{X}_{00}^*,\ X_{0f}=\bar{X}_{0f}^*,\ X_f=\bar{X}_f^*} = \begin{bmatrix} \mathbf{J}_{00} & \mathbf{J}_{01} & 0 \\ \mathbf{J}_{10} & \mathbf{J}_{11} & \mathbf{J}_{12} \\ 0 & 0 & \mathbf{J}_{22} \end{bmatrix} \tag{20}$$

$$\left(A_e - X_e^{(0)}S\right)\left(X_e^{(1)} - X_e\right) + \left(X_e^{(1)} - X_e\right)\left(A_e - X_e^{(0)}S\right)^T + \left(X_e - X_e^{(0)}\right)S\left(X_e - X_e^{(0)}\right) = \left(A_e - X_e^{\rm app}S\right)\left(X_e^{(1)} - X_e\right) + \left(X_e^{(1)} - X_e\right)\left(A_e - X_e^{\rm app}S\right)^T + \left(X_e - X_e^{\rm app}\right)S\left(X_e - X_e^{\rm app}\right) = 0.$$

Before proving this theorem, we introduce the following useful lemma [12].

Lemma 4: Consider the iterative algorithm which is based on the Kleinman algorithm

$$\left(A - X^{(n)}S\right)X^{(n+1)T} + X^{(n+1)}\left(A - X^{(n)}S\right)^T + X^{(n)}SX^{(n)T} + U = 0, \qquad n = 0, 1, 2, \ldots \quad (25)$$

where

$$X^{(0)} = X^{\rm app} = \begin{bmatrix} \bar{X}_{00}^* & \bar{X}_{0f}^* \\ 0 & \bar{X}_f^* \end{bmatrix}$$
$$X^{(n)} = \begin{bmatrix} X_{00}^{(n)} & X_{0f}^{(n)} \\ \Pi_e X_{0f}^{(n)T} & X_f^{(n)} \end{bmatrix}.$$

Under Assumptions 1–3, there exists a small $\bar{\sigma}$ such that for all $\|\mu\| \in (0, \bar{\sigma})$, $\bar{\sigma} \le \sigma^*$ the iterative algorithm (24) converges to the exact solution of X_e with the rate of quadratic convergence, where $X_e^{(n)} = \Phi_e^{-1}X^{(n)} = X^{(n)T}\Phi_e^{-1}$. That is, the following conditions are satisfied:

$$\|X^{(n)} - X\| = O\left(\|\mu\|^{2^n}\right), \qquad n = 0, 1, 2, \ldots. \quad (26)$$

Now, let us prove Theorem 2.

Proof: Subtracting (6) from (24) we find that $V_e = W_e - X_e$ satisfies the following MALE:

$$\left(A_e - X_e^{\rm app}S\right)V_e + V_e\left(A_e - X_e^{\rm app}S\right)^T + \left(X_e - X_e^{\rm app}\right)S\left(X_e - X_e^{\rm app}\right) = 0. \quad (27)$$

Similarly, subtracting (6) from (25) we also get the MALE

$$\left(A_e - X_e^{(n)}S\right)\left(X_e^{(n+1)} - X_e\right) + \left(X_e^{(n+1)} - X_e\right) \cdot \left(A_e - X_e^{(n)}S\right)^T + \left(X_e - X_e^{(n)}\right)S\left(X_e - X_e^{(n)}\right) = 0 \quad (28)$$

where $X_e^{(n)} = \Phi_e X^{(n)}$. When $n = 0$, we have the equation shown at the top of the page. Therefore, it is easy to verify that $V_e = X_e^{(1)} - X_e$ [9] because $A_e - X_e^{\rm app}S$ is stable from Theorem 1 in [7]. Consequently, we obtain that

$$\|V_e\| = \|W_e - X_e\| = \left\|X_e^{(1)} - X_e\right\| \le \left\|\Phi_e^{-1}\right\| \cdot \left\|X^{(1)} - X\right\| \le \|\mu\|^{-1} \cdot \left\|X^{(1)} - X\right\| = O(\|\mu\|).$$

Hence $V_e = W_e - X_e = O(\|\mu\|)$, which implies (23). ■

Remark 3: Taking the special case of $n = 0$ into account, Theorem 2 has a simpler proof via implicit function theorem. The proof can be done by using a similar technique as in [10].

IV. Conclusion

The near-optimal Kalman filtering problem for MSPS has been investigated. The new design method of the near-optimal Kalman filters has been proposed. As a result, solving the high-dimensional ill-conditioned MARE has been replaced by solving the low-order singular perturbation parameter independent ARE. Furthermore, the proposed filters can be implemented even if the fast state matrices are singular.

Acknowledgment

The author would like to thank one anonymous reviewer whose comments greatly improved this presentation.

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